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Fuzzy logic, continuity and effectiveness

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Abstract. It is shown the complete equivalence between the theory of continuous (enumeration) fuzzy closure operators and the theory of (effective) fuzzy deduction systems in Hilbert style. Moreover, it is proven that any truth-functional semantics whose connectives are interpreted in [0,1] by continuous functions is axiomatizable by a fuzzy deduction system (but not by an effective fuzzy deduction system, in general).

1. Introduction

Fuzzy logic (in narrow sense) is a new promising chapter of formal logic whose basic ideas have been formulated by L. Zadeh (see [27] and [28]), J. A. Goguen (see [12]) and J. Pavelka (see [18], [19], and [20]). Such a logic, whose aim is to formalize the “approximate reasoning” we use in everyday life, is strictly related with multivalued logic where truth values besides 0 (false) and 1 (true) are allowed. In the recent years fuzzy logic was extensively investigate and several deep results where established in the spirit of mathematical logic (for a good overview see [16]). As a contribution in such a direction, in a series of papers we proposed to extend to fuzzy logic some general tools which are very useful in classical logic. Indeed, in account of the fact that effectiveness is a basic feature of any deduction apparatus, we extended to fuzzy sets some basic notions of recursion theory (see [2], [3], [4], [5] and [11]). Successively, in order to extend to fuzzy logic Tarski’s point of view (see, e.g. [22] and [24]), we defined an *abstract* fuzzy logic as a fuzzy closure operator in the lattice of all fuzzy subsets of the set of the formulas of a given language (see [6], [7], [8], [9] and [11]).

This paper is devoted mainly to prove that the abstract approach to fuzzy logic is equivalent to the Hilbert style approach as proposed by Pavelka in [18] and Goguen in [12]. Also, we examine the continuous truth-functional multivalued logics, i.e. propositional logics whose logical connectives are interpreted by continuous functions in [0,1]. We prove that any continuous truth-functional logic is axiomatizable, i.e. its associate logical consequence operator is the deduction operator

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of a suitable fuzzy Hilbert system. Moreover, we prove a constructive version of the equivalence between the abstract approach and Hilbert's approach. Indeed, we propose a notion of *enumeration fuzzy operator* extending the definition given in [21] for crisp operators and we call *effective abstract fuzzy deduction system* any enumeration fuzzy closure operator. Also, we call *effective* a fuzzy Hilbert system whose fuzzy inference rules are, in a sense, computable and whose fuzzy set of logical axioms is recursively enumerable. Then, we prove that the notions of effective abstract fuzzy deduction system and effective fuzzy Hilbert system coincide. Finally, we give an example of (axiomatizable) continuous truth-functional logic which is not axiomatizable by an effective fuzzy Hilbert system.

2. Preliminaries

Let S be a nonempty set and denote by U the real interval [0,1]. Then a *fuzzy subset* of S is any map $s : S \to U$ (see [25]). The *support* of s is defined by setting $Supp(s) = \{x \in S : s(x) \neq 0\}$ and we say that s is *finite* if $Supp(s)$ is finite. A fuzzy subset s is called *crisp* if $s(x) \in \{0, 1\}$ for every $x \in S$. We identify any subset X of S with a crisp subset, namely with its characteristic function c_X defined by setting $c_X(x) = 1$ if $x \in S$ and $c_X(x) = 0$ otherwise. As an example, we identify the empty set $\emptyset$ with the map $s^0 : S \to U$ constantly equal to 0 and the whole set S with the map $s^1 : S \to U$ constantly equal to 1. Let $\ddot{U}$ be the set of rational numbers in U, then we denote

- by $\mathcal{P}(S)$ the class of all the subsets of S,
- by $\mathcal{P}_f(S)$ the class of all the subsets of S which are finite or empty,
- by $\mathcal{F}(S)$ the class of all the fuzzy subsets of S,
- by $\mathcal{F}_f(S)$ the class of all the fuzzy subsets of S with values in $\ddot{U}$ which are finite or empty.

The *inclusion* is the binary relation in $\mathcal{F}(S)$ defined by setting, for every pair s and s' of fuzzy subsets of S,

$$s \subseteq s' \Leftrightarrow s(x) \leq s'(x) \text{ for every } x \in S.$$

Given $x, y \in U$, we write $x \vee y$ to denote $\max\{x, y\}$ and $x \wedge y$ to denote $\min\{x, y\}$. The *union* $s \cup s'$ and the *intersection* $s \cap s'$ of two fuzzy subsets s and s' are defined by setting, for every $x \in S$,

$$(s \cup s')(x) = s(x) \vee s'(x) \ ; \ (s \cap s')(x) = s(x) \wedge s'(x).$$

More generally, given a family $(s_i)_{i \in I}$ of fuzzy subsets of S, the *union* $\bigcup_{i \in I} s_i$ and the *intersection* $\bigcap_{i \in I} s_i$ are defined by

$$\bigcup_{i \in I} s_i(x) = \sup\{s_i(x) : i \in I\} \ ; \ \bigcap_{i \in I} s_i(x) = \inf\{s_i(x) : i \in I\}.$$

We define the *complement* -s of s by setting, for every $x \in S$,

$$(\text{-}s)(x) = 1 - s(x).$$

The usual language for set theory is used for fuzzy set theory, too. For example, if $s \subseteq s'$, we say that s is *contained* in s' or that s is a *part* of s'. Let L be a complete lattice. We call *operator* in L any map $J : L \to L$. We say that $x \in L$ is a *fixed point* of J if $J(x) = x$. We say that J is a *closure operator* if the following properties are satisfied:
(i) $x \leq x' \Rightarrow J(x) \leq J(x')$ (order-preserving),
(ii) $x \leq J(x)$ (inclusion),
(iii) $J(J(x)) = J(x)$ (idempotence).
It is easy to prove that if J is a closure operator, then, for any $x \in L$, $J(x)$ is the least fixed point of J greater or equal to x. Consequently, two closure operators with the same fixed points coincide. If S is a set, then we call *closure operator* in S any closure operator in the lattice $\mathcal{P}(S)$. We call *fuzzy closure operator* in S any closure operator in the lattice $\mathcal{F}(S)$. We say that a closure operator J in S is *compact* if

$$J(X) = \bigcup\{J(X_f) : X_f \text{ finite and } X_f \subseteq X\}.$$

To give a natural counterpart of such a notion in any complete lattice L, we call *upward directed* a nonempty class $\mathcal{C}$ of elements in L such that

$$x_1 \in \mathcal{C} \text{ and } x_2 \in \mathcal{C} \Rightarrow \exists x_3 \in \mathcal{C}, x_1 \leq x_3 \text{ and } x_2 \leq x_3.$$

The notion of *downward directed* class is defined by duality. The chains are examples of classes that are both upward and downward directed. If $\mathcal{C}$ is upward directed, the element $s = \bigcup \mathcal{C}$ is called *the limit* of $\mathcal{C}$ and we write $s = \lim \mathcal{C}$. If J is an order-preserving operator and $\mathcal{C}$ is directed, then the image $J(\mathcal{C}) = \{J(x) : x \in \mathcal{C}\}$ is also directed, obviously. This enables us to propose the following definition.

Definition 1. *An order-preserving operator $J : L \to L$ is continuous if, for every upward directed class $\mathcal{C}$,*

$$J(\lim \mathcal{C}) = \lim J(\mathcal{C}). \tag{1}$$

It is well known that an operator J in a set S is continuous if and only if it is compact. Likewise, it is also possible to characterize the continuity of the fuzzy closure operators in terms of finite fuzzy subsets (see [17]). To this purpose we define the relation $\ll$ by setting, given two fuzzy subsets s_1 and s_2, $s_1 \ll s_2$ provided that $s_1(x) < s_2(x)$ for every $x \in Supp(s_1)$. A basic feature of the relation $\ll$ is that, for any upward directed class $\mathcal{C}$ and s_f finite fuzzy subset,

$$s_f \ll \bigcup \mathcal{C} \Rightarrow s \in \mathcal{C} \text{ exists such that } s_f \ll s,$$

while the analogous implication is not true for the inclusion.

Theorem 1. *A fuzzy operator J is continuous iff, for every fuzzy subset τ,*

$$J(\tau) = \bigcup\{J(s_f) : s_f \in \mathcal{F}_f(S) \textit{ and } s_f \ll \tau\}. \tag{2}$$

Consequently, τ is a fixed point of a continuous closure operator J iff

$$s_f \in \mathcal{F}_f(S) \textit{ and } s_f \ll \tau \;\Rightarrow\; J(s_f) \subseteq \tau. \tag{3}$$

Proof. Let J be continuous. Then, since $\{s_f \in \mathcal{F}_f(S) \text{ and } s_f \ll \tau\}$ is a directed class, (2) is immediate. Assume that (2) is satisfied and let $\mathcal{C}$ be an upward directed class. Then, since J is order-preserving, $J(\bigcup \mathcal{C}) \supseteq \bigcup J(\mathcal{C})$. Moreover,

$$\begin{aligned} J(\bigcup \mathcal{C}) &= \bigcup \{J(s_f) : s_f \in \mathcal{F}_f(S) \text{ and } s_f \ll \bigcup \mathcal{C}\} \\ &= \bigcup \{J(s_f) : s_f \in \mathcal{F}_f(S), s \in \mathcal{C} \text{ exists such that } s_f \ll s\} \\ &\subseteq \bigcup \{J(s) : s \in \mathcal{C}\} = \bigcup J(\mathcal{C}). \end{aligned}$$

This proves that J is continuous.

Note that in (2) it is not necessary to confine ourselves to fuzzy subsets s_f assuming rational values. Then J is continuous iff, for any fuzzy subset τ,

$$J(\tau) = \bigcup \{J(s_f) : s_f \text{ finite fuzzy subset of } S \text{ such that } s_f \ll \tau\}. \tag{4}$$

3. Abstract (crisp) logic and Hilbert (crisp) systems

In the sequel we assume that $\mathbb{F}$ is a set whose elements we call *formulas* and that an effective coding for $\mathbb{F}$ exists. Then, the formulas in $\mathbb{F}$ can be listed in a sequence $\alpha_1, \alpha_2, \ldots$. We call *abstract deduction system* any pair $(\mathcal{P}(\mathbb{F}), \mathcal{D})$ where $\mathcal{D}$ is a compact closure operator in the lattice $\mathcal{P}(\mathbb{F})$ (see, e.g., [22] and [24]). We define a (crisp) *Hilbert system* as a pair $\mathcal{S} = (\mathbb{A}, \mathbb{R})$ where $\mathbb{A}$ is a subset of $\mathbb{F}$, the *set of logical axioms*, and $\mathbb{R}$ is a set of rules of inference. In turn, *a rule of inference* is a partial n-ary operation r on $\mathbb{F}$ whose domain we denote by $Dom(r)$. A *proof* π *of a formula* α *under a set* X *of hypothesis* is a sequence $\alpha_1, ..., \alpha_m$ of formulas such that $\alpha_m = \alpha$ and, for any $1 \leq i \leq m$, either

(i) α_i is an element in $\mathbb{A}$, or
(ii) α_i is an element in X, or
(iii) $\alpha_i = r(\alpha_{s(1)}, , \alpha_{s(n)})$ where $r \in \mathbb{R}$ and $s(1) < i, ..., s(n) < i$.

Given a Hilbert system $\mathcal{S}$, we call *deduction operator associated with* $\mathcal{S}$ the operator $\mathcal{D} : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ defined by setting,

$$\mathcal{D}(X) = \{\alpha \in \mathbb{F} : \text{ there exists a proof } \pi \text{ of } \alpha \text{ under hypothesis } X\}.$$

We call *equivalent* two Hilbert systems whose deduction operators coincide.

We say that a set T of formulas is *closed with respect to the inference rule* r if, for any $(x_1, ..., x_n) \in Dom(r)$,

$$x_1 \in T, ..., x_n \in T \;\Rightarrow\; r(x_1, ..., x_n) \in T.$$

We say that T is a *theory* if T contains the set of logical axioms and it is closed with respect to all the inference rules. It is immediate to prove that the class of theories coincides with the class of fixed points of $\mathcal{D}$. Consequently, two Hilbert systems are equivalent iff they have the same theories.

Proposition 1. *Any Hilbert system is equivalent to a Hilbert system whose inference rules are everywhere defined. Consequently, the notion of Hilbert system is equivalent to the notion of algebraic structure.*

Proof. Let $\mathcal{S}$ be a Hilbert system and define, for any rule r in $\mathcal{S}$ the rule r' defined by setting

$$r'(x_1, ..., x_n) = \begin{cases} r(x_1, ..., x_n) & \text{if } (x_1, ..., x_n) \in Dom(r) \\ x_1 & \text{otherwise.} \end{cases}$$

Obviously, a set T of formulas is closed with respect to r iff it is closed with respect to r'. Then the system $\mathcal{S}'$ obtained by substituting any rule r with r', is equivalent with $\mathcal{S}$.

Let $\mathcal{S}$ be a Hilbert system and $\mathcal{S}'$ a system equivalent to $\mathcal{S}$ whose inference rules are everywhere defined. Then $\mathcal{S}'$ defines an algebraic structure whose set of constants is the set $\mathbb{A}$ of logical axioms and whose operations are the inference rules in $\mathcal{S}'$. Conversely, any algebraic structure $\mathcal{A}$ defines a Hilbert system $\mathcal{S}(\mathcal{A})$ whose set of logical axioms and inference ruels coincide with the set of constants and the set of operations of $\mathcal{A}$.

Observe that in the deduction system $\mathcal{S}(\mathcal{A})$ associated with the algebraic structure $\mathcal{A}$, the theories coincide with the subalgebras and therefore, given $X \subseteq \mathcal{A}$, $\mathcal{D}(X)$ is the subalgebra $\langle X \rangle$ generated by X. In accordance with Proposition 1 and a result of G. Birkhoff and O. Frink, we have the following theorem (see [23]):

Theorem 2. *Let $\mathcal{S}$ be a Hilbert system and $\mathcal{D}$ the related deduction operator. Then $\mathcal{D}$ is a compact closure operator and therefore $(\mathcal{P}(\mathbb{F}), \mathcal{D})$ is an abstract deduction system. Conversely, let $(\mathcal{P}(\mathbb{F}), \mathcal{D}))$ be an abstract deduction system. Then a Hilbert system $\mathcal{S}$ exists whose deduction operator coincides with $\mathcal{D}$.*

Notice that the inference rules of the system given in Theorem 2 are commutative. Consequently, it is not restrictive to assume that in a crisp Hilbert system all the inference rules are commutative operations.

4. Abstract fuzzy logic and fuzzy Hilbert systems

Passing to the fuzzy case, we define an *abstract fuzzy deduction system* as a pair $(\mathcal{F}(\mathbb{F}), \mathcal{D})$ where $\mathcal{D}$ is a continuous closure operator in $\mathcal{F}(\mathbb{F})$. We call *fuzzy Hilbert system* a pair $\mathcal{S} = (a, \mathbb{R})$ where a is a fuzzy subset of $\mathbb{F}$, the *fuzzy subset of logical axioms*, and $\mathbb{R}$ is a set of fuzzy rules of inference. In turn, a *fuzzy rule of inference* is a pair $r = (r', r'')$, where

- r' is a partial n-ary operation on $\mathbb{F}$ whose domain we denote by $Dom(r)$,
- r'' is an n-ary operation on U preserving the least upper bound in each variable, i.e., given any family $(y_i)_{i \in I}$ of elements of U,

$$r''(x_1, ..., \sup_{i \in I} y_i, ..., x_n) = \sup_{i \in I} r''(x_1, ..., y_i, ..., x_n). \tag{5}$$

In other words, an inference rule r consists of

– a *syntactical component* r' that operates on formulas (in fact, it is a rule of inference in the usual sense),

– a *valuation component* r'' that operates on truth-values to calculate how the truth-value of the conclusion depends on the truth-values of the premises (see [12], [26] and [18]).

Condition (5) is also called *continuity condition* since it implies that the deduction operator associated with a fuzzy Hilbert system is continuous. We indicate an application of an inference rule r by

$$\frac{\alpha_1, \ldots, \alpha_n}{r'(\alpha_1, \ldots, \alpha_n)} \quad , \quad \frac{\lambda_1, \ldots, \lambda_n}{r''(\lambda_1, \ldots, \lambda_n)}.$$

The meaning of such a rule is that:
IF you know that the formulas $\alpha_1, ..., \alpha_n$ are true (at least) to the degree $\lambda_1, ..., \lambda_n$, THEN you can conclude that $r'(\alpha_1, ..., \alpha_n)$ is true (at least) to the degree $r''(\lambda_1, ..., \lambda_n)$.

We say that a fuzzy set τ of formulas is *closed with respect to an inference rule* $r = (r', r'')$ if, for any $(\alpha_1, ..., \alpha_n) \in Dom(r)$,

$$\tau(r'(\alpha_1, ..., \alpha_n)) \geq r''(\tau(\alpha_1), ..., \tau(\alpha_n)).$$

We say that τ is a *theory* of $\mathcal{S}$ if τ contains the fuzzy set of logical axioms and it is closed with respect to any inference rule. A *proof* π of a formula α is a sequence $\alpha_1, ..., \alpha_m$ of formulas such that $\alpha_m = \alpha$, together with a sequence of related "justifications". This means that, given any formula α_i, we must specify whether

- α_i is assumed as a logical axiom; or
- α_i is assumed as a hypothesis; or
- α_i is obtained by an inference rule (in this case we must indicate also the rule and the formulas $\alpha_{s(1)}, ..., \alpha_{s(n)}$ used to obtain α_i, where $s(1) < i, ..., s(n) < i$).

Differently from the crisp case, the justifications are necessary since different justifications of the same formula give rise to different valuations. We call an *initial valuation* any subset $v : \mathbb{F} \to U$ of formulas. In accordance with Pavelka's definition of semantics for a fuzzy logic (see Section 5), the information carried on by v is that, given any formula α, "the actual truth value of α is at least $v(\alpha)$". Let π be a proof and v an initial valuation. Then the *valuation* $Val(\pi, v)$ *of* π *with respect to* v is defined by induction on the length m of π by setting

$$Val(\pi, v) = \begin{cases} a(\alpha_m) & \text{if } \alpha_m \text{ is assumed as a logical axiom,} \\ v(\alpha_m) & \text{if } \alpha_m \text{ is assumed as a hypothesis,} \\ r''(Val(\pi_{s(1)}, v), ..., Val(\pi_{s(n)}, v)) & \text{if } \alpha_m = r'(\alpha_{s(1)}, ..., \alpha_{s(n)}) \end{cases}$$

where, for any $i \leq m$, π_i denotes the proof $\alpha_1, ..., \alpha_i$ of the formula α_i. If α is the formula proved by π, we interpret $Val(\pi, v)$ as a constraint on the truth value of α. Namely, the information furnished by π is that: "the actual truth value of α is at least $Val(\pi, v)$". Now, given a formula α, it should be possible to find another proof π' of α such that $Val(\pi', v) > Val(\pi, v)$. This happens, for instance, if the assumptions used in π' are more true than the assumptions used in π. In other words, unlike the usual Hilbert systems, in a fuzzy Hilbert system different proofs of the same formula α can give different constraints on the degree of validity of α. This suggests that, given a fuzzy set of axioms v (the available fuzzy information), in order to evaluate α we must refer to the whole set of proofs of α.

Definition 2. *Given a fuzzy Hilbert system $\mathcal{S}$, we call deduction operator associated with $\mathcal{S}$ the operator $\mathcal{D} : \mathcal{F}(\mathbb{F}) \to \mathcal{F}(\mathbb{F})$ defined by setting,*

$$\mathcal{D}(v)(\alpha) = \sup\{Val(\pi, v) : \pi \text{ is a proof of } \alpha\}, \tag{6}$$

for every initial valuation v and every formula α.

The intended meaning of $\mathcal{D}(v)(\alpha)$ is still that "the actual truth value of α is at least $\mathcal{D}(v)(\alpha)''$, but we have also that $\mathcal{D}(v)(\alpha)$ is the best possible "constraint" on the actual truth value of α we can draw from the information v. One proves that the theories of a fuzzy Hilbert system $\mathcal{S}$ coincide with the fixed points of the associated deduction operator $\mathcal{D}$. We say that two fuzzy Hilbert systems are *equivalent* provided that their deduction operators coincide. Since two closure operators coincide iff they have the same fixed points, two systems are equivalent iff they have the same theories.

Proposition 2. *Let $\mathcal{S}$ be a fuzzy Hilbert system and assume that one of the following conditions is satisfied*

(a) a tautology χ exists, i.e. $\mathcal{D}(\emptyset)(\chi) = 1$,

(b) given an inference rule (r', r'') and $\lambda_1, ..., \lambda_n$ in U,

$$r''(\lambda_1, ..., \lambda_n) \le \lambda_1 \wedge ... \wedge \lambda_n.$$

Then $\mathcal{S}$ is equivalent to a system whose inference rules are everywhere defined.

Proof. Assume (a) and, given any rule $r = (r', r'')$ in $\mathcal{S}$, denote by $t = (t', t'')$ the rule such that $t'' = r''$ and t' is the extension of r' obtained by setting $t'(\alpha_1, ..., \alpha_n) = \chi$ if r' is not defined in $(\alpha_1, ..., \alpha_n)$. Let $\mathcal{S}'$ be the system obtained by the so defined rules and with the same fuzzy subset of logical axioms as $\mathcal{S}$. Then it is immediate that every theory of $\mathcal{S}'$ is a theory of $\mathcal{S}$. Moreover, let τ be a theory of $\mathcal{S}$. Then we have that $\tau(\chi) \ge \mathcal{D}(\emptyset)(\chi) = 1$ and therefore $\tau(\chi) = 1$. Consequently, given any inference rule (t', t'') in $\mathcal{S}'$ arising from the rule (r', r'') in $\mathcal{S}$, we have that the inequality $\tau(t'(\alpha_1, ..., \alpha_n)) \ge t''(\tau(\alpha_1), ..., \tau(\alpha_n))$ is satisfied for $(\alpha_1, ..., \alpha_n) \in Dom(r)$ since in this case $t'(\alpha_1, ..., \alpha_n) = r'(\alpha_1, ..., \alpha_n)$ and it is satisfied in the case $(\alpha_1, ..., \alpha_n) \notin Dom(r)$ since $\tau(t'(\alpha_1, ..., \alpha_n)) = \tau(\chi) = 1$.

Assume (b) and, given a rule $r = (r', r'')$, denote by $t = (t', t'')$ the rule such that $t'' = r''$ and t' is the extension of r' obtained by setting $t'(\alpha_1, ..., \alpha_n) = \alpha_1$ if r' is not defined in $(\alpha_1, ..., \alpha_n)$. Moreover, let $\mathcal{S}'$ be the system defined by these extensions. Then it is immediate that every theory of $\mathcal{S}'$ is a theory of $\mathcal{S}$. Conversely, let τ be a theory of $\mathcal{S}$, and (t', t'') a rule in $\mathcal{S}'$ arising from the rule (r', r'')

in $\mathcal{S}$. Then it is immediate that $\tau(t'(\alpha_1, ..., \alpha_n)) \geq t''(\tau(\alpha_1), ..., \tau(\alpha_n))$ for any $(\alpha_1, ..., \alpha_n) \in Dom(r)$. Also, in the case $(\alpha_1, ..., \alpha_n) \notin Dom(r)$, since by condition (b) $\tau(\alpha_1) \geq r''(\tau(\alpha_1), ..., \tau(\alpha_n))$, we have that $\tau(t'(\alpha_1, ..., \alpha_n)) = \tau(\alpha_1) \geq t''(\tau(\alpha_1), ..., \tau(\alpha_n))$. This proves that τ is a theory of $\mathcal{S}'$. Thus, $\mathcal{S}'$ is equivalent to $\mathcal{S}$.

Notice that if the valuation part r'' of the inference rules is obtained from a triangular norm $\otimes$ by the equality $r''(\lambda_1, ..., \lambda_n) = \lambda_1 \otimes ... \otimes \lambda_n$, then, condition (b) is satisfied.

The following theorem shows that the abstract approach to fuzzy logic is equivalent to the Hilbert-style approach. This shows that, in a sense, Pavelka's definition of fuzzy logic is adequate for any monotone fuzzy logic.

Theorem 3. *Let $\mathcal{S}$ be a fuzzy Hilbert system whose deduction operator we denote by $\mathcal{D}$. Then $\mathcal{D}$ is a continuous fuzzy closure operator and therefore $(\mathcal{F}(\mathbb{F}), \mathcal{D})$ is an abstract fuzzy deduction system. Conversely, let $\mathcal{D}$ be a continuous fuzzy closure operator, i.e. let $(\mathcal{F}(\mathbb{F}), \mathcal{D})$ be an abstract fuzzy deduction system. Then, a fuzzy Hilbert system exists whose deduction operator coincides with $\mathcal{D}$.*

Proof. For the first part of the proposition see for example [11]. Assume that $\mathcal{D} : \mathcal{F}(\mathbb{F}) \to \mathcal{F}(\mathbb{F})$ is a continuous fuzzy closure operator and, for any $\alpha \in \mathbb{F}$ and $s_f \in \mathcal{F}_f(\mathbb{F})$ such that $Supp(s_f) = \{\alpha_1, ..., \alpha_n\} \neq \emptyset$, define the rule (r', r'') by setting

$$r'(x_1, ..., x_n) = \begin{cases} \alpha & \text{if } \{x_1, ..., x_n\} = Supp(s_f), \\ \text{undefined} & \text{otherwise,} \end{cases}$$

and

$$r''(\lambda_1, ..., \lambda_n) = \begin{cases} \mathcal{D}(s_f)(\alpha) & \text{if } \lambda_1 > s_f(\alpha_1), ..., \lambda_n > s_f(\alpha_n), \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that r'' satisfies (5) and therefore that $r = (r', r'')$ is a fuzzy inference rule. Let $\mathcal{S}$ be the Hilbert system whose fuzzy inference rules are the so defined rules and whose fuzzy subset of logical axioms is $\mathcal{D}(\emptyset)$. Denote by $\mathcal{D}'$ the related deduction operator. To prove that $\mathcal{D}$ coincides with $\mathcal{D}'$, we prove that the fixed points of $\mathcal{D}$ coincide with the fixed points of $\mathcal{D}'$.

Let τ be a fixed point of $\mathcal{D}'$, i.e., τ is closed with respect to the inference rules and $\tau \supseteq \mathcal{D}(\emptyset)$. Moreover, given any $s_f \in \mathcal{F}_f(\mathbb{F})$, $s_f \neq \emptyset$, and $\alpha \in \mathbb{F}$, let (r', r'') be the corresponding rule. Then, in the case that $\tau(\alpha_1) > s_f(\alpha_1), ..., \tau(\alpha_n) > s_f(\alpha_n)$, we have that $\tau(\alpha) = \tau(r'(\alpha_1, ..., \alpha_n)) \geq r''(\tau(\alpha_1), ..., \tau(\alpha_n)) = \mathcal{D}(s_f)(\alpha)$. Consequently, $\tau \supseteq \mathcal{D}(s_f)$ for any $s_f \in \mathcal{F}_f(\mathbb{F})$ such that $s_f \ll \tau$, and, by Theorem 1, τ is a fixed point of $\mathcal{D}$.

Conversely, let τ be a fixed point of $\mathcal{D}$, then $\tau = \mathcal{D}(\tau) \supseteq \mathcal{D}(\emptyset)$. Moreover, let $r = (r', r'')$ be any rule and assume that (r', r'') is defined by s_f such that $Supp(s_f) = \{\alpha_1, ..., \alpha_n\} \neq \emptyset$ and $\alpha \in \mathbb{F}$. We claim that, for any $x_1, ..., x_n \in \mathbb{F}$ such that $(x_1, ..., x_n) \in Dom(r)$,

$$\tau(r'(x_1, ..., x_n)) \geq r''(\tau(x_1), ..., \tau(x_n)).$$

Indeed, if $r''(\tau(x_1), ..., \tau(x_n)) \neq 0$, then $\tau(x_1) > s_f(\alpha_1), ..., \tau(x_n) > s_f(\alpha_n)$. Consequently,

$$\tau(r'(x_1, ..., x_n)) = \tau(\alpha) = \mathcal{D}(\tau)(\alpha) \geq \mathcal{D}(s_f)(\alpha) = r''(\tau(x_1), ..., \tau(x_n)).$$

This proves that τ is a theory of $\mathcal{S}$ and therefore a fixed point of $\mathcal{D}'$.

5. Truth-functional fuzzy semantics

By following [18], we call *fuzzy semantics* any class $\mathcal{M}$ of fuzzy subsets of formulas such that the map s_1 constantly equal to 1 is not in $\mathcal{M}$. The meaning of such a condition is obvious: no world in which every formula is true exists. The elements in $\mathcal{M}$ are named *fuzzy models*. Given an initial valuation $v : \mathbb{F} \to U$, we say that an element m in $\mathcal{M}$ is a *model of* v provided that $m \supseteq v$. We say that v is *satisfiable* if a model for v exists. The *logical consequence operator Lc* is defined by setting,

$$Lc(v) = \bigcap\{m \in \mathcal{M} : m \supseteq v\},$$

for any $v \in \mathcal{F}(\mathbb{F})$. Like the initial valuation v, the meaning of $Lc(v)(\alpha)$ is still "α is true at least at degree $Lc(v)(\alpha)$", further, we have also that $Lc(v)(\alpha)$ gives the best possible lower constraint for the truth value of α we can draw from v. Observe that, while $Lc(v)(\alpha) = 1$ entails that α is true in any model of v, $Lc(\mathrm{v})(\alpha) = 0$ does not mean that α is false but that the available information v does not say anything to support α. In accordance with Pavelka's formalisms, we have the following definition:

Definition 3. *We say that a fuzzy semantics $\mathcal{M}$ is axiomatizable by an Hilbert system $\mathcal{S}$ if the deduction operator $\mathcal{D}$ of $\mathcal{S}$ coincides with the logical consequence operator Lc associated with $\mathcal{M}$.*

In account of Theorem 3, a fuzzy semantics $\mathcal{M}$ is axiomatizable by a Hilbert system iff the associated logical consequence operator Lc is continuous. We say that a fuzzy semantics $\mathcal{M}$ is *logically compact* if the class $Sat(\mathcal{M})$ of satisfiable initial valuations is inductive, i.e. if $Sat(\mathcal{M})$ is closed with respect to the limits of upward directed families.

Theorem 4. *A fuzzy semantics is logically compact iff, for any initial valuation v,*

$$v \text{ satisfiable } \Leftrightarrow \text{ every finite } v_f \ll v \text{ is satisfiable.} \tag{7}$$

Proof. Let $\mathcal{M}$ be logically compact. Then, it is trivial that v satisfiable implies that every finite fuzzy subset v_f such that $v_f \ll v$ is satisfiable. In order to prove the converse implication, assume that every finite fuzzy subset v_f such that $v_f \ll v$ is satisfiable. Then, since v is the inductive limit of the class $\{v_f \in \mathcal{F}_f(S) : v_f \ll v\}$, v is satisfiable. This proves (7). Conversely, assume (7) and let $\mathcal{H}$ be an inductive class of satisfiable fuzzy subsets. We have to prove that $v = \bigcup \mathcal{H}$ is satisfiable. Now, for every finite fuzzy set v_f such that $v_f \ll v$ an element $s \in \mathcal{H}$ exists such that $v_f \subseteq s$. Since s is satisfiable, v_f is satisfiable too. Thus, by (7) v is satisfiable.

A very important class of fuzzy semantics is given by the truth-functional valuations. Assume that $\mathbb{F}$ is the set of formulas in a zero order language. This means that we start from

- an infinite set $VAR = \{p_1, ..., p_n, ...\}$ of propositional variables ;
- a set of logical connectives containing the usual ones, namely $\wedge$, $\vee$, $\neg$.

As usual, $\mathbb{F}$ is defined assuming that the propositional variables are formulas and that if h is an n-ary connective and $\alpha_1, ..., \alpha_n$ are formulas, then $h(\alpha_1, ..., \alpha_n)$ is a formula. Also, we assume that every n-ary connective h is interpreted by a suitable n-ary operation $\underline{h}$ in the lattice U and that the interpretations of $\vee$, $\wedge$, and $\neg$ satisfy the following very general conditions:

- (i) they extend the related classical interpretations,
- (ii) the interpretation $\text{-} : U \to U$ of $\neg$ is order-reversing,
- (iii) the interpretations of $\vee$ and $\wedge$ are order-preserving.

A *truth-functional valuation* is any fuzzy subset m of $\mathbb{F}$ such that

$$m(h(\alpha_1, ..., \alpha_n)) = \underline{h}(m(\alpha_1), ..., m(\alpha_n)) \tag{8}$$

for every n-ary logical connective h and $\alpha_1, ..., \alpha_n$ in $\mathbb{F}$. Let $v : VAR \to U$ be a valuation of the propositional variables, i.e. an assignment of truth values to the propositional variables. Then there exists exactly one truth-functional valuation m_v such that $m_v(p_i) = v(p_i)$ for every $i \in N$. In this case we say that m_v is the *truth-functional extension* of v. The class $\mathcal{M}$ of the truth-functional valuations is a fuzzy semantics since $s^1 \notin \mathcal{M}$. Indeed, otherwise, given any propositional variable p_1, by condition (i),

$$1 = s^1(\neg p_1) = \text{-}s^1(p_1) = \text{-}1 = 0.$$

Definition 4. *Let $\mathcal{M}$ be the class of truth-functional valuations. Then $\mathcal{M}$ is called a truth-functional semantics. We say that $\mathcal{M}$ is continuous if all the connectives are interpreted by continuous functions.*

We can extend any assignment $v : VAR \to U$ into an initial valuation by setting $v(\alpha) = 0$ everywhere α is not a propositional variable. In such a case we have that, for any propositional variable p_i,

$$Lc(v)(p_i) = v(p_i) = m_v(p_i).$$

Indeed, trivially, $Lc(v)(p_i) \geq v(p_i)$, and, since m_v is a model of v, $v(p_i) = m_v(p_i) \geq Lc(v)(p_i)$. Nevertheless, $Lc(v)$ is different from m_v, in general. This since if v is assumed as an initial valuation, then $v(p_i)$ carries on the information "the actual truth value of p_i is at least $v(p_i)$" while if v is assumed as an assignment of truth values, then the resulting information is that "the actual value of p_i is exactly $v(p_i)$". The following proposition gives the correct way to relate the truth-functional extensions with Lc.

Proposition 3. *Assume that the interpretation* - *of the negation is injective and let* $v : VAR \to U$ *be a valuation of the propositional variables. Moreover, denote by* v' *the extension of* v *to the whole set* $\mathbb{F}$ *of formulas obtained by setting*

$$v'(x) = \begin{cases} v(p_i) & \text{if } x = p_i, \\ \text{-}v(p_i) & \text{if } x = \neg p_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $Lc(v') = m_v$ *and* v' *is an initial valuation whose only model is* m_v.

Proof. It is immediate that m_v is a model of v'. Moreover, if m is a model of v', then $m(p_i) \geq v'(p_i)$. Also, since $\text{-}(m(p_i)) = m(\neg p_i) \geq \text{-}(v'(p_i))$ and - is injective, we have that $m(p_i) \leq v'(p_i)$ and $m(p_i) = v'(p_i) = m_v(p_i)$. This proves that $m = m_v$.

6. Any continuous truth-functional semantics is axiomatizable

The search for a characterization of the truth-functional logics whose corresponding fuzzy semantics are axiomatizable is an interesting open question. A first basic step in this direction was made in [20], where J. Pavelka proves that of all the $2^{\aleph}$ isomorphism types of residuated lattice carried on by [0,1], only the ones whose interpretations of the logical connectives are continuous serve as a basis for a semantically complete propositional calculus. To give a further contribution, firstly we will show that if the truth-functional semantics $\mathcal{M}$ is continuous, then a formula exists suggesting simple procedures to compute the fuzzy subset $Lc(v)$ of logical consequences of an initial valuation v. To simplify our treatment, we assume that in the language under consideration there are only a finite number of propositional variables $p_1, ..., p_k$. In such a case, a model is an assignment of truth values $\lambda_1, ..., \lambda_k$ to the variables $p_1, ..., p_k$ and therefore a point in U^k. In accordance, $\mathcal{M}$ coincides with the set of points in the space U^k. As usual, any formula α is associated with a (polynomial) function $\underline{\alpha} : U^k \to U$. Namely, $\underline{\alpha}$ is defined by induction on the complexity of α by setting, for any $\lambda_1, ..., \lambda_k$ in U,

$$\underline{\alpha}(\lambda_1, ..., \lambda_k) = \begin{cases} \lambda_i, & \text{if } \alpha = p_i, \\ \underline{h}(\underline{\alpha}_1(\lambda_1, ..., \lambda_k), ..., \underline{\alpha}_t(\lambda_1, ..., \lambda_k)), & \text{if } \alpha = h(\alpha_1, ..., \alpha_t). \end{cases}$$

Observe that if m is a truth-functional model, then

$$m(\alpha) = \underline{\alpha}(m(p_1), ..., m(p_k)).$$

Obviously, if the truth functional semantics is continuous, then any $\underline{\alpha} : U^k \to U$ is a continuous function.

Definition 5. *Let* $\gamma_1, \gamma_2, ...$ *be a complete enumeration of all the formulas. Then for every initial valuation* v *we denote by* $M(v)$ *the set of solutions* $(\lambda_1, ..., \lambda_k) \in U^k$ *of the system*

$$\begin{aligned} &\underline{\gamma}_1(\lambda_1, ..., \lambda_k) \geq v(\gamma_1) \\ &\underline{\gamma}_2(\lambda_1, ..., \lambda_k) \geq v(\gamma_2) \\ &\quad ... \end{aligned} \tag{9}$$

In the following, given a real-valued function f and a subset X of the domain of f, we denote by $\inf(f, X)$ and $\min(f, X)$ the values $\inf\{f(\lambda) : \lambda \in X\}$ and $\min\{f(\lambda) : \lambda \in X\}$, respectively.

Proposition 4. *The models of an initial valuation v coincide with the points in M(v) and, for every formula* α*,*

$$Lc(v)(\alpha) = \inf(\underline{\alpha}, M(v)). \tag{10}$$

Moreover,

$$M(v) = \bigcap\{M(v_f) : v_f \text{ is finite and } v_f \ll v\}. \tag{11}$$

If $\mathcal{M}$ *is continuous, then* $M(v)$ *is compact and*

$$Lc(v)(\alpha) = \min(\underline{\alpha}, M(v)). \tag{12}$$

Proof. Equality (10) is obvious. In order to prove (11), observe that it is immediate that $M(v) \subseteq M(v_f)$ for every $v_f \ll v$. Conversely, suppose $(\lambda_1, ..., \lambda_k) \in M(v_f)$, i.e.,

$$\underline{\gamma}_1(\lambda_1, ..., \lambda_k) \geq v_f(\gamma_1)$$
$$\underline{\gamma}_2(\lambda_1, ..., \lambda_k) \geq v_f(\gamma_2)$$
$$\dots$$

for every v_f finite such that $v_f \ll v$. Then,

$$\underline{\gamma}_1(\lambda_1, ..., \lambda_k) \geq \sup\{v_f(\gamma_1) : v_f \text{ is finite and } v_f \ll v\} = v(\gamma_1)$$
$$\underline{\gamma}_2(\lambda_1, ..., \lambda_k) \geq \sup\{v_f(\gamma_2) : v_f \text{ is finite and } v_f \ll v\} = v(\gamma_2)$$
$$\dots$$

and $(\lambda_1, ..., \lambda_k) \in M(v)$. The remaining part of the proposition is obvious.

Theorem 5. *Any continuous truth-functional semantics is logically compact and axiomatizable by a fuzzy Hilbert system.*

Proof. Let $\mathcal{M}$ be a continuous truth-functional semantics, $\mathcal{C}$ an upward directed class of satisfiable initial valuations and $v = \lim(\mathcal{C})$. Then, $\{M(s) : s \in \mathcal{C}\}$ is a downward directed class of nonempty compact sets and, by the finite intersection property,

$$M(v) = \bigcap\{M(s) : s \in \mathcal{C}\} \neq \emptyset.$$

This proves that v is satisfiable. In order to prove that $\mathcal{M}$ is axiomatizable by a Hilbert system, by Theorem 3 it is enough to prove that Lc is continuous. To this aim, by Theorem 1, we have to prove that

$$Lc(v)(\alpha) = \sup\{Lc(v_f)(\alpha) : v_f \text{ is finite and } v_f \ll v\},$$

i.e., by (12), that

$$\min(\underline{\alpha}, M(v)) = \sup\{\min(\underline{\alpha}, M(v_f)) : \; v_f \text{ is finite and } v_f \ll v\}.$$

To prove such an equality, set

$$M = \{\min(\underline{\alpha}, M(v_f)) : v_f \text{ is finite and } v_f \ll v\}.$$

Then, it is immediate that $\min(\underline{\alpha}, M(v))$ is an upper bound of M. Let l be any upper bound of M, we will prove that $\min(\underline{\alpha}, M(v)) \leq l$, i.e. that $(\delta_1, ..., \delta_k) \in M(v)$ exists such that $\underline{\alpha}(\delta_1, ..., \delta_k) \leq l$. Indeed, for any v_f finite such that $v_f \ll v$, set

$$T(v_f) = \{(\lambda_1, ..., \lambda_k) \in U^k : 0 \leq \underline{\alpha}(\lambda_1, ..., \lambda_k) \leq l\} \cap M(v_f).$$

Since $\{v_f : v_f$ finite and $v_f \ll v\}$ is upward directed, the class $\mathcal{C} = \{T(v_f) : v_f$ finite and $v_f \ll v\}$ of subsets of U^k is downward directed. It is immediate that each $T(v_f)$ is compact. Moreover, since at least the points of $M(v_f)$ in which $\underline{\alpha}$ assumes the value $\min(\underline{\alpha}, M(v_f))$ belong to $T(v_f)$, $T(v_f) \neq \emptyset$. Then

$$\bigcap \mathcal{C} = \{(\lambda_1, ..., \lambda_k) \in U^k : 0 \leq \underline{\alpha}(\lambda_1, ..., \lambda_k) \leq l\} \cap M(v) \neq \emptyset, \qquad (13)$$

and this proves that $(\delta_1, ..., \delta_k) \in M(v)$ exists such that $\underline{\alpha}(\delta_1, ..., \delta_k) \leq l$.

7. Recursively enumerable fuzzy sets

We say that a set S has an *effective coding* if an one-one map $c : S \to N$ from S into the set N of natural numbers exists. We assume that both c and c^{-1} are given by suitable (informal) algorithms (see [21] pag. 27). For example, the free semigroup generated by a finite alphabet, the set of formulas of a logic are sets with an effective coding. It is immediate that the Cartesian product of two sets with an effective coding is a set with an effective coding. By identifying the sets with an effective coding with N, we can extend all the notions of recursion theory (usually defined in N) to these sets. As an example, assume that S_1 and S_2 are codified by the one-one-computable functions $c_1 : S_1 \to N$ and $c_2 : S_2 \to N$. Then we say that a partial map $f : S_1 \to S_2$ is *partial recursive* provided that a partial recursive function $f' : N \to N$ exists such that, given any x in S_1, f is defined in x if and only if f' is defined in $c_1(x)$ and

$$f(x) = c_2^{-1}(f'(c_1(x))),$$

i.e., the following diagram commutes:

$$\begin{array}{ccc} & f & \\ S_1 & \longrightarrow & S_2 \\ c_1 \downarrow & & \uparrow c_2^{-1} \\ N & \longrightarrow & N \\ & f' & \end{array}$$

Recall that a subset X of S is called *recursively enumerable*, in brief r.e., if a partial recursive function exists whose domain is X. It is immediate that if $R \subseteq S_1 \times S_2$ is a recursive relation from a set S_1 to a set S_2, then the set

$$\{x \in S_1 : y \in S_2 \text{ exists such that } (x, y) \in R\}$$

is recursively enumerable. Also, a set X is recursively enumerable iff either X is empty or a recursive function $f : N \to S$ exists such that X is the codomain of f, that is $X = \{f(1), f(2), ...\}$. The following rather obvious proposition shows that it is possible to define the classical notion of recursive enumerability in terms of limit.

Proposition 5. *A subset X of S is recursively enumerable if and only if there exists a total recursive map $h : S \times N \to \{0, 1\}$, increasing with respect to the second variable, such that, for any $x \in S$,*

$$c_X(x) = \lim_{n\to\infty} h(x, n). \tag{14}$$

Proof. Assume that X is recursively enumerable. Then, if $X = \emptyset$ equation (14) is satisfied by setting h equal to the function constantly equal to zero. In the case $X \neq \emptyset$, let f be a total recursive function whose codomain is X and define h by

$$h(x, n) = \begin{cases} 0 & \text{if } x \notin \{f(1), ..., f(n)\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then, $c_X(x) = \lim_{n\to\infty} h(x, n)$. Conversely, assume that a total recursive function $h : S \times N \to \{0, 1\}$ exists which is increasing with respect to the second variable and such that $c_X(x) = \lim_{n\to\infty} h(x, n)$. Then the map

$$f(x) = \begin{cases} 1, & \text{if } h(x, n) > 0 \text{ for a suitable } n, \\ \text{undefined}, & \text{otherwise} \end{cases}$$

is a partial recursive function whose domain is X.

Such a proposition suggests how the notion of the recursive enumerability can be extended to fuzzy subsets. We assume that a Gödel coding for $\ddot{U}$ is fixed and therefore that the notion of recursive map from $S \times N \to \ddot{U}$ is defined.

Definition 6. *A fuzzy subset $s : S \to U$ is recursively enumerable (in brief, r.e.) if a recursive map $h : S \times N \to \ddot{U}$ exists such that, for every $x \in S$, $h(x, n)$ is increasing with respect to n and*

$$s(x) = \lim_{n\to\infty} h(x, n). \tag{15}$$

Note that, since h is increasing with respect to n, (15) is equivalent to

$$s(x) = \sup\{h(x, n) : n \in N\}. \tag{16}$$

Let h be any recursive map (not necessarily increasing with respect to the second variable) and define s by (16). Then, s is recursively enumerable. In fact, by setting

$$k(x, n) = h(x, 1) \vee ... \vee h(x, n)$$

we obtain a recursive function, increasing with respect to n and such that, for any $x \in S$, $s(x) = \lim_{n\to\infty} k(x, n)$. More generally, if $h : S \times N^p \to \ddot{U}$ is any recursive map and s is the fuzzy subset of S defined by setting

$$s(x) = \sup\{h(x, n_1, ..., n_p) : n_1 \in N, ..., n_p \in N\},$$

then s is recursively enumerable. Indeed, it is sufficient to consider a Gödel numbering $c : N \to N^p$ and to set $k(x, n) = h(x, c(n))$. The following definition and proposition extend the classical ones.

Definition 7. *A fuzzy subset s of S is recursively co-enumerable if its complement -s is recursively enumerable, s is decidable if it is both recursively enumerable and recursively co-enumerable.*

Proposition 6. *A fuzzy subset s of S is recursively co-enumerable iff a recursive map $k : S \times N \to Ü$ exists such that, for every $x \in S$, $k(x, n)$ is decreasing with respect to n and*

$$s(x) = \lim_{n\to\infty} k(x, n).$$

Proof. Assume that a recursive function $d : S \times N \to Ü$ exists such that $d(x, n)$ is increasing with respect to n and $-s(x) = \lim_{n\to\infty} d(x, n)$. Then, by setting $k(x, n) = 1 - d(x, n)$,we have that

$$s(x) = 1 - \lim_{n\to\infty} d(x, n) = \lim_{n\to\infty} 1 - d(x, n) = \lim_{n\to\infty} k(x, n),$$

where $k(x, n)$ is decreasing with respect to n. In the same way one proves the converse implication.

The proof of the following theorem is evident.

Theorem 6. *A fuzzy subset s of S is decidable iff for every $x \in S$, $s(x)$ is the limit of an effectively computable nested sequence of intervals, that is iff two recursive maps $h : S \times N \to Ü$ and $k : S \times N \to Ü$ exist such that, for any $x \in S$, $h(x, n)$ is increasing and $k(x, n)$ is decreasing with respect to the second variable, for every $n \in N$,*

$$h(x, n) \leq s(x) \leq k(x, n)$$

and

$$\lim_{n\to\infty} h(x, n) = s(x) = \lim_{n\to\infty} k(x, n).$$

We say that a recursive function $f : S \times N \to Ü$ is *recursively convergent to* s if $s(x) = \lim_{n\to\infty} f(x, n)$ and a recursive function $e : S \times N \to N$ exists such that, for every $x \in S$ and $p \in N$,

$$|f(x, n) - f(x, m)| < 1/p, \text{ for any } n, m \geq e(x, p).$$

Theorem 7. *A fuzzy subset s of S is decidable iff there exists a recursive function $f : S \times N \to Ü$ recursively convergent to s.*

8. Effectiveness in crisp logic

It is natural to require that the deduction operator of a fuzzy logic satisfies some kind of "computability" property beside continuity. In this section we will examine the crisp case and we start from an interesting definition proposed in [21]. Observe that if S is codified, then both $\mathcal{P}_f(S)$ and $S \times \mathcal{P}_f(S)$ can be codified. Consequently, the notion of recursively enumerable subsets of $S \times \mathcal{P}_f(S)$ is defined.

Definition 8. *An operator $H : \mathcal{P}(S) \to \mathcal{P}(S)$ is called an enumeration operator if a recursively enumerable subset W of $S \times \mathcal{P}_f(S)$ exists such that, for any subset X of S,*

$$H(X) = \{x \in S : X_f \text{ exists such that } (x, X_f) \in W \text{ and } X_f \subseteq X\}. \tag{17}$$

It is immediate to prove that if H is an enumeration operator, then

$$X \text{ recursively enumerable } \Rightarrow H(X) \text{ recursively enumerable.}$$

Proposition 7. *An operator $H : \mathcal{P}(S) \to \mathcal{P}(S)$ is an enumeration operator iff H is compact and*

$$W_H = \{(x, X_f) \in S \times \mathcal{P}_f(S) : x \in H(X_f)\}$$

is a recursively enumerable relation.

Proof. Let H be an enumeration operator, then it is obvious that H is compact. Let W be as in Definition 8 and let g be a recursive function whose codomain is W. Then, since for any $X_f \in \mathcal{P}_f(S)$,

$$H(X_f) = \{x \in S : X'_f \text{ exists such that } (x, X'_f) \in W \text{ and } X'_f \subseteq X_f\},$$

we have also that

$$(x, X_f) \in W_H \Leftrightarrow n \in N \text{ exists such that } g(n) = (x, X'_f) \text{ with } X'_f \subseteq X_f.$$

This proves that W_H is recursively enumerable. The converse implication is immediate.

The following definition extends the definition given in [1].

Definition 9. *We call effective an abstract deduction system $(\mathcal{P}(\mathbb{F}), \mathcal{D})$ such that $\mathcal{D}$ is an enumeration closure operator.*

A first attempt to define the effective Hilbert systems is to consider Hilbert systems whose set $\mathbb{A}$ of logical axioms is decidable and such that every rule $r \in \mathbb{R}$ is a partial recursive function. This is O.K. in the case of a finite number of inference rules, but we have to be a little more accurate in the general case. Indeed, the following paradoxical result holds.

Proposition 8. *Let H be any compact closure operator. Then, a system $\mathcal{S}$ exists with computable inference rules and a decidable subset of logical axioms whose deduction operator coincides with H.*

Proof. For any $X = \{\alpha_1, ..., \alpha_n\} \in \mathcal{P}_f(\mathbb{F})$ and α in $\mathbb{F}$, we denote by r_α^X the n-ary inference rule defined by assuming that r_α^X is the first projection, i.e. $r_\alpha^X(x_1, ..., x_n) = x_1$, in the case that $\alpha \notin H(\{\alpha_1, ..., \alpha_n\})$. Moreover, in the case that $\alpha \in H(\alpha_1, ..., \alpha_n)$ we define r_α^X by setting

$$r_\alpha^X(x_1, ..., x_n) = \begin{cases} \alpha & \text{if } x_1 = \alpha_1, ..., x_n = \alpha_n, \\ x_1 & \text{otherwise.} \end{cases}$$

Also, for any formula ψ in $H(\emptyset)$, denote by $r_\psi : \mathbb{F} \to \mathbb{F}$ the map constantly equal to ψ. Both the rule r_α^X and r_ψ are recursive functions. Let ϕ be a fixed element in $H(\emptyset)$, then we denote by $\mathcal{S}$ the crisp Hilbert system with the just considered rules and whose set of logical axioms is $\{\phi\}$. Then the system of logical axioms is decidable and each inference rule is computable. It is evident that the deduction operator of $\mathcal{S}$ coincides with H.

Since compact operators exist with no level of computability, the proposition shows that a more correct condition of effectiveness for a Hilbert system is necessary. The question is that in the proof we have now exposed the algorithms for the inference rules are not given in a uniform way. Indeed, it is not possible to establish effectively what kind of algorithm a given pair X and α is associated with. Then, we propose the following definition where we denote by $r_1, r_2, ...$ an effective enumeration of all the partial recursive operations in $\mathbb{F}$.

Definition 10. *We say that a crisp Hilbert system $\mathcal{S}$ is effective if*
(a) a recursive function $h : N \to N$ exists such that $\mathbb{R} = \{r_{h(i)} : i \in N\}$,
(b) the set $\mathbb{A}$ of logical axioms is recursively enumerable.

In the case $\mathbb{R}$ finite, (a) is equivalent to say that all the rules in $\mathbb{R}$ are partial recursive. Notice that it is not restrictive to assume that the rules $r_{h(i)}$ in $\mathbb{R}$ are total recursive functions. Indeed, let $\mathcal{S}$ be any Hilbert system satisfying a) and b) of Definition 10 and define the rule $r_{t(i,j)}$ by setting

$$r_{t(i,j)}(\alpha_1, ..., \alpha_n) = \begin{cases} r_{h(i)}(\alpha_1, ..., \alpha_n), & \text{if } r_{h(i)} \text{converges in } \alpha_1, ..., \alpha_n \\ & \text{in less than } j \text{ steps,} \\ \alpha_1, & \text{otherwise .} \end{cases}$$

Since such an algorithm depends in an uniform way from i and j, we can assume that t is a recursive function. Then, the system defined by this family of inference rules and whose set of logical axioms is $\mathbb{A}$ is equivalent to $\mathcal{S}$. The constructive point of view imposes a more precise definition of proof. Indeed, recall that in a proof $\alpha_1, ..., \alpha_n$ a justification of a formula α_j is one of the following claims
i) α_j belongs to the set $\mathbb{A}$ of logical axioms
ii) α_j is a hypothesis
iii) α_j is obtained by the rule $r_{h(i)}$ applied to the early proved formulas $\alpha_{s(1)}, ..., \alpha_{s(n)}$.

We can represent a justification as an element in the set

$$\{la, hy\} \cup \{\{i, s(1), ..., s(n)\} : i \in N, h(i) \text{ is the index of an } n\text{-ary rule }\},$$

and therefore we can assign a code number to any justification. Then a *proof* is a sequence $\langle\alpha_1, i_1\rangle, ..., \langle\alpha_n, i_n\rangle$ of elements in $\mathbb{F} \times N$ such that, for $j = 1, ..., n$,

- if i_j is the code number of la, then $\alpha_j \in \mathbb{A}$
- if i_j is the code number of a justification like $\{i, s(1), ..., s(n)\}$, then $s(1) < j, ..., s(n) < j$ and $\alpha_j = r_{h(i)}(\alpha_{s(1)}, ..., \alpha_{s(n)})$.

It is immediate that an algorithm exists such that, given any sequence $\langle\alpha_1, i_1\rangle, ..., \langle\alpha_n, i_n\rangle$, the algorithm converges if such a sequence is a proof, it diverges otherwise. Then, the set of proofs is the domain of a partial recursive function and therefore it is recursively enumerable (but not decidable, in general). If all the inference rules are total recursive and $\mathbb{A}$ is decidable, then the set of proofs is decidable.

Theorem 8. *Let $\mathcal{D}$ be the deduction operator of an effective crisp Hilbert system. Then $\mathcal{D}$ is an enumeration closure operator and therefore $(\mathcal{P}(\mathbb{F}), \mathcal{D})$ is an effective deduction system. Conversely, let $(\mathcal{P}(\mathbb{F}), \mathcal{D})$ be an effective deduction system. Then an effective crisp Hilbert system exists whose deduction operator coincides with $\mathcal{D}$.*

Proof. By Proposition 7, in order to prove that $\mathcal{D}$ is an enumeration operator we have to prove that the relation

$$W = \{(x, X_f) \in \mathbb{F} \times \mathcal{P}_f(\mathbb{F}) : x \in \mathcal{D}(X_f)\}$$

is recursively enumerable. Let $(\pi_i)_{i\in N}$ be an effective enumeration of all the proofs, then

$$W = \{(x, X_f) : \text{ there is a proof } \pi_i \text{ of } x \text{ under hypotheses } X_f\}.$$

This proves that W is recursively enumerable. Conversely, assume that $\mathcal{D} : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ is an enumeration closure operator. Then, by definition, a recursively enumerable subset $W \subseteq \mathbb{F} \times \mathcal{P}_f(\mathbb{F})$ exists such that

$$\mathcal{D}(X) = \{x \in \mathbb{F} : X_f \text{ exists such that } (x, X_f) \in W \text{ and } X_f \subseteq X\}.$$

Consider, for any $n, i \in N$, the rule r_i^n such that, for any $\alpha_1, ..., \alpha_n$ in $\mathbb{F}$,

- $r_i^n(\alpha_1, ..., \alpha_n)$ is the i-th formula in the effective generation of $\mathcal{D}(\alpha_1, ..., \alpha_n)$ if $|\mathcal{D}(\alpha_1, ..., \alpha_n)| \geq i$,
- $r_i^n(\alpha_1, ..., \alpha_n)$ is undefined if $|\mathcal{D}(\alpha_1, ..., \alpha_n)| < i$.

It is easy to prove that such a rule is a partial recursive function. Let $\mathcal{S}$ be the effective crisp Hilbert system whose set of logical axioms is the recursively enumerable set $\mathcal{D}(\emptyset)$ and with the inference rules we have now defined. Then T is a theory of $\mathcal{S}$ iff $T \supseteq \mathcal{D}(\emptyset)$ and, for every rule r_i^n and $\alpha_1, ..., \alpha_n$ in T, $r_i^n(\alpha_1, ..., \alpha_n) \in T$. In turn, this is equivalent to say that $\mathcal{D}(X_f) \subseteq T$ for every $X_f \in \mathcal{P}_f(T)$ and therefore, due to the compactness of $\mathcal{D}$, that T is a fixed point of $\mathcal{D}$. Thus, the theories of $\mathcal{S}$ coincide with the fixed points of $\mathcal{D}$. This proves that $\mathcal{D}$ is the deduction operator of $\mathcal{S}$.

9. Enumeration fuzzy operators

Passing to the fuzzy case, in order to extend the definition of enumeration operator observe that $S \times \mathcal{F}_f(S)$ can be codified and therefore that the notion of recursively enumerable fuzzy subset of $S \times \mathcal{F}_f(S)$ is defined.

Definition 11. *We say that a fuzzy operator $H : \mathcal{F}(S) \to \mathcal{F}(S)$ is an enumeration fuzzy operator if a recursively enumerable fuzzy relation $w : S \times \mathcal{F}_f(S) \to U$ exists such that*

$$H(s)(x) = \sup\{w(x, s_f) : s_f \ll s\}. \tag{18}$$

Equivalently, H is an enumeration fuzzy operator if and only if a recursive map $k : S \times \mathcal{F}_f(S) \times N \to \ddot{U}$ exists such that, for any $x \in S$ and $s \in \mathcal{F}(S)$,

$$H(s)(x) = \sup\{k(x, s_f, m) : m \in N \text{ and } s_f \ll s\},$$

Theorem 9. *Any enumeration fuzzy operator $H : \mathcal{F}(S) \to \mathcal{F}(S)$ is continuous. Moreover,*

$$s \textit{ recursively enumerable} \Rightarrow H(s) \textit{ recursively enumerable} .$$

Proof. Let $(s_i)_{i\in I}$ be an upward directed family of fuzzy subsets. In order to prove that $H(\bigcup_{i\in I} s_i) = \bigcup_{i\in I} H(s_i)$, observe that,

$$\begin{aligned} H(\bigcup_{i\in I} s_i)(x) &= \sup\{w(x, s_f) : s_f \ll \bigcup_{i\in I} s_i\} \\ &= \sup\{w(x, s_f) : \text{ there is } i \in I \text{ such that } s_f \ll s_i\} \\ &= \sup_{i\in I} H(s_i)(x). \end{aligned}$$

Assume that s is recursively enumerable and therefore that a recursive map $h : S \times N \to \ddot{U}$ exists which is increasing with respect to the second variable and $s(x) = \lim_{n\to\infty} h(x, n)$. For any $n \in N$, we denote by h_n the fuzzy subset of S defined by setting $h_n(x) = h(x, n)$ for any $x \in S$. Then, since H is continuous and $(h_n)_{n\in N}$ is upward directed, we have that

$$H(s)(x) = H(\bigcup_{n\in N} h_n)(x) = \sup_{n\in N} H(h_n)(x).$$

By observing that

$$H(h_n)(x) = \sup\{w(x, s_f) : s_f \ll h_n\},$$

and that a recursive map $k : S \times \mathcal{F}_f(S) \times N \to \ddot{U}$ exists such that

$$w(x, s_f) = \sup_{m\in N} k(x, s_f, m)$$

for any $x \in S$ and $s_f \in \mathcal{F}_f(S)$, we have that

$$H(s)(x) = \sup\{k(x, s_f, m) : n, m \in N \text{ and } s_f \ll h_n\}.$$

Define r by setting $r(x, i, n, m) = k(x, s_f, m)$ if i is the code number of s_f and $s_f \ll h_n$ and $r(x, i, n, m) = 0$ otherwise. Then r is a recursive map and

$$H(s)(x) = \sup_{i\in N}(\sup_{n\in N}(\sup_{m\in N} r(x, i, n, m))).$$

This proves that $H(s)$ is recursively enumerable.

Theorem 10. *Let $H : \mathcal{F}(S) \to \mathcal{F}(S)$ be a fuzzy operator and define $w_H : S \times \mathcal{F}_f(S) \to U$ by setting, for any $x \in S$ and $s_f \in \mathcal{F}_f(S)$,*

$$w_H(x, s_f) = H(s_f)(x). \tag{19}$$

Then H is an enumeration operator iff H is continuous and w_H is a recursively enumerable fuzzy relation.

Proof. Assume that H is an enumeration operator, let w be as in Definition 11 and $k : S \times \mathcal{F}_f(S) \times N \to \ddot{U}$ a recursive map such that

$$w(x, s'_f) = \sup_{m\in N} k(x, s'_f, m)$$

for any $x \in S$ and $s'_f \in \mathcal{F}_f(S)$. Then

$$w_H(x, s_f) = H(s_f)(x) = \sup\{k(x, s'_f, m) : s'_f \ll s_f, m \in N\}.$$

Define the function h by setting $h(x, s_f, s'_f, m) = k(x, s'_f, m)$ if $s'_f \ll s_f$ and $h(x, s_f, s'_f, m) = 0$ otherwise. Then

$$w_H(x, s_f) = \sup\{h(x, s_f, s'_f, m) : s'_f \in \mathcal{F}_f(S) \text{ and } m \in N\}$$

and this proves that w_H is recursively enumerable. Conversely, if w_H is recursively enumerable, then, since H is continuous,

$$H(s)(x) = \sup\{H(s_f)(x) : s_f \ll s\} = \sup\{w_H(x, s_f) : s_f \ll s\}.$$

This proves that H is an enumeration operator.

Such a theorem entails that in Definition 11 it is not restrictive to assume that w is order-preserving with respect to the second variable, i.e. that $w(x, s) \leq w(x, s')$ whenever s and s' are finite fuzzy subsets such that $s \subseteq s'$.

10. Effective fuzzy logic

We define an *abstract effective fuzzy deduction system* as any abstract fuzzy deduction system $(\mathcal{F}(\mathbb{F}), \mathcal{D})$ such that $\mathcal{D}$ is a fuzzy enumeration operator. We define an *effective fuzzy Hilbert system* as a fuzzy system whose fuzzy subset of logical axioms is recursively enumerable and whose fuzzy inference rules are computable in a uniform way. To make precise such a definition, we denote by $r'_1, r'_2, ...$ an effective coding of all the partial recursive operations in $\mathbb{F}$, i.e. all the partial recursive functions from a Cartesian product $\mathbb{F}^n$ of $\mathbb{F}$ to $\mathbb{F}$ (where n varies in N). Moreover, we denote by $d_1, d_2, ...$ an effective coding of all the partial recursive functions from a Cartesian product $\ddot{U}^n$ of $\ddot{U}$ to $\ddot{U}$ (where n varies in N). Also, we set

$$r''_i(x_1, ..., x_n) = \sup\{d_i(y_1, ..., y_n) : y_1 < x_1, ..., y_n < x_n\}. \tag{20}$$

It is easy to prove that each r''_i is a total function satisfying (5). Moreover, if d_i satisfies (5), then r''_i is an extension of d_i and therefore $r''_i(\lambda_1, ..., \lambda_n)$ is a rational number whenever $\lambda_1, ..., \lambda_n$ are rational numbers. The usual triangular norms can be obtained in such a way.

Definition 12. *A fuzzy Hilbert system* $(a, \mathbb{R})$ *is effective provided that :*

(a) two recursive maps $h : N \to N$ *and* $k : N \to N$ *exist such that for any* $i \in N$, $d_{k(i)} : \ddot{U}^n \to \ddot{U}$ *is a total function satisfying (5) and*

$$\mathbb{R} = \{(r'_{h(i)}, r''_{k(i)}) : i \in N\},$$

(b) the fuzzy set a of logical axioms is recursively enumerable.

In the fuzzy case, the notion of proof is slightly different from the crisp case. In fact, since every formula belongs to the fuzzy subset of logical axioms (possibly to the degree zero), it is not necessary to control condition i). Then we define a proof as a sequence $\langle \alpha_1, i_1 \rangle, ..., \langle \alpha_n, i_n \rangle$ such that, for $j = 1, ..., n$, if i_j is the code number of a justification like $\{i, s(1), ..., s(n)\}$, then $\alpha_j = r'_{h(i)}(\alpha_{s(1)}, ..., \alpha_{s(n)})$. Again, the set of proofs can be enumerated in an effective way. The following theorem extends Theorem 8 to fuzzy deduction systems.

Theorem 11. *The deduction operator* $\mathcal{D}$ *of an effective fuzzy Hilbert system* $\mathcal{S}$ *is an enumeration fuzzy operator and therefore it defines an effective abstract fuzzy deduction system* $(\mathcal{F}(\mathbb{F}), \mathcal{D})$. *Conversely, let* $(\mathcal{F}(\mathbb{F}), \mathcal{D})$ *be an effective abstract fuzzy deduction system. Then an effective fuzzy Hilbert system exists whose deduction operator coincides with* $\mathcal{D}$.

Proof. Let $\mathcal{D}$ be the deduction operator of an effective fuzzy Hilbert system $\mathcal{S}$. Then, since $\mathcal{D}$ is continuous, by Theorem 10, to prove that $\mathcal{D}$ is an enumeration operator, it is enough to prove that the fuzzy subset $w_{\mathcal{D}}$ of $\mathbb{F} \times \mathcal{F}_f(\mathbb{F})$ defined by setting $w_{\mathcal{D}}(x, s_f) = \mathcal{D}(s_f)(x)$ is recursively enumerable. Indeed, let $a' : \mathbb{F} \times N \to \ddot{U}$ be a total recursive function, increasing with respect to n, such that, for every formula x,

$$a(x) = \sup\{a'(x, n) : n \in N\} \tag{21}$$

and, for every $n \in N$, define the fuzzy subset a_n by setting $a_n(x) = a'(x, n)$. Moreover, given a proof π, denote by $Val(\pi, s_f, n)$ the valuation of π in the fuzzy Hilbert system obtained by assuming as a fuzzy set of logical axioms a_n instead of a. Since the values of s_f and a_n are rational numbers, $Val(\pi, s_f, n)$ is a rational number. We will prove that

$$Val(\pi, s_f) = \sup\{Val(\pi, s_f, n) : n \in N\} \tag{22}$$

by induction on the length $l(\pi)$ of $\pi = \alpha_1, \alpha_2, ..., \alpha_m$. In fact, if $l(\pi) = 1$, and more generally if the last formula α in π is assumed either as a hypothesis or as a logical axiom, then (22) is immediate. Suppose that α is obtained by an inference rule r, namely that $\alpha = r'(\alpha_{s(1)}, ..., \alpha_{s(p)})$. Then

$$\begin{aligned}
&Val(\pi, s_f)\\
&\quad = r''(Val(\pi_{s(1)}, s_f), ..., Val(\pi_{s(p)}, s_f))\\
&\quad = r''(\sup\{Val(\pi_{s(1)}, s_f, n) : n \in N\}, ..., \sup\{Val(\pi_{s(p)}, s_f, n) : n \in N\})\\
&\quad = \sup\{r''(Val(\pi_{s(1)}, s_f, n_1), ..., Val(\pi_{s(p)}, s_f, n_p)) : n_1, ..., n_p \in N\}
\end{aligned}$$

$$= \sup\{r''(Val(\pi_{s(1)}, s_f, n), ..., Val(\pi_{s(p)}, s_f, n)) : n \in N\}$$
$$= \sup\{Val(\pi, s_f, n) : n \in N\},$$

where we used the inductive hypothesis, the fact that r'' preserves the joins and the fact that the quantities $Val(\pi_{s(j)}, s_f, n)$ are increasing with respect to n. From (22) it follows that

$$\mathcal{D}(s_f)(\alpha) = \sup\{Val(\pi, s_f) : \pi \text{ is a proof of } \alpha\}$$
$$= \sup\{Val(\pi, s_f, n) : \pi \text{ is a proof of } \alpha \text{ and } n \in N\}.$$

Let $\pi_1, \pi_2, ...$ be an effective enumeration of all the proofs and define the function $h : \mathbb{F} \times \mathcal{F}_f(\mathbb{F}) \times N \times N \to \ddot{U}$ by setting $h(\alpha, s_f, i, n) = Val(\pi_i, s_f, n)$ if π_i is a proof of α and $h(\alpha, s_f, i, n) = 0$ otherwise. Then h is a total recursive function whose values are rational numbers. In fact, since the valuation part of any inference rule is computable on the rational numbers, we can compute the rational number $Val(\pi, s_f, n)$ in an effective way. Thus,

$$\mathcal{D}(s_f)(\alpha) = \sup\{h(\alpha, s_f, i, n) : i \in N \text{ and } n \in N\},$$

and this proves that $w_{\mathcal{D}}$ is recursively enumerable and therefore that $\mathcal{D}$ is an enumeration fuzzy operator.

Conversely, let $(\mathcal{F}(\mathbb{F}), \mathcal{D})$ be an effective abstract deduction system. Then a recursive map $g : \mathcal{F}_f(\mathbb{F}) \times \mathbb{F} \times N \to \ddot{U}$ exists such that

$$\mathcal{D}(s_f)(x) = \sup\{g(s_f, x, n) : n \in N\}, \text{ for any } s_f \in \mathcal{F}_f(\mathbb{F}) \text{ and } x \in \mathbb{F}.$$

To define a suitable Hilbert system, we associate with any $\alpha \in \mathbb{F}$, $m \in N$ and $s_f \in \mathcal{F}_f(\mathbb{F})$, $s_f \neq \emptyset$, the fuzzy inference rule (r', r'') where r' is defined by setting

$$r'(x_1, ..., x_n) = \begin{cases} \alpha, & \text{if } x_1 = \alpha_1, ..., x_n = \alpha_n, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

where $\alpha_1, ..., \alpha_n$ are the formulas in $Supp(s_f)$. Also, r'' is the function associated with the map $d : \ddot{U}^n \to \ddot{U}$ defined by setting

$$d(\lambda_1, ..., \lambda_n) = \begin{cases} g(s_f, \alpha, m), & \text{if } \lambda_1 > s_f(\alpha_1), ..., \lambda_n > s_f(\alpha_n), \\ 0, & \text{otherwise.} \end{cases}$$

Both r' and d are partial recursive functions whose algorithms depend uniformly on s_f, α, m. Consequently, two recursive functions $h : N \to N$ and $k : N \to N$ exist such that $r'_{h(i)} = r'$ and $r''_{k(i)} = d$ where i is the code number of (α, s_f, m). Moreover d satisfies condition (5). We indicate by $\mathcal{S}$ the effective Hilbert fuzzy system whose fuzzy set of logical axioms is $\mathcal{D}(\emptyset)$ and such that $\mathbb{R} = \{(r'_{h(i)}, r''_{k(i)}) : i \in N\}$. To prove that $\mathcal{D}$ is the deduction operator of $\mathcal{S}$, we prove that a fuzzy set of formulas τ is a theory of $\mathcal{S}$ iff τ is a fixed point of $\mathcal{D}$, i.e., $\tau \supseteq \mathcal{D}(s_f)$ for any $s_f \in \mathcal{F}_f(\mathbb{F})$ such that $s_f \ll \tau$. Indeed, let τ be a theory. Then in the case $s_f = \emptyset$ we have that $\tau \supseteq \mathcal{D}(s_f)$ by hypothesis. If $s_f \neq \emptyset$, let $\alpha_1, ..., \alpha_n$ be the elements in

$Supp(s_f)$, $m \in N$, $\alpha \in \mathbb{F}$ and let (r', r'') be the inference rule associated with α, s_f and m. Then,

$$\begin{aligned}\tau(\alpha) &= \tau(r'(\alpha_1, ..., \alpha_n)) \geq r''(\tau(\alpha_1), ..., \tau(\alpha_n)) \\ &= \sup\{d(\lambda_1, ..., \lambda_n) : \lambda_1 < \tau(\alpha_1), ..., \lambda_n < \tau(\alpha_n)\}.\end{aligned}$$

Since $\lambda_1, ..., \lambda_n$ exist such that $\tau(\alpha_1) > \lambda_1 > s_f(\alpha_1), ..., \tau(\alpha_n) > \lambda_n > s_f(\alpha_n)$, we have $\tau(\alpha) \geq d(\lambda_1, ..., \lambda_n) = g(s_f, \alpha, m)$. Consequently,

$$\tau(\alpha) \geq \sup\{g(s_f, \alpha, n) : n \in N\} = \mathcal{D}(s_f)(\alpha)$$

and $\tau \supseteq \mathcal{D}(s_f)$. Let τ be a fixed point of $\mathcal{D}$, then $\tau = \mathcal{D}(\tau) \supseteq \mathcal{D}(\emptyset)$. Moreover, let (r', r'') be any rule and assume that (r', r'') is defined by $s_f \neq \emptyset$, $m \in N$ and $\alpha \in \mathbb{F}$. We claim that $\tau(r'(\alpha_1, ..., \alpha_n)) \geq d(\lambda_1, ..., \lambda_n)$ for any $\lambda_1 < \tau(\alpha_1), ..., \lambda_n < \tau(\alpha_n)$ and therefore that $\tau(r'(\alpha_1, ..., \alpha_n)) \geq r''(\tau(\alpha_1), ..., \tau(\alpha_n))$, where $\alpha_1, ..., \alpha_n$ are the elements in $Supp(s_f)$. Indeed, if $d(\lambda_1, ..., \lambda_n) \neq 0$, then $\lambda_1 > s_f(\alpha_1), ..., \lambda_n > s_f(\alpha_n)$. Consequently, since $s_f(\alpha_1) < \tau(\alpha_1), ..., s_f(\alpha_n) < \tau(\alpha_n)$ and therefore $s_f \subseteq \tau$, we have that

$$\begin{aligned}\tau(r'(\alpha_1, ..., \alpha_n)) = \tau(\alpha) = \mathcal{D}(\tau)(\alpha) &\geq \mathcal{D}(s_f)(\alpha) \\ &\geq g(s_f, \alpha, m) = d(\lambda_1, ..., \lambda_n).\end{aligned}$$

This means that τ is a theory of $\mathcal{S}$.

We call *axiomatizable* a fuzzy theory admitting a decidable fuzzy subset of axioms. Moreover, we call *negation* any computable map $\neg : \mathbb{F} \to \mathbb{F}$ and we say that a theory τ is *complete* if $\tau(\alpha) + \tau(\neg\alpha) = 1$, for any $\alpha \in \mathbb{F}$.

Theorem 12. *Let $\mathcal{S}$ be an effective fuzzy Hilbert system. Then any axiomatizable theory is recursively enumerable. Moreover, if $\mathcal{S}$ is with a negation, then any axiomatizable and complete theory is decidable.*

Proof. The first part of the theorem is an immediate consequence of the fact that $\mathcal{D}$ is an enumeration operator. Assume that τ is axiomatizable and complete and let $h : \mathbb{F} \times N \to \ddot{U}$ be a recursive map increasing with respect to the second variable and such that $\tau(x) = \lim_{n\to\infty} h(x, n)$ for any $x \in \mathbb{F}$. Then, given any formula α, $\tau(\alpha) = 1 - \tau(\neg\alpha) = 1 - \lim_{n\to\infty} h(\neg\alpha, n) = \lim_{n\to\infty} 1 - h(\neg\alpha, n) = \lim_{n\to\infty} k(\alpha, n)$ where we have set $k(\alpha, n) = 1 - h(\neg\alpha, n)$. Since k is recursive and decreasing with respect to the second variable, τ is recursively co-enumerable and therefore decidable.

Note that in [13] it was proved that, with reference to Pavelka's extension of Lukasiewicz's propositional logic, an axiomatizable theory τ exists such that the cut $C(\tau, 1)$ is Π_2-complete and therefore not recursively enumerable. This result is not in contradiction with the fact that τ is recursively enumerable. It means that, given any formula α, while we are able to produce an increasing sequence of rational numbers converging to $\tau(\alpha)$, we are not able to decide if $\tau(\alpha)$ is equal to 1 or not. These difficulties are not a characteristic of fuzzy logic but they arise everywhere we must give a constructive approach to a theory involving real numbers. As an example, recall that it is not decidable if two recursive real numbers are equal or not. Then it is not surprising that, at the same time, we know an algorithm to compute the real number $\tau(\alpha)$ and we are not able to decide if $\tau(\alpha)$ is equal to 1 or not.

Theorem 13. *There exists a continuous (and therefore axiomatizable by a fuzzy Hilbert system) truth-functional fuzzy semantics which is not axiomatizable by an effective fuzzy Hilbert system.*

Proof. Let W be any non recursively enumerable subset of N and let $\underline{\nu} : U \to U$ be the function which is linear in each interval $[1-1/2^{i-1}, 1-1/2^i]$, $i \in N$, such that $\underline{\nu}(0) = 1$, $\underline{\nu}(1) = 0$ and

$$\underline{\nu}(x) = \begin{cases} 1/2^i, & \text{if } x = 1 - 1/2^i \text{ and } i \in W, \\ 3/(4 \cdot 2^i), & \text{if } x = 1 - 1/2^i \text{ and } i \notin W. \end{cases}$$

Since $\underline{\nu}(1 - 1/2^{i-1}) \geq 3/(4 \cdot 2^i) \geq 1/2^i \geq \underline{\nu}(1 - 1/2^i)$, $\underline{\nu}$ is a (continuous) strictly decreasing function. Consider any truth-functional semantics in which $\neg$ is interpreted as usual by the function $\text{-}(x) = 1 - x$ and with a further "negation" ν interpreted by $\underline{\nu}$. Moreover, let $v : \mathbb{F} \to U$ be the initial valuation defined by setting

$$v(x) = \begin{cases} 1 - 1/2^i & \text{if } \; x = p_i, \\ 1/2^i & \text{if } \; x = \neg p_i, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 3, such a valuation has only a model m_v and therefore $Lc(v) = m_v$. Obviously, $m_v(\nu(p_i)) = \underline{\nu}(m_v(p_i)) = \underline{\nu}(1 - 1/2^i)$ and therefore

$$Lc(v)(\underline{\nu}(p_i)) = \begin{cases} 1/2^i & \text{if } i \in W, \\ 3/(4 \cdot 2^i) & \text{otherwise}. \end{cases}$$

Assume that $\mathcal{M}$ is axiomatizable by an effective fuzzy Hilbert system. Then the related deduction operator $\mathcal{D}$ coincides with Lc. Since v is decidable, we have that $Lc(v) = \mathcal{D}(v)$ is recursively enumerable. Then a recursive function $h : \mathbb{F} \times N \to \ddot{U}$ exists such that h is increasing with respect to the second variable and $Lc(v)(\alpha) = \lim_{n\to\infty} h(\alpha, n)$ for any formula α. In particular, $\lim_{n\to\infty} h(\underline{\nu}(p_i), n) = 1/2^i$ if $i \in W$ and $\lim_{n\to\infty} h(\underline{\nu}(p_i), n) = 3/(4 \cdot 2^i)$ if $i \notin W$. Consequently, given $i \in N$, we can compute the increasing sequence

$$h(\underline{\nu}(p_i), 1), \; h(\underline{\nu}(p_i), 2), \; h(\underline{\nu}(p_i), 3), \; ...$$

In the case we find an index n such that $h(\underline{\nu}(p_i), n) > 3/4(1/2^i)$ we can conclude that $i \in W$. So, W should be recursively enumerable, against the hypothesis.

It is an open question to find conditions for the effective axiomatizability of a truth-functional semantics. We conjecture that any truth-functional semantics whose logical connectives are interpreted by "computable" operations in U is effectively axiomatizable.

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