Research papers

Struct Multidisc Optim 22, 335–350

Extensions of design potential concept for reliability-based design optimization to nonsmooth and extreme cases

K.K. Choi, J. Tu and Y.H. Park

Abstract The reliability-based design optimization (RBDO) can be described by the design potential concept in a unified system space, where the probabilistic constraint is identified by the design potential surface of the reliability target that is obtained analytically from the first-order reliability method (FORM). This paper extends the design potential concept to treat nonsmooth probabilistic constraints and extreme case design in RBDO. In addition, refinement of the design potential surface, which yields better optimum design, can be obtained using more accurate second-order reliability method (SORM). By integrating performance probability analysis into the iterative design optimization process, the design potential concept leads to a very effective design potential method (DPM) for robust system parameter design. It can also be applied effectively to extreme case design (ECD) by directly representing a probabilistic constraint in terms of the system performance function.

Key words reliability-based, optimization, design potential method, extreme case design, nonsmooth

1 Nomenclature

- $P(\bullet)$ probability function
- $\overline{P}_f$ prescribed failure probability target
- $\Phi(\bullet)$ standard normal cumulative distribution function (CDF)
- $F_G(\bullet)$performance function CDF, $F_G(g)=P[G(\mathbf{x})<g]$
- g^* target probabilistic performance measure, $g^* = F_G^{-1}[\Phi(-\beta_t)]$
- β_s reliability index, $\beta_s = -\Phi^{-1}[F_G(0)]$
- β_t reliability target index, $\beta_t = -\Phi^{-1}(\overline{P}_f)$
- β_t^e equivalent reliability target index for nonsmooth probabilistic constraint
- $\beta_{t,e}$ equivalent reliability target index in RBDO using SORM
- $\mathbf{d}_P^k$ design potential point (DPP) in DPM corresponding to design $\mathbf{d}^k$

Received July 25, 2000

K.K. Choi, J. Tu and Y.H. Park

Center for Computer-Aided Design, and Department of Mechanical Engineering, College of Engineering, The University of Iowa, Iowa City, IA 52242, USA
e-mail: kkchoi@ccad.uiowa.edu, jtu@ccad.uiowa.edu, ypark@ccad.uiowa.edu

2 Introduction

In the conventional reliability-based design optimization (RBDO) methodology (Enevoldsen and Sorensen 1994; Chandu and Grandhi 1995; Grandhi and Wang 1998), the probabilistic constraint is directly prescribed by the reliability index obtained from the first-order reliability method (FORM). The RBDO problem is then solved often by well-developed search methods for the constrained nonlinear optimization, where the search direction is evaluated by solving an optimization subproblem with linearized probabilistic constraints at the current design. However, the prohibitive computational cost prevented RBDO from broader engineering applications (Frangopol and Corotis 1996).

Some simplistic techniques (Parkinson *et al.* 1993) using linear statistical analysis have been developed for practical robust system parameter design, where the system parameter variability is often described only by parameter tolerance limits. Sundaresan *et al.* (1993) proposed the concept of the corner space evaluation (CSE) to better understand the design robustness. Yu and Ishii (1998) further studied the interdependency among parameter tolerances and proposed the manufacturing variation pattern (MVP) that is obtained through design of experiments (DOE) for robust system parameter design.

Tu and Choi (1997, 1999) studied performance probability analysis from the design optimization perspective and proposed a general approach for robust and efficient probabilistic constraint evaluation. The RBDO is described in the unified system space Tu *et al.* (1999b), where design potential surfaces, which are obtained analytically from FORM, are used to identify the proba-

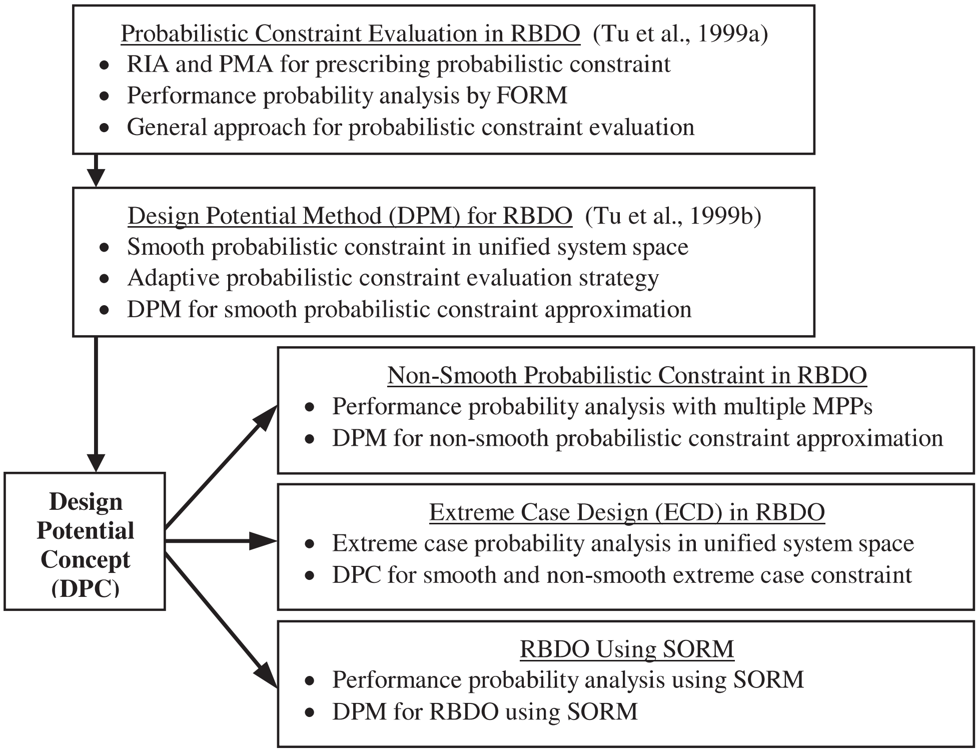

Fig. 1 Design potential concept for RBDO

bilistic constraint. More important, by comprehending the close interconnection between probability analysis and probabilistic constraint approximation in the iterative RBDO process, the highly effective design potential method (DPM) is developed for RBDO with smooth probabilistic constraints.

This paper introduces a general design potential concept (DPC) for RBDO with smooth and nonsmooth probabilistic constraints. In the unified system space, the probabilistic constraint can then be measured by design potential surfaces that are obtained from both the more accurate second-order reliability method (SORM) and the extreme case probability analysis. The DPC provides an in-depth understanding of RBDO and leads to the effective DPM for solving general RBDO problems. In addition, DPC can be applied to the extreme case design (ECD) of RBDO by directly representing the extreme case probabilistic constraint in terms of the system performance function. As shown in Fig. 1, the new RBDO methodology that combines DPM with a robust adaptive probabilistic constraint evaluation strategy can thus be used effectively and efficiently for broader engineering applications.

3 Review of probabilistic constraint evaluation in RBDO

An engineering system can be described by a set of continuous random system parameters $\mathbf{X} = [X_i]^T$ $(i = 1, \ldots, n)$, which represent the collectively exhaustive sets of outcomes $\mathbf{x} = [x_i]^T$ that can take on real values over specified tolerance limits, i.e. $\mathbf{x}^L \le \mathbf{x} \le \mathbf{x}^U$, and are completely characterized by the system parameter joint probability density function (JPDF) $f_{\mathbf{X}}(\mathbf{x})$ (Ayyub and McCuen 1997).

In the RBDO model for robust system parameter design (Enevoldsen and Sorensen 1994; Chandu and Grandhi 1995), mean values of random system parameters are often chosen as independent design variables, $\mathbf{d} = [d_i]^T \equiv [\mu_i]^T$, and the cost function is minimized subject to prescribed probabilistic constraints. For the given design, a system performance criterion is described by a system performance function $G(\mathbf{x})$ where the system fails if $G(\mathbf{x}) < 0$, and a target performance failure probability limit $\overline{P}_f$ is prescribed in a probabilistic constraint as

$$P[G(\mathbf{x}) < 0] \le \overline{P}_f \,. \tag{1}$$

The performance probability analysis of $G(\mathbf{x})$ is to evaluate its cumulative distribution function (CDF) F_G in terms of the probabilistic performance measure g as

$$F_G(g) = P[G(\mathbf{x}) < g] = \int\limits_{G(\mathbf{x})<g} \cdots \int f_{\mathbf{X}}(\mathbf{x}) \mathbf{d}x_1 \ldots \mathbf{d}x_n \,, \quad \mathbf{x}^L \le \mathbf{x} \le \mathbf{x}^U \,, \tag{2}$$

where the probability integration domain is bounded by the parameter tolerance limits. The target failure probability limit can also be represented by the reliability target index, $\beta_t = -\Phi^{-1}(\overline{P}_f)$, and the generalized probability index β_G (Madsen *et al.* 1986) is often used to measure the performance probability as

$$\beta_G(g) = -\Phi^{-1}[F_G(g)]\,. \tag{3}$$

Tu and Choi (1997, 1999) showed that the comprehensive probabilistic constraint in (1) can be consistently described by the reliability index approach (RIA) and the performance measure approach (PMA), respectively, as

$$\beta_S(\mathbf{d}) \geq \beta_t\,, \quad g^*(\mathbf{d}) \geq 0\,, \tag{4}$$

where the reliability index $\beta_S = -\Phi^{-1}[F_G(0)]$ is evaluated by reliability analysis and the target probabilistic performance measure $g^* = F_G^{-1}[\Phi(-\beta_t)]$ is evaluated by inverse reliability analysis. The PMA is inherently robust and more efficient in evaluating the inactive probabilistic constraint, while RIA could be more efficient for the violated probabilistic constraint but may not yield a solution if the design has zero failure probability.

Performance probability analysis in probabilistic constraint evaluation can be performed effectively using FORM, or more accurately by SORM (Madsen *et al.* 1986). The FORM/SORM represent an analytical approach for approximate probability integration, where general transformations (Hohenbichler and Rackwitz 1981; Madsen *et al.* 1986) between the often dependent non-normal system parameters $\mathbf{X}$ ($\mathbf{x}$-space) and the independent standardized normal variables $\mathbf{U}$ ($\mathbf{u}$-space) at the design $\mathbf{d}^k$ (Tu *et al.* 1999b) are

$$\mathbf{u} = [u_i]^T = \mathbf{T}(\mathbf{x};\mathbf{d}^k) = [T_i(\mathbf{x};\mathbf{d}^k)]^T\,,$$

$$\mathbf{x} = [x_i]^T = \mathbf{T}^{-1}(\mathbf{u};\mathbf{d}^k) = [T_i^{-1}(\mathbf{u};\mathbf{d}^k)]^T\,,$$

$$i = 1,\ldots,n\,. \tag{5}$$

Thus, the performance function $G(\mathbf{x})$ can then be expressed in the $\mathbf{u}$-space as

$$G(\mathbf{x}) = G[\mathbf{T}^{-1}(\mathbf{u};\mathbf{d}^k)] = G_U(\mathbf{u})\,. \tag{6}$$

In the $\mathbf{u}$-space, the point on surface $G_U(\mathbf{u}) = g$ with the maximum joint probability density is the point with the minimum distance β from the origin and is called the most probable point (MPP) $\mathbf{u}_g^*$ (or $\mathbf{x}_g^*$ in the $\mathbf{x}$-space). The minimum distance is defined in FORM as the first-order probability index $\beta_{\mathrm{FORM}}(g)$ as

$$\beta_{\mathrm{FORM}}(g) = \beta = \|\mathbf{u}_g^*\| = \|\mathbf{T}(\mathbf{x}_g^*;\mathbf{d}^k)\|\,. \tag{7}$$

The principal curvatures κ_ℓ^g $(\ell = 1,2,\ldots,n-1)$ at the MPP $\mathbf{u}_g^*$ are used in the more accurate SORM in defining the second-order probability index $\beta_{\mathrm{SORM}}(g)$ as

$$\beta_{\mathrm{SORM}}(g) = \Psi\left(\beta_{\mathrm{FORM}}(g), \kappa_\ell^g\right)\,, \tag{8}$$

where the SORM operator $\Psi(\bullet)$ is explicitly defined, such as the one-term formula of Breitung (1984):

$$\Psi\left(\beta_{\mathrm{FORM}}(g), \kappa_\ell^g\right) = \Phi^{-1}\left\{1 - \Phi\left[-\beta_{\mathrm{FORM}}(g)\right]\prod_{\ell=1}^{n-1}\left[1 - \beta_{\mathrm{FORM}}(g)\kappa_\ell^g\right]^{-1/2}\right\} \tag{9}$$

and a more accurate three-term formula that was proposed by Tvedt (1990).

4 Design potential concept for RBDO in unified system space

In the RBDO model, probabilistic constraints (4) are defined on the design variable space ($\mathbf{d}$-space), while system performance functions are defined in the system parameter sample space ($\mathbf{x}$-space). For the design d^k, the probability integration domain is the subset $x^L(d^k) \leq \mathbf{x} \leq \mathbf{x}^U(\mathbf{d}^k)$ in the $\mathbf{x}$-space, which corresponds to the standardized normal reference space ($\mathbf{u}$-space) in FORM/SORM.

The unified system space is then defined (Tu and Choi 1999) by mapping the $\mathbf{d}$-space into the $\mathbf{x}$-space as

$$\mathbf{x} = \mathbf{T}^{-1}(\mathbf{0};\mathbf{d})\,. \tag{10}$$

Thus, the design $\mathbf{d}^k$ in the unified system space will correspond to the origin $\mathbf{T}^{-1}(\mathbf{0};\mathbf{d}^k)$ of the $\mathbf{u}$-space, and the design potential surface that corresponds to the specific probability measurement β_P around $\mathbf{d}^k$ is defined as $\|\mathbf{T}(\mathbf{x};\mathbf{d}^k)\| = \beta_P$.

4.1 Design potential concept for probabilistic constraint

At the design $\mathbf{d}^k$, if first-order reliability analysis finds a unique MPP $\mathbf{x}_{g=0}^*$ on the performance function limit-state surface $G(\mathbf{x}) = 0$, there must be a design potential point (DPP) $\mathbf{d}_P^k$ that renders the probabilistic constraint active so that its corresponding design potential surface of the reliability target, $\|\mathbf{T}(\mathbf{x};\mathbf{d}_P^k)\| = \beta_t$, is tangential to the performance function limit-state surface at the same MPP, as shown in Fig. 2.

Thus, the limit-state surface of the probabilistic constraint $\beta_S(\mathbf{d}) = \beta_t$ [also $g^*(\mathbf{d}) = 0$] can be constructed conceptually in the unified system space by tangentially sweeping the design potential surface of the reliability target along the feasible side of the performance function limit-state surface $G(\mathbf{x}) = 0$.

The RBDO process can then be described by the design potential concept (DPC) as searching for the minimum cost design $\mathbf{d}_{\beta_t}^{\mathrm{opt}}$ that can fit the corresponding design potential surface of the reliability target, $\|\mathbf{T}(\mathbf{x};\mathbf{d}_{\beta_t}^{\mathrm{opt}})\| = \beta_t$, into the feasible side of all performance function limit-state surfaces, as shown in Fig. 3.

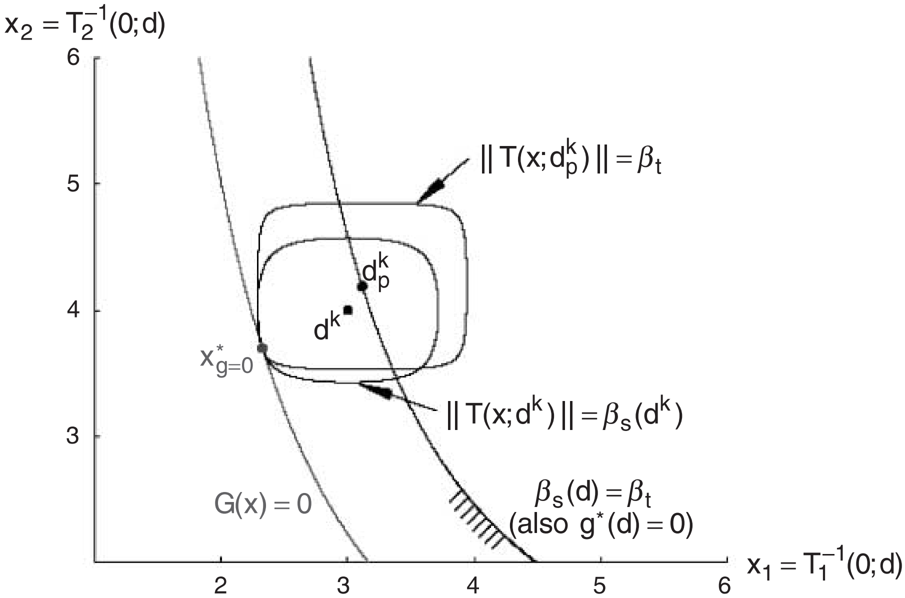

Fig. 2 Illustration of DPC in unified system space

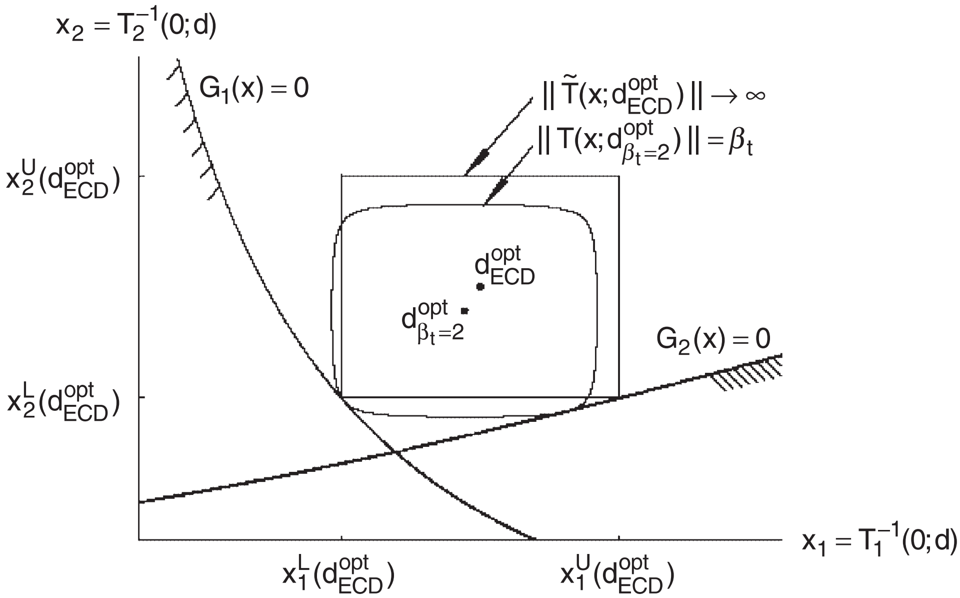

Fig. 3 Optimality of RBDO in unified system space

If FORM is used for performance probability analysis, one probabilistic constraint is identified active at the design $\mathbf{d}_P^k$ if $\beta_{S,\mathrm{FORM}}(\mathbf{d}_P^k) = \|\mathbf{T}(\mathbf{x}_{g=0}^*; \mathbf{d}_P^k)\| = \beta_t$. Thus, the corresponding surface of the reliability target is explicitly defined as $\|\mathbf{T}(\mathbf{x}; \mathbf{d}_P^k)\| = \beta_t$ and is called the surface of the reliability target.

In the extreme case design (ECD) of RBDO where the target failure probability is zero, i.e. $\beta \to \infty$, the design potential surface of the reliability target is called the surface of tolerance limits $\|\tilde{T}(\mathbf{x}; \mathbf{d}_P^k)\| \to \infty$, which encloses the corresponding probability integration domain for design $\mathbf{d}_P^k$, i.e. $\mathbf{x}^L(\mathbf{d}_P^k) \le \mathbf{x} \le \mathbf{x}^U(\mathbf{d}_P^k)$.

If SORM is used for more accurate probability analysis, one probabilistic constraint is active if $\beta_{S,\mathrm{SORM}}(\mathbf{d}_P^k) = \Psi(\beta_{S,\mathrm{FORM}}(\mathbf{d}_P^k), \kappa_\ell^0) = \beta_t$, where κ_ℓ^0 are principal curvatures of the limit-state surface in the **u**-space. Thus, the corresponding design potential surface of the reliability target is $\|\mathbf{T}(\mathbf{x}; \mathbf{d}_P^k)\| = \beta_{t,e} = \Psi^{-1}(\beta_t, \kappa_\ell^0)$, where the implicitly defined $\beta_{t,e}$ is called the equivalent target reliability index.

4.2 Design potential method for probabilistic constraint approximation

The DPC provides an in-depth understanding of RBDO and illustrates that performance probability analysis in probabilistic constraint evaluation yields important system design information. This leads to the design potential method (DPM) that can significantly accelerate the RBDO convergence by taking advantage of the close coupling of performance probability analysis and the iterative design optimization process.

At the design $\mathbf{d}^k$, if the probabilistic constraint can be evaluated in RIA by FORM to find the MPP $\mathbf{x}^*_{g=0}$ (or $\mathbf{u}^*_{g=0}$ in its corresponding $\mathbf{u}$-space) on the performance function limit-state surface $G(\mathbf{x})=0$, then the DPP $\mathbf{d}^k_P$ that is on the probabilistic constraint limit-state surface and its corresponding MPP $\mathbf{u}^{**}$ in the $\mathbf{u}$-space must satisfy

$$\mathbf{x}^*_{g=0} = \mathbf{T}^{-1}(\mathbf{u}^*_{g=0};\mathbf{d}^k) = \mathbf{T}^{-1}(\mathbf{u}^{**};\mathbf{d}^k_P)\,,$$

$$\mathbf{u}^{**} = -\beta_t \cdot \frac{\nabla_{\mathbf{u}} G_U(\mathbf{u}^{**})}{\|\nabla_{\mathbf{u}} G_U(\mathbf{u}^{**})\|}\,. \tag{11}$$

Thus, $\mathbf{d}^k_P$ and $\mathbf{u}^{**}$ can be determined by solving the nonlinear equation system (11) by using the Newton's method (Atkinson 1989). If SORM is required for more accurate probability integration, β_t in (11) is replaced by $\beta_{t,e} = \Psi^{-1}(\beta_t, \kappa^0_\ell)$.

Note that DPP is in effect obtained as a by-product of performance probability analysis without requiring additional costly performance function evaluation (Tu and Choi 1999). The RBDO problem is often solved using gradient-based search methods (Arora 1989), such as SLP, SQP, and MFD. Instead of the conventional linearization at the current design $\mathbf{d}^k$, the probabilistic constraint is linearized by DPM at the DPP $\mathbf{d}^k_P$ as

$$\nabla^T_{\mathbf{d}} \beta_S(\mathbf{d}^k_P)\cdot(\mathbf{d}-\mathbf{d}^k_P) \geq 0 \tag{12}$$

Since the DDP $\mathbf{d}^k_P$ is located on the limit-state surface of the probabilistic constraint, the constraint approximation in DPM becomes exact at $\mathbf{d}^k_P$. Therefore, DPM provides better constraint approximation without additional costly evaluation of the system performance function. Consequently, a higher rate of convergence in solving the RBDO problem can be achieved (Tu *et al.* 1999b).

5 Nonsmooth probabilistic constraint in RBDO

Even for a smooth system performance function, the corresponding probabilistic constraint could be nonsmooth if reliability analysis by FORM/SORM finds multiple MPPs. In such case, since each MPP is a local minimum in performance probability analysis, the corresponding system JPDF has multiple local maxima in the probability integration domain. The performance probability can then be represented (Madsen *et al.* 1986) as

$$\Phi\left[\beta_s(\mathbf{d}^k)\right] = \Theta\Big\{\Phi\left[\beta^a_s(\mathbf{d}^k)\right], \Phi\left[\beta^b_s(\mathbf{d}^k)\right], \ldots, \Phi\left[\beta^{nm}_s(\mathbf{d}^k)\right]\Big\}\,, \tag{13}$$

where $\Theta(\bullet)$ is a transformation operator and MPPs and their corresponding probability indexes are identified from a to nm. When multiple MPPs at the design $\mathbf{d}^k$ are sparsely distributed on the performance function limit-state surface, the transformation operator $\Theta(\bullet)$ can be simplified as

$$\Theta\left[\Phi(\beta^a_s), \Phi(\beta^b_s), \ldots, \Phi(\beta^{nm}_s)\right] = \Phi(\beta^a_s)\cdot\Phi(\beta^b_s)\cdot \ldots \cdot \Phi(\beta^{nm}_s)\,. \tag{14}$$

In practical applications, however, two local MPPs that correspond to the two lowest probability indexes, β^a_s and β^b_s, are often adequate to approximate the reliability index of one performance function. For illustration purpose, a case with two MPPs is used in a two-dimensional example in this paper.

5.1 DPC for nonsmooth probabilistic constraint

At the design $\mathbf{d}^k$, the reliability index function with two local MPPs, i.e. $\mathbf{x}^*_a$ and $\mathbf{x}^*_b$ on the performance function limit-state surface $G(\mathbf{x})=0$ in the unified system space as shown in Fig. 4, or $\mathbf{u}^*_a$ and $\mathbf{u}^*_b$ on $G_U(\mathbf{u})=0$ in the corresponding $\mathbf{u}$-space as shown in Fig. 5, can be computed in RIA as

$$\beta_s(\mathbf{d}^k) = \Phi^{-1}\left\{\Phi\left[\beta^a_s(\mathbf{d}^k)\right]\cdot\Phi\left[\beta^b_s(\mathbf{d}^k)\right]\right\} \tag{15}$$

where

$$\beta^a_s(\mathbf{d}^k) = \|\mathbf{u}^*_a\| = \|\mathbf{T}(\mathbf{x}^*_a;\mathbf{d}^k_P)\|\,,$$

and

$$\beta^b_s(\mathbf{d}^k) = \|\mathbf{u}^*_b\| = \|\mathbf{T}(\mathbf{x}^*_b;\mathbf{d}^k_P)\|\,.$$

Using DPC, the nonsmooth RBDO can be viewed as searching for the minimum cost design that can fit the surface of the equivalent reliability target, $\|\mathbf{T}(\mathbf{x};\mathbf{d}^{\mathrm{opt}}_{\beta_t})\| = \beta^e_t$, to the feasible side of the performance function limit-state surface. The equivalent reliability target index, $\beta^e_t = \min\{\beta^a_s(\mathbf{d}^{\mathrm{opt}}_{\beta_t}), \beta^b_s(\mathbf{d}^{\mathrm{opt}}_{\beta_t})\}$, is used so that the true reliability index at the optimum design is maintained as

$$\beta_s(\mathbf{d}^{\mathrm{opt}}_{\beta_t}) = \Phi^{-1}\left\{\Phi\left[\beta^a_s(\mathbf{d}^{\mathrm{opt}}_{\beta_t})\right]\cdot\left[\beta^b_s(\mathbf{d}^{\mathrm{opt}}_{\beta_t})\right]\right\} = \beta_t\,.$$

If the RBDO optimum $\mathbf{d}^{\mathrm{opt}}_{\beta_t}$ is at the nonsmooth point $\mathbf{d}^{\mathrm{nsp}}_{\beta_t}$, as shown in Fig. 6, then $\beta^e_t = \beta^a_s(\mathbf{d}^{\mathrm{nsp}}_{\beta_t}) = \beta^b_s(\mathbf{d}^{\mathrm{nsp}}_{\beta_t})$ and thus

$$\beta_s(\mathbf{d}^{\text{nsp}}_{\beta_t}) = \Phi^{-1}\left\{\Phi\left[\beta^e_s(\mathbf{d}^{\text{nsp}}_{\beta_t})\right]\cdot\left[\beta^e_s(\mathbf{d}^{\text{nsp}}_{\beta_t})\right]\right\} = \beta_t\,.$$

The conventional RBDO methodology cannot deal with the nonsmooth probabilistic constraint because it is not differentiable with respect to design variables. Instead, DPM can be applied to construct multiple simultaneously linearized probabilistic constraints by (12) at all DPPs corresponding to multiple MPPs in defining the search direction determination subproblem, such as in the example with two MPPs as

$$\nabla^T_{\mathbf{d}} G(\mathbf{x}^*_a)\cdot(\mathbf{d}-\mathbf{d}^k_{P,a}) \geq 0\,,$$

$$\nabla^T_{\mathbf{d}} G(\mathbf{x}^*_b)\cdot(\mathbf{d}-\mathbf{d}^k_{P,b}) \geq 0\,, \tag{16}$$

where $\mathbf{d}^k_{P,a}$ and $\mathbf{d}^k_{P,b}$ are DPPs that can be obtained by first substituting β_t in (11) with β^e_t and then solving the equation system that is comprised of (11).

Note that the equivalent reliability target index β^e_t cannot be pre-determined if $\mathbf{d}^{\text{opt}}_{\beta_t}$ is not the nonsmooth point of the probabilistic constraint. However, it is known

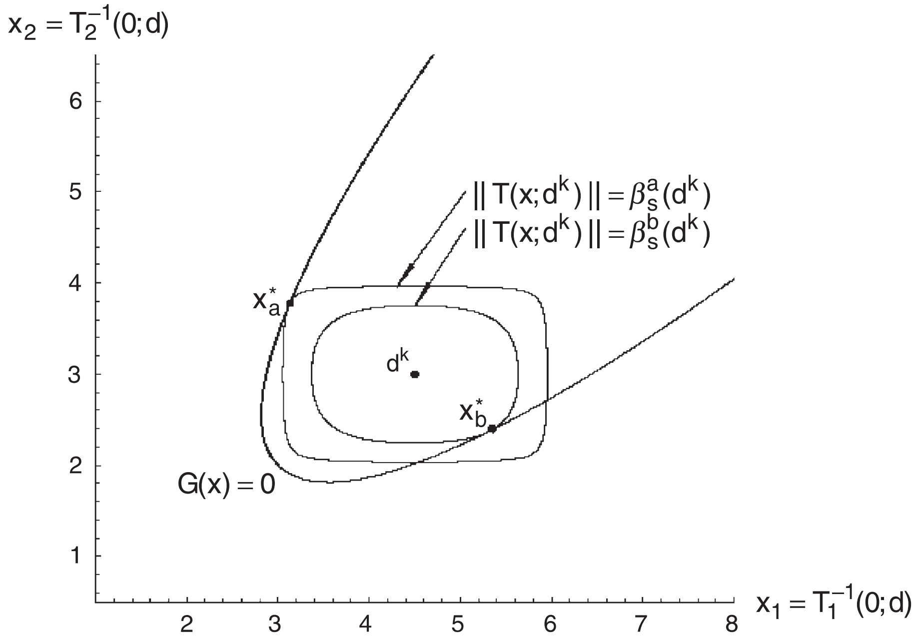

Fig. 4 Performance probability analysis with multiple MPPs

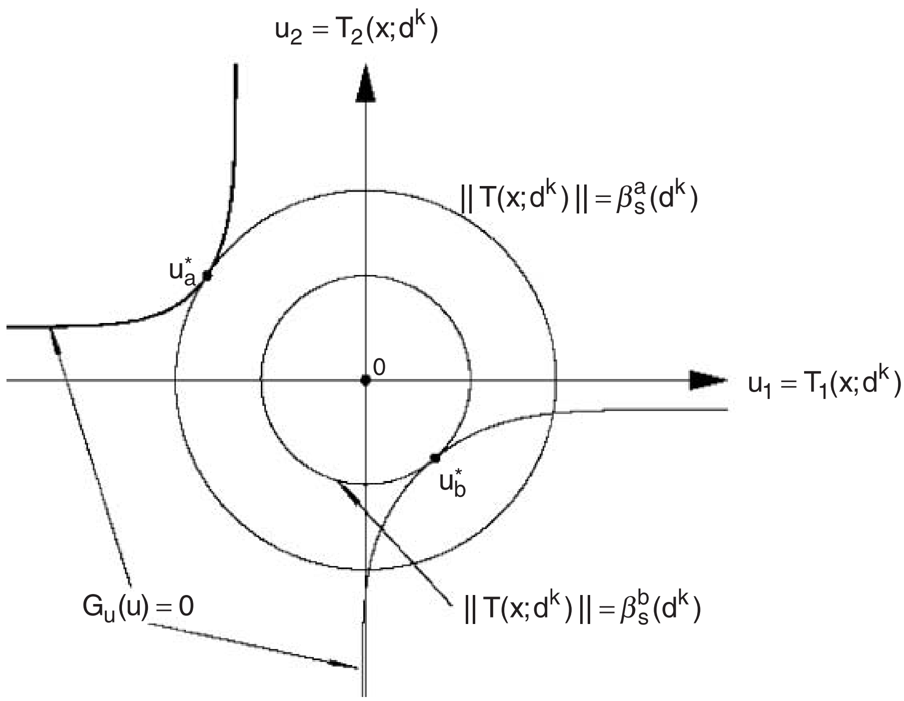

Fig. 5 Multiple MPPs in the **u**-space

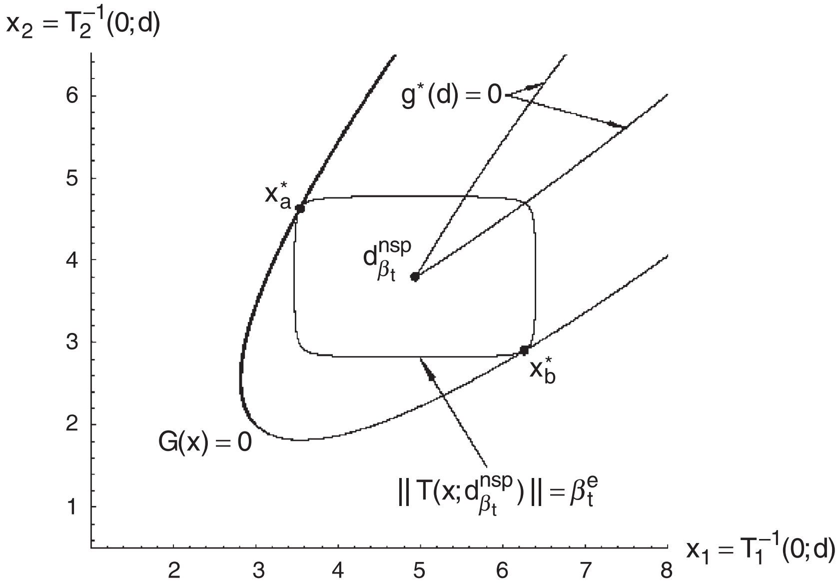

Fig. 6 Nonsmooth probabilistic constraint in RBDO

that $\beta_t \le \beta_t^e \le \Phi^{-1}[\sqrt{\Phi(\beta_t)}]$ because

$$\beta_t^e = \min\{\beta_s^a(\mathbf{d}_{\beta_t}^{\text{opt}}), \beta_s^b(\mathbf{d}_{\beta_t}^{\text{opt}})\}$$

and

$$\Phi(\beta_t) = \Phi\left[\beta_s^a(\mathbf{d}_{\beta_t}^{\text{opt}})\right] \cdot \Phi\left[\beta_s^b(\mathbf{d}_{\beta_t}^{\text{opt}})\right] .$$

A two-step strategy with iterative DPM for multidimensional case is proposed here.

Step 1: Set $\beta_t^e = \Phi^{-1}[\sqrt{\Phi(\beta_t)}]$ and perform RBDO using (16) to converge to a feasible design $\mathbf{d}_{\text{DPM}}^e$ with $\min\{\beta_s^a(\mathbf{d}_{\text{DPM}}^e), \beta_s^b(\mathbf{d}_{\text{DPM}}^e)\} = \beta_t^e$.

Step 2: If $\beta_s^a(\mathbf{d}_{\text{DPM}}^e) = \beta_s^b(\mathbf{d}_{\text{DPM}}^e) = \beta_t^e$, then the RBDO optimum is $\mathbf{d}_{\beta_t}^{\text{opt}} = \mathbf{d}_{\text{DPM}}^e = \mathbf{d}_{\beta_t}^{\text{nsp}}$ that renders both of (16) active, as shown in Fig. 6. Otherwise,

- if $\beta_s^b(\mathbf{d}_{\text{DPM}}^e) > \beta_s^a(\mathbf{d}_{\text{DPM}}^e) = \beta_t^e$, the RBDO optimum will only render the first of (16) active. Thus, continue RBDO iteration by resetting $\beta_t^e(\mathbf{d}^k) = \Phi^{-1}\{\Phi(\beta_t)/\Phi[\beta_s^b(\mathbf{d}^k)]\}$ until it converges to $\mathbf{d}_{\beta_t}^{\text{opt}}$ so that $\beta_S(\mathbf{d}_{\beta_t}^{\text{opt}}) = \beta_t$;
- if $\beta_s^a(\mathbf{d}_{\text{DPM}}^e) > \beta_s^b(\mathbf{d}_{\text{DPM}}^e) = \beta_t^e$, the RBDO optimum will only render the second of (16) active. Thus, continue RBDO iteration by resetting $\beta_t^e(\mathbf{d}^k) = \Phi^{-1}\{\Phi(\beta_t)/\Phi[\beta_s^b(\mathbf{d}^k)]\}$ until it converges to $\mathbf{d}_{\beta_t}^{\text{opt}}$.

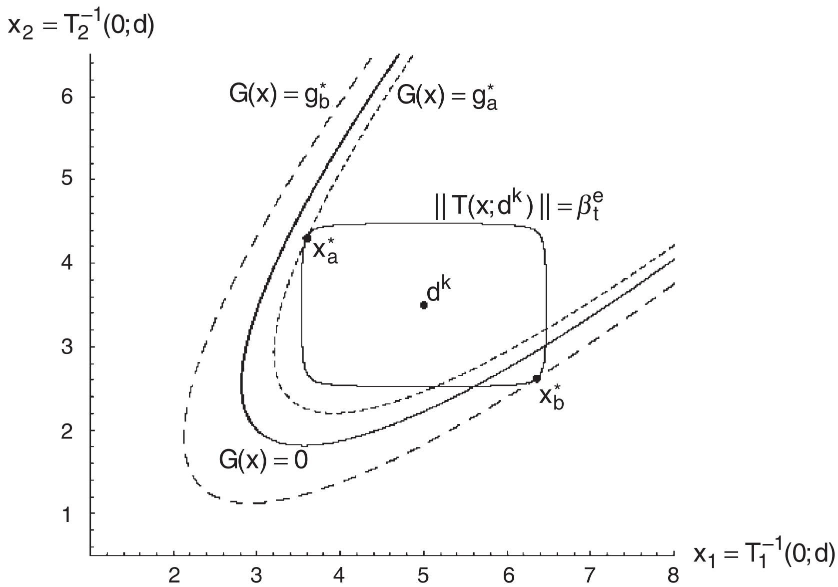

Fig. 7 Extended PMA for nonsmooth probabilistic constraint approximation

5.2 Robust nonsmooth probabilistic constraint approximation

The DPM requires RIA for reliability analysis of the performance function. The RIA is more efficient for violated probabilistic constraints but it may not yield a solution if the design has zero failure probability. In such case, a robust performance measure approach (PMA) can be extended to apply for nonsmooth probabilistic constraint approximation by assuming the equivalent reliability target index $\beta_t^e = \Phi^{-1}\left(\sqrt{\Phi(\beta_t)}\right)$.

At the design $\mathbf{d}^k$, the probabilistic constraint is approximated by the extended PMA, where the equivalent target probabilistic performance measures corresponding to multiple MPPs on the design potential surface, $\|\mathbf{T}(\mathbf{x};\mathbf{d}^k)\| = \beta_t^e$, are obtained as $g_a^*(\mathbf{d}^k)$ and $g_b^*(\mathbf{d}^k)$, as illustrated in Fig. 7. Similar to PMA for smooth probabilistic constraint (Tu *et al.* 1999a), two simultaneously linearized probabilistic constraints can be obtained at each MPP as

$$g_a^*(\mathbf{d}^k) + \nabla_{\mathbf{d}}^T G(\mathbf{x}^{*,a}) \cdot (\mathbf{d} - \mathbf{d}^k) \geq 0\,,$$

$$g_b^*(\mathbf{d}^k) + \nabla_{\mathbf{d}}^T G(\mathbf{x}^{*,b}) \cdot (\mathbf{d} - \mathbf{d}^k) \geq 0\,. \tag{17}$$

The extended PMA is inherently robust because it is well defined in the probability integration domain, and it is more efficient than the first step in the proposed iterative DPM with RIA if the nonsmooth probabilistic constraint is inactive. Because PMA will converge to the same feasible design as in the first step of the DPM with RIA, PMA can be used as an alternative that is chosen adaptively depending on the estimated marginal status of the nonsmooth probabilistic constraint in the iterative RBDO process (Tu and Choi 1999).

6 Extreme case design in RBDO

In some engineering applications where system parameter tolerance limits are well controlled by the quality assurance procedure in the manufacturing process, the optimum design with zero failure probability may be required and RBDO becomes an extreme case design (ECD). An extreme case probabilistic constraint is expressed as

$$P\left[G(\mathbf{x}) < 0\right] = 0\,, \tag{18}$$

which does not yield a solution for RIA but can still be represented in PMA as

$$g^L(\mathbf{d}) = F_G^{-1}(0) \geq 0\,, \tag{19}$$

and the evaluation of the minimum probabilistic performance measure is extreme case analysis that can be performed by solving an unconstrained nonlinear optimization problem as

$$g^L(\mathbf{d}^k) = \min[G(\mathbf{x})]\,, \quad \mathbf{x}^L(\mathbf{d}^k) \leq \mathbf{x} \leq \mathbf{x}^U(\mathbf{d}^k)\,, \tag{20}$$

where the optimum solution $\tilde{\mathbf{x}}^*$ is called the extreme value point (EVP) and thus $g^L(\mathbf{d}^k) = G(\tilde{x}^*)$.

6.1 Extreme case analysis in unified system space

In ECD, the system parameter tolerance limits are often directly represented in terms of the nominal mean value $\tilde{\mu}_i$ and nominal tolerance Δ_i as $x_i^L = \tilde{\mu}_i - \Delta_i$ and $x_i^U = \tilde{\mu}_i + \Delta_i$. The nominal mean value is then often used as an independent design variable, while the nominal tolerance is either constant or depends on the nominal mean value, e.g., $\mathbf{d} = [d_i]^T \equiv [\tilde{\mu}_i]^T$ and $\Delta(\mathbf{d}) = [\Delta_i(\tilde{\mu}_i)]^T$ $(i = 1, 2, \ldots, n)$. Thus, the mapping from the $\mathbf{d}$-space to the $\mathbf{x}$-space becomes simply as $\mathbf{x} = \mathbf{d}$.

As shown in Fig. 3, the ECD optimum is identified by DPC as the minimum cost design that can fit the surface of tolerance limits, $\|\tilde{T}(\mathbf{x};\mathbf{d}_{\mathrm{ECD}}^{\mathrm{opt}})\| \to \infty$, to the feasible side of all system performance function limit-state surfaces. It is clear that $\mathbf{d}_{\mathrm{ECD}}^{\mathrm{opt}}$ is more conservative than the RBDO optimum $\mathbf{d}_{\beta_t}^{\mathrm{opt}}$ with finite β_t, where the surface of the reliability target expands as β_t increases.

At the design $\mathbf{d}^k$, evaluations of extreme case probabilistic constraints are illustrated in the unified system space in Fig. 8, where two EVPs, $\tilde{x}^{*,1}$ and $\tilde{x}^{*,2}$, are vertices of the surface of tolerance limits, $\|\tilde{\mathbf{T}}(\mathbf{x};\mathbf{d}^k)\| \to \infty$, and minimum probabilistic performance measures of two probabilistic constraints are $g_j^L(\mathbf{d}^k) = G_j(\tilde{x}^{*,j})$ $(j = 1, 2)$. Thus, $\tilde{x}^{*,1}$ is on the first performance function iso-surface $G_1(\mathbf{x}) = G_1(\tilde{x}^{*,1}) = g_1^L(\mathbf{d}^k)$ and $\tilde{x}^{*,2}$ is on the second performance function iso-surface $G_2(\mathbf{x}) = G_2(\tilde{x}^{*,2}) = g_2^L(\mathbf{d}^k)$.

Extreme case analysis represents the extreme case of PMA and the surface of tolerance limits represents the upper-limit of the design potential surface, which also helps to explain the nonexistence of a solution for RIA. For the second probabilistic constraint where the minimum probabilistic performance measure is $g_2^L(\mathbf{d}^k) > 0$ and thus $P\left[G_2(\mathbf{x}) < 0\right] = 0$, RIA may not have a solution and thus reliability analysis by FORM/SORM may not yield a solution because the performance limit-state surface $G_2(\mathbf{x}) = 0$ has no definition in the corresponding probability integration domain $\mathbf{x}^L(\mathbf{d}^k) \leq \mathbf{x} \leq \mathbf{x}^U(\mathbf{d}^k)$.

Equation (19) can then be directly linearized at design $\mathbf{d}^k$ in defining the search direction determination subproblem as

$$g^L(\mathbf{d}^k) + \nabla_{\mathbf{d}}^T g^L(\mathbf{d}^k) \cdot (\mathbf{d} - \mathbf{d}^k) \geq 0\,, \tag{21}$$

where

$$\nabla_{\mathbf{d}}^T g^L(\mathbf{d}^k) = \nabla_{\mathbf{x}}^T G(\tilde{x}^*)\,. \tag{22}$$

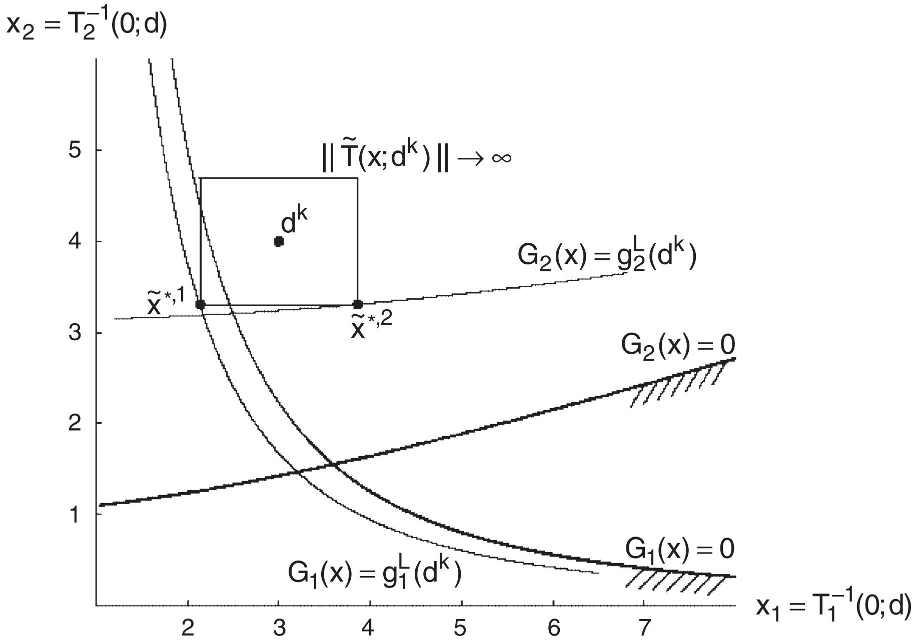

Fig. 8 Illustration of extreme case analysis in unified system space

Note that (21) and (22) can be used robustly in the general ECD (Tu and Choi 1999). However, DPC can be directly applied to represent an extreme case probabilistic constraint in terms of the system performance function and the ECD problem can then be solved more effectively by treating it as a traditional deterministic optimization problem.

6.2
Direct application of DPC in smooth ECD

Because EVPs are often at vertices of the corresponding probability integration domain that is enclosed by the surface of tolerance limits, as shown in Fig. 8, the extreme case probabilistic constraint can be simplified by DPC. For convenience, a design potential vector is defined as

$$\varDelta_P(\mathbf{d}) = -\mathbf{S}_G^T(\mathbf{d}) \cdot \varDelta(\mathbf{d}) \,, \tag{23}$$

where $\mathbf{S}_G$ is defined as the signum vector of the performance function gradient,

$$\mathbf{S}_G(\mathbf{d}) \quad = \left[\frac{\partial G(\mathbf{d})}{\partial d_i} \cdot \left| \frac{\partial G(\mathbf{d})}{\partial d_i} \right|^{-1} \right]^T \,. \tag{24}$$

If the EVP $\tilde{x}^*$ at the design $\mathbf{d}^k$ is the vertex of the surface of tolerance limits, it can be expressed as

$$\tilde{x}^* = \mathbf{d}^k + \varDelta_P(\mathbf{d}^k) \,, \tag{25}$$

and if $\tilde{x}^*$ is on the performance function limit-state surface, then $\mathbf{d}^k$ must render the extreme case probabilistic constraint active, i.e.

$$g^L(\mathbf{d}^k) = G(\tilde{x}^*) = G[\mathbf{d}^k + \varDelta_P(\mathbf{d}^k)] = 0 \,. \tag{26}$$

Thus, the extreme case probabilistic constraint in (19) can be directly expressed by DPC in terms of the system performance function as

$$g^L(\mathbf{d}) = G[\mathbf{d} + \varDelta_P(\mathbf{d})] \geq 0 \,. \tag{27}$$

If $G(\mathbf{x})$ is monotonic in terms of all system parameters, then the signum vector $\mathbf{S}_G$ is a constant vector and EVP is always at a vertex of the surface of tolerance limits (Sundaresan *et al.* 1993). In such case, (27) is the exact representation of the extreme case probabilistic constraint and ECD can essentially be solved as a traditional deterministic optimization problem.

Derived from linear statistic analysis by calculating the transmitted variation, so-called worst case tolerance analysis (Parkinson *et al.* 1993) essentially used a first-order Taylor expansion of (27) to prescribe the probabilistic constraint as

$$G(\mathbf{d}) + \nabla_{\mathbf{d}}^T G(\mathbf{d}) \cdot \varDelta_P(\mathbf{d}) \geq 0 \,, \tag{28}$$

which often yields overly conservative design or even infeasible design in practical applications because the system performance functions are often highly nonlinear in terms of system parameters (Sundaresan *et al.* 1993; Yu and Ishii 1998).

6.3
Direct application of DPC in nonsmooth ECD

If $G(\mathbf{x})$ is not monotonic in terms of some system parameters and its limit-state surface encloses a convex corner, then there exists a nonsmooth point $\mathbf{d}_{\mathrm{ECD}}^{\mathrm{nsp}}$ on the limit-state surface of the extreme case probabilistic constraint. For example, in a two-dimensional example as shown in

Fig. 9, two EVPs on the performance function limit-state surface correspond to the design $\mathbf{d}_{\mathrm{ECD}}^{\mathrm{nsp}}$ that is the nonsmooth point on the limit-state surface of the probabilistic constraint. In case that all EVPs are at vertices of the surfaces of tolerance limits, the extreme case probabilistic constraint can always be completely described in DPC by multiple simultaneous constraints in terms of the performance function, whose intersection is the nonsmooth point $\mathbf{d}_{\mathrm{ECD}}^{\mathrm{nsp}}$. For the example in Fig. 9, two simultaneous constraints corresponding to two EVPs are expressed explicitly as

$$G[\mathbf{d}+\Delta_P^a(\mathbf{d})] \geq 0\,, \quad G[\mathbf{d}+\Delta_P^b(\mathbf{d})] \geq 0\,. \tag{29}$$

If the nonsmooth point $\mathbf{d}_{\mathrm{ECD}}^{\mathrm{nsp}}$ is the ECD optimum $\mathbf{d}_{\mathrm{ECD}}^{\mathrm{opt}}$, then both constraints in (29) are active, as shown in Fig. 9. Two different signum vectors of the performance function gradient at EVPs can be obtained by comparing signum vectors at all vertices of the surface of tolerance limits.

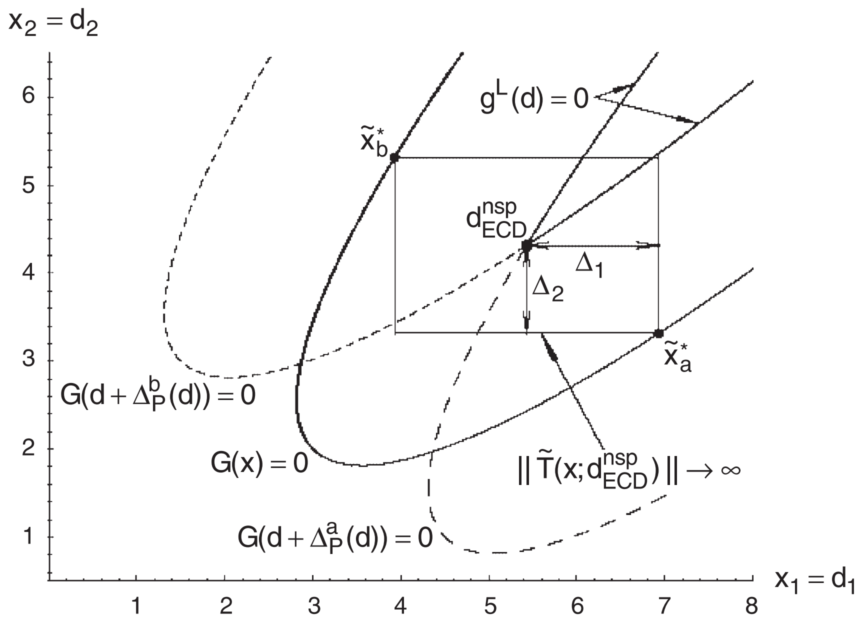

Fig. 9 DPC for nonsmooth extreme case probabilistic constraint

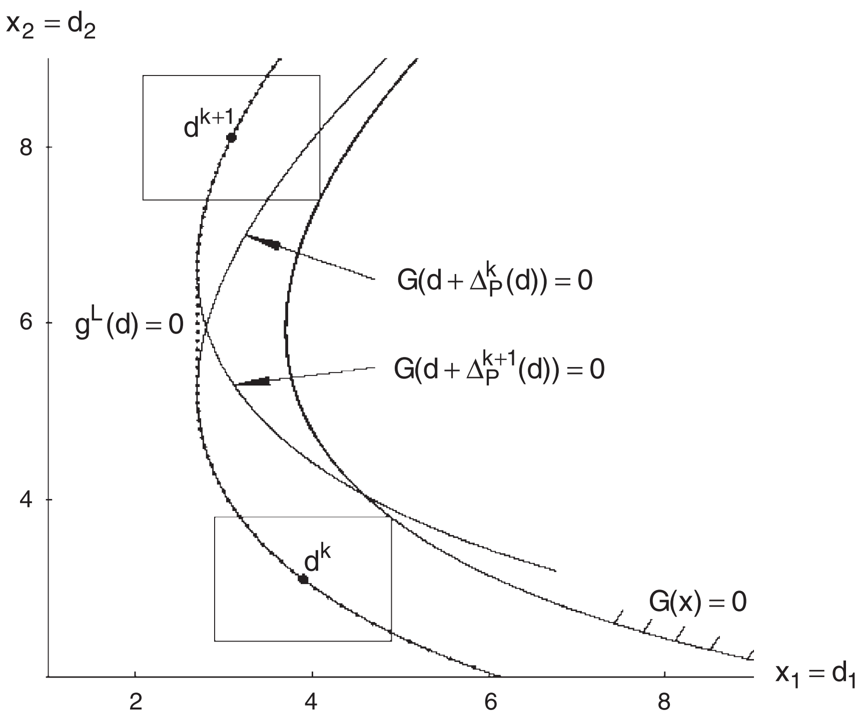

Fig. 10 Approximation error from direct application of DPC in ECD

6.4 Approximation error from direct application of DPC in ECD

If $G(\mathbf{x})$ is not monotonic in terms of some system parameters and its limit-state surface encloses a concave corner, then the EVP may not be a vertex of the surface of tolerance limits. In such case, (27) only represents an approximation of the extreme case probabilistic constraint.

As shown in Fig. 10 where $G(\mathbf{x})$ is not monotonic in terms of x_2, the two possible signum vectors are $\mathbf{S}_G(\mathbf{d}^k) = [+1, +1]^T$ and $\mathbf{S}_G(\mathbf{d}^{k+1}) = [+1, -1]^T$. Thus, (27) can be completely represented by the combination of two constraints as

$$G[\mathbf{d} + \Delta_P^k(\mathbf{d})] \geq 0 \,,$$

$$G[\mathbf{d} + \Delta_P^{k+1}(\mathbf{d})] \geq 0 \,, \tag{30}$$

where $\Delta_P^k(\mathbf{d}) = -\mathbf{S}_G^T(\mathbf{d}^k) \cdot \Delta(\mathbf{d})$ and $\Delta_P^{k+1}(\mathbf{d}) = -\mathbf{S}_G^T(\mathbf{d}^{k+1}) \cdot \Delta(\mathbf{d})$.

Note that a gap exists between the limit-state surface of the exact probabilistic constraint $g^L(\mathbf{d}) = 0$ (dotted curve) and the limit-state surfaces of the combined constraints of (30). This is because the EVP corresponding to the design in the gap region is not a vertex of the surface of tolerance limits. Although (30) provides a correct description of the extreme case probabilistic constraint other than the gap region, the approximation error of DPC in the gap region always yields an infeasible design. In contrast, extreme case analysis can always be used robustly to perform probabilistic constraint evaluation and to solve the ECD problem.

Therefore, a two-step strategy is suggested for ECD. First, the direct application of DPC is used because it yields a higher rate of convergence. Then, extreme case analysis of (20) is used to check the feasibility of the optimum design obtained from the direct application of DPC and (21) can be used iteratively to correct the approximation error if necessary.

7 Examples

Using FORM/SORM for performance probability analyses, a smooth RBDO problem is solved using DPM, and its ECD is performed by direct application of DPC. Then, a nonsmooth RBDO problem is solved using DPM, and its ECD is also given for comparison.

7.1 Smooth RBDO and ECD

Consider a system described by two independent uniformly distributed system parameters $\mathbf{X} = [X1, X_2]^T$ with constant standard deviations, $\sigma_1 = 1/2$ and $\sigma_2 = 2/5$. The design variable is chosen as $\mathbf{d} = [d_1, d_2]^T \equiv [\mu_1, \mu_2]^T$, and system parameter CDFs are expressed in terms of design variables as

$$F_{X_i}(x_i) = \frac{x_i - d_i}{2\sqrt{3}\sigma_i + 1/2} \,,$$

$$d_i - \sqrt{3}\sigma_i \leq x_i \leq d_i + \sqrt{3}\sigma_i \,, \quad i = 1, 2 \,. \tag{31}$$

The transformations between the $\mathbf{u}$-space and the $\mathbf{x}$-space at design $\mathbf{d}^k$ are nonlinear as

$$u_i = T_i(x_i; d_i^k) = \Phi^{-1}[F_{X_i}(x_i)] =$$

$$\Phi^{-1}\left(\frac{x_i - d_i}{2\sqrt{3}\sigma_i + 1/2}\right) \,, \quad i = 1, 2 \,, \tag{32}$$

$$x_i = T_i^{-1}(u_i; d_i^k) = F_{X_i}^{-1}[\Phi(u_i)] =$$

$$d_i - \sqrt{3}\sigma_i + 2\sqrt{3}\sigma_i \Phi(u_i) \,, \quad i = 1, 2 \,, \tag{33}$$

and the mapping from the $\mathbf{d}$-space to the $\mathbf{x}$-space can be simplified in this example as

$$\mathbf{x} = \mathbf{T}^{-1}(0; \mathbf{d}) = \mathbf{d} \,. \tag{34}$$

An RBDO problem with prescribed reliability target $\beta = 2$ can be formulated as

$$\min \ \text{Cost}(\mathbf{d}) = d_1 + d_2 \,,$$

$$\text{subject to } P(G_j(\mathbf{x}) < 0) \leq \Phi(-\beta_t) \,, \quad j = 1, 2, 3 \,,$$

$$1 \leq d_1 \leq 10 \,, 1 \leq d_2 \leq 10 \,, \tag{35}$$

where three smooth nonlinear performance functions are

$$G_1(\mathbf{x}) = \frac{1}{20} x_1^2 x_2 - 1 \,,$$

$$G_2(\mathbf{x}) = \frac{1}{10}(10x_2^3 - x_1^2 x_2 - 2x_1) - 1 \,,$$

$$G_3(\mathbf{x}) = \frac{80}{x_1^2 + 8x_2 + 5} - 1 \,. \tag{36}$$

7.1.1 Smooth RBDO using FORM

A special case of the RBDO problem with $\beta = 0$ can essentially be solved as a deterministic optimization problem whose optimum often is an effective initial design

Table 1 RBDO History with DPM ($\mathbf{d}^0 = [3.593, 1.549]^T$)

k-th Iteration	Cost	$\mathbf{d}^k$ d_1^k	d_2^k	$j=1$ $\beta^1_{s,\mathrm{FORM}}$	$\mathbf{d}_P^k$		$j=2$ $\beta^2_{s,\mathrm{FORM}}$	$\mathbf{d}_P^k$		$j=3$ $g_3^*(\mathbf{d}^k)$
0	5.142	3.593	1.549	0.000	4.327	2.129	0.000	2.970	2.178	0.962
1	6.412	4.004	2.408	1.877	4.025	2.430	1.918	3.981	2.417	0.557
2	6.455	4.027	2.428	2.000	4.027	2.428	2.000	4.027	2.428	0.546
Optimum	6.455	4.027	2.428	active	4.027	2.428	active	4.027	2.428	inactive

Table 2 RBDO History using SORM ($\mathbf{d}^0_{\mathrm{SORM}} = [4.027, 2.428]^T$)

k-th Iteration	Cost	$\mathbf{d}^k$ d_1^k	d_2^k	$j=1$ $\beta^1_{s,\mathrm{FORM}}$	$\mathbf{d}_P^k$		$j=2$ $\beta^2_{s,\mathrm{FORM}}$	$\mathbf{d}_P^k$		$j=3$ $g_3^*(\mathbf{d}^k)$
0	6.455	4.027	2.428	2.212	3.988	2.391	2.201	4.088	2.406	0.546
1	6.380	3.997	2.383	1.992	3.999	2.384	1.980	3.989	2.385	0.565
2	6.383	3.996	2.387	1.997	3.997	2.387	2.001	3.997	2.387	0.565
3	6.384	3.997	2.387	2.000	3.997	2.387	2.000	3.997	2.387	0.565
Optimum	6.384	3.997	2.387	active	3.997	2.387	active	3.997	2.387	inactive

for general RBDO. In such case, the surface of the reliability target shrinks to a point in the unified system space and the probabilistic constraint can be directly expressed in terms of the system performance function as $g^*(\mathbf{d}) = G(\mathbf{T}^{-1}(\mathbf{0};\mathbf{d}))ge0$. Thus, the initial design of this problem, $\mathbf{d}^0 = [3.593, 1.549]^T$, can be obtained from

$$\min \mathrm{Cost}(\mathbf{d})\,,$$

$$\text{subject to } G_j(\mathbf{d}) \geq 0\,, \quad j = 1, 2, 3\,,$$

$$1 \leq d_1 \leq 10\,, \quad 1 \leq d_2 \leq 10\,, \tag{37}$$

which can be obtained by using SLP, SQP, or MFD. Using DPM with adaptive probabilistic constraint evaluation by FORM, the RBDO problem is solved using SLP and the optimum design is $\mathbf{d}^{\mathrm{opt}}_{\mathrm{FORM}} = [4.027, 2.428]^T$ with the RBDO history shown in Table 1.

7.1.2 Smooth RBDO using SORM

Solving the RBDO problem by FORM is easier than using SORM because the performance function Hessian matrix,

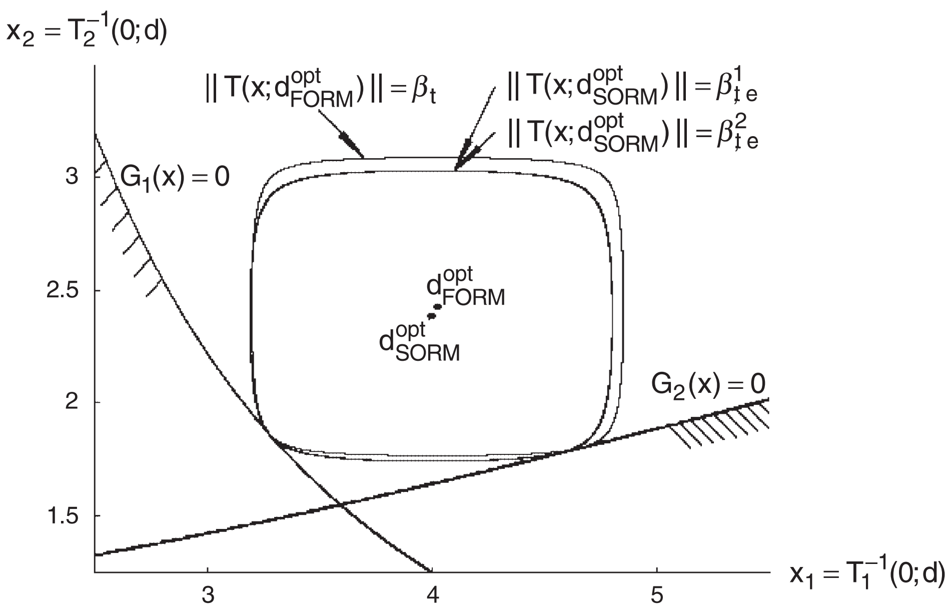

Fig. 11 Comparison of SORM and FORM in RBDO

that is often difficult to evaluate, is required in SORM to compute principle curvatures of the performance function. Thus, it is suggested here to use the RBDO/FORM optimum design as the RBDO/SORM initial design, i.e. $\mathbf{d}^0_{\mathrm{SORM}} = [4.027, 2.428]^T$.

Using DPM with adaptive probabilistic constraint evaluation by SORM, this case is easily solved using SLP. The $\mathbf{d}^{\mathrm{opt}}_{\mathrm{FORM}}$ is compared with $\mathbf{d}^{\mathrm{opt}}_{\mathrm{SORM}} = [3.997, 2.387]^T$ in Fig. 11 and the RBDO history is listed in Table 2. Notice that design $\mathbf{d}^{\mathrm{opt}}_{\mathrm{FORM}}$ fits the surface of the reliability target, $\|\mathbf{T}(\mathbf{x}; \mathbf{d}^{\mathrm{opt}}_{\mathrm{FORM}})\| = \beta_t$, into the feasible side of the active performance function limit-state surfaces, $G_1(\mathbf{x}) = 0$ and $G_2(\mathbf{x}) = 0$, while design $\mathbf{d}^{\mathrm{opt}}_{\mathrm{SORM}}$ fits both surfaces of the equivalent reliability target, $\|\mathbf{T}(\mathbf{x}; \mathbf{d}^k)\| = \beta^j_{t,e}$ $(j = 1, 2)$.

Although two active performance functions have the same reliability target index $\beta = 2$, the equivalent reliability target indices for performance functions are generally different as the curvatures of different performance function limit-state surfaces are different. In this example, $\beta^1_{t,e} = 1.804$ and $\beta^2_{t,e} = 1.817$, and the two surfaces of equivalent reliability target are incidentally very close.

7.1.3 Smooth ECD example

The nominal mean values of the system parameters are used as independent design variable, $\mathbf{d} = [d_i]^T = [\tilde{\mu}_i]^T$, with constant nominal tolerances $\Delta_i = \sqrt{3}\sigma_i$ $(i = 1, 2)$.

Because three performance functions are monotonic in terms of two system parameters, their signum vectors are constant and therefore three design potential vectors are also constant as $\Delta^1_P = [-\sqrt{3}\sigma_1, -\sqrt{3}\sigma_2]^T$, $\Delta^2_P = [\sqrt{3}\sigma_1, -\sqrt{3}\sigma_2]^T$, and $\Delta^3_P = [\sqrt{3}\sigma_1, \sqrt{3}\sigma_2]^T$, where $\sigma_1 = 1/2$ and $\sigma_2 = 2/5$. The ECD model can then be expressed as

$$\min \mathrm{Cost}(\mathbf{d}) = d_1 + d_2 \,,$$

$$\text{subject to } G_j(\mathbf{d} + \Delta^j_P) \geq 0 \,, \quad j = 1, 2, 3 \,,$$

$$1 \leq d_1 \leq 10 \,, \quad 1 \leq d_2 \leq 10 \,, \tag{38}$$

which can be solved as a deterministic optimization problem and the ECD optimum is $\mathbf{d}^{\mathrm{opt}}_{\mathrm{ECD}} = [4.124, 2.577]^T$. The direct application of DPC for ECD is illustrated in Fig. 12 and $\mathbf{d}^{\mathrm{opt}}_{\mathrm{ECD}}$ has been compared with the RBDO optimum $\mathbf{d}^{\mathrm{opt}}_{\beta_t=2}$ in Fig. 3.

Because system parameters are independent and uniformly distributed in their tolerance limits in this example, $\mathbf{d}^{\mathrm{opt}}_{\mathrm{ECD}}$ is just slightly more conservative than $\mathbf{d}^{\mathrm{opt}}_{\beta_t=2}$. However, it must be pointed out that if the distributions of system parameters are biased and/or interdependent, ECD may yields significantly more conservative designs than RBDO with a finite β_t (Tu and Choi 1999).

7.2 Nonsmooth RBDO and ECD

Consider another system with two independent, uniformly distributed system parameters whose standard deviations are constants as $\sigma_1 = \sqrt{3}/2$ and $\sigma_2 = \sqrt{3}/3$.

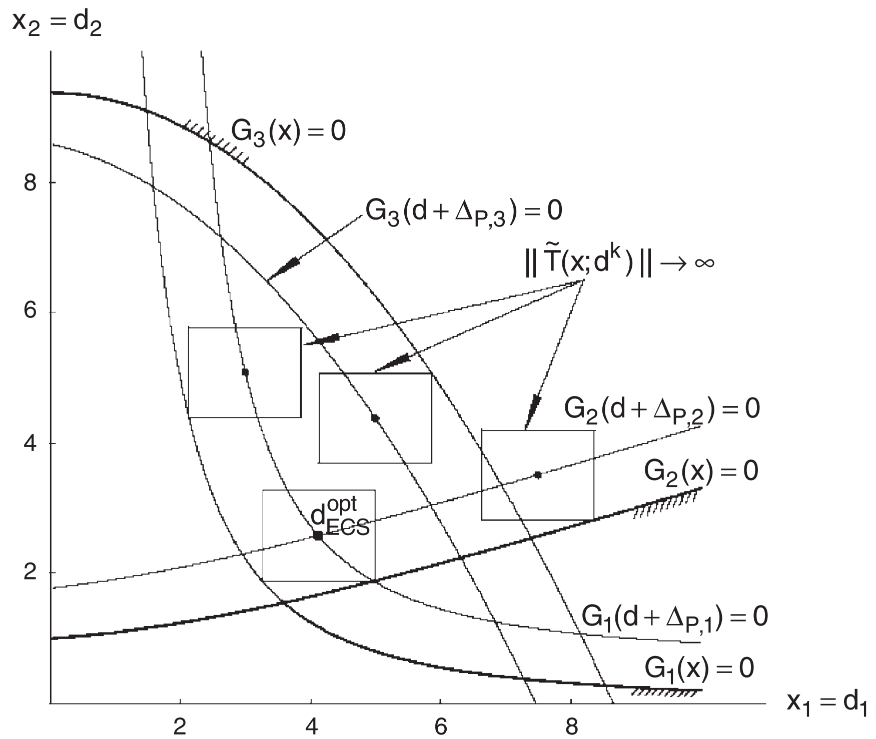

Fig. 12 Illustration of DPC for smooth ECD

A system performance function is defined as

$$G(\mathbf{x}) = 1 - \frac{1}{15}(x_1 - x_2 - 1)^2 - \frac{1}{400}(x_1 + x_2 - 25)^2 . \quad (39)$$

7.2.1 Nonsmooth RBDO using FORM

An RBDO problem with prescribed reliability target $\beta_t = 2$ can be formulated as

$$\min \mathrm{Cost}(\mathbf{d}) = d_1 + d_2 ,$$

$$\text{subject to } P[G(\mathbf{x}) \le 0] \le \Phi(-\beta_t) ,$$

$$1 \le d_1 \le 10 , \quad 1 \le d_2 \le 10 . \quad (40)$$

If the standard RBDO procedure is used, the optimization algorithm will oscillate around design $[5.0, 4.0]^T$ and the two distinct MPPs will be found. Thus, the equivalent reliability target index in the first step of DPM with RIA can be determined as

$$\beta_t^e = \Phi^{-1}\left[\sqrt{\Phi(\beta_t)}\right] = \Phi^{-1}\left[\sqrt{\Phi(2)}\right] = 2.275 . \quad (41)$$

Starting from the initial design $\mathbf{d}^0 = [5.0, 4.0]^T$, SLP is used to solve this RBDO problem by using DPM with RIA of (29), and it converges after 5 iterations and the RBDO history is listed in Table 3, where two DPPs corresponding to two active probabilistic constraints, $\mathbf{d}_{P,a}^k$ and $\mathbf{d}_{P,b}^k$, also converge to the RBDO optimum.

The RBDO optimum $\mathbf{d}_{\beta_t=2}^{\mathrm{opt}} = [4.934, 3.798]^T$ is shown in Fig. 6, where the MPPs at the optimum are $\mathbf{x}_a^* = [3.538, 4.627]^T$ and $\mathbf{x}_b^* = [6.266, 2.902]^T$. In this example, the extended PMA can also be used to obtain the true RBDO optimum. The optimum corresponding to $\beta_t = 3$ (i.e. $\beta_t^e = 3.205$) is obtained as $\mathbf{d}_{\beta_t=3}^{\mathrm{opt}} = [5.311, 4.191]^T$ and compared with the ECD result in Fig. 13.

Table 3 Design potential method for nonsmooth RBDO

k-th iteration	Cost	$\mathbf{d}_k$ d_1^k	d_2^k	MPP $\mathbf{x}_a^{*,k}$ $\beta_S^a(\mathbf{d}^k)$	$\mathbf{d}_{P,a}^k$		MPP $\mathbf{x}_b^{*,k}$ $\beta_S^b(\mathbf{d}^k)$	$\mathbf{d}_{P,b}^k$	
0	9.000	5.000	4.000	2.138	5.022	3.962	3.236	5.134	3.918
1	8.836	4.969	3.867	2.271	4.970	3.866	2.461	5.012	3.844
2	8.772	4.947	3.825	2.275	4.949	3.827	2.345	4.965	3.816
3	8.747	4.939	3.808	2.275	4.940	3.810	2.304	4.944	3.802
4	8.734	4.934	3.800	2.275	4.936	3.803	2.284	4.938	3.799
5	8.732	4.934	3.798	2.275	4.934	3.798	2.275	4.934	3.798
Optimum	8.732	4.934	3.798	active	4.934	3.798	active	4.934	3.798

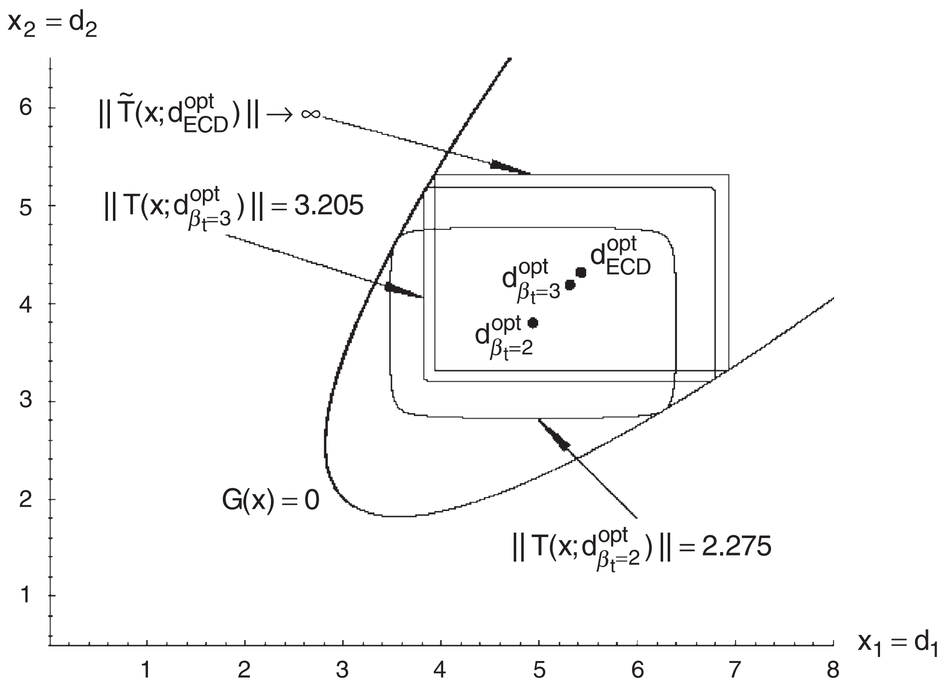

Fig. 13 Comparison of nonsmooth RBDO and ECD

In addition, the nonsmooth probabilistic constraint evaluation by RIA at the design $\mathbf{d}^k = [4.5, 3.0]^T$ is illustrated in the unified system space and the corresponding $\mathbf{u}$-space, respectively, in Figs. 4 and 5. The PMA at the design $\mathbf{d}^k = [5.0, 3.5]^T$ is illustrated in Fig. 7, where RIA does not find one of the two MPPs on $G(\mathbf{x}) = 0$ because only one piece of the performance function limit-state surface is defined inside the corresponding probability integration domain.

7.2.2 Nonsmooth ECD by DPC

In the ECD model, the nominal tolerances of the system parameters are constants as $\Delta_1 = 1.5$ and $\Delta_2 = 1.0$, and nominal mean values are used as independent design variables $\mathbf{d} = [d_i]^T \equiv [\tilde{\mu}_i]^T$ $(i = 1, 2)$.

If the standard extreme case analysis is carried out using (21), the optimization algorithm will oscillate around certain design and the two distinct signum vectors of the performance function can be obtained as $\mathbf{S}_G^a = [+1, -1]^T$ and $\mathbf{S}_G^b = [-1, +1]^T$. Thus, the extreme case probabilistic constraint, $g^L(\mathbf{d}) \geq 0$, can be completely represented by (29) as

$$G(x_1 - \Delta_1, x_2 + \Delta_2) = 1 - \frac{1}{15}\left(x_1 - x_2 - \frac{7}{2}\right)^2 - \frac{1}{400}\left(x_1 + x_2 - \frac{51}{2}\right)^2 \geq 0\,,$$

$$G(x_1 + \Delta_1, x_2 - \Delta_2 = 1 - \frac{1}{15}\left(x_1 - x_2 + \frac{3}{2}\right)^2 - \frac{1}{400}\left(x_1 + x_2 - \frac{49}{2}\right)^2 \geq 0\,, \quad (42)$$

and the ECD problem can be solved as a deterministic optimization problem. The SLP is used to obtain the optimum $\mathbf{d}_{\mathrm{ECD}}^{\mathrm{opt}} = [5.429, 4.315]^T$, which is illustrated in Fig. 9 and compared with RBDO results in Fig. 13.

Note that the reliability target index β_t is generally a highly nonlinear function of the design in RBDO. In this example, independent random system parameters are evenly distributed within their tolerance limits and therefore $\mathbf{d}_{\beta_t=3}^{\mathrm{opt}}$ and $\mathbf{d}_{\mathrm{ECD}}^{\mathrm{opt}}$ are rather close. In such case, ECD can be used as an effective alternative of RBDO for robust system parameter design and parameter tolerance design. However, ECD can yield overly conservative design if distributions of system parameters are biased or interdependent and the related study can be found in the reference (Tu and Choi 1999).

8 Summary

The DPC provides an in-depth understanding of the interconnection between performance probability analysis by FORM/SORM and the iterative RBDO process in the unified system space. Taking advantage of the inherent design information from probabilistic constraint evaluation, DPC leads to an effective DPM for probabilistic constraint approximation in smooth and nonsmooth RBDO. The DPC can also be effectively applied to ECD by directly representing smooth and nonsmooth extreme case probabilistic constraints in terms of system performance functions. Moreover, the adaptive probabilistic constraint evaluation strategy can be used to ensure the robustness in RBDO and ECD.

In conclusion, the proposed RBDO methodology that combines DPM with the adaptive probabilistic constraint evaluation strategy can be effectively and efficiently used for broader engineering applications.

Acknowledgements Research is partially supported by the Automotive Research Center sponsored by the U.S. Army TARDEC.

References

Arora, J.S. 1989: *Introduction to optimum design*. New York: McGraw-Hill

Atkinson, K.E 1989: *An introduction to numerical analysis*. New York: John Wiley & Sons

Ayyub, B.M.; McCuen, R.H. 1997: *Probability, statistics, & reliability for engineers*. New York: CRC Press

Breitung, K. 1984: Asymptotic approximations for multinormal integrals. *J. Eng. Mech.* **110**, 357–366

Chandu, S.V.L.; Grandhi, R.V. 1995: General purpose procedure for reliability based structural optimization under parametric uncertainties. *Advances Eng. Software* **23**, 7–14

Enevoldsen, I.; Sorensen, J.D. 1994: Reliability-based optimization in structural engineering. *Structural Safety* **15**, 169–196

Frangopol, D.M.; Corotis, R.B. 1996: Reliability-based structural system optimization: state-of-the-art versus state-of-the-practice. In: Cheng, F.Y. (ed.) *Analysis and computation* (Proc. 12th Conf., held in conjunction with Structures Congr. XIV), pp. 67–78

Grandhi, R.V.; Wang, L.P. 1998: Reliability-based structural optimization using improved two-point adaptive nonlinear approximations. *Finite Elements Analysis Design* **29**, 35–48

Hohenbichler, M.; Rackwitz, R. 1981: Nonnormal dependent vectors in structural reliability. *J. Eng. Mech. Div., ASCE* **107**, 1227–1238

Madsen, H.O.; Krenk, S.; Lind, N.C. 1986: *Methods of structural safety*. Englewood Cliffs, NJ: Prentice-Hall

Parkinson, A.; Sorensen C.; Pourhassan, N. 1993: A general approach for robust optimal design. *J. Mech. Des., ASME* **115**, 74–80

Sundaresan, S.; Ishii, K.; Houser, D.R. 1993: A robust optimization procedure with variations on design variables and constraints. *Advances in Design Optimization, ASME* pp. 249–256

Tu, J.; Choi, K.K. 1997: A performance measure approach in reliability-based structural optimization. *Technical Report R97-02*, Center for Computer-Aided Design, The University of Iowa, Iowa City, IA

Tu, J.; Choi, K.K. 1999: New progress in reliability-based design optimization. *Technical Report R99-07*, Center for Computer-Aided Design, The University of Iowa, Iowa City, IA

Tu, J.; Choi, K.K.; Park, Y.H. 1999a: A new study on reliability-based design optimization. *J. Mech. Des., ASME* **121**, 557–564

Tu, J.; Choi, K.K.; Park, Y.H. 1999b: Design potential method for reliability-based system parameter design using adaptive probabilistic constraint evaluation. *AIAA J.* (submitted)

Tvedt, L. 1990: Distribution of quadratic forms in normal space-application to structural reliability. *J. Eng. Mech.* **116**, 1183–1197

Yu, J.C.; Ishii, K. 1998: Design for robustness based on manufacturing variation patterns. *J. Mech. Des., ASME* **120**, 196–202