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# Natural deduction with general elimination rules

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**Abstract.** The structure of derivations in natural deduction is analyzed through isomorphism with a suitable sequent calculus, with twelve hidden convertibilities revealed in usual natural deduction. A general formulation of conjunction and implication elimination rules is given, analogous to disjunction elimination. Normalization through permutative conversions now applies in all cases. Derivations in normal form have all major premisses of elimination rules as assumptions. Conversion in any order terminates.

Through the condition that in a cut-free derivation of the sequent $\Gamma \Rightarrow C$, no inactive weakening or contraction formulas remain in $\Gamma$, a correspondence with the formal derivability relation of natural deduction is obtained: All formulas of $\Gamma$ become open assumptions in natural deduction, through an inductively defined translation. Weakenings are interpreted as vacuous discharges, and contractions as multiple discharges. In the other direction, non-normal derivations translate into derivations with cuts having the cut formula principal either in both premisses or in the right premiss only.

## 1. Introduction

We shall analyze the structure of derivations in natural deduction through isomorphic correspondence with derivations in a suitable sequent calculus. The key insight is to formulate all elimination rules of natural deduction in the manner of disjunction elimination. The standard conjunction and implication elimination rules come out as special cases: it is seen that these rules stand behind the failure of unique correspondence between natural deduction and sequent calculus derivations. In particular, twelve cases of failure of normalization in propositional logic are identified. When conjunction and implication elimination rules are formulated as *general* elimination rules, derivations permit conversion to *full normal form*. The characteristic of this form is that *all major premisses of elimination rules are assumptions*. Normalization holds for any order of conversions.

In full normal form for intuitionistic logic, also premisses of falsity elimination, or the rule 'ex falso quodlibet,' are assumptions. Thus, a normal intuitionistic derivation of a formula $C$ begins with assumptions and inferences of the form $\frac{\perp}{A}$, followed by subderivations in minimal logic. The usual conjunction and implication elimination rules do not permit this, which created a discrepancy between natural

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deduction and sequent calculus. In the former, falsity elimination can occur in the middle of a derivation, but in the latter, falsity elimination always is in the beginning of a derivation.

The concept of full normal form is extended to intuitionistic predicate logic by a general elimination rule for the universal quantifier, analogous to the elimination rule for the existential quantifier. This will bring forth twelve more cases of hidden convertibilities in natural deduction.

Our analysis is based on translations establishing isomorphism between natural deduction derivations and suitable sequent calculus derivations. The formal derivability relation of sequent calculus, written $\Gamma \Rightarrow C$, is usually related to a meta-level derivability relation for natural deduction, written $\Gamma \vdash C$. This latter is defined through the existence of a natural deduction derivation of $C$ from open assumptions *contained* in $\Gamma$. We give a correspondence with the formal derivability relation of natural deduction: If in the derivation of $\Gamma \Rightarrow C$ there remain no inactive weakening or contraction formulas in the context $\Gamma$, *all* formulas of $\Gamma$ become open assumptions in the translation to a natural deduction derivation. Equivalence between natural deduction and sequent calculus only obtains when inactive weakenings and contractions are absent in the latter.

In the sequent calculus we use, weakening is an explicit structural rule. Weakening by a formula that is active in a logical rule in a sequent calculus derivation corresponds to a vacuously discharged formula in natural deduction. To study contraction, we treat contexts as multisets. A sequent calculus derivation has contractions whenever more formula occurrences are discharged in a natural deduction rule than is indicated in the schematic rule, say, more than one in implication introduction. It was not possible to see fully what weakening and contraction amount to in terms of natural deduction before the general elimination rules were available.

The proof of cut elimination for the sequent calculus corresponding to natural deduction with general elimination rules is a straightforward induction on length of cut formula and height of derivation of the premisses of cut. When contexts are treated as multisets, a case of cut elimination is encountered in which the right premiss has been derived by contraction. To obtain cut elimination for this case, a multi-cut rule, as in Gentzen's original proof, can be used. But a direct proof is also available, through consideration of how the premiss of contraction was derived.

The translations we give also apply to non-normal derivations. Normalization can be achieved through translation to sequent calculus followed by cut elimination and translation back. The normal form thus obtained is not unique as cut elimination is not unique. Direct normalization through detour and permutation conversions, instead, will give strong normalization and uniqueness of normal form for natural deduction with general elimination rules.

## 2. Hidden convertibilities in natural deduction

Normal derivations with the usual natural deduction rules for conjunction and implication have a pleasant property: In each step of inference, the formula below is an immediate subformula of a formula above, or the other way around. With disjunction elimination, this simple subformula structure along all branches of a normal derivation tree is lost. But on the other hand, if the major premiss of an elimination step is concluded by disjunction elimination, the derivation converts into a more direct form. For example, if both steps are disjunction eliminations, we have

$$\dfrac{\dfrac{A \vee B \quad \begin{matrix}\overset{1.}{[A]}\\ \vdots \\ C\vee D\end{matrix} \quad \begin{matrix}\overset{2.}{[B]}\\ \vdots\\ C\vee D\end{matrix}}{C \vee D}{\scriptstyle \vee E,1.,2.} \quad \begin{matrix}\overset{3.}{[C]}\\ \vdots\\ E\end{matrix} \quad \begin{matrix}\overset{4.}{[D]}\\ \vdots\\ E\end{matrix}}{E}{\scriptstyle\vee E,3.4.}$$

This derivation converts into

$$\dfrac{A \vee B \quad \dfrac{\begin{matrix}\overset{1.}{[A]}\\ \vdots \\ C\vee D\end{matrix} \quad \begin{matrix}\overset{3.}{[C]}\\ \vdots\\ E\end{matrix} \quad \begin{matrix}\overset{4.}{[D]}\\ \vdots\\ E\end{matrix}}{E}{\scriptstyle\vee E,3.,4.} \quad \dfrac{\begin{matrix}\overset{2.}{[B]}\\ \vdots \\ C\vee D\end{matrix} \quad \begin{matrix}\overset{3.}{[C]}\\ \vdots\\ E\end{matrix} \quad \begin{matrix}\overset{4.}{[D]}\\ \vdots\\ E\end{matrix}}{E}{\scriptstyle\vee E,3.,4.}}{E}{\scriptstyle\vee E,1.,2.}$$

If disjunction elimination is used to conclude a major premiss of conjunction or implication elimination, translations similar to the above apply. These *permutation conversions* were found by Prawitz in 1965. It is possible that the last step in the derivation of $C \vee D$ from $A$ or $B$ is $\vee I$. Elimination with major premiss $A \vee B$ separates the introduction of $C \vee D$ from an elimination of $C \vee D$. A permutation conversion can reveal such a 'hidden' *detour convertibility*.

In terms of sequent calculus, where the rule corresponding to $\vee E$ is the left disjunction rule $L\vee$, the first derivation is

$$\dfrac{\dfrac{A \Rightarrow C \vee D \quad B \Rightarrow C \vee D}{A \vee B \Rightarrow C \vee D}{\scriptstyle L\vee} \quad \dfrac{C \Rightarrow E \quad D \Rightarrow E}{C \vee D \Rightarrow E}{\scriptstyle L\vee}}{A \vee B \Rightarrow E}{\scriptstyle Cut}$$

The second derivation corresponds to

$$\dfrac{\dfrac{A \Rightarrow C \vee D \quad \dfrac{C \Rightarrow E \quad D \Rightarrow E}{C \vee D \Rightarrow E}{\scriptstyle L\vee}}{A \Rightarrow E}{\scriptstyle Cut} \quad \dfrac{B \Rightarrow C \vee D \quad \dfrac{C \Rightarrow E \quad D \Rightarrow E}{C \vee D \Rightarrow E}{\scriptstyle L\vee}}{B \Rightarrow E}{\scriptstyle Cut}}{A \vee B \Rightarrow E}{\scriptstyle L\vee}$$

Thus, the conversion of the natural deduction derivation into a more direct form corresponds to a step of cut elimination, where the cut is permuted with $L\vee$, to move it upwards in the derivation.

In Schroeder-Heister (1984), the following *general* conjunction elimination rule is presented,

$$\dfrac{A\&B \quad \begin{matrix}[A, B]\\ \vdots\\ C\end{matrix}}{C}{\scriptstyle \&E}$$

The standard rules come out as special cases when $C = A$ and $C = B$, respectively:

$$\frac{A\&B \quad [A]}{A}\,\&E_1 \qquad \frac{A\&B \quad [B]}{B}\,\&E_2$$

In the other direction, leaving out the dummy discharged assumptions in these special cases, if $C$ is derivable from $A, B$, we have

$$\begin{matrix}\dfrac{A\&B}{A} \quad \dfrac{A\&B}{B} \\ \vdots \\ C\end{matrix}$$

But the structural properties of these two *special* elimination rules are quite different from those of the general elimination rule. To give an example, with the special rules we have the derivation

$$\cfrac{\cfrac{(A\&B)\&C}{A\&B}}{A}$$

With the general rule, this becomes the derivation

$$\cfrac{\cfrac{(A\&B)\&C \quad \overset{1.}{[A\&B]}}{A\&B}\,\&E,1. \qquad \overset{2.}{[A]}}{A}\,\&E,2. \tag{1}$$

Here the major premiss of the second elimination is itself a conclusion of general conjunction elimination and a permutation conversion can be made:

$$\cfrac{(A\&B)\&C \qquad \cfrac{\overset{1.}{[A\&B]} \quad \overset{2.}{[A]}}{A}\,\&E,2.}{A}\,\&E,1. \tag{2}$$

Now the major premisses of both instances of the elimination rule have become assumptions.

To write derivations in terms of sequent calculus, we make explicit the rules of *weakening* and *contraction*: $\Gamma, \Delta, \Theta, \ldots$ are finite multisets (lists without order) of formulas, with $\Gamma, \Delta$ indicating multiset union and $A, \Gamma$ addition of one copy of formula $A$ to multiset $\Gamma$.

$$\frac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\,W \qquad \frac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\,Ctr$$

Sequent calculus derivations start with sequents of the form $A \Rightarrow A$ or $\bot \Rightarrow C$.

Schroeder-Heister's general conjunction elimination rule corresponds to the left conjunction rule of sequent calculus, through the correspondence

$$\frac{A, B, \Gamma \Rightarrow C}{A\&B, \Gamma \Rightarrow C}\,L\& \quad \rightsquigarrow \quad \frac{A\&B \qquad \begin{matrix}[A, B], \Gamma \\ \vdots \\ C\end{matrix}}{C}\,\&E$$

Derivation (1) with the general elimination rule corresponds, in a way to be made exact below, to

$$\dfrac{\dfrac{\dfrac{A\&B \Rightarrow A\&B}{A\&B, C \Rightarrow A\&B}{\scriptstyle W}}{(A\&B)\&C \Rightarrow A\&B}{\scriptstyle L\&} \qquad \dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}{\scriptstyle W}}{A\&B \Rightarrow A}{\scriptstyle L\&}}{(A\&B)\&C \Rightarrow A}{\scriptstyle Cut}$$

Derivation (2) corresponds to

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}{\scriptstyle W}}{A\&B \Rightarrow A}{\scriptstyle L\&}}{A\&B, C \Rightarrow A}{\scriptstyle W}}{(A\&B)\&C \Rightarrow A}{\scriptstyle L\&}$$

It can be obtained from the first one by permuting the cut up twice, first with $L\&$ and then with weakening in the left premiss. We observe that the elimination of cut corresponds to the conversion of major premisses of $\&E$ rules into assumptions.

With the standard implication elimination rule, or modus ponens, we observe the same phenomenon: A derivation such as

$$\dfrac{\dfrac{A \supset (B \supset C) \quad A}{B \supset C} \qquad B}{C}$$

does not convert. But if in a sequent calculus derivation the last rule is $L\supset$ and it is translated analogously to rules $L\vee$ and $L\&$ a *general* implication elimination rule is found:

$$\dfrac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow C}{A \supset B, \Gamma, \Delta \Rightarrow C}{\scriptstyle L\supset} \quad \rightsquigarrow \quad \dfrac{A \supset B \quad \begin{matrix}\Gamma \\ \vdots \\ A\end{matrix} \quad \begin{matrix}[B], \Delta \\ \vdots \\ C\end{matrix}}{C}{\scriptstyle \supset E}$$

Again, we obtain the standard elimination rule as a special case, by setting $C = B$. In the other direction, if $C$ is derivable from $B$, we have

$$\begin{matrix}\dfrac{A \supset B \quad A}{B} \\ \vdots \\ C\end{matrix}$$

With the general rule, our example derivation is:

$$\dfrac{\dfrac{A \supset (B \supset C) \quad A \quad \overset{1.}{[B \supset C]}}{B \supset C}{\scriptstyle \supset E,1.} \qquad B \quad \overset{2.}{[C]}}{C}{\scriptstyle \supset E,2.}$$

It converts into the derivation

$$\cfrac{A \supset (B \supset C) \quad A \qquad \cfrac{\overset{1.}{[B \supset C]} \quad B \quad \overset{2.}{[C]}}{C}{\scriptstyle \supset E,2.}}{C}{\scriptstyle \supset E,1.}$$

Translations of these derivations into sequent calculus are: For the first, we have

$$\cfrac{\cfrac{A \Rightarrow A \quad B \supset C \Rightarrow B \supset C}{A \supset (B \supset C), A \Rightarrow B \supset C}{\scriptstyle L\supset} \qquad \cfrac{B \Rightarrow B \quad C \Rightarrow C}{B \supset C, B \Rightarrow C}{\scriptstyle L\supset}}{A \supset (B \supset C), A, B \Rightarrow C}{\scriptstyle Cut}$$

The second one gives instead the cut-free derivation

$$\cfrac{A \Rightarrow A \quad \cfrac{B \Rightarrow B \quad C \Rightarrow C}{B \supset C, B \Rightarrow C}{\scriptstyle L\supset}}{A \supset (B \supset C), A, B \Rightarrow C}{\scriptstyle L\supset}$$

There are altogether twelve cases of hidden convertibilities in natural deduction for propositional logic with special elimination rules.

For quantifiers, the standard elimination rules are

$$\cfrac{\forall x A}{A(t/x)}{\scriptstyle \forall E} \qquad \cfrac{\exists x A \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix}}{C}{\scriptstyle \exists E}$$

where $t$ is a term free for $x$ in $A$ and usual variable restrictions for $\exists E$ apply. Similarly to the case of propositional logic, if the major premiss of an elimination step is derived by $\forall E$, the derivation does not convert. This brings out twelve new cases of hidden convertibilities, all eliminable by the use of the *general* elimination rule for the universal quantifier,

$$\cfrac{\forall x A \quad \begin{matrix}[A(t/x)] \\ \vdots \\ C\end{matrix}}{C}{\scriptstyle \forall E}$$

This rule will permit a full normal form for derivations in intuitionistic first-order logic. The special elimination rule follows by setting $C = A(t/x)$. In the other direction, if $C$ is derivable from $A(t/x)$, we have the derivation

$$\begin{matrix}\cfrac{\forall x A}{A(t/x)} \\ \vdots \\ C\end{matrix}$$

The detailed treatment of quantifiers brings no essential new aspects and is left to another occasion.

What has been said of conjunction and implication elimination extends to falsity elimination $\frac{\perp}{C}$. In full normal form, its major premiss $\perp$ is an assumption. Thus, in intuitionistic derivations in full normal form instances of rule $\perp E$ are top inferences, followed by a derivation in minimal logic. A typical case of conversion is

$$\dfrac{\dfrac{A \supset \perp \quad A \quad \overset{1.}{[\perp]}}{\perp}{\scriptstyle \supset E,1.}}{C}{\scriptstyle \perp E} \qquad \rightsquigarrow \qquad \dfrac{A \supset \perp \quad A \quad \dfrac{\overset{1.}{[\perp]}}{C}{\scriptstyle \perp E}}{C}{\scriptstyle \supset E,1.}$$

that cannot be done with the modus ponens rule.

## 3. A sequent calculus isomorphic to natural deduction

We shall introduce a sequent calculus, to be called *G0i*, corresponding precisely to natural deduction with logical introduction and general elimination rules.

**G0i**

**Logical axiom:**

$A \Rightarrow A$

**Logical rules:**

$$\dfrac{A, B, \Gamma \Rightarrow C}{A\&B, \Gamma \Rightarrow C}{\scriptstyle L\&} \qquad \dfrac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A\&B}{\scriptstyle R\&}$$

$$\dfrac{A, \Gamma \Rightarrow C \quad B, \Delta \Rightarrow C}{A \vee B, \Gamma, \Delta \Rightarrow C}{\scriptstyle L\vee} \qquad \dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}{\scriptstyle R\vee_1} \qquad \dfrac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}{\scriptstyle R\vee_2}$$

$$\dfrac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow C}{A \supset B, \Gamma, \Delta \Rightarrow C}{\scriptstyle L\supset} \qquad \dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}{\scriptstyle R\supset}$$

$$\dfrac{}{\perp \Rightarrow C}{\scriptstyle L\perp}$$

$$\dfrac{A(t/x), \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}{\scriptstyle L\forall} \qquad \dfrac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \forall x A}{\scriptstyle R\forall}$$

$$\dfrac{A(y/x), \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}{\scriptstyle L\exists} \qquad \dfrac{\Gamma \Rightarrow A(t/x)}{\Gamma \Rightarrow \exists x A}{\scriptstyle R\exists}$$

**Rules of weakening and contraction:**

$$\dfrac{\Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}{\scriptstyle W} \qquad \dfrac{A, A, \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}{\scriptstyle Ctr}$$

The restriction in $R\forall$ and in $L\exists$ is that $y$ does not occur free in the conclusion. The first axiom applies to arbitrary formulas. Therefore, in particular, it gives $\perp \Rightarrow \perp$ as an instance, where falsity $\perp$ is not an atomic formula but a logical constant of length 0. To emphasize that $L\perp$ is a logical rule, we have written it as a zero-premiss left rule. If it is left out, a sequent calculus for *minimal* logic is obtained.

Each rule has a *context*, a finite multiset of formulas designated by $\Gamma$, $\Delta$ in the above rules, *active* formulas designated by $A$ and $B$, and a *principal* formula that is introduced by the rule in question. Corresponding to the treatment of assumptions in natural deduction, two-premiss rules have *independent* contexts, both collected in the antecedent of the conclusion.

The rule of cut,

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}{\scriptstyle Cut}$$

is proved admissible in *G0i* in von Plato (2001).

We shall give an inductive definition of a translation from cut-free derivations in *G0i* to natural deduction derivations with general elimination rules. It is sometimes thought that natural deduction is not able to express the rule of weakening. One defines instead derivability in natural deduction by: $C$ is derivable from $\Gamma$ if there is a derivation with open assumptions *contained* in $\Gamma$. Here we shall consider the more strict, formal derivability relation.

**Definition 3.1.** *A formula in a sequent calculus derivation is* used *if it is active in an antecedent in a rule.*

Rules that use a formula make it disappear from an antecedent, so these are the left rules and $R\supset$. In natural deduction, use of formulas corresponds to the discharge of assumptions. The little numbers written next to the mnemonic symbol for the rule applied and on top of formulas indicate what is discharged and where. We shall adjust the translation from sequent calculus to natural deduction accordingly, by adding *labels* to used formulas. Labels on top of discharged formulas are called *assumption labels* and those next to the rules *discharge labels*.

**Principle 3.2. Unique discharge.** *No two rules in a derivation must have the same discharge labels.*

The translation of a cut-free sequent calculus derivation of a sequent $\Gamma \Rightarrow C$ to natural deduction, where $\Gamma$ has no unused weakening or contraction formulas, starts with the last step and works root-first step by step until it reaches axioms and instances of $L\bot$. In this process, it is crucial to keep track of how formulas in the antecedents turn into assumptions. To satisfy Principle 3.2, each rule that discharges assumptions must have *fresh discharge labels*. Below, in each case of translation, we write the result of the first step of translation with a rule in natural deduction notation, and the premisses from which the translation continues in sequent calculus notation. We also add square brackets and treat labelled and bracketed formulas in the same way as other formulas when continuing the translation. The natural deduction derivation comes out from the translation all finished:

If the last rule to be translated is logical, we have

$$\frac{\begin{matrix}\vdots\\ A, B, \Gamma \Rightarrow C\end{matrix}}{A\&B, \Gamma \Rightarrow C}{\scriptstyle L\&} \quad\rightsquigarrow\quad \frac{A\&B \quad \begin{matrix}\vdots\\ \overset{1.}{[A]}, \overset{2.}{[B]}, \Gamma \Rightarrow C\end{matrix}}{C}{\scriptstyle \&E,1.,2.}$$

$$\frac{\begin{matrix}\vdots\\ \Gamma \Rightarrow A\end{matrix} \quad \begin{matrix}\vdots\\ \Delta \Rightarrow B\end{matrix}}{\Gamma, \Delta \Rightarrow A\&B}{\scriptstyle R\&} \quad\rightsquigarrow\quad \frac{\begin{matrix}\vdots\\ \Gamma \Rightarrow A\end{matrix} \quad \begin{matrix}\vdots\\ \Delta \Rightarrow B\end{matrix}}{A\&B}{\scriptstyle \&I}$$

$$\dfrac{\overset{\vdots}{A, \Gamma \Rightarrow C} \quad \overset{\vdots}{B, \Delta \Rightarrow C}}{A \lor B, \Gamma, \Delta \Rightarrow C}{\scriptstyle L\lor} \quad \rightsquigarrow \quad \dfrac{A \lor B \quad \overset{1.}{[A]}, \overset{\vdots}{\Gamma \Rightarrow C} \quad \overset{2.}{[B]}, \overset{\vdots}{\Delta \Rightarrow C}}{C}{\scriptstyle \lor E,1.,2.}$$

$$\dfrac{\overset{\vdots}{\Gamma \Rightarrow A}}{\Gamma \Rightarrow A \lor B}{\scriptstyle R\lor_1} \quad \rightsquigarrow \quad \dfrac{\overset{\vdots}{\Gamma \Rightarrow A}}{A \lor B}{\scriptstyle \lor I_1} \qquad \dfrac{\overset{\vdots}{\Gamma \Rightarrow B}}{\Gamma \Rightarrow A \lor B}{\scriptstyle R\lor_2} \quad \rightsquigarrow \quad \dfrac{\overset{\vdots}{\Gamma \Rightarrow B}}{A \lor B}{\scriptstyle \lor I_2}$$

$$\dfrac{\overset{\vdots}{\Gamma \Rightarrow A} \quad \overset{\vdots}{B, \Delta \Rightarrow C}}{A \supset B, \Gamma, \Delta \Rightarrow C}{\scriptstyle L\supset} \quad \rightsquigarrow \quad \dfrac{A \supset B \quad \overset{\vdots}{\Gamma \Rightarrow A} \quad \overset{1.}{[B]}, \overset{\vdots}{\Delta \Rightarrow C}}{C}{\scriptstyle \supset E,1.}$$

$$\dfrac{\overset{\vdots}{A, \Gamma \Rightarrow B}}{\Gamma \Rightarrow A \supset B}{\scriptstyle R\supset} \quad \rightsquigarrow \quad \dfrac{\overset{1.}{[A]}, \overset{\vdots}{\Gamma \Rightarrow B}}{A \supset B}{\scriptstyle \supset I,1.}$$

$$\dfrac{}{\bot \Rightarrow C}{\scriptstyle L\bot} \quad \rightsquigarrow \quad \dfrac{\bot}{C}{\scriptstyle \bot E}$$

If the last rule is weakening or contraction, we have

$$\dfrac{\overset{\vdots}{\Gamma \Rightarrow C}}{\overset{n.}{[A]}, \Gamma \Rightarrow C}{\scriptstyle W} \quad \rightsquigarrow \quad \overset{\vdots}{\Gamma \Rightarrow C} \qquad \dfrac{\overset{\vdots}{A, A, \Gamma \Rightarrow C}}{\overset{n.}{[A]}, \Gamma \Rightarrow C}{\scriptstyle Ctr} \quad \rightsquigarrow \quad \overset{n.}{[A]}, \overset{n.}{[A]}, \overset{\vdots}{\Gamma \Rightarrow C}$$

If the last rule is an axiom, we have

$$A \Rightarrow A \quad \rightsquigarrow \quad A$$

By the assumption of no unused weakening or contraction formulas, the translation can only reach weakening or contraction formulas indicated as discharged by square brackets. The topsequents of derivations are axioms or instances of $\bot \Rightarrow C$. If the translation arrives to these sequents and they do not have labels, their antecedents turn into *open* assumptions of the natural deduction derivation. When a formula is used, the translation produces formulas with labels and we can reach topsequents $\overset{n.}{[A]} \Rightarrow A$ and $\overset{n.}{[\bot]} \Rightarrow C$ with a label in the antecedent. These are translated into $\overset{n.}{[A]}$ and $\frac{\overset{n.}{[\bot]}}{C}\bot E$, with discharged assumptions. Note that if a labelled formula gets decomposed further up in the derivation, the labelled formula itself becomes a major premiss of an elimination rule that has been assumed. The components, instead, do not inherit that label but only those indicated in the above translations. Two different labels must be used for assumptions $A$ and $B$ in rules $\&E$ and $\lor E$. The translation produces derivations in which the major premisses of elimination rules always are (open or discharged) assumptions:

**Definition 3.3.** *A derivation in natural deduction is in* full normal form *if all major premisses of E-rules are assumptions.*

We shall refer to such derivations briefly as normal. Notice that $\bot$ in $\bot E$ is counted as a major premiss of an $E$-rule.

Translation of derivations with cuts will be discussed in Section 4.

The translation is an algorithm that works its way up from the endsequent in a local way, reflecting the local character of sequent calculus rules. It produces syntactically correct derivation trees with discharges fully formalized.

The translation of applications of the rule of weakening into natural deduction may seem somewhat surprising, but it will lead to a useful insight about the nature of this rule. Natural deduction rules permit to discharge formulas that have not occurred in a derivation. Similarly, natural deduction rules permit to discharge any number of occurrences of an assumption, not just the occurrence indicated in the schematic rule.

**Definition 3.4.** *Rule $\supset I$ and the elimination rules produce a* vacuous (multiple) *discharge whenever*

1. *In $\supset I$ concluding $A \supset B$ no occurrence (more than one occurrence) of assumption $A$ was discharged.*
2. *In $\&E$ and $\vee E$ with major premisses $A\&B$ and $A \vee B$, no occurrence of $A$ or $B$ (more than one occurrence of $A$ or $B$, or more than two if $A = B$), was discharged.*
3. *In $\supset E$ with major premiss $A \supset B$ no (more than one) occurrence of $B$ was discharged.*

A *weakening formula* (resp. *contraction formula*) is a formula $A$ introduced by weakening (contraction) in a derivation. There can be applications of weakening that have no correspondence in natural deduction: Whenever we have a derivation with weakening formulas that are not used, the endsequent is of the form $A, \Gamma \Rightarrow C$, with $A$ an *inactive* weakening formula throughout.

The condition of no inactive weakening or contraction formulas in a sequent calculus derivation permits a correspondence with the formal derivability relation of natural deduction:

**Theorem 3.5.** *Given a derivation of $\Gamma \Rightarrow C$ with no inactive weakening or contraction formulas, there is a natural deduction derivation of $C$ from $\Gamma$ with each formula of $\Gamma$ an open assumption.*

*Proof.* The proof is by induction on the height of derivation, using the translation from sequent calculus. If $\Gamma \Rightarrow C$ is an axiom or instance of $L\bot$, $\Gamma = C$ or $\Gamma = \bot$, and the translation gives the natural deduction derivations $C$ and $\frac{\bot}{C}\bot E$ with open assumptions $C$ and $\bot$, respectively. If the last rule is $L\&$, we have $\Gamma = A\&B, \Gamma'$ and the translation gives

$$\frac{A\&B \qquad \overset{1.}{[A]}, \overset{2.}{[B]}, \Gamma' \overset{\vdots}{\Rightarrow} C}{C}\,{\scriptstyle \&E,1.,2.}$$

If there are no inactive weakenings or contractions in the derivation of $A, B, \Gamma' \Rightarrow C$, there is by inductive hypothesis a natural deduction derivation of $C$ from open assumptions $A, B, \Gamma'$. Now assume $A\&B$ and apply $\&E$ to obtain a derivation of $C$ from $A\&B, \Gamma'$.

If there is an inactive weakening or contraction formula in the derivation of $A, B, \Gamma' \Rightarrow C$ it is by assumption not in $\Gamma'$ so it is $A$ or $B$ or both. Deleting the weakenings and contractions with unused formulas we obtain a derivation of $A^m, B^n, \Gamma' \Rightarrow C$, with $m, n \geqslant 0$ copies of $A$ and $B$, respectively. By the inductive hypothesis, there is a corresponding natural deduction derivation with open assumptions $A^m, B^n, \Gamma'$. Application of $\&E$ now gives a derivation of $C$ from $A\&B, \Gamma'$. All the other cases of logical rules are dealt with similarly.

The last step cannot be weakening or contraction by the assumption about no inactive weakening or contraction formulas. □

By the translation, the natural deduction derivation in Theorem 3.5 is normal. Later we show the converse result. Equivalence of derivability between sequent calculus and natural deduction only applies if unused weakenings and contractions are absent.

**Theorem 3.6.** *Given a derivation of* $\Gamma \Rightarrow C$ *with no inactive weakening or contraction formulas, if* $A$ *is a weakening (contraction) formula in the derivation, then* $A$ *is vacuously (multiply) discharged in the corresponding natural deduction derivation.*

*Proof.* Formula $A$ can be used in left rules and $R\supset$ only. Applying the translation to natural deduction, $A$ becomes a labelled formula in the antecedent, and translating further, it disappears when a weakening with $A$ is translated, and is multiplied when a contraction on $A$ is translated. □

Perhaps the simplest example is, with the corresponding natural deduction at right,

$$\cfrac{\cfrac{\cfrac{A \Rightarrow A}{A, B \Rightarrow A}\,W}{A\&B \Rightarrow A}\,L\&}{\Rightarrow A\&B \supset A}\,R\supset \qquad \cfrac{\cfrac{\overset{1.}{[A\&B]} \quad \overset{2.}{[A]}}{A}\,\&E,2.,3.}{A\&B \supset A}\,\supset I,1.$$

In the natural deduction derivation, $B$ is vacuously discharged. The translation produces, as a trace of the weakening, the discharge label 3 to which no assumption label corresponds. An intermediate stage of the translation just before the disappearance of the weakening formula is

$$\cfrac{\cfrac{\overset{1.}{[A\&B]} \quad \cfrac{A \Rightarrow A}{\overset{2.}{[A]}, \overset{3.}{[B]} \Rightarrow A}\,W}{A}\,\&E,2.,3.}{A\&B \supset A}\,\supset I,1.$$

In Gentzen's original sequent calculus there were two left rules for conjunction:

$$\cfrac{A, \Gamma \Rightarrow C}{A\&B, \Gamma \Rightarrow C}\,L\&_1 \qquad \cfrac{B, \Gamma \Rightarrow C}{A\&B, \Gamma \Rightarrow C}\,L\&_2$$

These left rules correspond to the standard elimination rules for conjunction, and the derivation of $A\&B \supset A$ and its translation become

$$\dfrac{\dfrac{A \Rightarrow A}{A\&B \Rightarrow A}{\scriptstyle L\&_1}}{\Rightarrow A\&B \supset A}{\scriptstyle R\supset} \qquad \dfrac{\dfrac{\overset{1.}{[A\&B]}}{A}{\scriptstyle \&E}}{A\&B \supset A}{\scriptstyle \supset I,1.}$$

Weakening is hidden in Gentzen's left conjunction rules and vacuous discharge in the special conjunction elimination rules. It is not possible to state fully the meaning of weakening in terms of natural deduction without using the general elimination rules.

The premiss of a contraction step can arise in essentially three ways: First, the duplication $A, A$ comes from a rule with two premisses each having one occurrence of $A$. Second, $A$ is the principal formula of a left rule and a premiss had $A$ already in the antecedent. Third, weakening is applied to a premiss having $A$ in the antecedent.

The simplest example of a multiple discharge should be the derivation of $A \supset A\&A$, given here both in *G0i* with a contraction and in translation to natural deduction with a double discharge:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A, A \Rightarrow A\&A}{\scriptstyle R\&}}{A \Rightarrow A\&A}{\scriptstyle Ctr}}{\Rightarrow A \supset A\&A}{\scriptstyle R\supset} \qquad \dfrac{\dfrac{\overset{1.}{[A]} \quad \overset{1.}{[A]}}{A\&A}{\scriptstyle \&I}}{A \supset A\&A}{\scriptstyle \supset I,1.}$$

In Definition 3.4, the clause about more than two occurrences in $\&E$ and $\vee E$ in case of $A = B$, is exemplified by the derivation of $A \vee A \supset A$:

$$\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A \vee A \Rightarrow A}{\scriptstyle L\vee}}{\Rightarrow A \vee A \supset A}{\scriptstyle R\supset} \qquad \dfrac{\dfrac{\overset{3.}{[A \vee A]} \quad \overset{1.}{[A]} \quad \overset{2.}{[A]}}{A}{\scriptstyle \vee E,1.2.}}{A \vee A \supset A}{\scriptstyle \supset I,3.}$$

Here there is no contraction even if two occurrences of $A$ are discharged at $\vee E$.

We now come to the translation from natural deduction to sequent calculus. It is essential to use multisets to see how natural deduction can keep track of contraction. This is no problem since it is well defined how many times open assumptions $A, B, C, \ldots$ appear above any given formula in a derivation.

Translation from fully normal natural deduction derivations with unique discharge to the calculus *G0i* is defined inductively according to the last rule used:

*1. The last rule is $\&I$:*

$$\dfrac{\begin{matrix}\Gamma & \Delta \\ \vdots & \vdots \\ A & B\end{matrix}}{A\&B}{\scriptstyle \&I} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Gamma & \Delta \\ \vdots & \vdots \\ A & B\end{matrix}}{\Gamma, \Delta \Rightarrow A\&B}{\scriptstyle R\&}$$

*2. The last rule is $\&E$:* The natural deduction derivation, with $m$-fold discharge on $A$ and $n$-fold on $B$, is

$$\dfrac{A\&B \qquad \begin{matrix} \overset{1.}{[A^m]}, \overset{2.}{[B^n]}, \Gamma \\ \vdots \\ C \end{matrix}}{C}\,{\scriptstyle \&E,1.,2.}$$

The translation is by cases according to values of $m$ and $n$:

$m = 0, n = 0$:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Gamma \\ \vdots \\ C\end{matrix}}{A, \Gamma \Rightarrow C}{\scriptstyle W}}{A, B, \Gamma \Rightarrow C}{\scriptstyle W}}{A\&B, \Gamma \Rightarrow C}{\scriptstyle L\&}$$

$m = 1, n = 1$:

$$\dfrac{\begin{matrix}A, B, \Gamma \\ \vdots \\ C\end{matrix}}{A\&B, \Gamma \Rightarrow C}{\scriptstyle L\&}$$

Note that the discharge labels and brackets have been removed. The cases of $m = 1, n = 0$ and $m = 0, n = 1$ have one weakening step before the $L\&$ inference.

$m > 1, n = 0$:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}A^m, \Gamma \\ \vdots \\ C\end{matrix}}{A, \Gamma \Rightarrow C}{\scriptstyle Ctr^m}}{A, B, \Gamma \Rightarrow C}{\scriptstyle W}}{A\&B, \Gamma \Rightarrow C}{\scriptstyle L\&}$$

Here $Ctr^m$ indicates an $m-1$ fold contraction, and discharges in $m$ occurrences of assumption $A$ have been removed. The rest of the cases for $\&E$ are similar.

*3. The last rule is* $\vee I$*:*

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ A\end{matrix}}{A \vee B}{\scriptstyle \vee I_1} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ A\end{matrix}}{\Gamma \Rightarrow A \vee B}{\scriptstyle R\vee_1} \qquad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ B\end{matrix}}{A \vee B}{\scriptstyle \vee I_2} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Gamma \\ \vdots \\ B\end{matrix}}{\Gamma \Rightarrow A \vee B}{\scriptstyle R\vee_2}$$

*4. The last rule is* $\vee E$*:* The natural deduction derivation is

$$\dfrac{A \vee B \qquad \begin{matrix}\overset{1.}{[A^m]}, \Gamma \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}\overset{2.}{[B^n]}, \Delta \\ \vdots \\ C\end{matrix}}{C}{\scriptstyle \vee E,1.,2.}$$

and the translation is again by cases according to the values of $m$ and $n$:

$m = 0, n = 0$:

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ C\end{matrix}}{A, \Gamma \Rightarrow C}\,W \qquad \dfrac{\begin{matrix}\Delta\\ \vdots\\ C\end{matrix}}{B, \Delta \Rightarrow C}\,W}{A \vee B, \Gamma, \Delta \Rightarrow C}\,L\vee$$

$m = 1, n = 1$:

$$\dfrac{\begin{matrix}A, \Gamma\\ \vdots\\ C\end{matrix} \quad \begin{matrix}B, \Delta\\ \vdots\\ C\end{matrix}}{A \vee B, \Gamma, \Delta \Rightarrow C}\,L\vee$$

Here again, the assumptions have been opened. The general case is $m > 1, n > 1$:

$$\dfrac{\dfrac{\begin{matrix}A^m, \Gamma\\ \vdots\\ C\end{matrix}}{A, \Gamma \Rightarrow C}\,Ctr^m \qquad \dfrac{\begin{matrix}B^n, \Delta\\ \vdots\\ C\end{matrix}}{B, \Delta \Rightarrow C}\,Ctr^n}{A \vee B, \Gamma, \Delta \Rightarrow C}\,L\vee$$

5. *The last rule is* $\supset I$*:* The general case with $m > 1$ is translated by

$$\dfrac{\begin{matrix}\overset{1.}{[A^m]}, \Gamma\\ \vdots\\ B\end{matrix}}{A \supset B}\,{\supset}I,1. \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}A^m, \Gamma\\ \vdots\\ B\end{matrix}}{A, \Gamma \Rightarrow B}\,Ctr^m}{\Gamma \Rightarrow A \supset B}\,R{\supset}$$

Again assumptions have been opened. If $m = 0$, there is a weakening instead of contraction, if $m = 1$, there is just rule $R\supset$.

6. *The last rule is* $\supset E$*:* The general case is translated as

$$\dfrac{A \supset B \qquad \begin{matrix}\Gamma\\ \vdots\\ A\end{matrix} \qquad \begin{matrix}\overset{1.}{[B^n]}, \Delta\\ \vdots\\ C\end{matrix}}{C}\,{\supset}E,1. \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\end{matrix} \qquad \dfrac{\begin{matrix}B^n, \Delta\\ \vdots\\ C\end{matrix}}{B, \Delta \Rightarrow C}\,Ctr^n}{A \supset B, \Gamma, \Delta \Rightarrow C}\,L{\supset}$$

The other cases are translated analogously to above.

7. *The last rule is* $\bot E$*:*

$$\dfrac{\bot}{C}\,\bot E \quad \rightsquigarrow \quad \dfrac{}{\bot \Rightarrow C}\,L\bot$$

8. *The last formula is an assumption:*

$$A \quad \rightsquigarrow \quad A \Rightarrow A$$

Notice that in a fully normal derivation, the premiss of rule $\perp E$ is an assumption and nothing remains to be translated in step *7*. If in *7* or *8* there are discharge labels and brackets they are removed.

**Theorem 3.7.** *Given a fully normal natural deduction derivation of $C$ from open assumptions $\Gamma$, there is a derivation of $\Gamma \Rightarrow C$ in G0i.*

*Proof.* By the translation defined. □

There are no unused weakenings or contractions in the derivation of $\Gamma \Rightarrow C$. By the translation, we obtain the converse to theorem 3.6:

**Theorem 3.8.** *If $A$ is vacuously (multiply) discharged in the derivation of $C$ from open assumptions $\Gamma$, then $A$ is a weakening (contraction) formula used in the derivation of $\Gamma \Rightarrow C$ in G0i.*

The usual explanation of contraction runs something like "if you can derive a formula using assumption $A$ twice you can also derive it using $A$ only once." But this is just a verbal statement of the rule of contraction. Logical rules of natural deduction that discharge assumptions vacuously or multiply are reproduced as weakenings or contractions plus a logical rule in sequent calculus, but the weakening and contraction rules in themselves have no proof-theoretical meaning, as was pointed out by Gentzen (1936, pp. 513–14) already.

By the translation of a derivation from natural deduction to sequent calculus, each formula in the former appears in the latter. We therefore have a somewhat surprising proof of the

**Corollary 3.9 (Subformula property).** *In a normal derivation of $C$ from open assumptions $\Gamma$, each formula in the derivation is a subformula of $\Gamma, C$.*

The translations we have defined from natural deduction to sequent calculus and the other way around do not quite establish an isomorphism between the two: the order of logical rules is preserved, but it is possible to permute weakenings and contractions on a formula $A$ as long as $A$ remains inactive so that isomorphism obtains modulo such permutations. This, however, is a minor point that can be handled by doing weakening and contraction on formula $A$ right before $A$ is used.

Translation of non-normal derivations will be discussed in Section 5.

## 4. Derivations with cuts

We show that derivations with cuts can be translated into natural deduction if the cuts are of a suitable kind: the *detour cuts* and *permutation cuts* corresponding to cuts with the cut formula principal in both premisses and right premiss only, respectively. These are the *principal* cuts, the rest are *nonprincipal* cuts. Principal cuts correspond, in terms of natural deduction, to instances of rules of elimination in which the major premisses are not assumptions.

A sequent calculus derivation has an equivalent in natural deduction only if it has no unused weakening or contraction formulas. By this criterion, there is no correspondence in natural deduction for many of the nonprincipal cuts of sequent calculus. In particular, if the right premiss of cut has been derived by contraction, the contraction formula is not used in the derivation and there is no corresponding natural deduction derivation. This is precisely the problematic case that led Gentzen to use the rule of multicut. If cut and contraction are permuted, the right premiss of a cut becomes derived by another cut and there is likewise no translation.

In translating derivations with cuts, if the left premiss is an axiom the cut is deleted. There are three detour cuts and another twelve permutation cuts with left premiss derived by a logical rule to be translated. We also translate principal cuts on $\perp$ as well as cases where the left premiss has been derived by a structural rule, but derivations with other cases of cuts will not be translated. Translation of rules other than cut have been given in Section 3.

*1.* Detour cut on $A\&B$, and we have the derivation

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma\Rightarrow A}\quad \overset{\vdots}{\Delta\Rightarrow B}}{\Gamma,\Delta\Rightarrow A\&B}{\scriptstyle R\&}\qquad \dfrac{\overset{\vdots}{A,B,\Theta\Rightarrow C}}{A\&B,\Theta\Rightarrow C}{\scriptstyle L\&}}{\Gamma,\Delta,\Theta\Rightarrow C}{\scriptstyle Cut}$$

The translation is:

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma\Rightarrow A}\quad \overset{\vdots}{\Delta\Rightarrow B}}{A\&B}{\scriptstyle \&I}\qquad \overset{1.}{[A]},\overset{2.}{[B]},\Theta\overset{\vdots}{\Rightarrow} C}{C}{\scriptstyle \&E,1.,2.}$$

Translation now continues from the premisses.

*2., 3.* Detour cuts on $A\vee B$ and $A\supset B$. The translations are analogous to *1*, with the left and right rules translated as in Section 3.

*4.* Permutation cut on $C\&D$ with left premiss derived by $L\&$:

$$\dfrac{\dfrac{\overset{\vdots}{A,B,\Gamma\Rightarrow C\&D}}{A\&B,\Gamma\Rightarrow C\&D}{\scriptstyle L\&}\qquad \dfrac{\overset{\vdots}{C,D,\Delta\Rightarrow E}}{C\&D,\Delta\Rightarrow E}{\scriptstyle L\&}}{A\&B,\Gamma,\Delta\Rightarrow E}{\scriptstyle Cut}$$

The translation is

$$\dfrac{\dfrac{A\&B\qquad \overset{1.}{[A]},\overset{2.}{[B]},\Gamma\overset{\vdots}{\Rightarrow} C\&D}{C\&D}{\scriptstyle \&E,1.,2.}\qquad \overset{3.}{[C]},\overset{4.}{[D]},\Delta\overset{\vdots}{\Rightarrow} E}{E}{\scriptstyle \&E,3.,4.}$$

The rest of the permutation cuts with $L\&$, $L\vee$ and $L\supset$ are translated analogously.

*5.* We also have permutation cuts on $\perp E$ but no detour cuts since $\perp$ can never be principal in the left premiss. The derivation and its translation are, where $L$ stands for a (one-premiss) left rule and $E$ for an elimination,

$$\dfrac{\dfrac{\overset{\vdots}{\Gamma' \Rightarrow \bot}}{\Gamma \Rightarrow \bot}{\scriptstyle L} \qquad \dfrac{}{\bot \Rightarrow C}{\scriptstyle L\bot}}{\Gamma \Rightarrow C}{\scriptstyle Cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\vdots}{\Gamma' \Rightarrow \bot}}{\bot}{\scriptstyle E}}{C}{\scriptstyle \bot E}$$

*6.* ‘Structural’ cuts with left premiss derived by weakening, contraction or cut. For weakening and contraction the translation reaches, by the condition of no unused weakening or contraction formulas, a conclusion of cut of the form $[\overset{n.}{A}], \Gamma, \Delta \Rightarrow C$. In the case of weakening, the left premiss of cut $A, \Gamma \Rightarrow B$ has been derived from $\Gamma \Rightarrow B$, in the case of contraction from $A, A, \Gamma \Rightarrow B$. The cuts are translated with left premiss replaced by $\Gamma \Rightarrow B$ and $[\overset{n.}{A}], [\overset{n.}{A}], \Gamma \Rightarrow B$, respectively.

For left premiss of cut derived by another cut the translation is modular and the upper cut is handled as before.

## 5. Non-normal derivations

We began in Section 2 with examples of non-normal natural deduction derivations corresponding to sequent calculus derivations with cuts. The latter are produced by translations defined inductively according to the last step. Derived major premisses are called *conversion formulas*. There are three cases of non-normality in which the major premiss of an elimination rule has been derived by the corresponding introduction rule:

*1.* The conversion formula has been derived by $\&I$ and the derivation is

$$\dfrac{\dfrac{\overset{\Gamma}{\overset{\vdots}{A}} \quad \overset{\Delta}{\overset{\vdots}{B}}}{A\&B}{\scriptstyle \&I} \qquad \overset{[\overset{1.}{A^m}], [\overset{2.}{B^n}], \Theta}{\overset{\vdots}{C}}}{C}{\scriptstyle \&E,1.,2.}$$

The translation is by cases according to values of $m$ and $n$. The general case is

$$\dfrac{\dfrac{\overset{\Gamma}{\overset{\vdots}{A}} \quad \overset{\Delta}{\overset{\vdots}{B}}}{\Gamma, \Delta \Rightarrow A\&B}{\scriptstyle R\&} \qquad \dfrac{\dfrac{\overset{A^m, B^m, \Theta}{\overset{\vdots}{C}}}{A, B, \Theta \Rightarrow C}{\scriptstyle Ctr^m, Ctr^n}}{A\&B, \Theta \Rightarrow C}{\scriptstyle L\&}}{\Gamma, \Delta, \Theta \Rightarrow C}{\scriptstyle Cut}$$

There is an $m + n - 2$ fold contraction in case $m, n > 1$.

*2., 3.* The conversion formula has been derived by $\vee I$ or $\supset I$ and the translation is analogous.

When detour conversions are applied, the open assumptions in a derivation can change. For example, the derivation

$$\dfrac{\dfrac{A \quad B}{A\&B}{\scriptstyle \&I} \qquad \overset{1.}{[A]}}{A}{\scriptstyle \&E,1.}$$

converts into the derivation $A$. Translation gives

$$\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow A\&B}{\scriptstyle R\&} \qquad \dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}{\scriptstyle W}}{A\&B \Rightarrow A}{\scriptstyle L\&}}{A, B \Rightarrow A}{\scriptstyle Cut}$$

Cut elimination produces the derivation

$$\dfrac{A \Rightarrow A}{A, B \Rightarrow A}{\scriptstyle W}$$

Deletion of the unused weakening gives the derivation $A \Rightarrow A$, corresponding to the result of the detour conversion.

Given a (cut-free) derivation of $\Gamma \Rightarrow C$, we can first delete the unused weakenings, then translate to natural deduction, and last add the unused weakening formulas of $\Gamma$ to the natural deduction derivation by the above trick on formula $B$, to obtain a non-normal derivation of $C$ from open assumptions $\Gamma$.

There is a good number of non-normalities with a permutation convertibility but we only show one typical case:

4. The conversion formula $C\&D$ has been derived by $\&E$ from $A\&B$:

$$\dfrac{\dfrac{A\&B \qquad \begin{matrix} \overset{1.}{[A^m]}, \overset{2.}{[B^m]}, \Gamma \\ \vdots \\ C\&D \end{matrix}}{C\&D}{\scriptstyle \&E,1.,2.} \qquad \begin{matrix} \overset{3.}{[C^k]}, \overset{4.}{[D^l]}, \Delta \\ \vdots \\ E \end{matrix}}{E}{\scriptstyle \&E,3.,4.}$$

The translation is by cases according to values of $m, n, k, l$, with the general case

$$\dfrac{\dfrac{\dfrac{\begin{matrix} A^m, B^n, \Gamma \\ \vdots \\ C\&D \end{matrix}}{A, B, \Gamma \Rightarrow C\&D}{\scriptstyle Ctr^m, Ctr^n}}{A\&B, \Gamma \Rightarrow C\&D}{\scriptstyle L\&} \qquad \dfrac{\dfrac{\begin{matrix} C^k, D^l, \Delta \\ \vdots \\ C \end{matrix}}{C, D, \Delta \Rightarrow E}{\scriptstyle Ctr^k, Ctr^l}}{C\&D, \Delta \Rightarrow E}{\scriptstyle L\&}}{A\&B, \Gamma, \Delta \Rightarrow E}{\scriptstyle Cut}$$

If $A\&B$ in turn is a conversion formula, a cut on $A\&B$ is inserted after the rule $L\&$ that concludes the left premiss of the cut on $C\&D$.

Translations when $\vee E$ and $\supset E$ have been used are analogous to the one for $\&E$. Translation when $\bot E$ has been used is the converse to translation 5 in Section 4. If the major premiss in the derivation of a conversion formula is again a conversion formula, another cut is inserted.

Consider a typical principal cut, say, on $A\&B$:

$$\dfrac{\Gamma \Rightarrow A\&B \qquad \dfrac{\begin{matrix}\vdots \\ A, B, \Delta \Rightarrow C\end{matrix}}{A\&B, \Delta \Rightarrow C}{\scriptstyle L\&}}{\Gamma, \Delta \Rightarrow C}{\scriptstyle Cut}$$

We see that the cut is redundant, in the sense that its left premiss is an axiom, precisely when $A\&B$ is an assumption in the corresponding natural deduction derivation. In this case, the cut is not translated but deleted. We have, in general:

*A non-normal instance of a logical rule in natural deduction is represented in sequent calculus by the corresponding left rule and a cut.*

Let us compare this explanation of cut to the presentation of cut as a combination of two lemmas $\Gamma \Rightarrow A$ and $A, \Delta \Rightarrow C$ into a theorem $\Gamma, \Delta \Rightarrow C$. Consider the derivation of $C$ from assumptions $A, \Delta$ in natural deduction. Obviously $A$ plays an essential role only if it is analyzed into components by an elimination rule, thus, $A$ is a major premiss of that elimination rule. If not, it acts just as a parameter in the derivation. Our explanation of cut makes more precise the idea of cut as a combination of lemmas: In terms of sequent calculus, the cut formula has to be principal in a left rule in the derivation of $A, \Delta \Rightarrow C$.

Given a non-normal derivation, translation to sequent calculus, followed by cut elimination and translation back to natural deduction, will produce a normal derivation:

**Theorem 5.1. Normalization.** *Given a natural deduction derivation of $C$ from $\Gamma$, the derivation converts to a normal derivation of $C$ from $\Gamma^*$ where each formula in $\Gamma^*$ is a formula in $\Gamma$.*

This process of normalization will not produce a unique result since cut elimination will not.

## 6. The structure of normal derivations

Theorem 5.1 gave a proof of normalization for intuitionistic natural deduction with general elimination rules, through a translation to sequent calculus, cut elimination and translation back to natural deduction. Strong normalization and uniqueness of normal form (modulo the choices in simplification convertibilities on disjunction, see below) for our system of natural deduction is given by Joachimski and Matthes (2001). Their proof uses a system of term assignment.

We consider three different types of non-normalities of a natural deduction derivation with general elimination rules that depend on how a major premiss of an elimination rule was derived. Then the subformula structure of normal derivations is detailed, with a direct proof of the subformula property.

**(a) Detour conversions:** Gentzen's original notion of a normal derivation in natural deduction was that no conclusion of an introduction rule must be the major premiss of an elimination rule. (This is seen from the example on implication in his 1934–35, Sec. II. §5.12.) Non-normal derivations are transformed into normal ones by detour conversions that delete each such pair of introduction and elimination rule instances. In a fully general form, a detour convertibility on the formula $A\&B$ obtains in a derivation whenever it has a part of the form

$$\dfrac{\dfrac{\begin{matrix}\vdots & & \vdots\\ [A] & & [B]\end{matrix}}{A\&B}\,{\&I} \qquad \begin{matrix}\overset{1.}{[A^m]},\overset{2.}{[B^n]}\\ \vdots\\ C\end{matrix}}{\begin{matrix}C\\ \vdots\end{matrix}}\,{\&E,1.,2.}$$

Detour conversion on $A\&B$ gives, through simultaneous substitution, the modified derivation

$$\begin{matrix}\begin{matrix}\vdots & & \vdots\\ A, & \overset{m\times}{\ldots}, & A\end{matrix} \quad \begin{matrix}\vdots & & \vdots\\ B, & \overset{n\times}{\ldots}, & B\end{matrix}\\ \vdots\\ C\\ \vdots\end{matrix}$$

A detour convertibility on $A \vee B$ is quite analogous. For implication, the situation is more complicated since a vacuous or multiple discharge is possible also in the introduction of the conversion formula:

$$\dfrac{\dfrac{\begin{matrix}\overset{1.}{[A^m]}\\ \vdots\\ B\end{matrix}}{A \supset B}\,{\supset I,1.} \qquad \begin{matrix}\vdots\\ A\end{matrix} \qquad \begin{matrix}\overset{2.}{[B^n]}\\ \vdots\\ C\end{matrix}}{\begin{matrix}C\\ \vdots\end{matrix}}\,{\supset E,2.}$$

Detour conversion on $A \supset B$ gives the modified derivation

$$\begin{matrix}\begin{matrix}\vdots & & \vdots\\ A, & \overset{m\times}{\ldots}, & A\end{matrix} \quad \begin{matrix}\vdots & & \vdots\\ A, & \overset{m\times}{\ldots}, & A\end{matrix}\\ \begin{matrix}\vdots & & \vdots\\ B, & \overset{n\times}{\ldots} & ,B\end{matrix}\\ \vdots\\ C\\ \vdots\end{matrix}$$

In detour conversions, open assumptions typically get multiplied or deleted.

**(b) Permutation conversions for general elimination rules:** There are four elimination rules which gives sixteen cases of permutation convertibilities, major premisses of elimination rules that are derived by another elimination rule. All of these act in a similar way on derivations and we only show one:

A permutation convertibility on major premiss $C\&D$ derived by $\&E$ on $A\&B$ obtains whenever a derivation has the part

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\vdots\\ A\&B\end{matrix}\qquad \begin{matrix}[A^m, B^n]\\ \vdots\\ C\&D\end{matrix}}{C\&D}{\scriptstyle \&E} \qquad \begin{matrix}[C^k, D^l]\\ \vdots\\ E\end{matrix}}{E}{\scriptstyle \&E}\\ \vdots\end{matrix}$$

After the permutation conversion the part is

$$\begin{matrix}\dfrac{\begin{matrix}\vdots\\ A\&B\end{matrix}\qquad \dfrac{\begin{matrix}[A^m, B^n]\\ \vdots\\ C\&D\end{matrix}\qquad \begin{matrix}[C^k, D^l]\\ \vdots\\ E\end{matrix}}{E}{\scriptstyle \&E}}{E}{\scriptstyle \&E}\\ \vdots\end{matrix}$$

The effect of the conversion is that the height of derivation of major premiss $C\&D$, as measured from the discharged assumptions $A$, $B$, is diminished by one.

**(c) Simplification conversions:** Other reductions of natural deduction derivations exist besides detour and permutation conversions. In Prawitz (1971), a simplification of derivations in natural deduction is suggested, called properly *simplification conversion*. The convertibility arises from disjunction elimination when in at least one of the auxiliary derivations, say the first one, a disjunct was not assumed:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A \vee B\end{matrix}\qquad \begin{matrix}\Delta\\ \vdots\\ C\end{matrix}\qquad \begin{matrix}\overset{1.}{[B]}, \Theta\\ \vdots\\ C\end{matrix}}{C}{\scriptstyle \vee E, 1.}$$

The elimination step is not needed, for $C$ is already concluded in the first auxiliary derivation. With general elimination rules for conjunction and implication, we analogously have:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\&B\end{matrix}\qquad \begin{matrix}\Delta\\ \vdots\\ C\end{matrix}}{C}{\scriptstyle \&E}\qquad\qquad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ A \supset B\end{matrix}\qquad \begin{matrix}\Delta\\ \vdots\\ A\end{matrix}\qquad \begin{matrix}\Theta\\ \vdots\\ C\end{matrix}}{C}{\scriptstyle \supset E}$$

In both inferences, $C$ is already concluded without the elimination rule, and simplification conversion extends to all elimination rules, quantifier rules included. The notion is captured by the

**Definition 6.1.** *A* simplification convertibility *in a derivation is an instance of an E-rule with no discharged assumptions, or an instance of* $\vee E$ *with no discharges of at least one disjunct.*

A simplification convertibility can prevent the normalization of a derivation, as is shown by the following:

$$\dfrac{\dfrac{\dfrac{\overset{1.}{[A]}}{A \supset A}{\scriptstyle \supset I,1.} \qquad \dfrac{\overset{2.}{[B]}}{B \supset B}{\scriptstyle \supset I,2.}}{(A \supset A)\&(B \supset B)}{\scriptstyle \&I} \qquad \dfrac{\overset{3.}{[C]}}{C \supset C}{\scriptstyle \supset I,3.}}{C \supset C}{\scriptstyle \&E}$$

There is a detour convertibility but due to the vacuous discharge in $\&E$, the pieces of derivation do not fit together in the right way to remove it. Instead a simplification conversion into the derivation

$$\dfrac{\overset{3.}{[C]}}{C \supset C}{\scriptstyle \supset I,3.}$$

will remove the detour convertibility.

It is possible that in a simplification convertibility with $\vee E$, both auxiliary assumptions are vacuously discharged. In this case, there are two converted derivations of the conclusion.

**(d) The subformula structure of general elimination rules:** With special elimination rules in the $\vee$- and $\exists$-free fragment, there is a simple subformula structure along all *branches* of a normal derivation, from assumptions to a minor premiss of rule $\supset E$ or to the conclusion. In fully normal derivations with general elimination rules, branches are replaced by sequences of formulas that jump from major premisses to their auxiliary assumptions. Contrary to first appearance, a greater uniformity in the structure of derivations, for the full language of predicate logic, is achieved.

The subformula property in natural deduction is more complicated than in sequent calculus, due to the nonlocal character of the rules of inference. It is obtained through the notion of *thread* (a term suggested to us by Dag Prawitz) where for simplicity we assume that no simplification convertibilities obtain:

**Definition 6.2.** *A* thread *in a natural deduction derivation of $C$ from open assumptions $\Gamma$ without simplification convertibilities is a sequence of formulas $A_1, \dots, A_n$ such that*

1. *$A_n$ is either $C$ or a minor premiss of $\supset E$.*
2. *$A_{i-1}$ is either a major premiss with auxiliary assumption $A_i$ in an $E$-rule, or a minor premiss in an $E$-rule with $A_{i-1} = A_{ig}$, or a premiss with conclusion $A_i$ in an $I$-rule.*
3. *$A_1$ is a top formula not discharged by an $E$-rule.*

Threads typically run through a sequence of major premisses of $E$-rules, until the conclusion of the innermost major premiss is built up by $I$-rules, and so on. If vacuous instances of elimination rules are admitted, there can be threads that stop at the major premiss.

Threads in a normal derivation, briefly, *normal threads*, have the following structure:

$$(\overbrace{A_1, \dots, A}^{E\text{-}part}\underbrace{{}_m, \dots, A_n}_{I\text{-}part})$$

In the $E$-part, the major premisses follow in succession and $A_{i+1}$ is an immediate subformula of $A_i$. In the $I$-part, either $A_{i+1}$ is equal to $A_i$ or $A_i$ is an immediate subformula of $A_{i+1}$.

We concluded in corollary 3.9 the subformula property of normal derivations with general elimination rules by a corresponding result that is immediate for the sequent calculus *G0i*. A more direct proof in terms of natural deduction sheds some light on the structure of threads:

**Direct proof of the subformula property:** Each formula $A$ is in at least one normal thread, and it is a subformula of the topformula or of the endformula of the thread. In the former case, the topformula is either an open assumption and the subformula property follows, else it is discharged by $\supset I$ and $A$ is a subformula of the endformula of the thread. If the endformula is the conclusion of the whole derivation the subformula property follows. If it is the endformula of a minor thread, it is also a subformula of the corresponding major premiss. The major premiss is either an open assumption and the subformula property follows. Else the major premiss is discharged by $\supset I$ and belongs to some normal thread with endformula further down in the derivation. If this endformula is the conlusion of the derivation the subformula property follows, if not, by repeating the argument the conclusion is reached. □

In sequent calculus, the rule of falsity elimination is represented by a sequent $\perp \Rightarrow C$ by which derivations can start. In usual natural deduction, instead, falsity elimination can apply at any stage of a derivation. This discrepancy is now explained as a hidden convertibility. In particular, if the conversion formula is $\perp$ derived by $\perp E$ we have a derivation with two non-normal instances of $\perp E$. Since $\perp E$ has only a major premiss, a permutation conversion just removes one of these instances:

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \perp\end{matrix}}{\perp}\,{\scriptstyle \perp E}}{C}\,{\scriptstyle \perp E} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ \perp\end{matrix}}{C}\,{\scriptstyle \perp E}$$

The first derivation has the translation to sequent calculus

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \perp\end{matrix} \quad \perp \Rightarrow \perp}{\Gamma \Rightarrow \perp}\,{\scriptstyle Cut} \qquad \dfrac{}{\perp \Rightarrow C}\,{\scriptstyle L\perp}}{\Gamma \Rightarrow C}\,{\scriptstyle Cut}$$

and the converted one

$$\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \perp\end{matrix} \quad \dfrac{}{\perp \Rightarrow C}\,{\scriptstyle L\perp}}{\Gamma \Rightarrow C}\,{\scriptstyle Cut}$$

Fully normal derivations do not have redundant iterations of $\bot E$. In Prawitz (1965, p. 20), the effect of the above permutation conversion is achieved by the ad hoc restriction that in $\bot E$ the conclusion be different from $\bot$.

In a typical application of $\bot E$ in natural deduction with the special elimination rules we have, using the modus ponens rule,

$$\dfrac{\dfrac{A \supset \bot \quad A}{\bot}\,Mp}{C}\,\bot E$$

With the more general $\supset E$ rule, the derivation and its permutation conversion are

$$\dfrac{\dfrac{A \supset \bot \quad A \quad \overset{1.}{[\bot]}}{\bot}\,\supset\! E,1.}{C}\,\bot E \qquad \rightsquigarrow \qquad \dfrac{A \supset \bot \quad A \quad \dfrac{\overset{1.}{[\bot]}}{C}\,\bot E}{C}\,\supset\! E,1.$$

Here the premiss $\bot$ is converted into a topformula of the derivation. The same applies in general and we thus obtain the following

**Proposition 6.3.** *A fully normal intuitionistic derivation begins with assumptions and instances of the intuitionistic $\bot E$ rule, followed by a subderivation in minimal logic.*

This fact will give a natural translation of intuitionistic into minimal logic: Consider an intuitionistic derivation of $C$ in full normal form. The conclusions of falsity elimination are derivable from falsity eliminations concluding atoms. By the subformula property, these are atoms of $C$, and let them be $P_1, \dots, P_n$. Each step $\frac{\bot}{P_i}$ is replaced by an assumption $\bot \supset P_i$, and $P_i$ is concluded from $\bot$ by $\supset E$ instead of $\bot E$. Collecting all the new assumptions, we obtain the

**Theorem 6.4.** *Formula $C$ is intuitionistically derivable if and only if*

$$(\bot \supset P_1)\& \cdots \&(\bot \supset P_n) \supset C$$

*is derivable in minimal logic.*

It would be a redundancy in a normal derivation if it had major premisses of elimination rules that are derivable formulas:

**Definition 6.5.** *A major premiss of an elimination rule is a* proper assumption *if it is underivable.*

**Theorem 6.6.** *Given a derivation, there is a derivation in which all major premisses of elimination rules are proper assumptions.*

*Proof.* Consider a derivable major premiss $A$. In a normal derivation of $A$, the last rule must be an $I$-rule since an $E$-rule would leave an open assumption. A substitution of assumption $A$ with a normal derivation creates a detour convertibility. From the conversion schemes, we observe that no conversion ever produces new major premisses of $E$-rules and that detour conversions produce shorter conversion formulas. Therefore the process of substituting derivable major premisses of $E$-rules with their derivations and subsequent normalization terminates in a derivation with proper assumptions. □

By the undecidability of predicate logic, the theorem does not give an effective proof transformation. A translation to sequent calculus gives the

**Corollary 6.7.** *If the sequent $\Gamma \Rightarrow C$ is derivable in G0i, it has a derivation in which all formulas principal in left rules are underivable.*

The eliminability of derivable principal formulas in left rules in sequent calculus derivations was discovered by Mints (1993). The formulation in terms of natural deduction with general elimination rules makes this result intuitive.

## 7. Concluding remarks

Gentzen found the rules of natural deduction through an analysis of actual mathematical proofs. How natural are the general elimination rules, in comparison? In an informal proof, a proposition of the form $A\&B$ is used by deriving consequences of the assumptions $A$ and $B$, without the intermediate logical steps of the usual conjunction elimination rules. Similarly, $A \supset B$ is used by deriving consequences from $B$, and if at some stage $A$ obtains, those consequences obtain. The same natural use of logic is found with $A \vee B$ in proofs by cases.

In Gentzen's original work, a translation of natural deduction derivations into sequent calculus is described (1934–35, sec. V. §4). Each formula $C$ is first replaced by a sequent $\Gamma \Rightarrow C$ in which $\Gamma$ is a list of open assumptions $C$ depends on, and then the rules are translated. Rules $\&I$ and $\vee I$ are translated in the obvious way. Translations of $\supset I$ and $\vee E$ involve possible weakenings and contractions, corresponding to vacuous and multiple discharges. Whenever in the natural deduction there are instances of $\&E$ and $\supset E$, the first phase of the translation gives steps such as

$$\frac{\Gamma \Rightarrow A\&B}{\Gamma \Rightarrow A} \qquad \frac{\Gamma \Rightarrow A \supset B \quad \Delta \Rightarrow A}{\Gamma, \Delta \Rightarrow B}$$

These are turned into sequent calculus inferences by the replacements

$$\frac{\Gamma \Rightarrow A\&B \quad \dfrac{A \Rightarrow A}{A\&B \Rightarrow A}{\scriptstyle L\&_1}}{\Gamma \Rightarrow A}{\scriptstyle Cut} \qquad \frac{\Gamma \Rightarrow A \supset B \quad \dfrac{\Delta \Rightarrow A \quad B \Rightarrow B}{A \supset B, \Delta \Rightarrow B}{\scriptstyle L\supset}}{\Gamma, \Delta \Rightarrow B}{\scriptstyle Cut}$$

Each instance of these $E$-rules leads to a cut. The cut is redundant if in natural deduction the major premiss is an assumption, but this need not be the case with $\&E$ or $\supset E$ even if the derivation is normal in Gentzen's sense of not containing detour convertibilities.

With the knowledge that the special elimination rules of natural deduction correspond to 'hidden cuts,' it is to be expected that a normal natural deduction in the old sense translates into a sequent calculus derivation with cuts. In Gentzen's work, the 'Hauptsatz' is proved in terms of sequent calculus, and the possibility of a formulation in terms of a normal form in intuitionistic natural deduction is only mentioned. No comment is made about the cuts that the translation to sequent calculus produces.

There is another way of arriving at the general elimination rule for conjunction than the translation from sequent calculus we have used, namely, constructive type theory. The general rule comes straight out by suppressing the proof objects in the typed rule, as in Martin-Löf (1984, p. 44). In the other direction, typing our general implication elimination rule will result in a new selector, *generalized application*:

$$\frac{c : A \supset B \quad a : A \quad \begin{matrix}[x : B]\\ \vdots \\ d : C\end{matrix}}{gap(c, a, (x)d) : C}$$

A full type-theoretical rule uses the function type $(A)B$ that has no correspondence in first-order logic. The usual first-order selector *ap* corresponding to *modus ponens* is defined, for $B = C$, by $ap(c, a) = gap(c, a, (x)x) : B$. Normality means that each selector term has a variable as first argument. A direct proof of strong normalization for natural deduction with general elimination rules was found by Joachimski and Matthes (2001), through a term assignment system. (They also suggested the term 'generalized application' for general implication elimination typed.)

In Ekman (1998), it is noticed that a derivation of the formula $\sim(P \supset\subset \sim P)$, where equivalence is implication in both directions, either is not normal or else has a subderivation of the form

$$\dfrac{P \supset \sim P \qquad \dfrac{\sim P \supset P \qquad \sim P}{P}{\scriptstyle \supset E}}{\sim P}{\scriptstyle \supset E}$$

The derivation has the reduncancy, or is 'indirect' in Ekman's terminology, that the derivation of the conclusion could be replaced by the derivation of the first occurrence of $\sim P$. But this will produce a non-normal derivation, for the top occurrence of $\sim P$ is the conclusion of $\supset I$ and the bottom occurrence a major premiss of $\supset E$. This problem is solved by the use of the general implication elimination rule, as shown in our (2000).

In our view, the essential difference between sequent calculus and natural deduction is not in this or that set of logical rules. The most important difference is that the logical rules of natural deduction permit non-normal instances, where sequent calculus has to use a left logical rule and a cut. We take this to be the characteristic feature of natural deduction. Secondly, there are no explicit structural rules of weakening or contraction, but vacuous and multiple discharges of assumptions instead. It is possible to devise a 'sequent calculus in natural deduction style,' exemplified by the rules

$$\frac{A^m, B^n, \Gamma \Rightarrow C}{A\&B, \Gamma \Rightarrow C}{\scriptstyle L\&} \qquad \frac{A^m, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}{\scriptstyle R\supset}$$

having instances for any values $m, n \geq 0$. There are no explicit rules of weakening or contraction but these are built into the logical rules that use assumptions. (See Negri and von Plato 2001 for the calculus and a proof of cut elimination.) With the calculus *G0i*, isomorphism with derivations in natural deduction is obtained modulo some permutations of weakenings and contractions. In sequent calculus in natural deduction style, the correspondence between the two main forms of logical calculi is perfect.

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