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Optimum shapes of bar cross-sections

N.V. Banichuk, F. Ragnedda and M. Serra

Abstract In this paper, the problems of optimization of cylindrical bar cross sections are formulated. The functional considered characterizes rigidities, maximum stress and the areas of the cross-section of the bar. The shape of the boundary of the cross-section is taken as a design variable and is found in the case of regular polygonal contours. Using minimax approaches optimal designs have been obtained for simply connected and doubly connected cross-sections having given convex holes. Investigations performed and complete solutions derived from the cross-sectional area minimization under rigidity and strength constraints show the changes of the optimal shapes as functions of the problem parameter.

Key words optimization, stress, rigidity, beam cross-sections

1 Introduction

The problems of shape optimization for cylindrical elastic bars arise in structural design in the context of improving their resistance characteristics with regards to bending, torsion, compression and other statical and dynamical loading conditions. Many theoretical investigations have been devoted to the rational and optimal design of bar cross-sections. But up to now there are some important aspects of optimal shape design which create difficulties for the effective application of existing techniques and are awaiting solution. This can be explained if we take into account that the finding of the best shapes of the bar cross-sections is reduced to problems with unknown boundaries, which are under investigation in modern mathematics. The most complicated questions in designing the shape of bar cross-sections are connected with the definition of the set of admissible cross-sections, investigation of the local extremum, finding of the global optimum and solution of more realistic multipurpose optimization problems. One of the earlier results of shape optimization of the bar cross-sections is contained in a fundamental work by Saint Venant (1961), where for the first time the problem of the optimal structural design was formulated as a problem with unknown boundaries, which must be found from the condition of maximization of torsional rigidity under the area constraint. A rigorous solution of this problem can be found in the text by Pólya and Szegő (1962). It was shown that the bar with circular cross-section has maximum torsional rigidity if it is compared with other convex cross-sections having the same area of the cross-section. The application of symmetrization theorems proved that the obtained optimum is not only a local one but a global one. Various generalizations of this problem in the case of doubly connected cross-sections have been studied by Weinberger and Serrin (1978), Banichuk (1975, 1976), Kurshin (1975), Kurshin and Onoprienko (1976) and Kurshin and Rastorguev (1979). The important problem of finding the shape of the cross-section of the strongest cylindrical column, i.e. the column which has the largest critical buckling load, was formulated by Keller (1960) for the set of convex cross-sections and investigated for the case of the regular polygonal domain. It was shown that the cross-section of the strongest column is not a circle but is instead an equilateral triangle and that changing the cross-section from a circle to an equilateral triangle increases the critical buckling load by 20.9 %. This triangularization is performed under the condition that the cross-sections con-

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sidered have the same area. The problem of the optimum convex shape for maximum bending stiffness (equilateral triangle), was solved before by Ting (1963) by purely geometrical means and later by Karihaloo and Hemp (1987a) using variational techniques. Banichuk and Karihaloo (1976) obtained the minimum-weight design of a solid cylindrical bar that was to act as a shaft or as a beam at different times during its design life and had to have certain minimum torsional and bending stiffness. The shape of a hollow cylindrical bar under the same conditions was found by Parbery and Karihaloo (1977). Parbery and Karihaloo (1980) investigated the problem of determining the cross-sectional shape of a thin-walled cylinder of constant (but unknown) wall thickness and given contour length that uses the least amount of material to achieve the prescribed minimum stiffness in bending and torsion. Another solution of the same problem was obtained by Karihaloo and Hemp (1982) in a closed form with the help of parametric representation of the unknown boundary.

Rational and optimal shapes of beam cross-sections have been found by Ishlinskii (1940) and Banichuk and Kobelev (1983) for the case when the plane of bending is given and the strength constraint is taken into account. In the same year Karihaloo and Hemp (1983) solved the problem of finding the shape of a cross-section of a doubly symmetrical thin-walled cylinder of constant wall thickness and given contour length, which uses the smallest possible wall thickness to provide, at least, certain given rigidities in torsion and bending in a principal plane. The solution depends on the ratio I_0/J_0 between given bending rigidity and torsional rigidity. If $I_0/J_0 \leq 1/2$ the solution is a circle; if $I_0/J_0 > 1/2$ it is like an ellipse. Since 1983 several other papers about optimum cross-sections have been published. Among them we can cite that of Karihaloo and Hemp (1987c), who solve numerically the problem of the optimum section (minimum area) of prescribed torsional and flexural rigidity; that of Karihaloo and Hemp (1987b) where an approximated analytical solution of the problem of finding the shape of a plane convex figure of minimum area with torsional constant and least moment of inertia prescribed is found.

Very recently Lipton (1998) solved the problem of finding the shaft and fiber cross-section that yield the maximum torsional rigidity (with imperfect bonding between matrix and fibers and fixed joint area of fiber cross-section); Lipton supposes also that the cross-sectional area is fixed.

Here we cite only results on the design of cylindrical bars, closely connected with our considerations, and by no means intend to give observations on shape optimization. Some additional results on shape optimization of bars can be found in the book of Banichuk (1983).

In this paper we initially discuss the formulation of the classical problem of the maximization of minimal bending rigidity of a bar, i.e. the maximization of the minimal moment of inertia of cross-sections. Some generalizations of the problem for doubly connected cross-sections are given. Then the problem of finding the best shape of the cross-section from the condition of minimization of the maximum stress is formulated and investigated. Special attention is devoted to the problem of minimum weight design of cylindrical bars under rigidity and strength constraints. The optimal solution is found for the family of admissible cross-sections having the shape of a regular polygon.

2
Maximization of bending rigidity

To find the best geometrical characteristics of beams, rods and bars is an important problem of structural design. Improvement of cross-section geometry gives us the possibility to decrease significantly the weight of the structural elements without losing rigidity and strength. In this paper we will mainly be interested in the optimization of such mechanical functionals as critical buckling load, strength and rigidity or integral stiffness. Consider first the cross-section rigidity properties which determine the resistance of rods against buckling and control lateral bending. As is well-known, in order to reduce the deflections of the beam under lateral bending or to increase buckling load (fundamental eigenvalue) it is necessary to increase the cross-section moment of inertia under some isoperimetric constraints. The area S of the cylindrical bar cross-section Ω ($S = meas\Omega$), which determines the volume V of the bar ($V = SL$, L is length of the bar), is supposed to be given. We will consider mainly convex simply connected cross-sections. The optimization problem considered consists in finding the boundary Γ of the cross-section Ω which maximizes the minimal moment of inertia $I(\Gamma, \alpha)$,

$$I_* = \max_{\Gamma} \min_{\alpha} I(\Gamma, \alpha) , \tag{1}$$

under the isoperimetric constraint

$$S(\Gamma) = \int_{\Omega} \mathrm{d}\Omega = S_0 , \tag{2}$$

where S_0 is a given value of the cross-sectional area, α is the angle that determines the orientation of the plane of bending (Fig. 1).

We will consider the case when the applied external forces act in the same bending plane (problems of lateral bending) but the orientation of this plane is unknown beforehand and can be taken arbitrarily. This case is the most important for buckling problems of compressed columns when loss of stability takes place in the plane with minimum bending rigidity (minimum moment of inertia). In this case, to optimize the rigidity of the rod (the critical buckling load of the column) it is necessary to maximize the minimum moment of inertia of the cross-section. As was noted by Keller (1960) and followed from

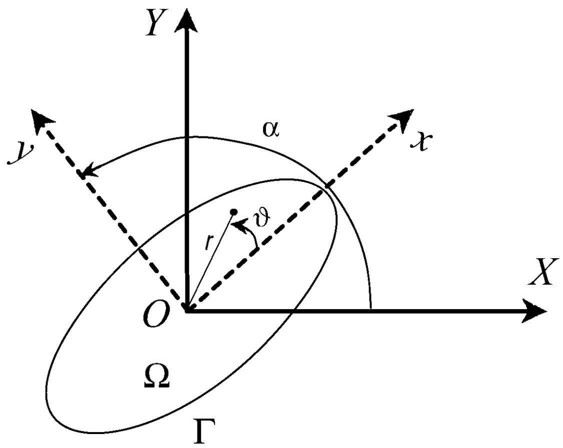

Fig. 1 Cross-section and reference systems

symmetrization theorems (Pólya and Szegő 1962), to optimize the cross-sectional shape of the rod it is required to consider the family of admissible symmetrical cross-sections having equal moment of inertia for any neutral line crossing the centroid. Note that any regular symmetrical polygon [from triangular ($n = 3$) to circle ($n = \infty$), n is the number of polygon sides] has equal moments of inertia with respect to any axis. In the following we will consider regular polygons and investigate their properties. For the polar moment of inertia I_p of a regular polygon with n axes of symmetry, we have the following formula:

$$I_p = \int_\Omega \rho^2 \, \mathrm{d}\Omega \,, \tag{3}$$

where $\rho^2 = x^2 + y^2$. It is supposed that the area of the cross-section satisfies the condition (2). Taking into account that for considered symmetrical cross-sections, moments of inertia with respect to arbitrary axes crossing the centroid are equal, we have

$$I = \frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\int_\Omega y^2 \, \mathrm{d}\Omega + \int_\Omega x^2 \, \mathrm{d}\Omega\right) = \frac{1}{2} I_p \,, \tag{4}$$

where I_x and I_y are the moments of inertia of the cross-section with respect to axes x and y of the orthogonal coordinate system (Oxy) where O is the centroid of the cross-section (Fig. 2).

In order to evaluate I_p it is sufficient to consider one elementary triangle OPQ (see Fig. 2) with the polar moment of inertia $(I_p)^e$, taking into account that $I_p = n(I_p)^e$).

If the local orthogonal coordinate system $O\eta\zeta$ is introduced in such a manner that the axis $O\eta$ is perpendicular to the side PQ, then we have $(I_p)^e = (I_\zeta)^e + (I_\eta)^e$. Here $(I_\eta)^e = b^3 a/48$, $(I_\zeta)^e = ba^3/4$ are the moments of inertia of the triangle OPQ with respect to the axes η and ζ, and a and b are, respectively, the lengths of the height OT and the side PQ. Using the expression for $(I_p)^e$, the isoperimetric equality $S_0 = n(ba/2)$, obtained from (2),

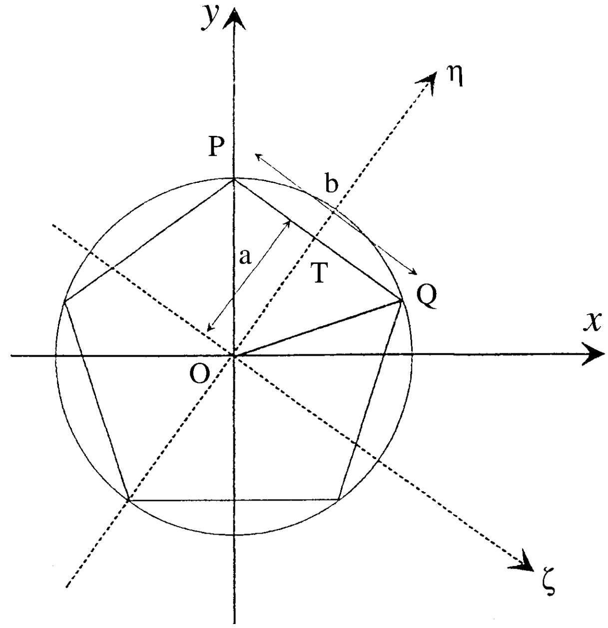

Fig. 2 Regular polygonal cross-section

and performing summation of $(I_p)^e$ ($e = 1, 2, \ldots, n$), we have

$$I = \frac{1}{2} I_p = \frac{n}{2}(I_p)^e = S_0^2 \varphi(n) \,, \tag{5}$$

$$\varphi(n) = \frac{\sin^2(\frac{\pi}{n}) + 3\cos^2(\frac{\pi}{n})}{12n \sin(\frac{\pi}{n})\cos(\frac{\pi}{n})} \,. \tag{6}$$

The dependence of the nondimensional moment of inertia $\tilde{I} = I/S_0^2$ on the number of sides n is presented in Fig. 3.

In the following tilde is omitted. As can be seen from Fig. 3, the nondimensional moment of inertia I monotonically decreases when n tends to infinity and the maximum of I is realized for $n = 3$. So the optimum is attained for an equilateral triangle and the worst case corresponds to the limiting case ($n = \infty$) of the circular cross-section. Optimality of the triangular cross-section between all regular polygonal domains (of the same area) was first shown by Keller (1960) in the context of investigation of the strongest column having the maximum critical buckling load. Optimality of a triangular cross-section for the class of symmetrical convex domains has been an unsolved problem up to now. But by using calculus of variations it is possible to prove one local property for any regular polygon with n sides. Apply to its contour Γ an arbitrary small symmetrical perturbation that does not violate the convexity of the domain and isoperimetric constraint $S = S_0$. For the sake of simplicity, the unperturbed domain PQR and the perturbed third part $P'KT'DQ'$ of the boundary are shown in Fig. 4 in the case of a triangular cross-section.

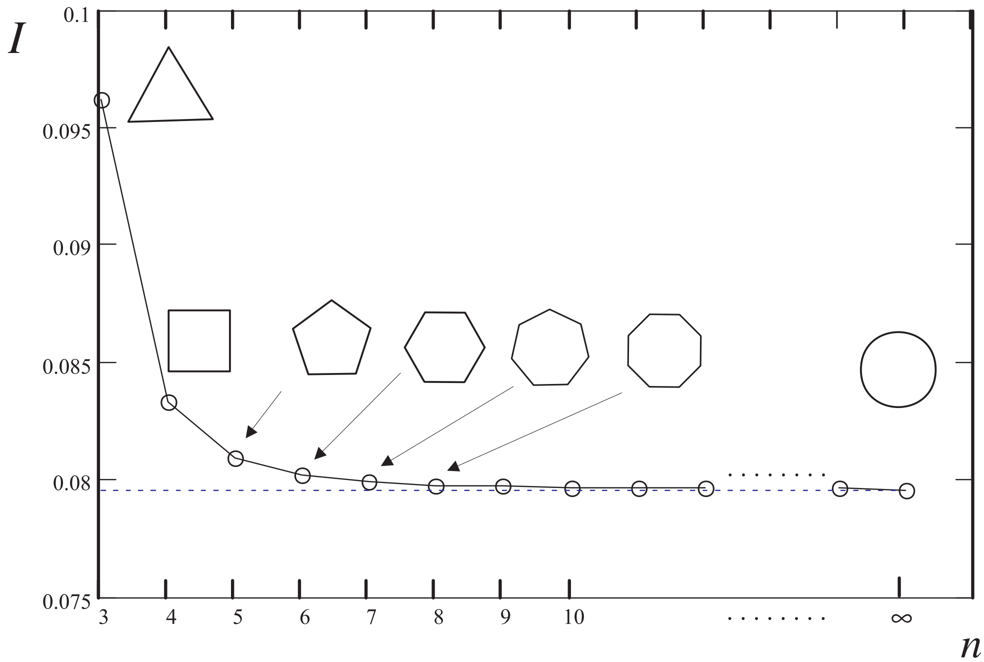

Fig. 3 Moment of inertia of a regular polygon versus the number of sides

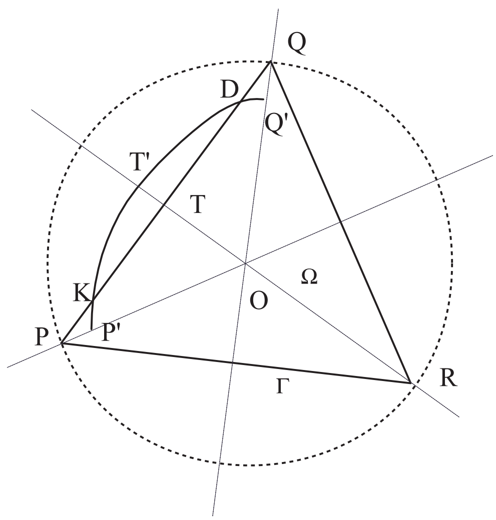

Fig. 4 Small convex perturbation of a regular polygon

Expression for small variation δI of the moment of inertia I is given by the formula

$$\delta I = \frac{1}{2}\delta \int\limits_{\Omega} \rho^2 \, \mathrm{d}\Omega = \frac{1}{2} \int\limits_{\Gamma} \rho^2 \delta f \, \mathrm{d}\Gamma \,, \tag{7}$$

where δf is the variation of the boundary Γ measured in the direction of the external normal to the boundary. Taking into account the symmetry of the unperturbed polygon and disturbed boundary with respect to the axis of symmetry shown in Fig. 4 by dashed lines, we will have the following representations for the variation of the moment of inertia:

$$\delta I = 3 \left(\int\limits_{P}^{K} \rho^2 \delta f \, \mathrm{d}\Gamma + \int\limits_{K}^{T} \rho^2 \delta f \, \mathrm{d}\Gamma \right) \,, \tag{8}$$

and the area of the cross-section

$$\delta S = 6 \left(\int\limits_{P}^{K} \delta f \, \mathrm{d}\Gamma + \int\limits_{K}^{T} \delta f \, \mathrm{d}\Gamma \right) \,. \tag{9}$$

Using (9) and the isoperimetric condition $S = S_0$ ($\delta S = 0$), we obtain the relation

$$\int\limits_{P}^{K} \delta f \, \mathrm{d}\Gamma = - \int\limits_{K}^{T} \delta f \, \mathrm{d}\Gamma \,. \tag{10}$$

The required estimation is performed with application of (10) and the inequalities

$$(\rho^2)_{PK} \geq (\rho^2)_K \,, \quad (\rho^2)_{KT} \leq (\rho^2)_K \,, \tag{11}$$

to (8) for δI in the following manner:

$$\delta I \leq 3 \left[(\rho^2)_K \int\limits_{P}^{K} \delta f \, \mathrm{d}\Gamma + (\rho^2)_K \int\limits_{K}^{T} \delta f \, \mathrm{d}\Gamma \right] =$$

$$3(\rho^2)_K \left(\int\limits_{P}^{K} \delta f \, \mathrm{d}\Gamma + \int\limits_{K}^{T} \delta f \, \mathrm{d}\Gamma \right) \,. \tag{12}$$

Analogous estimations can be performed for any regular polygon. Thus $\delta I \le 0$ and consequently every regular polygon gives the local maximum for the functional considered.

Previously we considered only bars with simply connected cross-sections. Suppose now that the cross-section is doubly connected (the bar has a hole) and the domain Ω is bounded by a given internal contour Γ_i (boundary of the hole) and an unknown external contour Γ. Suppose also that the internal domain Ω_i, bounded by the contour Γ_i, is convex and its moments of inertia with respect to arbitrary axes, lying in the plane of the cross-section and crossing the centroid, are equal. Assume that the area S of the cross-section Ω is given ($S = S_0$) and that the boundary Γ of the equilateral triangle, having the area $S = S_i + S_0$ $[S_i + S_0 = meas(\Omega_i + \Omega_0)]$ and the same centroid as the domain Ω_i, does not touch the boundary Γ_i. The last condition is essential in order to consider free variations of the external contour Γ for the class of regular polygons. For this set of considered boundaries Γ, the minimal distance between Γ and the centroid, as is known, is realized for the equilateral triangle. Under the formulated assumptions, the unknown external boundary Γ must be found from the condition of optimization of the moment of inertia I of the considered cross-section Ω. Take into consideration a regular polygon for Γ and use the following representation for the moment of inertia I of the bar cross-section Ω:

$$I(n) = I_s(n) - I_i \,, \tag{13}$$

where the moment of inertia I_s of simply connected domain $\Omega_i + \Omega$ depends on the number of sides of the polygon and the moment I_i of inertia of the internal convex domain Ω_i does not depend on n. The optimization problem is reduced to maximization of $I_s(n)$; we have

$$I_* = \max_n I(n) = \max_n [I_s(n)] - I_i = I_s(3) - I_i \,. \tag{14}$$

Thus, the optimal cross-section is an equilateral triangle, having the centre at the centroid of the domain Ω_i.

3
Strength maximization

Consider now the problem of stress minimization for cylindrical bars having symmetrical convex cross-sections. We suppose that the plane of the bending of the bar and applications of external forces is unknown beforehand, but the limiting value M_0 of the bending moment, acting in the considered cross-section, is given. Only the normal stresses, arising in the cross-section are taken into consideration and the following formula is used for their calculation:

$$\sigma(x,y,\Gamma,\beta) = M_0 \frac{h(x,y,\beta)}{I(\Gamma,\beta)} \,, \tag{15}$$

where the angle β characterizes the orientation of the neutral line of the bending with respect to the global coordinate system, and h is the distance between the considered point (x, y) of the cross-section and the neutral line. The optimization problem consists in finding the boundary Γ of the cross-section Ω, which minimizes the maximal stress

$$\sigma_* = \min_\Gamma \max_\beta \max_{(x,y)\in\Omega} \sigma(x,y,\Gamma,\beta) \,, \tag{16}$$

under the isoperimetric constraint (2).

Define by h_m and σ_m, respectively, the maximal value of h and σ on Ω, for fixed boundary Γ and angle β, i.e.

$$h_m = h_m(\Gamma,\beta) = \max_{(x,y)\in\Omega} h(x,y,\beta) \,, \tag{17}$$

$$\sigma_m = \sigma_m(\Gamma,\beta) = \frac{M_0}{I(\Gamma,\beta)} h_m \,. \tag{18}$$

As in Sect. 2, we confine our considerations to the case of symmetrical convex cross-sections bounded by regular polygons. In this case, the moment of inertia does not depend on β $[I = I(\Gamma)]$ and the maximum value of h with respect to β $(0 \le \beta \le 2\pi)$ and $(x, y) \in \Omega$ is given by

$$\max_\beta h_m(\Gamma,\beta) = \max \beta \max_{(x,y)\in\Omega} h(x,y,\beta) =$$

$$\frac{S_0^{1/2}}{\sqrt{\left[n \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)\right]}} \,, \tag{19}$$

where n equals the number of sides of the polygon considered. Note that (19) was obtained with the help of isoperimetric conditions (2). Using the corresponding formulae (5) and (19) we find the expression for the maximal value of stress σ in the following form:

$$\sigma_M = \max_\beta \sigma_m = \frac{M_0}{I} \max_\beta h_m(\Gamma,\beta) = \frac{M_0}{S_0^{3/2}} \psi(n) \,, \tag{20}$$

$$\psi(n) \equiv \frac{12\sqrt{n \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)}}{\sin^2\left(\frac{\pi}{n}\right) + 3\cos^2\left(\frac{\pi}{n}\right)} \,. \tag{21}$$

The dependence of the maximum stress σ_M on the number n of the sides of the polygon is shown in Fig. 5 for the nondimensional variable $\tilde{\sigma}_M = \sigma_M M_0^{-1} S_0^{3/2}$.

In the following tilde is omitted. As can be seen from Fig. 5, the nondimensional maximum $\sigma_M = \Psi(n)$ monotonically decreases when n tends to infinity and the minimum of $\sigma_M(n)$ with respect to n is realized for $n = \infty$. So the optimum is reached for the circular cross-section and the worst case corresponds to the equilateral triangle ($n = 3$).

It is possible to show that a circular cross-section gives the local minimum for the optimized functional for small variations which do not violate the convexity of the domain. To do this, let us consider a circular cross-section

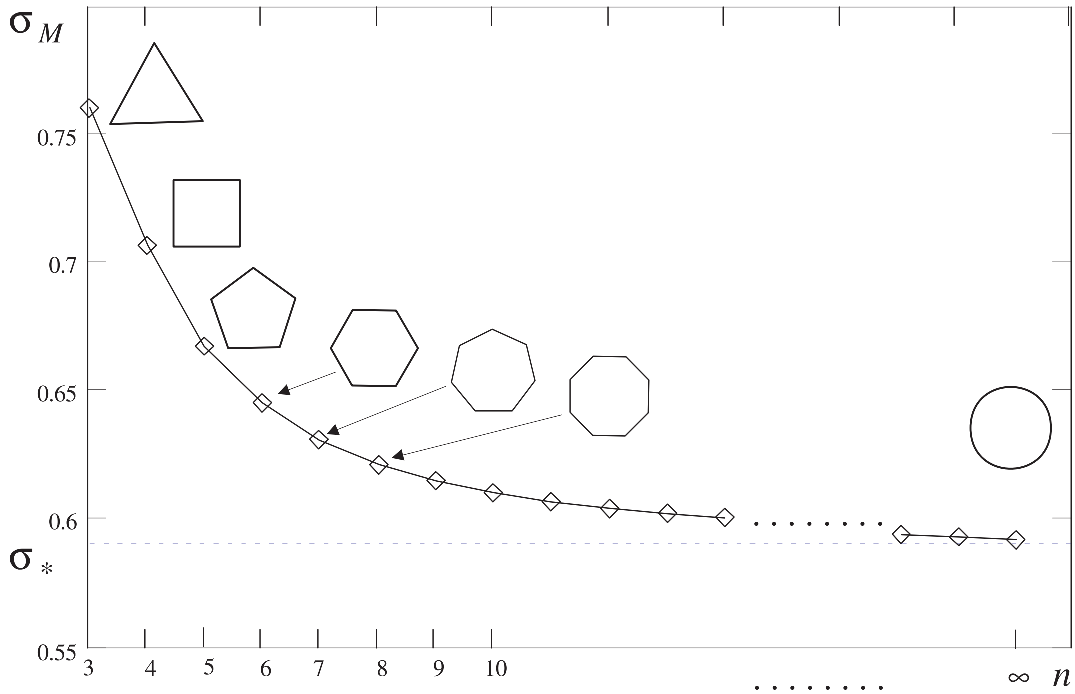

Fig. 5 Maximum stress versus the number of sides of a regular polygon

Ω and apply to its contour Γ an arbitrary small symmetrical perturbation that does not violate the convexity of the domain and isoperimetric constraint (2) ($S = S_0$). For the sake of simplicity and without loss of generality, the perturbed contour in Fig. 6 has three axes of symmetry, shown by dashed lines.

A small variation of the optimized functional is evaluated with the help of the following expression:

$$\delta\sigma_M = \frac{1}{I}\delta h_m - \frac{h_m}{I^2}\delta I\,. \tag{22}$$

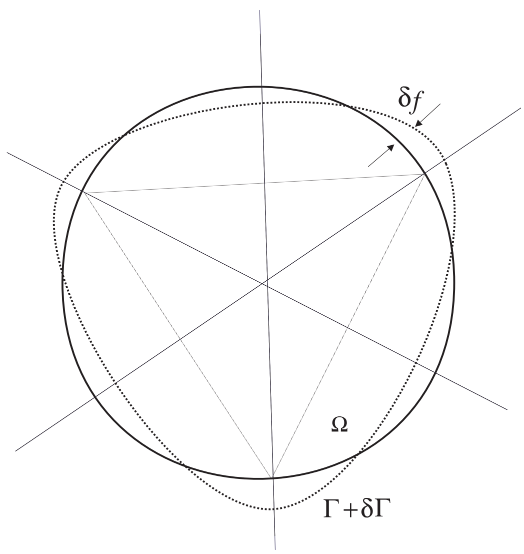

Fig. 6 Small convex perturbation of a circle

Taking into account the formula for a small variation of the area S of the considered circular domain and the isoperimetric condition we will have

$$\int\limits_{\Gamma} \delta f \,\mathrm{d}\Gamma = 0\,. \tag{23}$$

Using (23) we can evaluate the first variation of the moment of inertia

$$\delta I = \frac{1}{2}\int\limits_{\Gamma} \rho^2 \delta f \,\mathrm{d}\Gamma = \frac{\rho^2}{2}\int\limits_{\Gamma} \delta f \,\mathrm{d}\Gamma = 0\,. \tag{24}$$

Here we used the property that the function $\rho(x, y)$ is a constant at the points of the boundary Γ (ρ is the radius of the considered cross-section). Thus, the second term in the representation (22) is cancelled. As for the first term in (22), the variation of h_m is nonpositive ($\delta h_m \leq 0$) as can be seen from Fig. 6 and consequently

$$\delta\sigma_M = \frac{1}{I}\delta h_m \leq 0\,, \tag{25}$$

for considered perturbations of the circular cross-section.

As in Sect. 2, it is possible to generalize results obtained in the case of doubly connected cross-sections. We will consider the bar, where the cross-section Ω is bounded by the given internal boundary Γ_i and by an unknown external boundary Γ, where the shape is found for the set of regular polygons. Making the same assumption as in Sect. 2 for the case of shape optimization of the doubly connected domain, we reduce the original problem of minimization of the maximum stress to the following one:

$$\sigma_* = \min_n \left[\frac{M_0 h_M(n)}{I_s(n) - I_i} \right] , \tag{26}$$

where I_i is the moment of inertia of the domain Ω_i, which does not depend on n. The expressions for the maximum h_M and for the moment of inertia $I_s(n)$ are given by (19), (5) and (6), in which it is necessary to substitute $S_i + S_0$ for S_0.

4
Minimum weight design of bars under strength and rigidity constraints

In previous chapters we investigated the problems of strength and rigidity optimization for cylindrical bodies having the same cross-sectional areas. We were constrained by considering the symmetrical polygonal cross-section. This family of cross-sections does not exhaust the class of symmetrical convex cross-sections, which have the same moment of inertia with respect to the arbitrary line crossing the centroid. So the problems of global shape optimization for the family of convex cross-sections are open up to now. Taking into account the practical importance of the family of symmetrical polygonal cross-sections, opposite tendencies in cross-sectional shape changing to maximize rigidity and minimize the maximum stress and also the meaningful effect of optimization, it is of interest to consider more general optimum solution problems for symmetrical polygonal cross-sections.

In this section we will study the following shape optimization problem for the family of symmetrical rods with polygonal cross-sections. It is necessary to minimize the cross-sectional area

$$J(\Gamma) = S(\Gamma) \to \min_{\Gamma} , \tag{27}$$

under the following rigidity and strength constraints:

$$I \equiv \varphi(n) S^2 \geq I_0 , \tag{28}$$

$$\sigma \equiv \frac{M_0}{S^{3/2}} \psi(n) \leq \sigma_0 \tag{29}$$

where M_0, I_0 and σ_0 are given values of applied bending moment and the limit values for admissible moments of inertia and stresses, respectively. Functions $\varphi(n)$ and $\psi(n)$ are defined by

$$\varphi(n) = \frac{\sin^2\left(\frac{\pi}{n}\right) + 3\cos^2\left(\frac{\pi}{n}\right)}{12 n \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)} ,$$

$$\psi(n) = \frac{12\sqrt{n \sin\left(\frac{\pi}{n}\right) \cos\left(\frac{\pi}{n}\right)}}{\sin^2\left(\frac{\pi}{n}\right) + 3\cos^2\left(\frac{\pi}{n}\right)} . \tag{30}$$

In the following we introduce the nondimensional cross-sectional area $\tilde{S}$, the optimized functional $\tilde{J}$, the moment of inertia $\tilde{I}$, the stress $\tilde{\sigma}$, and the problem parameter k by the expressions

$$\tilde{S} = \frac{S}{\sqrt{I_0}} , \quad \tilde{J} = \frac{J}{\sqrt{I_0}} , \quad \tilde{I} = \frac{I}{\sqrt{I_0}} ,$$
$$\tilde{\sigma} = \frac{\sigma}{\sigma_0} , \quad k = \frac{M_0}{\sigma_0 I_0^{3/4}} , \tag{31}$$

and then omit the tilde.

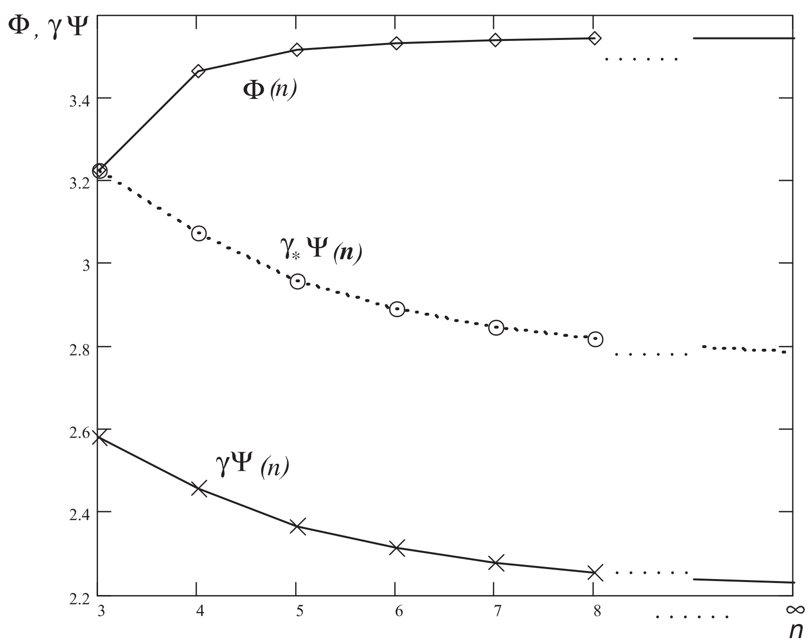

Fig. 7 Area minimization of a regular polygon under stress and rigidity constraints: situation (a)

Under the performed transformations, (27) does not change and the inequalities (28) and (29) take the form

$$I \equiv \varphi(n) S^2 \geq 1\,, \tag{32}$$

$$\sigma \equiv k\psi(n) S^{-3/2} \leq 1\,. \tag{33}$$

Now we conclude that the solution of the considered optimization problem is dependent on a problem parameter k. To find the solution of the problem of area S minimization under rigidity and strength constraints, it is convenient to transform the inequalities (32) and (33) into the form

$$S \geq \Phi(n)\,, \quad \Phi(n) \equiv \phi(n)^{-1/2}\,, \tag{34}$$

$$S \geq \gamma\Psi(n)\,, \quad \Psi(n) \equiv \psi(n)^{2/3}\,, \quad \gamma = k^{2/3}\,. \tag{35}$$

It is clear that the optimum value of the minimized functional for each given γ can be presented in the following manner:

$$S_{\text{opt}}(\gamma) = \min_n \max\left\{\Phi(n), \gamma\Psi(n)\right\}\,, \tag{36}$$

where the internal maximization consists in finding for any fixed n the maximum of two numbers written in brackets, and the external minimization consists in finding the optimal $n_{\text{opt}} = n_{\text{opt}}(\gamma)$ which realizes the optimal value $S_{\text{opt}}(\gamma)$ of the functional.

In constructing the optimal solution we can encounter three different situations (a), (b) and (c) as shown in Figs. 7, 8 and 9, respectively.

The first situation (a), shown in Fig. 7, corresponds to the case when $\gamma \in B_I = [0, \gamma_*]$ and $\Phi(n) \geq \gamma\Psi(n)$, for $n = 3, 4, 5, \ldots$. Taking into account that $\Phi(n)$ and $\Psi(n)$ are, respectively, monotonically increasing and monotonically decreasing functions of the integer argument n, we determine $\gamma_* = \Phi(3)/\Psi(3) = 0.74$. For the considered interval $0 \leq \gamma \leq \gamma_*$ the internal maximum in (36) is realized for $\Phi(n)$ and the external minimum with respect to n is attained for $n = 3$. Thus for $\gamma \in B_I$ we have the following optimal solution:

$$n_{\text{opt}} = 3\,, \quad S_{\text{opt}} = \Phi(3) = 3.22\,. \tag{37}$$

The second situation (b) shown in Fig. 8, corresponds to the case when $\gamma \in B_\sigma = [\gamma_{**}, \infty]$ and $\Phi(n) \leq \gamma\Psi(n)$ for $n = 3, 4, 5, \ldots$. In this case the properties of monotonically decreasing $\Psi(n)$ and monotonically increasing $\Phi(n)$ and the inequality presented show that $\gamma_{**} = \Phi(\infty)/\Psi(\infty) = 0.96$. For the considered interval $\gamma_{**} \leq \gamma \leq \infty$ the internal maximum in (36) is realized for $\gamma\Psi(n)$ and the external minimum with respect to n is obtained for $n = \infty$. Thus for $\gamma \in B_\sigma$ we have that the optimal solution corresponds to the circular cross-section

$$n_{\text{opt}} = \infty\,, \quad S_{\text{opt}} = \gamma\Psi(\infty) = 3.69\gamma\,. \tag{38}$$

The third situation (c), shown in Fig. 9, corresponds to the case when $\gamma \in B_{I\sigma} = [\gamma_* \leq \gamma \leq \gamma_{**}]$. For this case, let us determine the values of $\gamma_i^0 \in B_{I\sigma}$ by

$$\gamma = \gamma_i^0 = \frac{\Phi(i)}{\Psi(i)}\,, \quad i = 3, 4, 5, \ldots\,. \tag{39}$$

If $\gamma = \gamma_i^0$, then the optimal solution corresponds to the symmetrical i-angular polygon and is given by the follow-

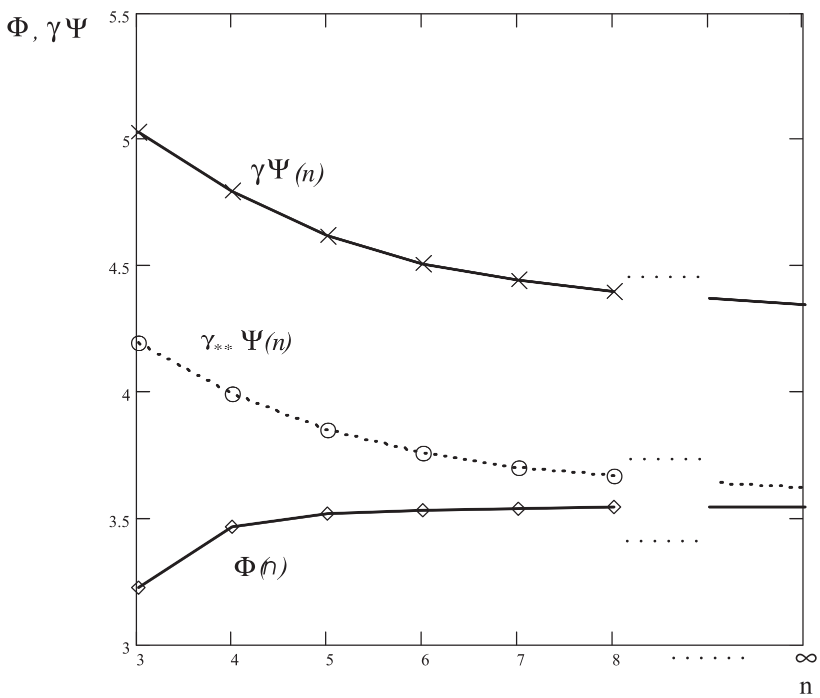

Fig. 8 Area minimization of a regular polygon under stress and rigidity constraints: situation (b)

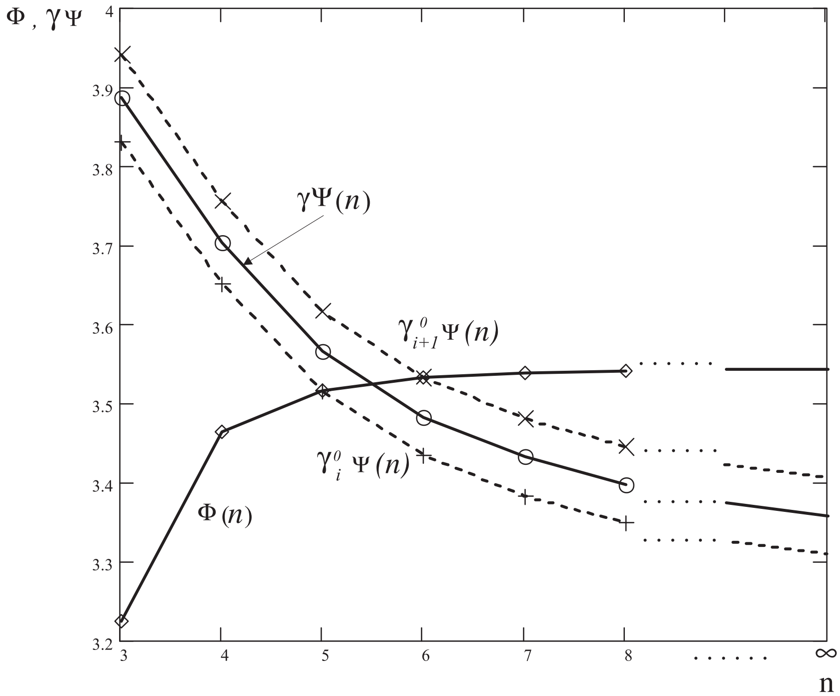

Fig. 9 Area minimization of a regular polygon under stress and rigidity constraints: situation (c)

ing formulae:

$$n_{\mathrm{opt}} = i\,, \quad S_{\mathrm{opt}} = \Phi(i) = \gamma_i^0 \Psi(i)\,. \tag{40}$$

In this case, the internal maximum in (36) is realized for $\Phi(n)$ if $n > i$ and for $\gamma_i^0 \Psi(i)$ if $n < i$. If $n = i$ then the internal maximum in (36) is realized under the condition that the first and second expressions are equal, i.e. $\Phi(n) = \gamma_i^0 \Psi(i)$. Taking into account that $\Phi(n)$ and $\Psi(n)$ are, respectively, monotonically increasing and monotonically decreasing functions, we obtain (40). Find now the optimal solution for the typical subinterval $\gamma_* \le \gamma_i^0 < \gamma < \gamma_{i+1}^0 \le \gamma_{**}$. To do this let us determine the switching point γ_i^s as

$$\gamma_i^s = \frac{\Phi(i+1)}{\Psi(i)}\,, \quad \gamma_i^0 < \gamma_i^s < \gamma_{i+1}^0\,. \tag{41}$$

This value satisfies the equalities

$$\gamma_i^s \Psi(i) = \Phi(i+1) = \gamma_{i+1}^0 \Psi(i+1)\,. \tag{42}$$

If $\gamma_i^0 < \gamma < \gamma_i^s$ and consequently

$$\gamma \Psi(i) < \Phi(i+1) = \gamma_{i+1}^0 \Psi(i+1)$$

then the optimal solution is given by

$$n_{\mathrm{opt}} = i\,, \quad S_{\mathrm{opt}} = \gamma \Psi(i)\,. \tag{43}$$

If $\gamma_i^s < \gamma < \gamma_{i+1}^0$ then

$$\gamma \Psi(i) > \Phi(i+1) = \gamma_{i+1}^0 \Psi(i+1)\,,$$

and for this case the optimal solution is presented by

$$n_{\mathrm{opt}} = i+1\,, \quad S_{\mathrm{opt}} = \Phi(i)\,. \tag{44}$$

The optimal solution is presented in Fig. 10 as a function of the problem parameter γ.

As is seen from Fig. 10 $S_{\mathrm{opt}}(\gamma)$ is constant and equal to $\Phi(3) = 3.22$ for the interval $0 \le \gamma \le \gamma_*$ and the optimal cross-section is an equilateral triangular of constant area. For $\gamma_{**} \le \gamma$ the dependence of S_{opt} on parameter γ is represented by the straight line. The tangent at the angle of inclination of this straight line with respect to γ-axis is equal to $\Psi(\infty)$. Optimal cross-sections have circular shapes. The corresponding cross-sectional area is proportional to the value of γ. For the interval $\gamma_* \le \gamma \le \gamma_{**}$ we have a piece-wise linear dependence of S_{opt} on γ. At the first subinterval $\gamma_* = \gamma_3^0 \le \gamma \le \gamma_4^0$ our continuous line $S_{\mathrm{opt}} = S_{\mathrm{opt}}(\gamma)$ has two parts. The first part ($\gamma_3^0 \le \gamma \le \gamma_3^s$) is characterized by the line that has a nonzero angle of inclination with respect to the axis γ. The tangent of the angle is $\Psi(3)$. For this part the optimal shape of the cross-section is the equilateral triangular of variable area. The distribution of the area is given by $\gamma \Psi(3)$. The second part ($\gamma_3^s \le \gamma \le \gamma_4^0$) is represented by the horizontal line with a squared cross-section of constant area (optimal solution). The same dependence of S_{opt} on problem parameter γ we have for the second $\gamma_4^0 \le \gamma \le \gamma_5^0$ and next

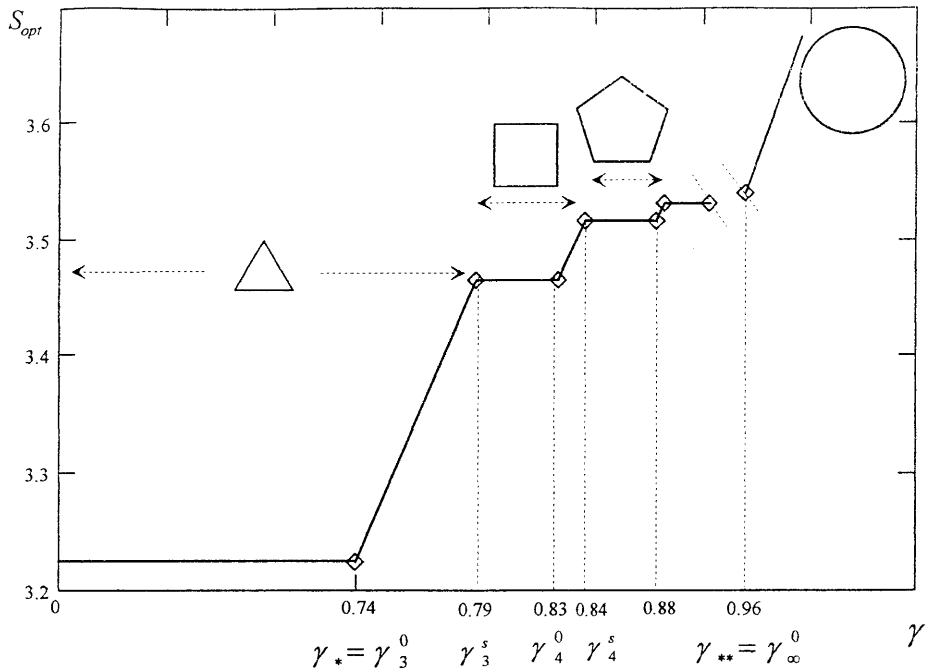

Fig. 10 Optimal regular polygon area S versus problem parameter γ

subintervals ($\gamma_i^0 \le \gamma \le \gamma_{i+1}^0$, $i = 5, 6, 7, \dots$). Note that the length of subinterval $\gamma_{i+1}^0 - \gamma_i^0$ is monotonically diminished with the increase of number i and tends to zero when $i \to \infty$.

5
Some notes and concluding remarks

In the present paper we have examined shape optimization problems for cylindrical bars and demonstrated some possibilities of the effective application of parametric analysis. For simplicity we have studied shape optimization problems taking into account only the normal stresses acting in the cross-section of a bent bar. But the optimization approaches described in this paper can also be applied to the problems of the shape optimization of the cross-section taking the maximum value of the stress intensity $[(\sigma^2 + m\tau^2)^{1/2}$ with τ as shear stress, m as given parameter] as a minimized functional.

We have confined our attention to the case of pure bending of the bar. It is clear that the problems investigated can be generalized when the torsional rigidity and torsional stress are also taken into consideration. Corresponding shape optimization problems can be interpreted as multipurpose design problems, which include two separate loading conditions: bending and torsion.

We have also focused on the case in which the M_0 moment does not change its value. Rigorously speaking, we have examined only statically determinate cases for which the equilibrium equations can be integrated separately and the bending moment M can be taken as known beforehand. We have also supposed that the distribution of the moment is constant along the bar. More complicated statically indeterminate cases require the taking into account of the behavioural equation as an additional part of the formulated optimization problem and will be investigated separately with the help of a proposed minimax approach, which will include additional stress maximization with respect to the positioning of the cross-section.

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