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Reliability-based topology optimization

G. Kharmanda, N. Olhoff, A. Mohamed and M. Lemaire

Abstract The objective of this work is to integrate reliability analysis into topology optimization problems. The new model, in which we introduce reliability constraints into a deterministic topology optimization formulation, is called Reliability-Based Topology Optimization (RBTO). Several applications show the importance of this integration. The application of the RBTO model gives a different topology relative to deterministic topology optimization. We also find that the RBTO model yields structures that are more reliable than those produced by deterministic topology optimization (for the same weight).

Key words reliability-based topology optimization, reliability-based design optimization, reliability analysis, sensitivity analysis, finite element analysis

1 Introduction

Reliability-based optimization aims to define the best compromise between cost and safety. One reliability-based model is Reliability-Based Design Optimization (RBDO), which allows structures to be designed. For the RBDO model, the coupling between geometrical modeling, mechanical simulation, reliability analyses, and optimization methods leads to very long computing times and weak convergence stability. Traditionally, the solution of the RBDO model is achieved by alternating reliability and optimization iterations (sequential approach). This approach leads to low numerical efficiency, which is disadvantageous for engineering applications on real structures. In order to avoid this difficulty, an efficient method was proposed by Kharmanda *et al.* (2001a, 2002c), which is called the hybrid (or concurrent) RBDO method. This method is based on the simultaneous solution of the reliability and the design optimization problem.

Furthermore, in the shape optimization case, when integrating the reliability analysis into the optimization of geometrical shape variables, CAD model updating is necessary during the design phase. Then, the parametrization step allows the search directions of the optimization process to be defined. Parameters defining points and directions are chosen among the design variables that define the geometry of the domain boundary. The shape optimization process is directed by the information corresponding to the geometrical boundary perturbation. The structural geometry to be modified during the optimization process can be defined by several descriptions, such as Bezier, B-spline, or NURBS models (Kharmanda *et al.* 2002a,b). This model is called Reliability-Based Shape Optimization (RBSO).

In deterministic topology optimization, the designer aims to obtain the solution without taking into account the effects of uncertainties concerning geometry and loading given by deterministic variables. The integration of the reliability or safety criteria into the topology optimization presents a new type of optimization called Reliability-Based Topology Optimization (RBTO) (Kharmanda and Olhoff 2001b). The purpose of RBTO is to take into account the randomness of the applied loads and the description of the geometry. It also provides the designer with several solutions. Using the RBTO model, we obtain different topologies compared with the deterministic topology optimization procedure. The resulting topologies depend on the target reliability levels. The shape optimization algorithm can be applied to better control the solution. The resulting structures present

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G. Kharmanda[1], N. Olhoff[2,✉], A. Mohamed[3] and M. Lemaire[3]

[1] Faculty of Mechanical Engineering, Aleppo, Syria
e-mail: gh_kharmanda@hotmail.com
[2] Institute of Mechanical Engineering, Aalborg University, DK-9220, Aalborg East, Denmark
e-mail: no@ime.auc.dk
[3] LaRAMA-IFMA/UBP, Campus de Clermont-Ferrand, BP 265, 63175 Aubiere, France
e-mail: alaa.mohamed@ifma.fr, maurice.lemaire@ifma.fr

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a better volume/reliability ratio than the deterministic ones. In this paper, we first present the reliability analysis and the importance of its integration with the topology optimization problem. The deterministic topology optimization formulation and the RBTO model will be presented next. The RBTO procedure will then be detailed and several examples will show the applicability of the new model. Some reliability-based topologies for different reliability levels will finally be presented as new resulting topologies.

2 Reliability analysis

The design of structures and the prediction that they function well lead to the verification of a number of rules resulting from the knowledge of physical and mechanical experience by designers and constructors. These rules explain the necessity to limit loading effects such as stresses and displacements. Each rule represents an elementary event and the occurrence of several events leads to a failure scenario. The objective is then to evaluate the failure probability corresponding to the occurrence of critical failure modes.

2.1 Importance of safety criteria

In deterministic structural optimization, the designer aims to reduce the construction cost without taking the effects of uncertainties concerning materials, geometry, and loading into account. In this way, the resulting optimal configuration may represent a lower reliability level and then lead to a higher failure rate. The balance between cost minimization and reliability maximization is a great challenge to the designer. The interest in reliability criteria in design optimization is to improve the reliability level of the system without increasing its weight significantly. But when integrating reliability into topology optimization problems, the interest is to provide the designer with several topologies by considering the randomness of the main variables of the structure. In RBDO models, we distinguish between two kinds of variables:

1) The design variables $\mathbf{x}$, which are deterministic variables to be defined in order to optimize the design. They represent the control parameters of the mechanical system (e.g. dimensions, materials, loads) and of the probabilistic model (e.g. means and standard-deviations of the random variables);
2) The random variables $\mathbf{y}$, which represent the structural uncertainties, identified by probabilistic distributions. These variables can be geometrical dimensions, material characteristics, or the applied external loading. These two kinds of variables ($\mathbf{x}$ and $\mathbf{y}$) can be related by a probabilistic transformation.

However, in the RBTO model, the random and design variables are not related, because the design variables $\mathbf{x}$ are the densities of material of the discretization elements, while the random variables $\mathbf{y}$ are related to known quantities. Hence, we have three kinds of variables:

1) The deterministic topology design variables $\mathbf{x}$, which are deterministic variables;
2) The random variables $\mathbf{y}$, which represent the structural uncertainties, identified by probabilistic distributions. These variables can be geometrical dimensions, material characteristics, or the applied external loading;
3) The normalized variables $\mathbf{u}$, which relate the random variables and their mean values and standard-deviations.

2.2 Failure probability

In addition to the vector of deterministic variables $\mathbf{x}$ to be used in the system design and optimization, the uncertainties are modeled by a vector of stochastic physical variables affecting the failure scenario. The knowledge of these variables is not, at best, more than statistical information and we admit a representation in the form of random variables. For a given design rule, the basic random variables are defined by their joint probability distribution associated with some expected parameters; the vector of random variables is denoted herein by $\mathbf{Y}$, the realizations of which are written as $\mathbf{y}$. Safety is the state in which the structure is able to fulfil all the functioning requirements (e.g. strength and serviceability) for which it is designed. To evaluate the failure probability with respect to a chosen failure scenario, a limit state function G($\mathbf{x}$,$\mathbf{y}$) is defined by the condition of good functioning of the structure. In Fig. 1, the limit between **the state of failure** $G(\mathbf{x},\mathbf{y}) < 0$ and **the state of safety** $G(\mathbf{x},\mathbf{y}) > 0$ is known as the **limit state surface** $G(\mathbf{x},\mathbf{y}) = 0$. The failure probability is then calculated by

$$P_f = P_r\left[G\left(\mathbf{x},\mathbf{y}\right) \le 0\right] = \int_{G(\mathbf{x},\mathbf{y})\le 0} f_{\mathbf{Y}}(\mathbf{y})\ \mathrm{d}y_1 \cdots \mathrm{d}y_n\,, \quad (1)$$

where P_f is the failure probability, $f_Y(\mathbf{y})$ is the joint density function of the random variables $\mathbf{Y}$, and $P_r[.]$ is the probability operator. The evaluation of the integral in (1) is not easy, because it represents a very small quantity and all the necessary information for the joint density function are not available. For these reasons, the First and Second Order Reliability Methods FORM/SORM (Ditlevsen and Madsen 1996) were developed. They are based on the reliability index concept, followed by an estimation of the failure probability. The invariant reliability index β was introduced by Hasofer and Lind (1974), who proposed working in the space of standard independent Gaussian variables instead of the space of physical variables.

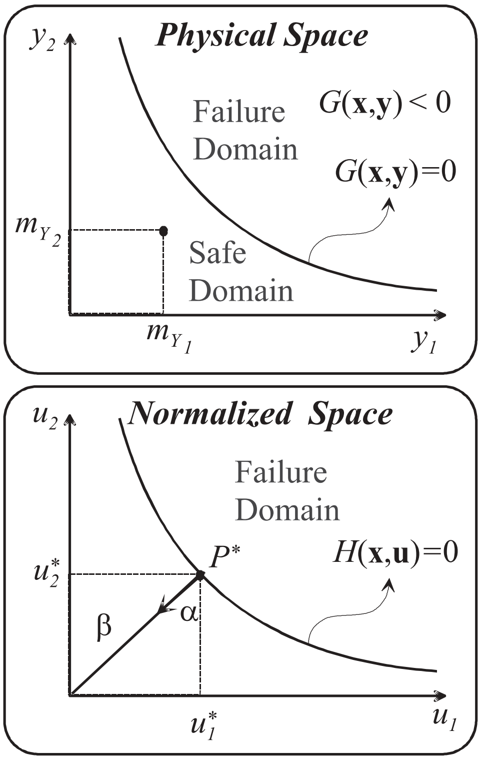

Fig. 1 Physical and normalized spaces

The transformation from the physical variables $\mathbf{y}$ to the normalized variables $\mathbf{u}$ is given by

$$\mathbf{u} = T(\mathrm{x}, \mathrm{y}) \quad \text{and} \quad \mathbf{y} = T^{-1}(\mathrm{x}, \mathrm{u})\,.$$

The operator $T(.)$ is called the *probabilistic transformation*. In this standard space, the limit state function takes the form

$$H(\mathbf{x}, \mathbf{u}) \equiv G(\mathbf{x}, \mathbf{y}) = 0\,. \tag{2}$$

In the FORM approximation, the failure probability is simply evaluated by:

$$P_f \approx \Phi(-\beta)\,, \tag{3}$$

where $\Phi(.)$ is the standard Gaussian cumulated function. For practical engineering, (3) gives a sufficiently accurate estimation of the failure probability.

2.3 Reliability evaluation

For a given failure scenario, the reliability index β is evaluated by solving a constrained optimization problem (Fig. 1).

The calculation of the reliability index can be realized by the following form:

$$\beta = \min\left(\sqrt{\mathbf{u}^T\mathbf{u}}\right) \quad \text{subject to} \quad H(\mathbf{x}, \mathbf{u}) \, \leq \, 0\,. \tag{4}$$

The solution of this problem is called the ***design point*** P^*, as illustrated in Fig. 1. When the mechanical model is defined by numerical methods such as the finite element method, the evaluation of the reliability implies a special coupling procedure between both reliability and mechanical models (Lemaire and Mohamed 2000).

3 Deterministic topology optimization

Deterministic topology optimization has a great impact on the performance of structures, and the last decade has seen an enormous interest in this important sub-area of structural optimization (Bendsøe and Kikuchi 1988; Bendsøe 1995; Olhoff *et al.* 1998; Rozvany 2000; Olhoff 2000; Rozvany and Olhoff 2000; Eschenauer and Olhoff 2001). For deterministic topologies, we can find several approaches for solving topology optimization problems. The homogenization approach was used in Bendsøe and Kikuchi (1988), based on studies of the existence of solutions. This approach has been adopted in many papers, but has the disadvantage that the determination and the evaluation of optimal microstructures and their orientations is cumbersome if not unresolved (for other than compliance problems) and, furthermore, the resulting structures cannot be built since no definite length-scale is associated with the microstructures. However, the homogenization approach to topology optimization is still important in the sense that it can provide bounds on the theoretical performance of structures. A topology optimization problem based on the homogenization approach, with the objective of minimizing the compliance (thereby maximizing the integral structural stiffness), subject to an upper bound on the volume, can be written as

$$\begin{array}{ll} \text{min} & L\,(w) \\ \text{subject to} & a_d\,(w, v) = L(v) \quad \text{for all} \quad v \in H \\ \text{and} & \text{volume} \leq V_0 \end{array} \tag{5}$$

with

$$L(v) = \int\limits_{\Omega} f\ v\ \mathrm{d}\Omega + \int\limits_{\Gamma} t\ v\ \mathrm{d}\Gamma \tag{6}$$

and

$$a_d(w, v) = \int\limits_{\Omega} C_{ijkl}(d)\varepsilon_{ij}(w)\varepsilon_{kl}(v)\,\mathrm{d}\Omega\,, \tag{7}$$

where f and t are, respectively, the body force and surface traction, ε_{ij} is the strain tensor, C_{ijkl} is the effective stiffness tensor of the microstructure cells, and H is the set of kinematically admissible displacement fields. The problem is defined on a fixed reference domain Ω and the stiffness C_{ijkl} depends on the design variables used.

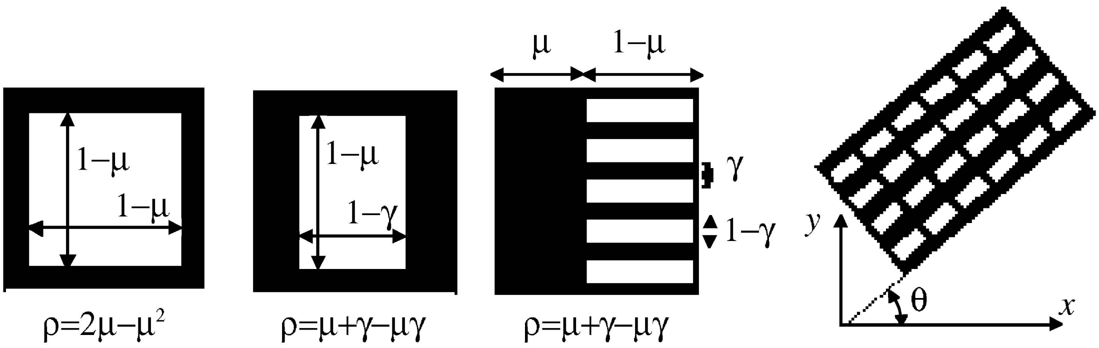

Fig. 2 Density variables in microstructure cells

For a so-called second-rank layering, such as the third cell in Fig. 2, we have the relationship

$$C_{ijkl} \equiv C_{ijkl}(\mu, \gamma, \theta)\,, \tag{8}$$

where μ and γ denote the dimensions of the layering and θ is the rotation angle of the layering. The relation (8) can be computed analytically and the volume is evaluated by

$$\text{volume} = \int_{\Omega} (\mu + \gamma - \mu\gamma)\,\mathrm{d}\Omega\,. \tag{9}$$

Alternative microstructures such as square or rectangular holes in square cells can also be used (such as the first two cells in Fig. 2), the important feature being the possibility of having density values covering the full interval [0,1]. The optimization problem can now be solved either by optimality criteria methods or by duality methods, where the advantage is to take into account the fact that the problem has just one constraint. The angle θ of layer rotation is controlled via the results on the optimal rotation of orthotropic materials as presented in Pedersen (1989, 1991) and Hassani and Hinton (1999).

Another approach is called the SIMP approach (Solid Isotropic Microstructure with Penalty) or power-law approach (Bendsøe 1989; Zhou and Rozvany 1991; Mlejnek 1992). Here, material properties are assumed to be constant within each element used to discretize the design domain and the variables are the relative densities of the elements. The material properties are modeled as the relative material properties of the solid material. This approach has been criticized, since it was argued that no physical material exists with properties described by the power-law interpolation. However, a recent paper by Bendsøe and Sigmund (1999) proved that the power-law approach is physically permissible as long as simple conditions for the power are satisfied (e.g. $p \geq 3$ for Poisson's ratio equal to 1/3). To ensure the existence of solutions, the power-law approach to topology optimization has been applied to problems with multiple constraints, multiple physics, and multiple materials.

A topology optimization problem based on the power-law approach, in which the objective is to minimize compliance, can be written as

$$\begin{aligned} \min \quad & C(\mathbf{x}) = \mathbf{q}^T\mathbf{K}\mathbf{q} = \sum_{e=1}^{N} (x_e)^p q_e^T k_0 q_e \\ \text{subject to} \quad & \frac{V(\mathbf{x})}{V_0} \leq f\,, \\ & \mathbf{K}\mathbf{q} = \mathbf{F}\,, \\ & \mathbf{0} < \mathbf{x}_{\min} \leq \mathbf{x} \leq \mathbf{1}\,, \end{aligned} \tag{10}$$

where $\mathbf{q}$ and $\mathbf{F}$ are the global displacement and force vectors, respectively, $\mathbf{K}$ is the global stiffness matrix, q_e and k_0 are the element displacement vector and stiffness matrix, respectively, $\mathbf{x}$ is the vector of design variables, $\mathbf{x}_{\min}$ is a vector of minimum relative densities (non-zero to avoid singularity), N is the number of elements to discretize the design domain, p is the penalization power, $V(\mathbf{x})$ and V_0 are the material volume and design domain volume, respectively, and f is a prescribed upper bound on the volume fraction. Whereas the solution of the above-mentioned approaches is based on mathematical programming techniques and continuous design variables, a number of papers have appeared on solving the topology optimization problem as an integer problem. Beckers successfully solved large-scale compliance minimization problems (Beckers 1999) using a dual-approach, but other approaches based on genetic algorithms or other heuristic approaches require thousands of function evaluations even for a small number of elements and must be considered impractical.

4 RBTO

Size, shape, and topology optimization problems address different aspects of a structural design problem. In a typical sizing problem the goal may be to find the optimal thickness distribution of a segmented linearly elastic plate. The optimal thickness distribution minimizes (or maximizes) a physical quantity such as the compliance (external work), peak stress, deflection, etc., while equilibrium and other constraints on the state and design variables are satisfied. The main feature of the sizing problem is that the domain of the design model and state variables is known a priori and is fixed throughout the optimization process.

On the other hand, in a shape optimization problem the goal is to find the optimum shape of this domain, that is, the shape problem is defined on a variable domain. Topology optimization of solid structures involves the determination of features such as the number and location of holes and the connectivity of the domain. The layout problem that shall be defined in the following combines several features of the traditional problems in structural design optimization (Bendsøe 1995). However, the integration of the reliability analysis in each step of the structural design optimization plays an important role by considering the variability of the most sensitive variables. This variability or randomness has to be considered in the optimization processes in order to reduce the structural weight in the uncritical regions and hence to produce structures that are reliable and economic. The applied loads are very often considered as random variables because they participate strongly in the failure or damage. Similarly, the geometry and the materials can be modeled as random variables. The purpose of reliability-based layout optimization is to find the reliable and optimal layout of a structure within a specified region. In the deterministic design optimization problem, the only known quantities are the applied loads, the possible support conditions, the volume of the structure to be constructed, and possibly some additional design specifications such as the location and size of prescribed holes. But the physical size and shape and the connectivity of the structure are unknown.

When introducing the reliability, the user chooses some known quantities that will be random variables. They are initially set to their mean values. Then they take random realizations during the optimization process.

4.1 Formulation

The main difference between the deterministic topology optimization procedure and the proposed RBTO model is that the randomness (variability) of the most important variables that exhibit a strong influence on the resulting optimal topology is taken into account. The deterministic topology problem allows for the prediction of the gross shape of the body and it is possible to predict the placement and shapes of holes in the structure. However, the RBTO model leads to a different set of optimal topologies with respect to that produced by the deterministic topology optimization procedure. In order to control the topologies produced, a reliability index β (see Hasofer and Lind 1974) is introduced with a normalized vector $\mathbf{u}$. In the case of a normal distribution, $\mathbf{u}$ is given by

$$u_j = \frac{y_j - m_{y_j}}{\sigma_{y_j}} \,, \tag{11}$$

with

$$\beta = \min \sqrt{u_1^2 + ... + u_j^2 + ... + u_J^2} \quad \text{subject to} \quad G \leq 0 \,,$$

where y_j is the j-th random variable, with mean value m_{y_j} and standard-deviation σ_{y_j}, and G is the limit state function. J is the number of selected random variables. In (11), the vector $\mathbf{u}$ defines the relationship between the random variables and the design variables.

The RBTO problem consists of minimizing the compliance subject to a given upper bound on the volume of material and the reliability constraints. The material density is used as a continuous design variable. The RBTO formulation based on the homogenization approach can be expressed as

$$\begin{aligned} &\text{min} && L\,(w) \\ &\text{subject to} && \beta(u) \geq \beta_t \,, \\ &&& a_d\,(w,v,u) = L(v,u) \quad \text{for all} \quad v \in H \,, \\ &&& \text{volume} \leq V_0 \,, \end{aligned} \tag{12}$$

with

$$L(v,u) = \int_{\Omega} f(u)\; v\; \mathrm{d}\Omega + \int_{\Gamma} t(u)\; v\; \mathrm{d}\Gamma$$

and

$$a_d(w,v,u) = \int_{\Omega} C_{ijkl}(d,u)\varepsilon_{ij}(w,u)\varepsilon_{kl}(v)\,\mathrm{d}\Omega \,.$$

However, using the SIMP approach, the problem can be written as

$$\begin{aligned} &\text{min} && C(\mathbf{x}) = \mathbf{q}^T\mathbf{K}\mathbf{q} = \sum_{e=1}^{N} (x_e)^p q_e^T k_0 q_e \\ &\text{subject to} && \beta(\mathbf{u}) \geq \beta_t \,, \\ &&& \mathbf{K}(\mathbf{x},\mathbf{y},\mathbf{u}) \cdot \mathbf{q}(\mathbf{x},\mathbf{y},\mathbf{u}) = \mathbf{F}(\mathbf{y},\mathbf{u}) \,, \\ &&& \frac{V(\mathbf{x},\mathbf{y},\mathbf{u})}{V_0} \leq f \,, \\ &&& \mathbf{0} < \mathbf{x}_{\min} \leq \mathbf{x} \leq \mathbf{1} \,, \end{aligned} \tag{13}$$

where β and β_t are the reliability index of the system and the target reliability index, respectively. (12) and (13) define problems called Reliability-Based Topology Optimization (RBTO) problems. For example, in (12) and (13), the loading and the geometry play an important role in the performance of the structure. Therefore, using the RBTO model, we consider the dimensions and loads as random variables. In this work, we apply the RBTO formulation in (13) based on the SIMP approach. The design variables $\mathbf{x}$ are the densities of material in each of the finite elements, while the random variables $\mathbf{y}$ are the external loads and the geometric dimensions. Since the volume fraction of the structure directly depends on the geometric dimensions, it can be included in the set of selected random variables.

4.2
RBTO algorithm

Figure 3 presents the RBTO algorithm as a sequential procedure that constitutes an iterative loop that is repeated until convergence. The RBTO model contains three principal successive processes: sensitivity analysis, reliability index evaluation, and the topology optimization process. Each one of these processes is considered as an independent loop that does not lead to a long computational time relative to the deterministic topology optimization scheme. This procedure is more appropriate than the coupled procedures used in the RBDO model, which contains nested optimization and reliability loops.

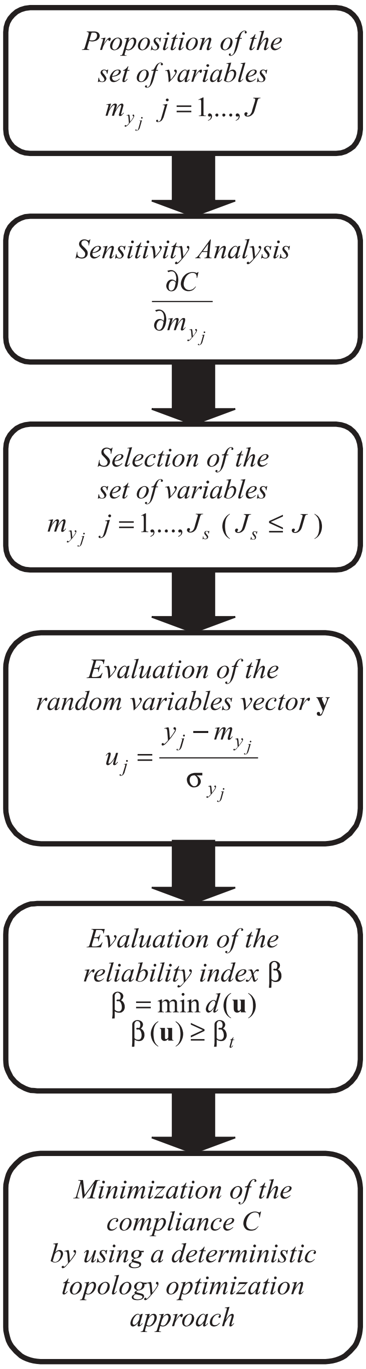

Fig. 3 RBTO procedure constituting an iterative loop that is repeated until convergence

In the RBDO field, several papers have studied the integration of reliability analysis into optimization problems (Stevenson 1967; Moses 1977; Feng and Moses 1986; Cheng *et al.* 1998). All the uncertain quantities can be modeled as random variables, and two spaces (Fig. 1) are required to carry out this integration. Hence, a lot of numerical computations are required in the space of the random variables in order to evaluate the system reliability. Furthermore, the optimization process itself is executed in the space of the design variables, which is deterministic. Consequently, in order to search for an optimal structure, the design variables are repeatedly changed, and each set of design variables corresponds to a new random variable space, which then needs to be manipulated to evaluate the structural reliability at that point. Because too many repeated searches are needed in the above two spaces, the computational time for such an optimization is a big problem.

When applying the principle of this approach to topology optimization, the problem becomes bigger because the computational time will increase significantly and the reliability analysis in each iteration of the topology optimization procedure will represent a very complex task. Thus, we define a new different strategy, which implies a coupling between the reliability analysis and the topology design problem without increasing the computational time.

First, we propose a set of variables assembled in the vector $\mathbf{m}_y$, which will be called the mean variable vector, and which concerns the applied loads and geometry of the structure. But, in order to select the most efficient variables, we study the sensitivity of the objective function with respect to the means of the proposed set of variables. The selected variables are assembled in the random variables vector $\mathbf{y}$. Secondly, the reliability index β is evaluated in satisfying the associated reliability constraint, and the resulting normalized vector $\mathbf{u}$ is used to formulate the random variables vector $\mathbf{y}$. Finally, using the resulting vector $\mathbf{y}$, we apply the SIMP approach to obtain the new reliable and optimal topology (Fig. 3).

4.2.1
Sensitivity analysis

In general, reliability-based optimization can be considered as a multi-objective (or -constraint) optimization because we have a principal objective function (cost, volume, etc.) and an associated one (reliability index or failure probability). In the case of our RBTO model, the proposed variables that are random play an important role in the reliability index function, but they do not necessarily play the same role in the compliance function.

Therefore, it is necessary to find the sensitivity of the compliance function with respect to the chosen variables.

Furthermore, the signs of the resulting gradients show the influence of each variable (negative or positive) on the objective function. The sensitivity analysis step can then improve the performance of the structure during the optimization process in order to yield a more reliable structure.

The sensitivity of the compliance with respect to the chosen means $\mathbf{m}_y$ of the geometry and applied loads can be calculated by several methods. The simplest one is the finite difference approach, e.g. considering that $\frac{\Delta m_{y_j}}{m_{y_j}} = 0.01$.

Using the classical finite difference approach, we can write

$$\frac{\partial C}{\partial m_{y_j}} = \frac{\Delta C}{\Delta m_{y_j}} = \frac{C\left(m_{y_j} + \Delta m_{y_i}\right) - C\left(m_{y_j}\right)}{\Delta m_{y_j}} \,. \qquad (14)$$

(14) is very simple to implement to provide the new set of selected variables.

4.2.2 Reliability index evaluation

The evaluation of the reliability index β can be carried out by a particular optimization procedure. This index is the minimum distance between the limit state and the origin in the normalized space (Fig. 1). The problem is given by

$$\beta = \min d(\mathbf{u}) = \sqrt{\sum u_j^2} \quad \text{subject to} \quad \beta(\mathbf{u}) \geq \beta_t \,. \qquad (15)$$

During the optimization procedure, we can analytically provide the derivative of the distance d with respect to u_j by

$$\frac{\partial d}{\partial u_j} = \frac{u_j}{d(\mathbf{u})} \,. \qquad (16)$$

The resulting vector $\mathbf{u}$ of the problem in (15) will be used to evaluate the random vector $\mathbf{y}$ by using (11) with the standard deviations given by $\sigma_{y_j} = 0.1 \times m_{y_j}$ and by considering that u_j has the same sign as the corresponding gradient $\partial C/\partial m_{y_j}$.

4.2.3 Topology optimization procedure

After satisfying the reliability constraints and determining the vector of the random variables, we call the topology optimization procedure with a new set of known quantities. The resulting optimal topology principally depends on the reliability index value. This procedure satisfies the other constraints but with consideration of the randomness of the principal variables of the structure. Figure 3 shows the different steps of the algorithm of the new Reliability-Based Topology Optimization model.

5 Applications

In order to illustrate the functioning and the importance of the proposed new model, we apply the deterministic topology optimization procedure and the RBTO model to several examples shown in Figs. 4a, 5a, 6a, and 7a. The first and the second examples represent a MBB-beam (see Olhoff *et al.* 1991) and a cantilever beam under a single external load (Figs. 4a and 5a, respectively). The third example is a cantilever beam with two load-cases (Fig. 6a), and the fourth example represents a cantilever beam with a prescribed hole (Fig. 7a). The objective of the following presentation is to show now the difference

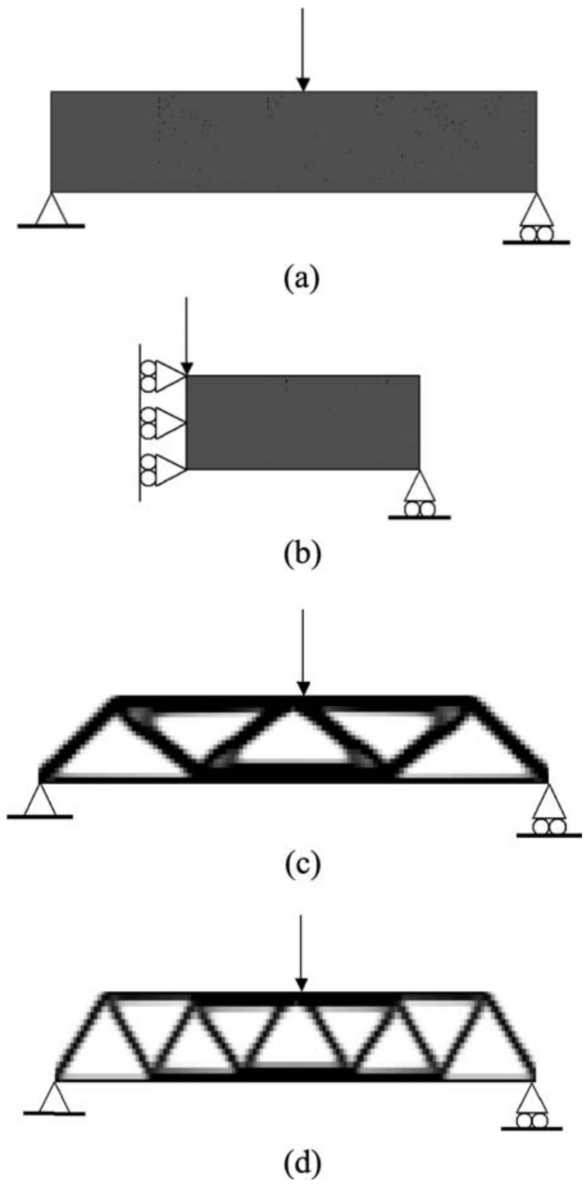

Fig. 4 Example 1, topology optimization and RBTO of the MBB-beam: (a) full design domain, (b) half design domain with symmetry boundary conditions, (c) resulting deterministic topology optimized beam, and (d) resulting RBTO structure

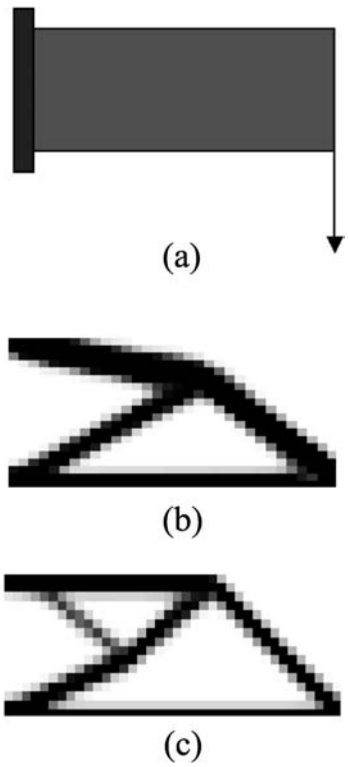

Fig. 5 Example 2, topology optimization of a cantilever beam: (a) design domain, (b) deterministic topology optimized beam, and (c) corresponding reliability-based topology

between the deterministic topologies and those obtained by the proposed new model.

5.1 Topology optimization approach

Sigmund (2001) presented a compact Matlab implementation of the deterministic topology optimization code for compliance minimization of statically loaded structures.

This Matlab code can be downloaded from the website http://www.topopt.dtu.dk. A number of simplifications are introduced to facilitate the implementation in Matlab. First, the design domain is assumed to be rectangular and discretized by square finite elements. This way the numbering of elements and nodes is simple (column by column starting in the upper left corner), and the aspect ratio of the structure is given by the ratio of elements in the horizontal and the vertical direction. A topology optimization problem based on the SIMP approach is implemented using (10). The optimization problem is solved by the use of a standard optimality criteria method. The main program is called from the Matlab prompt by the line

top(nelx,nely,volfrac,penal,rmin) ,

where *nelx* and *nely* are the numbers of elements in the horizontal and vertical directions, respectively, *volfrac* is the volume fraction, *penal* is the penalization power, and *rmin* is the filter size (divided by element size). Figures 4c, 5b, 6b, and 7b show the resulting deterministic topologies obtained by this code and with the data presented in Table 1.

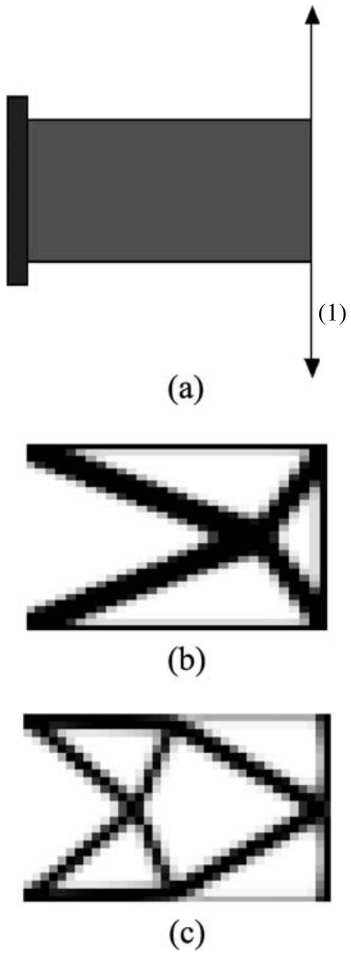

Fig. 6 Example 3, topology optimization of a cantilever beam with two load-cases: (a) design domain, (b) deterministic topology optimized beam, and (c) optimized beam with RBTO model using two load-cases

5.2 RBTO approach

In order to implement the RBTO model, we decided to create a new subroutine integrating the reliability constraint into a standard topology optimization program. For simplicity, we used the Matlab code written by Sigmund (2001). The main RBTO program is called from the Matlab prompt by the line

RBTO(nelx,nely,volfrac,penal,rmin,Bet) ,

where *Bet* is the target reliability level (or the variability measure). This subroutine, which is given in the Appendix, provides the optimal normalized vector **u** in satisfying the reliability constraint $\beta \geq \beta_t$. It also calculates

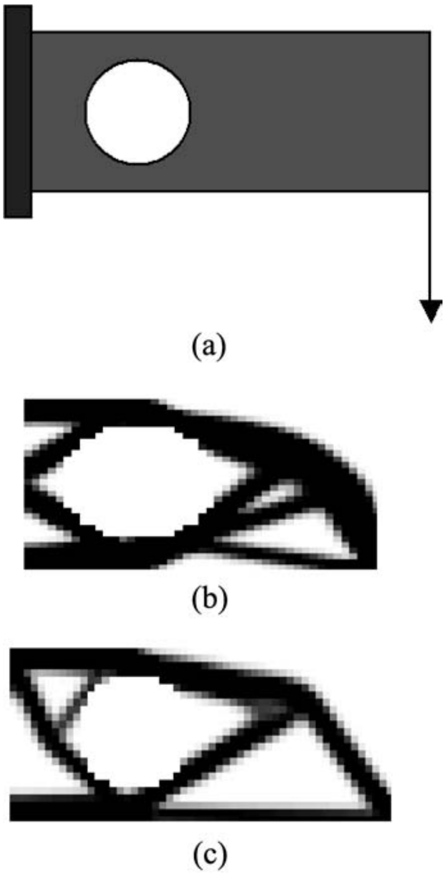

Fig. 7 Example 4, topology optimization of a cantilever beam with a prescribed hole: (a) design domain, (b) deterministic topology optimized beam, and (c) optimized beam with RBTO model

the corresponding normalized variables that concern the numbers of elements as integers. Finally, in order to satisfy the other constraints, $\mathbf{K}(\mathbf{x},\mathbf{y},\mathbf{u})\mathbf{q}(\mathbf{x},\mathbf{y},\mathbf{u}) = \mathbf{F}(\mathbf{y},\mathbf{u})$ and $V(\mathbf{x},\mathbf{y},\mathbf{u})/V_0 \leq f$, with a new set of known quantities (geometry and loads), it calls the topology optimization subroutine *top()* in adding the force as a new known quantity on several lines:

Line 02 has to be replaced by:

top(nelx,nely,volfrac,penal,rmin,newF) ;

Line 12 has to be replaced by:

[U] = FE(nelx,nely,x,penal,newF) ;

Line 66 has to be replaced by:

function[U] = FE(nelx,nely,x,penal,newF) ;

Line 79 has to be replaced by:

F(2,1) = newF.

In order to apply the described approach (Fig. 3), the geometry of the structure and the applied loads are first considered as the proposed set of variables. This set contains the number of elements used to discretize the design domain in the horizontal and vertical directions (*nelx* and *nely*), the volume fraction (*volfrac*), the filter size (*rmin*), and the applied load (F). The vector of mean values is then $\mathbf{m}_y = [mF\ mnelx\ mnely\ mvolfrac\ mrmin]$. In order to introduce the reliability analysis, we have to take into account the influence of these mean values by evaluating the sensitivity of the compliance with respect to each of these variables.

Table 1 Input and output values for the deterministic topologies

Example	1	2	3	4
nelx	60	32	30	45
nely	20	20	30	30
volfrac	0.5	0.4	0.4	0.5
penal	3.0	3.0	3.0	3.0
rmin	1.5	1.2	1.2	1.5
Compliance	203.3061	57.3492	61.2880	52.1080

5.2.1 Gradient evaluation

Table 2 provides the sensitivity of the compliance with respect to all the proposed mean values. We find that the filter size (*rmin*) has the least influence on the compliance. This result leads us to eliminate this variable from the means set. Furthermore, it is clear from this table that the gradients with respect to the means (*mnely*) and (*mvolfrac*) are negative, which means that the number of elements in the vertical direction and the volume fraction have a positive influence on the structural performance (when increasing these two values, we obtain a stiffer structure). On the other hand, the gradients with respect to the mean values (*mF*) and (*mnelx*) are positive. Therefore, when increasing these values, the compliance will increase (negative influence).

Table 2 Compliance sensitivities with respect to the proposed set

Example	1	2	3	4
$\partial C/\partial m_F$	408.90	115.48	123.14	140.00
$\partial C/\partial m_{nelx}$	519.26	118.39	77.68	55.91
$\partial C/\partial m_{nely}$	−465.05	−110.11	−77.21	−52.07
$\partial C/\partial m_{volfrac}$	−345.62	−139.70	−161.05	−56.32
$\partial C/\partial m_{rmin}$	18.76	7.29	9.06	2.32

5.2.2 β calculation

From the sensitivity analysis step, we can consider that the component u_j of the normalized vector has the same

Table 3 The normalized values, the reliability index, and the resulting objective function for RBTO

Example	1	2	3	4
u_F	1.886	1.869	1.801	1.801
u_{nelx}	1.833	1.875	2.000	2.000
u_{nely}	−2.000	−2.000	−2.000	−2.000
$u_{volfrac}$	−1.886	−1.869	−1.801	−1.801
β	3.805	3.808	3.806	3.806
Compliance	980.2142	248.3102	240.7449	187.0253

sign as the corresponding gradient $\partial C/\partial m_{y_j}$. Table 3 provides the normalized vector and the reliability index in considering the target reliability index $\beta_t = 3.8$. For simplicity, we consider herein the limit state function as a linear combination of the random variables.

5.2.3 Topology optimization approach

When calling the program RBTO( ), we consider the same input values of *nelx, nely, volfrac, penal*, and *rmin,* as presented in Table 1, but the input target reliability level has to be introduced (for these four examples: *Bet* = 3.8). After having evaluated the normalized vector **u**, we compute the new values of the selected random variable vector $\mathbf{y} = [F\ nelx\ nely\ volfrac]$ using (11) and considering the standard-deviations $\sigma_{y_j} = 0.1 m_{y_j}$.

Finally, in order to apply the topology optimization based on the SIMP approach (13), the program RBTO( ) will call the topology optimization code *top( )*, with a new set of values (data), that corresponds to the selected random variable vector **y**. Figures 4d, 5c, 6c, and 7(c) show the optimal topologies based on the usage of the RBTO algorithm. It is clear that the reliability-based topologies and their resulting compliance values are different from those obtained by the deterministic cases (see Tables 1 and 3). In order to show the importance of this study, we can apply a design optimization procedure to both reliability-based topologies and the deterministic ones under similar conditions.

5.3 Importance of the RBTO model

For the topology optimization, many efficient numerical methods have been developed and applied to different kinds of structures. But for RBTO problems, the coupling between mechanical modeling, reliability analysis and topology optimization represents a very complex task when applying the same classical procedure as of the RBDO model, especially for the first iterations. For the beam structure illustrated in Fig. 4a, when considering a half beam, the first and second iterations are

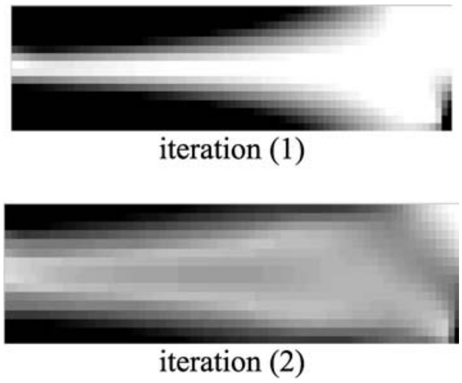

Fig. 8 First and second iterations for the first example (Fig. 4)

presented in Fig. 8. In this case, it is very difficult to analyze the reliability level at each iteration because the structure topology changes significantly between the iterations and this necessitates a very high computational time.

In general an optimized topology with rugged surfaces is not appropriate for use in industry, but needs to be post-processed using shape or size optimization. When applying the classical procedure of reliability-based optimization, if we obtain a structure that satisfies the reliability constraint and is similar to that presented in Fig. 8 (iteration 2), it is difficult to identify the form of the structure. The presented RBTO model does not need a lot of time and it is easy to implement. Furthermore, the resulting forms of the reliability-based topologies can easily be identified (Figs. 4d, 5c, 6c, and 7c).

In order to demonstrate the importance of the integration of reliability constraints into the deterministic topology optimization, we consider the results of one of the examples (Fig. 5) as truss structures (discrete models). We now calculate analytically the bar areas of the structures obtained by Deterministic Design Optimization and Reliability-Based Design Optimization, respectively. The truss structures are illustrated in Fig. 9. Figure 9a shows the optimal topology when using the deterministic procedure and Fig. 9b presents the topology when integrating the reliability in the topology procedure (RBTO). The structure is loaded by a vertical force F. The mean value of this force is $m_F = 8\,\text{KN}$, the safety factor is

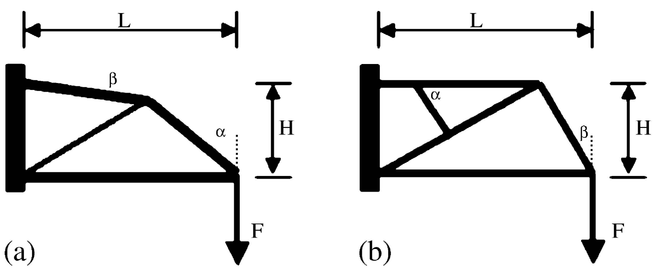

Fig. 9 Truss models for (Fig. 5): (a) deterministic topology and (b) reliability-based topology

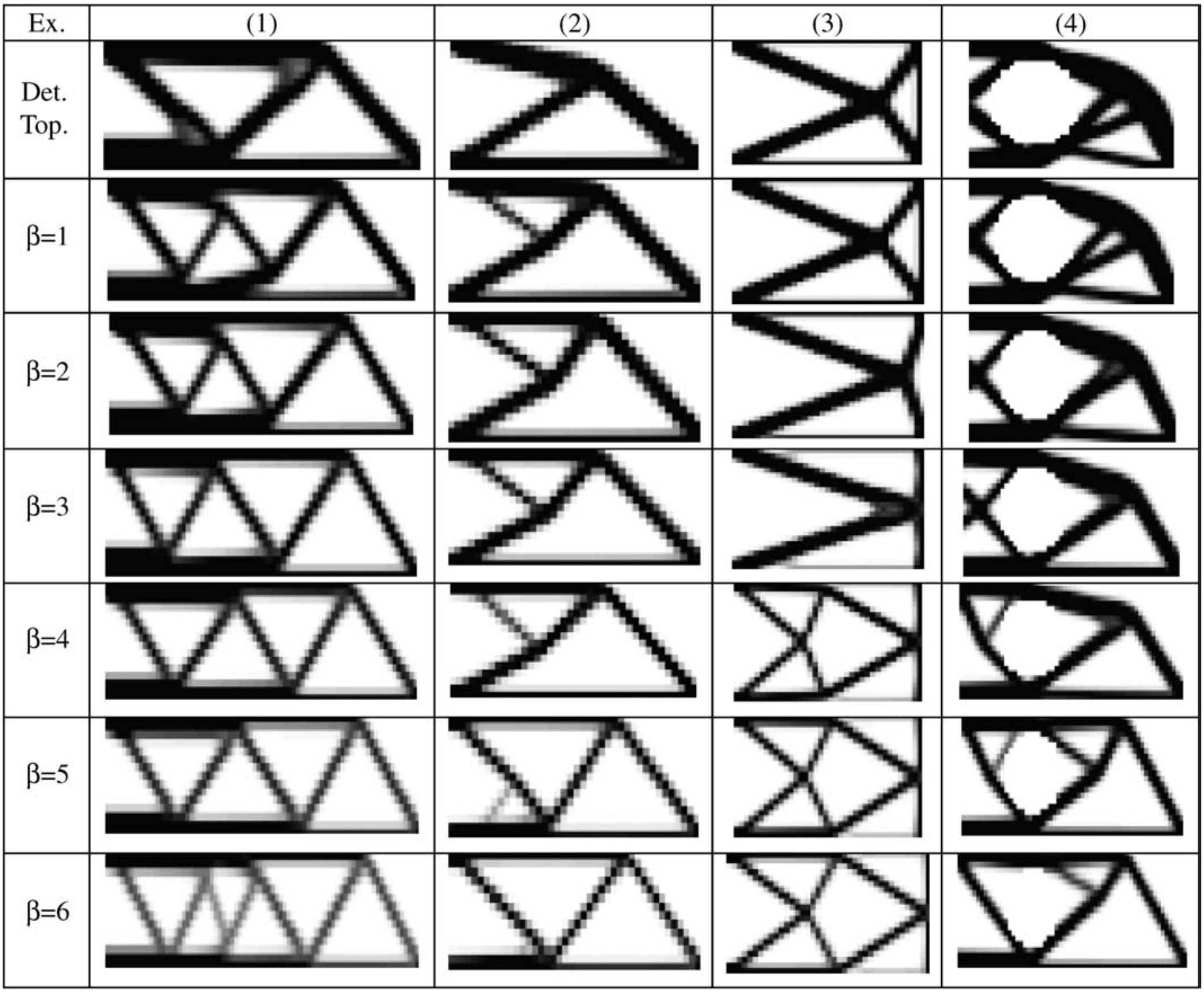

Fig. 10 Resulting topologies for different reliability levels

$S_f = 1.25$, and the allowable stress is $\sigma_w = 235\,\text{MPa}$. The dimensions and the angles in the figure are: $L = 1000\,\text{mm}$, $H = 875\,\text{mm}$, $\alpha = 45°$, and $\beta = 30°$.

For the deterministic design, the structural volume of the deterministic topology is $V_c = 132\,367\,\text{mm}^3$, but when introducing the reliability the structure volume will only be $V = 114\,325\,\text{mm}^3$. The weight reduction is thus

$$WR = 1 - \frac{V}{V_c} = 13.6\%\,. \tag{17}$$

This example shows that the new topology reduces the structural weight by 13.6% for the same conditions. This result means that the introduction of the reliability analysis during the topology optimization reduces the structural weight when using a shape optimization module (for deterministic design). Now if we consider that the force is the only random variable, according to the normal distribution law, we can evaluate the reliability of the structure by calculating the normalized variable

$$u = \frac{F - m_F}{\sigma_F}\,, \tag{18}$$

and when applying (18) in considering that the standard-deviation $\sigma_F = 0.1 m_F$, we then obtain

$$\beta = |u| = 2.5\,.$$

However, for the Reliability-Based Design Optimization model, when considering a simple case of one random variable (the applied force F) and the target reliability level $\beta_t = 3.0$, we have $u = 3.0$ and then from (18), $F = 10.4\,\text{KN}$.

For the same example, if we replace $F = 10\,\text{KN}$ by $F = 10.4\,\text{KN}$, and seeing that the relation of the stress $\sigma = N/A$ is linear, the structural volume of the deterministic topology procedure is $V_c^{\text{RBDO}} = 137\,662\,\text{mm}^3$. But when the reliability is introduced, the structural volume becomes $V^{\text{RBDO}} = 118\,898\,\text{mm}^3$ while, at the same time, the compliance increases. The weight reduction is 13.6%, the same as the deterministic one. This reduction demonstrates the importance of the reliability in the topology optimization. This importance can also be verified when considering shape and size optimization for deterministic optimization as well as for reliability-based optimization. In Fig. 10, the reader can see different topologies for different reliability levels of the four Examples 1–4 in Figs. 4–7.

6 Conclusion

The proposed Reliability-Based Topology Optimization (RBTO) model aims to consider randomness (variability) of the most important quantities of a structure, such as the geometry and the applied loads. The coupling between the reliability analysis and the topology optimization is carried out by the use of a particular optimization procedure. The importance of the RBTO model is in reducing the weight of structures for the same conditions. This weight reduction manifests itself in deterministic design optimization as well as in reliability-based design optimization. The first advantage of the RBTO model is that the resulting optimal topologies are more reliable than the deterministic topologies for the same weight of the structure (but associated with larger compliance values). The second advantage is that RBTO presents a new strategy for generating different topologies subject to different target reliability levels. This work can be extended to different topology optimization methods, such as the homogenization approach. It is also possible to implement new limit state functions in order to take into account the randomness of compliance, densities, and element dimensions.

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Appendix: Matlab code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%RELIABILITY-BASED TOPOLOGY OPTIMIZATION%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function RBTO(nelx,nely,volfrac,penal,rmin,Bet);
U = [3. 3. −3. −3.];
oldnelx = nelx; oldnely = nely;
d = sqrt(U(1) 2+U(2) 2+U(2) 2+U(3) 2);
lop = 0; lops = 0;
lp = 0; mi = 0;
diff = d−Bet;
% % START ITERATION % %
alpha = 1.;
while diff > 0.01
lp = lp + 1;
[Z] = [U];
dDdU = [U(1)/d U(2)/d U(2)/d U(3)/d];
[U] = [U]− alpha*[dDdU];
d = sqrt(U(1) 2+U(2) 2+U(2) 2+U(3) 2);
if d < Bet;
[U] = [Z]; alpha = alpha/2;
else
alpha = alpha;
end
d = sqrt(U(1) 2+U(2) 2+U(2) 2+U(3) 2); diff = d− Bet;
dDdU = [U(1)/d U(2)/d U(2)/d U(3)/d];
end
mnelx = nelx * (1+0.1*U(2));
while mnelx− nelx >= 1
lop = lop + 1;
nelx = nelx + 1;
end
if mnelx− nelx >= 0.5
nelx = nelx + 1;
end
mnely = nely * (1+0.1*U(2));
while nely− mnely >= 1
lops = lops + 1;
nely = nely− 1;
end
if nely− mnely >= 0.5
nely = nely− 1;
end
U = [3. (nelx-oldnelx)/0.1/oldnelx (nely-oldnely)/0.1/oldnely
−3.];
d = sqrt(U(1) 2+U(2) 2+U(2) 2+U(3) 2); diff = d− Bet;
dDdU = [U(1)/d 0.0 0.0 U(3)/d];
alpha = 1.;
while diff > 0.01
lp = lp + 1;
[Z] = [U];
dDdU = [U(1)/d 0.0 0.0 U(3)/d];
[U] = [U]-alpha*[dDdU];
d = sqrt(U(1) 2+U(2) 2+U(2) 2+U(3) 2);
if d < Bet;
[U] = [Z]; alpha = alpha/2;
else
alpha = alpha;
end
d = sqrt(U(1) 2+U(2) 2+U(2) 2+U(3) 2); diff = d− Bet;
dDdU = [U(1)/d 0.0 0.0 U(3)/d];
end
newF = −1 *(1+0.1*U(1));
volfrac = volfrac * (1+0.1*U(3));
disp([' For Beta = 'sprintf('%2.3f',d)])
disp([' U(1) = ' sprintf('%2.3f',U(1)) ...
' U(2) = ' sprintf('%2.3f',U(2)) ...
' U(2) = ' sprintf('%2.3f',U(2)) ...
' U(3) = ' sprintf('%2.3f',U(3))])
%CALL OF TOPOLOGY OPTIMIZATION FUNCTION%
top(nelx,nely,volfrac,penal,rmin,newF);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```