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Interpolation in fuzzy logic

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Abstract. We investigate interpolation properties of many-valued propositional logics related to continuous t-norms. In case of failure of interpolation, we characterize the minimal interpolating extensions of the languages. For finite-valued logics, we count the number of interpolating extensions by Fibonacci sequences.

1. Introduction

1.1. Outline

In this paper, we investigate interpolation theorems for fuzzy and many-valued logics whose truth functions are defined by continuous triangular norms [18, 19]. Those logics (which we shall call *triangular logics* for short) have attracted a lot of research in recent years since on the one hand they retain a mathematically appealing theory akin to the theory of Boolean algebras in classical logic, while on the other hand they subsume major fuzzy formalisms such as Łukasiewicz logic, Product logic, and Gödel Logic, as well as Classical Boolean Logic. In turn, every t-norm is the ordinal sum of Łukasiewicz, Product and Gödel norms [12]. The most thorough and recent treatment of triangular logics can be found in [6].

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1.2. Interpolation

Ever since Craig's seminal result on interpolation for classical predicate logic, interpolation properties have been recognized as important desiderata of logical systems. In propositional logic, we consider two interpolation properties of increasing strength.

Interpolation Property. If $A \longrightarrow B$, then there exists a formula $I(A, B)$ which contains only common variables of A and B such that

$$A \longrightarrow I(A, B) \longrightarrow B$$

Uniform Interpolation Property. If $A \longrightarrow B$, and V is the set of common variables in A and B, then there exist a *pre-interpolant* (or *left interpolant*) $I(A, V)$ and a *post-interpolant* (or *right interpolant*) $J(B, V)$ such that $I(A, V)$ and $J(B, V)$ depend only on A, V and B, V respectively, both contain only variables from V and

$$A \longrightarrow I(A, V) \longrightarrow J(B, V) \longrightarrow B$$

Remark. The presence of the (uniform) interpolation property is strongly connected with the existence of *analytic properties* of the deduction system. Suppose that we have a Hilbert style calculus for the logic, where the rule

$$\frac{A \longrightarrow B \quad B \longrightarrow C}{A \longrightarrow C}$$

is derivable. Then interpolation implies that there are tree-like proofs of $A \longrightarrow B$ and $B \longrightarrow C$ which involve only variables which are already in A and C respectively.

Classical propositional logic has the uniform interpolation property. Let $A(\mathbf{x}, \mathbf{y}), B(\mathbf{x}, \mathbf{z})$ be like above, i.e. V contains just the variables from the tuple $\mathbf{x}$. Then the pre-interpolant and the post-interpolant can be written as quantified boolean formulas:

$$J(A, V) = \forall \mathbf{y} A \;=\; \bigwedge_{\mathbf{y} \in \{0,1\}^{|\mathbf{y}|}} A(\mathbf{x}, \mathbf{y}) \tag{1}$$

$$I(B, V) = \exists \mathbf{z} B \;=\; \bigvee_{\mathbf{z} \in \{0,1\}^{|\mathbf{z}|}} B(\mathbf{x}, \mathbf{z}) \tag{2}$$

Since the propositional quantifier in quantified boolean logic can be eliminated by long disjunctions and conjunctions as indicated above, the interpolants can be easily constructed.

For triangular logics, one can define interpolants likewise, by using *fuzzy existential and universal quantifiers* [6] instead of ordinary propositional quantifiers. Universal (existential) fuzzy quantifiers correspond to taking the infimum (supremum respectively) of the quantified formula with respect to the quantified variable. Thus, in a two-valued framework, a fuzzy quantifier collapses to an ordinary propositional quantifier.

The uniform interpolation in triangular logics can be seen as quantifier elimination of fuzzy quantifiers in fuzzy formulas. Similarly, interpolation in modal and intuitionistic logics is related to amalgamation and, thus, quantifier elimination of first order quantifiers [13]. The main difference of the methods applied in the abovementioned papers is their disregard of syntax; yet the mere existence or absence of an interpolant (or amalgam) does not yield explicit and feasible interpolants.

For fuzzy quantifiers, a straightforward elimination of quantifiers as in formula 1 is not possible even if the language contains truth constants for all truth values, since such a translation yields infinitary formulas. For finitely-valued triangular logics, however, it is trivially possible to gain interpolation by adding all truth constants to the language because the binary truth functions for minimum and maximum are definable; here, it is rather of interest to consider which minimal sets of truth constants are necessary for interpolation.

This remark extends to logics in general: it is not so much the question *if* a logic interpolates, but rather *where* it interpolates, i.e., in which extension of the language. For logics with first-order definable semantics such as modal logics, interpolation indicates that the language in question has certain closure properties akin to first order logic; consider for example a modal logic whose Kripke structures are first order definable. Then every modal propositional statement can be translated into a first order statement, and thus, a modal implication $A \longrightarrow B$ turns into a first order implication $\gamma(A) \longrightarrow \gamma(B)$, yielding, by Craig interpolation, a first order interpolant $I_\gamma(A,B)$. Semantically, $I_\gamma(A,B)$ is an interpolant, yet the question arises whether this interpolant can be translated back into the language of modal logic.

Summarizing, the main questions addressed in this paper are:

1. Which triangular logics have (uniform) interpolation?
2. How can the interpolants be constructed?
3. How can a non-interpolating logic be extended to gain interpolation?
4. Which normal forms can be obtained for the logic and for the interpolants?
5. Which extensions of finite-valued triangular logics by truth constants have interpolation?

Table 1 Interpolation in infinite-valued logics

	Pure	With $\Box$	With $\triangle$	With $\triangle$ and $\Box$
Gödel	Interpolation	Interpolation	Interpolation	Interpolation
Łukasiewicz	x	Interpolation	x	Interpolation
Product	x		x	

1.3. Results

The paper is divided into two parts, the first dealing with infinite-valued logics, the second with finite-valued logics.

For **infinite-valued logics** we first show (Section 3.1) that regardless of the presence of additional truth constants, Gödel logic is the unique triangular logic with interpolation, and in fact, uniform interpolation.

Moreover (Theorem 4) Gödel logic has a weak disjunctive normal form. Note that ordinary interpolation of Gödel logic has been shown before in [13] in a non-constructive manner. The failure of interpolation for Łukasiewicz logic was previously demonstrated in [11].

Starting from those negative results, we investigate extensions of triangular logics by $0-1$ projections [1] and root operators [8]. The $0-1$ projection operator is defined by

$$\triangle : x \mapsto \lfloor x \rfloor \tag{3}$$

The projection $\triangle$ introduces a weak kind of introspection which allows to express quantifier-free statements about terms in the underlying logic. For a theory with quantifier-elimination, this means that the addition of $\triangle$ is quite powerful.

The root operators $\Box_i(x) = y$ are defined as the maximum solution of the equation

$$\underbrace{y \mathbin{\&} \ldots \mathbin{\&} y}_{i \text{ times}} = x \tag{4}$$

Let $\mathcal{L} + \Box$ denote the logic obtained by adding all countably many operators $\Box_1, \Box_2, \ldots$ to $\mathcal{L}$.

Table 1 summarizes the interpolation results for infinite-valued logics.

Finite-valued logics are treated here as finitely generated sublogics of the abovementioned logics; they approximate their infinite-valued counterpart as a Fraïssé-limit [2].

In Sections 4 and 5, general criteria are given when an extension of an $n-$valued Gödel or Łukasiewicz logic (G_n and $Ł_n$ respectively) by a set S of additional truth constants has uniform interpolation. Apart from G_3, no G_n or $Ł_n$ has interpolation, while we have seen above that adding all truth

Table 2 The number and characterization of finite-valued intermediate logics

	$G_{n+1}+S$	$\text{Ł}_{n+1}+S$
$\lvert S\rvert$ fixed	$\binom{\lvert S\rvert+1}{n-(\lvert S\rvert+1)}$	$R_{\lvert S\rvert}(n)=\sum_{d\mid n}\mu(\frac{n}{d})\binom{d-1}{\lvert S\rvert-1}$
$\lvert S\rvert$ arbitrary	F_n	$\sum_{d\mid n}\mu(\frac{n}{d})2^{d-1}$
Characterization	$\gcd(n\overline{S})=1$	$\max(n\overline{S})\leq 2$

constants trivially leads to uniform interpolation. Therefore, the structure of the *intermediate* logics is of interest. To this end, consider the set $\overline{S}$ which is the set of gaps between truth values from S and $0, 1$. Interpolation can be characterized in terms of $\overline{S}$, and the number of intermediate logics can be determined as summarized in figure 2. Again, we obtain normal forms for the case of Gödel logic.

2. Triangular logics and extensions

2.1. Metamathematics of fuzzy logic revisited

This section contains important definitions and background knowledge about triangular logics. Most of the presentation is based on [6].

2.1.1. Triangular logics In this paper, the truth functions of the logic (*the matrix of the logic*) are defined in a generic way from a *continuous triangular norm* t, i.e., a two-place function defined on a subset d_t of $[0, 1]$ which satisfies the following axioms [10]:

T1 $t(0, x) = 0$ and $t(1, x) = x$ T3 $t(x, y) = t(y, x)$

T2 $x \leq u$ & $y \leq z \Longrightarrow t(x, y) \leq t(u, z)$ T4 $t(t(x, y), z) = t(x, t(y, z))$

In triangular logics, a t-norm t defines the truth function of conjunction; implication then is defined by the right adjunct of t, i.e., by the unique truth function i such that

I1 $z \leq i(x, y)$ iff $t(x, z) \leq y$

If t is continuous, we obtain that $i(x, y) = \max\{z|t(x, z) \leq y\}$. Negation is defined by $i(x, 0)$.

Definition. A logic $\mathcal{L}$ is *triangular*, if it is defined by the connectives

$$\&\ , \longrightarrow, \neg, 0, 1 \tag{5}$$

Table 3 The truth functions of well-known triangular logics

	Constraint	Gödel (G)	Łukasiewicz (Ł)	Product (Π)
$x \,\&\, y$		$\min(x,y)$	$\max(0, x+y-1)$	$x \times y$
$x \longrightarrow y$	$x \leq y$	1	1	1
$x \longrightarrow y$	$x > y$	y	$1-x+y$	$\frac{y}{x}$
$\neg 0$		1	1	1
$\neg x$	$x > 0$	0	$1-x$	0

induced by a continuous t-norm. The tautologies $\mathsf{Taut}_{\mathcal{L}}$ of $\mathcal{L}$ is the set of formulas ϕ which evaluate to 1 for all assignments to the variables $var(\phi)$. □

Proviso. We do not distinguish between the truth functions and the syntactical connectives of triangular logics, i.e., we treat formulas as polynomials in the matrix algebra. When we speak of the *first order theory of a triangular logic*, then we mean the first order theory of the matrix of the logic. In such a first order logic, the propositional formulas are the first order terms. In Section 3.4, we shall see how the introspection operator $\triangle$ embeds first order quantifier-free expressions into the propositional logic.

We shall write $x \equiv y$ to abbreviate $(x \longrightarrow y) \,\&\, (y \longrightarrow x)$.

2.1.2. Ling's theorem Triangular logics comprise Łukasiewicz, Gödel, and Product Logic as special cases; the truth functions of those logics are summarized in Table 3. The following important result shows that they are the easiest triangular logics, and that they are a basis for all other triangular logics:

For a given norm t, let E be the set $\{x \in [0,1] : t(x,x) = x\}$. The complement $[0,1] - E$ is is the union of countably many non-overlapping open intervals, whose closure $cl([0,1]-E)$ corresponds to a class $\{I_i, i \in \mathcal{I}\}$ of closed intervals. Ling's theorem [12] says that the triangular logics are characterized by norms which are isomorphic to either Ł or Π within the intervals I_j, and behave like the minimum norm (Gödel norm) for arguments from different intervals. In short:

Theorem 1. *The triangular logics are obtained from the ordinal sums of the Ł, Π and G t-norms.*

Therefore, any investigation of triangular logics will focus on those prototypical cases first.

2.1.3. Disjunction and McNaughton's theorem What about disjunction? In the framework of t-norms, disjunction can be defined by the co-norm $1-t(1-x, 1-y)$. However, following previous work [6], we do *not* include disjunction as a basic operation. The following result shows that Gödel conjunction and disjunction (i.e., arithmetic minimum and maximum) are definable in all triangular logics, and thus, disjunction is definable in Gödel logic:

Fact 1 ([6]) *. There exists two formulas* $\min(x,y)$ *and* $\max(x,y)$ *which are identic to Gödel conjunction and Gödel disjunction in all triangular logics.*

Therefore, we can use $\min(x,y)$ and $\max(x,y)$ as abbreviations in propositional formulas. In [6], the minimum and maximum operations are denoted by $\wedge$ and $\vee$ respectively. Typically, we shall also use $\vee$ for Gödel logic (where the maximum indeed is used as a disjunction), but $\min$, $\max$ for the other logics, where more algebraic constructions are employed. Moreover, $\bigvee$ means iterated maximum, i.e., *iterated Gödel disjunction*.

As for disjunction in Łukasiewicz logic, we use a very deep and useful algebraic characterization due to McNaughton:

Theorem 2 ([14]). *The truth functions of Łukasiewicz logic coincide with the continuous piecewise linear functions with integer coefficients on* $[0,1]$.

As a consequence, Łukasiewicz logic defines Łukasiewicz disjunction. For product logic, disjunction is not definable, as can be easily seen from the truth functions.

By McNaughton's Theorem [14] it is easy to see that cut-off addition and subtraction can be defined in Ł:

$$x \dot{-} y \equiv \max(0, x-y) \tag{6}$$

$$x \dot{+} y \equiv \min(1, x+y) \tag{7}$$

$$\tag{8}$$

2.1.4. Deduction theorem It is known from [6] that G has a deduction theorem like classical propositional logic; i.e., if all assignments which make true ϕ make true ψ as well, then $\phi \longrightarrow \psi$ is a G tautology, and vice versa. Note however, that *no other* triangular logic has a classical deduction theorem; but each triangular logic has the generalized deduction thorem (see [6]).

2.1.5. Finite-valued triangular logics For Gödel logic G and Łukasiewicz logic *Ł*, there exist finite-valued logics which can be constructed as finitely generated subalgebras of the matrices. Indeed, for each natural number $i \geq 2$, there exist unique finitely generated subalgebras of size i. The corresponding $i-$valued logics are denoted G_i and $Ł_i$ respectively. It is clear that product logic does not have nontrivial finite subalgebras. Therefore, our treatment of finite-valued logics is confined to Łukasiewicz and Gödel logic.

2.2. Extensions of triangular logics

2.2.1. Projection operators As additional connectives, we introduce the 0-1-Projections $\bigtriangledown, \bigtriangleup$ from [1]. They are defined by $\bigtriangleup x = \lfloor x \rfloor$ and $\bigtriangledown x = \lceil x \rceil$.

Since $\bigtriangledown$ and $\bigtriangleup$ are mutually definable in L (by $\bigtriangleup x = \neg \bigtriangledown \neg x$) and $\bigtriangledown$ is definable already in G and Π, it suffices to consider extensions of those logics by $\bigtriangleup$.

$$x \leq y = \bigtriangleup(x \longrightarrow y) \tag{9}$$
$$x = y = \bigtriangleup(x \longrightarrow y) \ \& \ \bigtriangleup (y \longrightarrow x) \tag{10}$$
$$x > y = (x \geq y) \ \& \ \neg(x = y) \tag{11}$$

It is easy to see that those formulas have the intended semantics in all triangular logics, and that they obtain only classical values (they are *crisp*). We say that a formula is *crisp* if every atomic subformula is in the scope of a $\bigtriangleup$ operator.

We see that the $\bigtriangleup$ operator is very powerful in that it allows to formulate quantifier free statements about terms in the underlying many-valued logic. Thus, $\bigtriangleup$ is a kind of *introspection* operator.

In particular, the crisp formulas of a logic with $\bigtriangleup$ can be considered as the quantifier free fragment of a first order logic, with the propositional variables serving as first order variables.

This is illustrated by the following observation:

Proposition 1. *All triangular logics with $\bigtriangleup$ express the first order theory of the linear order over* $[0, 1]$.

Proof. Since the first order theory of linear order has quantifier elimination, it suffices to show that all quantifier-free formulas are expressible. This follows immediately by virtue of (9) to (11). □

2.3. Divisibility witnesses

Fact 2. *The divisiblity witnesses for L are defined by* $\Box_k(x) = \frac{y}{k} + \frac{k-1}{k}$.

Proof. Obvious. □

Fact 3. *The divisiblity witnesses for Π are defined by* $\Box_k(x) = x^{\frac{1}{k}}$.

Proof. Obvious. □

The following Lemma shows that we do not have to deal with $\Box_k$, but can use the much simpler function $\Box_k^* = \frac{x}{k}$.

Lemma 1. *$\Box_k$ and $\Box_k^*$ are mutually expressible in Ł.*

Proof. Consider $\Box_k(0) = \frac{k-1}{k}$. Since $\dot{-}$ is expressible in Ł, we can define $\Box_k^*(x) = \Box_k(x) \dot{-} \Box_k(0)$. On the other hand,

$$\Box_k(x) = \Box_k^*(x) \dot{+} \underbrace{\Box_k^*(1) \dot{+} \ldots \dot{+} \Box_k^*(1)}_{k-1 \text{ times}} .$$

$\Box$

In the rest of the paper, by misuse of notation, we shall always tacitly use $\Box_k^*$ instead of $\Box_k$ without further notice.

3. Infinite-valued interpolation

3.0.1. Semantic interpolation Let $\mathcal{L}$ be a triangular logic. For a formula ϕ, let $var(\phi)$ denote the set of propositional variables occurring in $\mathcal{L}$.

Definition. $\mathcal{L}$ has *interpolation* if for every tautology $(\phi \longrightarrow \psi) \in \mathsf{Taut}_{\mathcal{L}}$ there exists a formula $\alpha \in \mathcal{L}$ (the *interpolant* of ϕ and ψ), such that

1. $var(\alpha) \subseteq var(\phi) \cap var(\psi)$
2. $(\phi \longrightarrow \alpha), (\alpha \longrightarrow \psi) \in \mathsf{Taut}_{\mathcal{L}}$

If α depends only on ϕ and $var(\psi)$, then α is called a *pre-interpolant* (or *left interpolant*). If α depends only on ψ and $var(\phi)$, then α is called a *post-interpolant* (or *right interpolant*).

If pre-interpolants and post-interpolants always exist, then $\mathcal{L}$ has *uniform interpolation*. If only pre-interpolants (post-interpolants) exist, we speak of left (right) uniform interpolation. $\Box$

Remark. Let λ and ρ be the left and right interpolants respectively. Since ϕ implies all interpolants, it follows that $\phi \longrightarrow \rho$. Since λ depends only on ϕ, it follows that λ is an interpolant of $\phi \longrightarrow \rho$, i.e., $\phi \longrightarrow \lambda$ and $\lambda \longrightarrow \rho$. We conclude that the left interpolant in fact implies the right interpolant.

Consider an $\mathcal{L}-$tautology

$$a(\mathbf{x}, \mathbf{z}) \longrightarrow b(\mathbf{y}, \mathbf{z}) \tag{12}$$

where the $\mathbf{z}$ are the common variables of a and b. Then we know that $a(\mathbf{x}, \mathbf{z}) \leq b(\mathbf{y}, \mathbf{z})$ under all variable assignments. Consider the truth function

$$T(\mathbf{z}) = \sup\{a(\mathbf{x}, \mathbf{z}) : \mathbf{x} \in [0, 1]^r\} \tag{13}$$

where r is the number of variables in $\mathbf{x}$. Clearly, $a(\mathbf{x}, \mathbf{z}) \leq T(\mathbf{z}) \leq b(\mathbf{y}, \mathbf{z})$, and thus $T(\mathbf{z})$ is an interpolant, if (13) is expressible in the underlying logic. It follows that *in every triangular logic, $T(\mathbf{z})$ is the truth function of a pre-interpolant*.

3.0.2. Fuzzy quantifiers In classical propositional logic, the supremum and infimum of truth functions is expressed by propositional quantification. This correspondence can be extended to triangular logics by using *fuzzy quantifiers*. Let $\phi(x_1, \ldots, x_n, y)$ be a formula in a triangular logic with variables $x_1, \ldots, x_n, y$. Then the truth functions of $\exists y.\phi$ and $\forall y.\phi$ are given by

$$\exists y.\phi = \sup\{\phi(x_1, \ldots, x_n, y) \mid y \in [0, 1]\} \tag{14}$$
$$\forall y.\phi = \inf\{\phi(x_1, \ldots, x_n, y) \mid y \in [0, 1]\} \tag{15}$$

Lemma 2. *If a triangular logic has elimination of fuzzy quantifiers, then it has uniform interpolation.*

3.1. Infinite-valued Gödel logic has left interpolation

The whole section is devoted to the proof of the following result; on the way, we obtain a new functional complete set of connectives for Gödel logic, and a normal form (Chain Normal Form) similar to the disjunctive normal form in classical logic.

Theorem 3. *Infinite-valued Gödel Logic G has left uniform interpolation.*

We consider the formula $x \prec y$ which is defined as follows:

$$(x \longrightarrow y) \;\&\; ((y \longrightarrow x) \longrightarrow x) \tag{16}$$

Intuitively, it almost defines the semantical linear order, i.e., we have

$$x \prec y = \begin{cases} 1 & \text{if } x < y \\ x \;\&\; y & \text{otherwise} \end{cases} \tag{17}$$

Moreover, the ordering $\prec$ induces an evaluation of formulas by virtue of the following tautologies:

$$x \prec y \quad \longrightarrow \quad (x \longrightarrow y) \equiv 1 \tag{18}$$
$$x \prec y \quad \longrightarrow \quad (y \longrightarrow x) \equiv x \tag{19}$$
$$x \prec y \quad \longrightarrow \quad (x \;\&\; y) \equiv x \tag{20}$$
$$x \prec y \quad \longrightarrow \quad (x \vee y) \equiv y \tag{21}$$
$$x \prec y \;\&\; y \prec z \quad \longrightarrow \quad x \prec z \tag{22}$$
$$\neg(x \equiv 1) \quad \longrightarrow \quad x \not\prec x \tag{23}$$
$$x \prec y \vee y \prec x \vee x \equiv y \tag{24}$$

Using the deduction theorem, those tautologies follow trivially from the definition of $\prec$.

Let $V = \{v_1, \ldots, v_n\}$ be a set of propositional variables. Then a $\prec$-*chain* over V is a formula of the form

$$(0 \bowtie_0 v_{\pi(1)}) \mathbin{\&} (v_{\pi(1)} \bowtie_1 v_{\pi(2)}) \mathbin{\&} \cdots \\ \mathbin{\&} (v_{\pi(n-1)} \bowtie_{n-1} v_{\pi(n)}) \mathbin{\&} (v_{\pi(n)} \bowtie_n 1) \tag{25}$$

such that π is a permutation of $\{1, \ldots, n\}$ and $\bowtie_i$ is either $\equiv$ or $\prec$. (The reader easily checks that there are $n!2^{n+1}$ different $\prec$-chains.) Every $\prec$-chain describes an order type of the variables V. For a formula ϕ, let ϕ^ζ denote the value of ϕ under an evaluation which has the same order type as described by ζ. Clearly, $\phi^\zeta \in V \cup \{0, 1\}$. By induction on the formula structure, one easily shows that for each $\prec$-chain ζ and formula ϕ, the following is a tautology:

$$\zeta \longrightarrow \phi \equiv \phi^\zeta \tag{26}$$

Lemma 3. *Let ϕ be a formula, such that $var(\phi) = V$, and let ζ be a $\prec$-chain over V. Then there exists an atomic formula $v \in V \cup \{0, 1\}$ such that*

$$\phi \mathbin{\&} \zeta \equiv v \mathbin{\&} \zeta \tag{27}$$

Proof. It is easy to check that

$$(a \mathbin{\&} b) \mathbin{\&} (a \longrightarrow (b \equiv c)) \longrightarrow (a \mathbin{\&} c) \tag{28}$$

is a tautology in G. We instantiate formula 28 by setting $a = \zeta, b = \phi$, and $c = \phi^\zeta$. Since $\zeta \longrightarrow \phi \equiv \phi^\zeta$ (formula 26) is a tautology, we obtain that $\zeta \mathbin{\&} \phi \longrightarrow \zeta \mathbin{\&} \phi^\zeta$. The converse implication follows analogously. □

Corollary 1. *Let $C(V)$ be the set of all $\prec$-chains for V. Then*

$$\bigvee_{\zeta \in C(V)} \zeta \tag{29}$$

is a tautology.

Proof. Every possible evaluation of the variables in V has an order type which is described by one of the ζ. Therefore, the corresponding ζ becomes true. □

Theorem 4. *(Chain Normal Form) Let ϕ be a formula, and $V = var(\phi)$. Then ϕ is equivalent to a formula*

$$\bigvee_{\zeta \in C(V)} \zeta \mathbin{\&} v_\zeta \tag{30}$$

such that $v_\zeta \in V \cup \{0, 1\}$.

Proof. By Corollary 1, $\phi \equiv \phi \mathbin{\&} \bigvee_{\zeta \in C(V)} \zeta$. By moving ϕ into the disjunction, and applying Lemma 3, the result follows. □

Remark. Note that Theorem 4 is valid for all Gödel logics, whether they are finite or not. It is the main tool to prove the interpolation results. Moreover, it says that every formula is equivalent to a *flat* formula, i.e., to a formula of depth 4, if we count the large disjunction as one operation.

Lemma 4. *Let ϕ, ψ, γ be formulas such that $x \notin var(\gamma)$. Then*

$$\exists x.\phi \vee \psi \equiv (\exists x.\phi) \vee (\exists x.\psi) \tag{31}$$
$$\exists x.\phi \mathbin{\&} \gamma \equiv (\exists x.\phi) \mathbin{\&} \gamma \tag{32}$$

Proof. These are obvious properties of the supremum. □

Lemma 5. *Gödel logic has interpolation if the quantifiers in*

$$\exists x.(a \bowtie_1 x) \mathbin{\&} (x \bowtie_2 b) \mathbin{\&} x \tag{33}$$
$$\exists x.(a \bowtie_1 x) \mathbin{\&} (x \bowtie_2 b) \tag{34}$$

can be eliminated, where a, b may be either variables or $0, 1$, and $\bowtie_1, \bowtie 2$ may be either $\prec$ or $\equiv$.

Proof. Consider a formula $\exists x.\phi$. By Theorem 4, we may without loss of generality suppose that ϕ is in chain normal form (30). By Lemma 4 we can distribute the fuzzy quantifier over the disjunction and over those conjuncts of (30), where x does not occur. Therefore, only formulas of the form (33) and (34) are of interest. □

Proof of Theorem 3. By Lemma 5, it remains to eliminate the quantifier from (33) and (34). If one of $\bowtie_1, \bowtie_2$ equals $\equiv$, we are done, because we can replace x by a or b respectively. Thus, suppose that both $\bowtie_1 = \bowtie_2 = \prec$.

Consider (33) first. It is easy to check that (33) is equivalent to $\exists x.b \mathbin{\&} x$. (To see this, one first shows that $z \prec b \mathbin{\&} z$ is equivalent to $b \mathbin{\&} z$, and then notes that $a \prec z$ is always greater than or equal to z.) Therefore, the supremum is b in this case, i.e.

$$\exists x.(a \prec x) \mathbin{\&} (x \prec b) \mathbin{\&} x \equiv b \tag{35}$$

Let us finally turn to (34). We shall argue that

$$\exists x.(a \prec x) \mathbin{\&} (x \prec b) \equiv a \prec b \tag{36}$$

To see this, consider first the case that $a < b$. Then $a \prec b$ evaluates to 1, which can be easily witnessed by any x between a and b, like $x = (a+b)/2$. (Note that this is the only time where we exploit the density of truth values in G. We shall need this fact later in the proof of Theorem 11.)

If $a \geq b$, then $a \prec b$ evaluates to b. Suppose $x \leq b$. Then $a \prec x$ evaluates to x, and thus the conjunction remains below b. If $b < x < a$, then $a \prec x$ evaluates to x, and $x \prec b$ evaluates to b, so the conjunction again remains below b. In the last case, $b \leq a \leq x$. Then $x \prec b$ evaluates to b, and $a \prec x$ evaluates to 1, and again the conjunction evaluates to b. Since in all cases the conjunction either reaches b or stays below, we have proved that the supremum equals b.

Remarks.

- In Gödel logic, implication $\longrightarrow$ and negation $\neg$ can be replaced by the set $\{\prec, \equiv, \vee\}$ of connectives. This is an immediate consequence of the Chain Normal Form Theorem 4.
- It is not possible to obtain an interpolant $\exists y.\phi(y, \mathbf{x})$ as a finite disjunction $\bigvee_t \phi(t, \mathbf{x})$ of instances of ϕ. To see this, consider formula 36 from above. As the proof shows, x must obtain an intermediate value between a and b at the supremum. Gödel logic, however, cannot define intermediate truth values. On the other hand, we shall see in Section 4.1 that G_3, the unique interpolating finite-valued Gödel logic, has an interpolant of the form $\bigvee_t \phi(t, \mathbf{x})$.

3.2. Gödel logic has right interpolation

Unlike classical logic or Łukasiewicz logic, Gödel logic is not symmetric, i.e., it is not possible to treat existential and universal quantifiers as dual concepts in a straightforward way.

However, we can use the normal form result in a fortunate manner.

Theorem 5. *Gödel logic has right uniform interpolation.*

Proof. Consider a formula $\phi(\mathbf{x}, \mathbf{y})$ where we are looking for a right interpolant using just variables from $\mathbf{x}$. To this end, we consider all Gödel logic formulas in variables $\mathbf{x}$. By the Chain Normal Form Theorem (Theorem 4) there exists an enumeration $\phi_1, \ldots, \phi_N$ of these formulas (where N is the number of $\prec$-chains using variables $\mathbf{x}$), and we collect in $I = \{i : \phi_i(\mathbf{x}) \longrightarrow \phi(\mathbf{x}, \mathbf{y})\}$ the indices of those among them which might serve as interpolants.

It follows that for the disjunction $\rho = \bigvee_{i \in I} \phi_i$ it holds that $\rho \longrightarrow \phi$, and therefore, there exists an $i_0 \in I$, such that $\rho \equiv \phi_{i_0}$. Thus, by construction ϕ_0 is the unique maximal formula containing variables $\mathbf{x}$ which implies ϕ.

It remains to show that ϕ is an interpolant, i.e. that for all $\psi(\mathbf{z}, \mathbf{x})$ where $\psi(\mathbf{z}, \mathbf{x}) \longrightarrow \phi(\mathbf{x}, \mathbf{y})$ actually $\psi(\mathbf{z}, \mathbf{x}) \longrightarrow \phi_{i_0}(\mathbf{x})$. Let $\lambda(\mathbf{x})$ denote the left interpolant obtained from ψ. Then $\lambda(\mathbf{x}) \longrightarrow \phi(\mathbf{x}, \mathbf{y})$, and thus, $\lambda(\mathbf{x})$ is among $\phi_i, i \in I$, whence $\lambda(\mathbf{x}) \longrightarrow \phi_{i_0}$. $\square$

3.3. Failure of interpolation for non-Gödel logics

In this section, we show that Gödel logic is the unique infinite-valued triangular logic with interpolation. We shall see in Section 3.4 that only a strong extension of the language makes logics like Ł and Π interpolate.

The following result states that interpolation of a logic can be checked in a very uniform manner for all triangular logics; we say that a tautology Ψ interpolates if (1) it is of the form $\phi_1 \longrightarrow \phi_2$, and (2) it has an interpolant.

Theorem 6. *There is a formula Ψ, such that*

- *Ψ is a tautology in every triangular logic.*
- *Ψ interpolates in G.*
- *Ψ does not interpolate in Ł.*
- *Ψ does not interpolate in Π.*

Proof. The result follows from the Lemmas 6, 7, 8 below and Theorem 3 above. □

Here is the formula Ψ:

$$\underbrace{\min(\max(x,p) \longrightarrow p, \max(x,p))}_{\phi_1} \longrightarrow \underbrace{\max(\max(y,p) \longrightarrow p, \max(y,p))}_{\phi_2} \tag{37}$$

Lemma 6. *Ψ is a tautology in every triangular logic.*

Proof. Suppose we have a non-satisfying assignment to the variables x, y, p. Then, $y > p$ since otherwise $\phi_2 = 1$, and the formula becomes true. Moreover, $x > p$, because if $x \leq p$, then $\phi_1 = p$, while ϕ_2 is always greater than p since it is computed as a maximum of p and other values. Also, $x \neq y$, because otherwise the minimum in ϕ_1 and the maximum in ϕ_2 are taken over the same elements.

Since $x > p$ and $y > p$ we can simplify Ψ to

$$\min(x \longrightarrow p, x) \longrightarrow \max(y \longrightarrow p, y) \tag{38}$$

There are two cases remaining:

1. $x > y > p$. Then $t(x,z) \geq t(y,z)$ by axiom T2, and $t(x,z) \leq p$ implies $t(y,z) \leq p$. Hence $\{z|t(x,z) \leq p\} \subseteq \{z|t(y,z) \leq p\}$, and $x \longrightarrow p = \max\{z|t(x,z) \leq p\} \leq \max\{z|t(y,z) \leq p\} = y \longrightarrow p$, since t is continuous. Therefore $\min(x \longrightarrow p, x) \leq x \longrightarrow p \leq y \longrightarrow p \leq \max(y \longrightarrow p, y)$.
2. $y > x > p$. Then $\min(x \longrightarrow p, x) \leq x < y \leq \max(y \longrightarrow p, y)$. □

Lemma 7. *Ψ does not interpolate in Ł.*

Proof. Set $p = 0$, then Ψ becomes $\min(1-x, x) \longrightarrow \max(1-y, y)$ by some obvious simplifications in Ł. The unique interpolant is $\frac{1}{2}$ which is not expressible in Ł. □

Lemma 8. *Ψ does not interpolate in Π.*

Proof. For $p \neq 0$, Ψ turns into $\min\left(\frac{p}{\max(x,p)}, \max(x,p)\right) \longrightarrow \max\left(\frac{p}{\max(y,p)}, \max(y,p)\right)$. Then the unique interpolant is $p^{\frac{1}{2}}$. Since Π does not define square roots, this concludes the lemma. □

By the mentioned characterization of triangular logics as ordinal sums of the three basic logics, it is easy to obtain the following characterization of interpolation in triangular logics:

Theorem 7. *Gödel Logic is the unique triangular logic with interpolation.*

Proof. From Theorem 1 we know that every triangular logic except Gödel's logic contains a closed interval I_i which is isomorphic to Ł or Π. Let p be an inner point of I_i, i.e., by definition a non-idempotent truth value. Then the interpolant of Ψ at p is not expressible by virtue of Lemmas 7 and 8. □

Corollary 2. *No extension of L by constants has interpolation.*

Proof. If we don't fix p, the (unique) interpolant becomes $\frac{1+p}{2}$. Since this function does not have integer coefficients, the result follows from McNaughton's Theorem. □

From the above it may appear that adding a square root operator [8] might facilitate interpolation. The following corollary shows that this is not the case. However, we shall see in Section 3.4 that adding all division operators makes interpolation possible.

Corollary 3. *No extension of L by a finite number of division operators has interpolation.*

Proof. For a number k, consider the function $f_k(x) = \max(1-(k-1)x, 0)$. From McNaughton's Theorem we know that all f_k are expressible in Ł. Then $\Psi_k = \min(f_k(x), x) \longrightarrow \max(f_k(y), y)$ is a tautology with interpolant $\frac{1}{k}$. Now let S be a finite set of division operators, and consider the number $k = 1 + \prod_{\Box_s \in S} s$ which is relative prime with respect to all $s \in S$. Then the tautology Ψ_k has interpolant $\frac{1}{k}$, and is not expressible using operators from S. □

Corollary 4. *No extension of Π by constants has interpolation.*

Corollary 5. *No extension of Π by a finite number of root operators has interpolation.*

Proof. Similarly to Corollaries 2 and 3. $\square$

3.4. *Extensions of triangular logics*

The last section leaves open the question how to extend triangular logics to obtain interpolation. We treat the case of the $\triangle$ operator, and of divisibility witnesses $\square_k$.

Theorem 8.

- *$G + \triangle$ has interpolation.*
- *$Ł + \triangle$ does not have interpolation.*
- *$\Pi + \triangle$ does not have interpolation.*

Proof. The failure of interpolation for $Ł+\triangle$ and $\Pi+\triangle$ is an easy observation from the proof of Theorem 7. Interpolation for $G+\triangle$ follows by an extreme simplification of the argument in Theorem 3. Consider a formula

$$\exists x.\phi(x, y_1, \ldots, y_n).$$

It only depends on the order of the variables $y_1, \ldots, y_n$, whether the supremum equals $1, 0$ or one of the input values $y_1, \ldots, y_n$. By Proposition 1, the different cases can be described by quantifier-free formulas ϕ_t, $t \in \{0,1\} \cup \{y_1, \ldots, y_n\}$ such that $\phi_t(y_1, \ldots, y_n)$ is true iff $t = (\exists x.\phi(x, y_1, \ldots, y_n))$. Since all ϕ_t are crisp, $\phi_t \,\&\, t$ evaluates to 0 if the condition described by t is true, and to t otherwise. Therefore, we can combine all ϕ_t into a single formula

$$\exists x.\phi(x, y_1, \ldots, y_n) = \bigvee_{t \in \{0,1\} \cup \{y_1, \ldots, y_n\}} \phi_t \,\&\, t \tag{39}$$

The case of $\forall x.\phi(x, y_1, \ldots, y_n)$ is analogous. $\square$

Example. Consider $\exists x.((y_1 \leq y_2) \,\&\, (x \leq y_3) \,\&\, (y_4 \vee y_5))$. Is is clear that this formula has supremum $\max(y_4, y_5)$ if $y_1 \leq y_2$ and $y_3 \neq 0$, and has supremum 0 otherwise. Therefore, ϕ_{y_4} is $(y_1 \leq y_2) \,\&\, (y_3 \neq 0) \,\&\, y_4 \geq y_5$, ϕ_{y_5} is $(y_1 \leq y_2) \,\&\, (y_3 \neq 0) \,\&\, y_4 < y_5$, and ϕ_0 equals $\neg\phi_{y_4} \,\&\, \neg\phi_{y_5}$.

Corollary 6. *Theorem 8 holds even, if the language is extended by arbitrary truth constants.*

Fact 4. *In* $Ł + \Box$*, every rational in* $[0, 1]$ *is expressible by a closed term.*

Proof. Let $\frac{p}{q}$ be a rational. Then $\underbrace{\Box_q 1 + \cdots + \Box_q 1}_{p\text{times}}$ has truth value $\frac{p}{q}$. $\Box$

Corollary 7. *For every rational* q*, the first order formulas* $x > q$*,* $x \geq q$*,* $x = q$*,* $x < q$*, and* $x \leq q$ *are expressible in* $Ł + \triangle + \Box$*. Therefore, for any (open or closed) subinterval* S *of* $[0, 1]$*,* $x \in S$ *is expressible.*

3.4.1. Piecewise linear topology For the following, we need some easy facts and definitions from piecewise linear topology: Let $B = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$, be a set of $n-$dimensional points. Then the *(convex) polytope* generated by B is the convex hull of the points in B. Every polytope is representable as a finite intersection of halfspaces, and we shall see that containments in polytopes over $[0, 1]$ is expressible in $Ł + \Box + \triangle$.

In the proof to follow, we need the following well-known fact from piecewise linear topology:

Theorem 9. *[17] Let* $f : P \to Q$ *be piecewise linear and continuous, then there is a locally finite decomposition of* P *into polytopes,* $P = \bigcup P_i$*, such that* $f \restriction P_i$ *is linear for each* i*.*

Note that in spaces of the form $[0, 1]^k$, locally finite is equivalent to finite. Moreover, the functions we are interested in are in general not continuous, and therefore the $f \restriction P_i$ may be discontinuous on the border of P_i, if P_i is closed. Therefore we need the following Corollary which generalizes the situation for piecewise, non-continuous functions.

Corollary 8. *Let* $f : [0, 1]^n \to [0, 1]$ *be a piecewise linear function, then there is a finite partition of* P *into open polytopes* P_i*, such that for all* i*,* $f \restriction P_i$ *is linear.*

The following lemma is the key to our interpolation results. It can be seen as a generalization of McNaughton's Theorem, in that it removes the restrictions on integer coefficients and continuity.

Lemma 9. *Let* f *be a piecewise linear function on* $[0, 1]^n$*. Then the following equivalences hold:*

1. f *is continuous with rational coefficients iff* f *is definable in* $Ł + \{\Box_1, \Box_2, \ldots\}$.
2. f *has rational coefficients iff* f *is definable in* $Ł + \triangle + \{\Box_1, \Box_2, \ldots\}$.

Proof.

1. $\Longleftarrow$: If f is in $Ł + \{\Box_1, \Box_2, \dots\}$, then an easy induction shows that f is continuous, and has rational coefficients.
   $\Longrightarrow$: Suppose that f is continuous with rational coefficients. Let l be the least common multiple of the denominators of the rational coefficients. (The least common multiple must exist, because f is described by a finite number of linear polynomials.)
   Let us now consider the function lf, i.e. the function obtained by multiplying f with a constant l. Obviously, lf has integer coefficients, and $lf([0,1]) \subseteq [0,l]$.
   Consider the functions $f_1, \dots, f_l$ which are defined as follows:
$$f_i(\mathbf{c}) = \begin{cases} f(\mathbf{c}) - (i-1) & \text{if} f(\mathbf{c}) \in [i-1, i] \\ 0 & \text{if} f(\mathbf{c}) < i-1 \\ 1 & \text{if} f(\mathbf{c}) > i \end{cases} \tag{40}$$
   It is easy to see that $lf = \sum_{i=1}^{l} f_i$, and therefore
$$f(\mathbf{c}) = \sum_{i=1}^{l} \frac{f_i(\mathbf{c})}{l} = \sum_{i=1}^{l} \Box_l(f_i(\mathbf{c})) \tag{41}$$
   All the f_i have range in $[0,1]$, and are continuous by construction. Therefore, there exist Ł formulas $\phi_1, \dots, \phi_l$ which realize them. From formula 41 we conclude that
$$\Box_l(\phi_1) \dot{+} \Box_l(\phi_2) \dot{+} \cdots \Box_l(\phi_l) \tag{42}$$
   is the formula we looked for.
2. $\Longleftarrow$: By induction on the formula structure, it is obvious that f is piecewise linear.
   $\Longrightarrow$: Consider a function $f(\mathbf{c}), \mathbf{c} = c_1, \dots, c_d$, and let $P_1, \dots, P_n$ be a partition of $[0,1]^d$ into polytopes, such that f is linear on every P_i. By the first statement of the Theorem, there exist $Ł + \Box$ formulas $\phi_1(\mathbf{c}), \dots, \phi_n(\mathbf{c})$, such that ϕ_i coincides with f on P_i. Since membership of a tuple $\mathbf{c}$ in a polytope P_i is expressible by a Boolean combination of formulas $f = g$ and $f > g$ where f, g are linear (and therefore, by the first part of this Theorem, expressible in $Ł + \Box$), there exist $Ł + \Box + \triangle$ formulas χ_{P_i} such that
$$f(\mathbf{c}) = \bigvee_{1 \le i \le n} \chi_{P_i}(\mathbf{c}) \mathbin{\&} \phi_i(\mathbf{c}) \tag{43}$$

□

Using Lemma 9, it is easy to prove interpolation for $Ł + \Box$ and $Ł + \Box + \triangle$.

Theorem 10. *Let* $\Box_i(x) = \frac{x}{i}$. *Then*

- $Ł + \{\Box_1, \Box_2, \ldots\}$ *has uniform interpolation.*
- $Ł + \triangle + \{\Box_1, \Box_2, \ldots\}$ *has uniform interpolation.*

Proof. It suffices to show that piecewise linear functions are closed under taking suprema and infima of a single coordinate y; geometrically, this means taking the upper boundary of the union of orthogonal projections of all polytopes on the hyperplane $y = 0$, [15]. Since any parallel projection of a polytope on a plane is obviously a polytope, and parallel projections retain continuousness, the theorem is proven. $\Box$

4. Finite-valued interpolation

For finite-valued logics, the situation is combinatorially more involved than for infinite-valued logics. We use both proof theoretic and model theoretic methods to prove the following characterizations.

For a truth value d, let d^- denote the preceding truth value, and d^+ denote the succeeding truth value.

For a set of truth values $S = \{\frac{i_1}{k-1}, \ldots \frac{i_n}{k-1}\}$ in $k-$valued logic, we define its *grade* $gr(S)$ by

$$gr(S) = \gcd(i_1, \ldots, i_n, k-1) \quad (44)$$

S is *dense* if it contains all truth values except possibly $0, 1$. S is *semidense* if

$$\neg\exists x \notin S. \left(x + \frac{1}{k-1}\right) \notin S \cup \{1\} \quad (45)$$

holds, i.e., if it does not contain gaps of size greater than one.

4.1. Gödel logic

Theorem 11. *Let* C *be a (possibly empty) subset of the truth values for* G_k. *Then the following are equivalent:*

- $G_k + C$ *has uniform interpolation.*
- $G_k + C$ *has interpolation.*
- C *is semidense.*

The result follows from the following lemmas.

Lemma 10. *All semidense finite-valued Gödel logics have left interpolation.*

Proof. This proof is similar to the proof of Theorem 3. Again, we construct chain formulas to enforce an evaluation of a given formula. Let $V = var(\phi) = \{v_1, \ldots, v_n\}$, let $C = \{c_1, \ldots, c_m\}$ be the set of truth constants in the language including $0, 1$, and let $P \subseteq C \times C$ be the set of all pairs (c_i, c_j) such that $c_i < c_j$, and there is exactly one truth value between them. For every variable v_m, we have two possibilities to enforce its value. Either it is equal to a truth constant

$$v_m \equiv c_j \tag{46}$$

or it is between two constants $(c_i, c_j) \in P$

$$c_i \prec v_m \;\&\; v_m \prec c_j \tag{47}$$

A semidense chain formula ζ therefore is a conjunction

$$\bigwedge_{v_m \in V} \zeta_{v_m} \tag{48}$$

such that every ζ_{v_m} is of the form (46) or (47).

Like in the proof of Theorem 3, a semidense chain formula corresponds to an order type of the variables, and therefore it enforces an evaluation

$$\zeta \longrightarrow \phi \equiv \phi^{\zeta} \tag{49}$$

Therefore, every formula in a semidense Gödel logic is equivalent to a formula in *semidense chain normal form*:

$$\bigvee_{\zeta \in SD(V,C)} \zeta \;\&\; v_{\zeta} \tag{50}$$

such that $v_\zeta \in V \cup C$ and $SD(V, C)$ is the set of semidense chains with truth constants from C.

By Lemma 4, we can move the supremum to formulas of the form $(46)\ \&\ v$ and $(47)\ \&\ v$. Those formulas are subcases of (33) and (34) in the proof of Theorem 3 (obtained by setting a and b constant). The only property needed in the proof of Theorem 3 was the existence (yet not the explicit definability) of a truth value between a and b. Since $(c_i, c_j) \in P$ there exists an intermediate element between them, and we conclude that semidense finite-valued Gödel logic has interpolation. □

Corollary 9. *All semidense finite-valued Gödel logics have chain normal forms.*

Corollary 10. *All semidense finite-valued Gödel logics have right interpolation.*

Proof. Analogous to the proof of Theorem 5, using Lemma 10 and Corollay 9. □

Lemma 11. $G_i, i \geq 4$ *does not have interpolation.*

Proof. Considering the fact that validity of formulas with n variables is equivalent in all $G_i, n+1 \leq i \leq \omega$, any counterexample ϕ_n for G_n has to contain $n-1$ variables at least. We construct

$$\phi_n = \ (A_n \equiv x_1) \longrightarrow ((z \longrightarrow x_1) \vee z) \tag{51}$$

where A_n is defined as

$$x_1 \vee (x_1 \longrightarrow x_2) \vee \ldots \vee (x_{n-3} \longrightarrow x_{n-2}) \vee (x_{n-2} \longrightarrow 0) \tag{52}$$

For simplicity of notation, we sometimes write x_{n-1} for 0. 1^- denotes the greatest element below 1, i.e., $\frac{n-2}{n-1}$, and similarly $0^+ = \frac{1}{n-1}$.

Claim: $(A_n \equiv x_1) \ = \begin{cases} 1 & x_1 > x_2 > \cdots > x_{n-2} > 0 \\ x_1 & \text{otherwise} \end{cases}$

Proof. In the first case, x_1 has to be either 1 or 1^-, because there are only n truth values available. Since x_1 is larger than all other x_i, A_n equals x_1, whence $A_n \equiv 1$ becomes 1.

In the second case, there exists some i, such that $x_i \leq x_{i+1}$, and thus the implication $x_i \longrightarrow x_{i+1}$ evaluates to 1. Therefore, $A_n = 1$, and we conclude that $A_n \equiv x_1$ evaluates to x_1. □

Claim: $(z \longrightarrow x_1) \vee z \ = \begin{cases} 1 & z \leq x_1 \\ z & z > x_1 \end{cases}$

Proof. Obvious. □

Claim: ϕ_n is a tautology of G_n.

Proof. From the above case distinction we see that the only interesting case appears when $z > x_1$. If $x_1 > x_2 > \cdots > x_{n-2}$, then $1 = z > 1^- = x_1$, and therefore the implication holds. Otherwise, $A_n \equiv x_1$ evaluates to x_1, and the implication holds, too. □

It remains to show that ϕ_n does not have an interpolant. From the construction it follows that the interpolant is a formula in x_1. Like in the three-valued case, every formula in one variable is equivalent to a formula from $F = \{0, 1, x_1, \neg x_1, \neg\neg x_1, x_1 \vee \neg x_1\}$.

Consider the case that $z > x_1$, and that $x_1 > x_2 > \cdots > x_{n-2} > 0$. Then $z = 1$ and $x_1 = 1^-$, and both sides of the implication evaluate to 1. Therefore, every interpolant must map 1^- to 1. The only functions from F with this property are 1 and $\neg\neg x_1$.

On the other hand, consider the case where $x_1 = 0^+$ and $z = \frac{2}{n-1}$. Then $A_n \equiv x_1$ becomes 0^+, because x_1 cannot be larger than $x_2 > 0$. On the other hand, the right hand side of ϕ_n becomes $\frac{2}{n-1}$. Therefore, the interpolant must map 0^+ to either 0^+ or $\frac{2}{n-1}$. Since the only functions with this property are x_1 and $x_1 \vee \neg x_1$, there can be no interpolant. □

Lemma 12. *Let $\phi(x)$ be a formula of $G_m + C$ in one variable x, and let $[a, b]$ be an interval of the truth values of G_m, such that $C \cap [a, b]$ is empty. Then $\phi(x)$ is either constant on $[a, b]$, such that $\phi(a) \in C$, or $\phi(x) = x$ for all $x \in [a, b]$.*

Proof. By induction on the formula structure. For atomic formulas, the statement is trivially true. Suppose that it holds for formulas $\phi(x), \psi(x)$, and consider $\delta(x) = \phi(x) \longrightarrow \psi(x)$. If both $\phi(x)$ and $\psi(x)$ are projections on $[a, b]$, then $\delta(x)$ is constant 1. If $\phi(x)$ and $\psi(x)$ are constant on $[a, b]$, then $\delta(x)$ is constant, too.

If $\phi(x) = d$ is constant and $\psi(x) = x$ on $[a, b]$, then d is either below $[a, b]$ or above $[a, b]$. In the first case, $d \leq x$ for $x \in [a, b]$, and thus, $\delta(x) = 1$. In the second case, $d > x$ for $x \in [a, b]$, and therefore, $\delta(x) = x$.

Finally, suppose that $\phi(x) = x$ and $\psi(x) = d$ on $[a, b]$. Again, if d is below $[a, b]$, then $\delta(x) = 1$, otherwise $\delta(x)$ equals d.

The induction steps for conjunction, negation and disjunction are easier, and left to the reader. □

Lemma 13. *If a finite-valued Gödel logic has interpolation, then it is semidense.*

Proof. We prove the contraposition of the statement. Suppose that the basic logic is $G_m + C$, such that C is not semidense, and let $c, d \in C$ be two truth constants, which have $n - 2$ intermediate values, but no intermediate truth constants, such that $n \geq 4$. Since C is not semidense, such a pair must exist.

We introduce a mapping of formulas which is intended to relativize the formula ϕ_n from the proof of Lemma 11 to the interval $[c, d]$.

$$a^{[c,d]} = \max(c, \min(a, d)) \tag{53}$$

$$(\phi \mathbin{\&} \psi)^{[c,d]} = \phi^{[c,d]} \mathbin{\&} \phi^{[c,d]} \tag{54}$$

$$(\phi \vee \psi)^{[c,d]} = \phi^{[c,d]} \vee \phi^{[c,d]} \tag{55}$$

$$(\phi \longrightarrow \psi)^{[c,d]} = \min(\phi^{[c,d]} \longrightarrow \psi^{[c,d]}), d) \tag{56}$$

Consider a formula ϕ in G_n, and the formula $\phi^{[c,d]}$ in $G_m + C$. Then every assignment to the variables of ϕ is mimicked by an assignment to the variables in $\phi^{[c,d]}$, such that c plays the role of 0, and d plays the role of 1.

On the other hand, every assignment of a value less than c is turned into c by virtue of the $\max$ operation in $\phi^{[c,d]}$, and analogously for d and $\min$.

We conclude that the formula $\phi_n^{[c,d]}$ is a tautology in $G_m + C$.

Since $\phi_n^{[c,d]}$ behaves like ϕ_n, an interpolant must map d^- to d and c^+ either to c^+ or the successor of c^+.

By Lemma 12, no such truth function exists. □

This concludes the proof of Theorem 11.

Corollary 11. – *G_3 has interpolation.*
– *$G_i, i \geq 4$ does not have interpolation.*

With the $\triangle$ operator, the situation becomes much easier. The reason is that the $\triangle$ operator allows to distinguish all truth values.

Theorem 12. *The following are equivalent:*

– *$G_{n+1} + \triangle + S$ has interpolation.*
– *S is dense.*

Proof. Trivially, density implies interpolation, as explained in Section 1. Now suppose that in a logic $G_{n+1} + \triangle + S$, S is not dense, and consider the formula

$$\Psi(x_1, \ldots, x_n) = (x_1 > 0) \ \& \ (x_2 > x_1) \ \& \ \ldots \ \& \ (x_n > x_{n-1})$$

(Recall from Section 2.2 that the crisp relation $>$ is definable in $G_{n+1} + \triangle$.) It is obvious, that Ψ is true if and only if $x_1 = 0^+$, $x_2 = 0^{++}$ etc. Let i be a truth value not in S, and consider the formula

$$\min(\Psi(x_1, \ldots, x_n), x_i) \longrightarrow \max(\neg\Psi(y_1, \ldots, y_n), y_i) \qquad (57)$$

It is easy to see that that left hand side of formula 57 obtains only values $0, i$, and the right hand side only values $i, 1$. Therefore, the unique interpolant is i. Since i is not contained in S, the result follows. □

4.1.1. Interpolation by instances We have seen in Section 3.1 that in contrast to classical logic, the interpolant of infinite-valued Gödel cannot be obtained as a disjunction of formula instances; i.e., the interpolant is necessarily more complicated than in classical logic.

For G_3, however, we obtain an interpolant in the classical way. Although interpolation in the case of G_3 follows from Lemma 10, we close this section with a direct proof which is completely algebraic and of independent interest.

Lemma 14. *Let ϕ be a formula in G_3, and let a be a new propositional variable. For every substitution $\rho : Var(\phi) \to \{0, 1, a\}$ it holds that $\phi\rho(\frac{1}{2}) \leq \phi\rho(1)$.*

Proof. Every G_3 formula in one variable a is equivalent to a formula in $F = \{1, 0, a, \neg a, \neg\neg a, \neg a \vee a\}$. The reader can easily check from the table that F is closed under truth functions .

1	0	a	$\neg a$	$\neg\neg a$	$\neg a \vee a$
1	0	0	1	0	1
1	0	$\frac{1}{2}$	0	1	$\frac{1}{2}$
1	0	1	0	1	1

Since $Var(\phi\rho) = \{a\}$, it suffices to check the claimed property by comparing the last two lines of the table. □

Consider like above a tautology

$$A(\mathbf{x}, \mathbf{y}) \longrightarrow B(\mathbf{x}, \mathbf{z}) \tag{58}$$

where we try to express the supremum $\sup \left\{A(\mathbf{x}, \mathbf{y}) | \mathbf{y} \in \{0, \frac{1}{2}, 1\}^{|\mathbf{y}|}\right\}$ or, in terms of fuzzy quantifiers, $\exists \mathbf{y} A(\mathbf{x}, \mathbf{y})$.

Let $\Gamma(A(\mathbf{x}, \mathbf{y}))$ denote the set of all formulas obtained from $A(\mathbf{x}, \mathbf{y})$ by consistently replacing the variables $\mathbf{y}$ by variables from $\mathbf{x}$ and $0, 1$.

Theorem 13. *G_3 has interpolation. In particular*

$$\exists \mathbf{y} A(\mathbf{x}, \mathbf{y}) \ = \bigvee_{I(\mathbf{x}) \in \Gamma(A(\mathbf{x}, \mathbf{y}))} I(\mathbf{x}) \tag{59}$$

Proof. By considering all possible assignments σ to the $\mathbf{x}$. If none of the $\mathbf{x}$ is assigned $\frac{1}{2}$, then by virtue of Lemma 14, every assignment to the $\mathbf{y}$ can be majorized by an assignment not involving $\frac{1}{2}$. Since all such assignments are in $\Gamma(A(\mathbf{x}, \mathbf{y}))$, the formula is correct.

Otherwise, there is a variable x among the $\mathbf{x}$, such that $\sigma(x) = \frac{1}{2}$. Therefore, the outcome of every possible assignment to the $\mathbf{y}$ can be found in $\Gamma(A(\mathbf{x}, \mathbf{y}))$. □

4.2. Łukasiewicz logic

For Łukasiewicz logic, the situation is different, because in contrast to Gödel logic, its underlying algebra is not locally uniformly finite. On the other hand, it is sufficient to consider left interpolation only, since right interpolation then follows from the symmetry of Ł.

It is well-known that for two natural numbers a, b, their greatest common divisor $gcd(a, b)$ is expressible in the form $ka + lb$, $l, b \in \mathbb{Z}$. The following folklore version of *Euclid's algorithm* shows that cut-off subtraction is sufficient to express the gcd of a, b.

```
function gcd(a,b)
      if a=b then return a.
      if a > b then return gcd(a∸b,a)
        else return gcd(a,b∸a).
```

Lemma 15. *Let* $D = \{\frac{p_1}{q_1}, \frac{p_2}{q_2}, \cdots, \frac{p_n}{q_n}\}$ *where* $gcd(p_i, q_i) = 1, q_i \neq 1$ *for all* $1 \leq i \leq n$. *Then* D *generates the subalgebra* $Ł_{l+1}$ *of Ł, where* l *is the least common multiple (*lcm*) of the denominators* $\{q_1, \ldots, q_n\}$.

Proof. Consider $\frac{p_i}{q_i}$. Using Euclid's algorithm, we obtain an equation of the form

$$1 = (q_i \dot{-} (\ldots)) \tag{60}$$

containing only q_i, p_i and $\dot{-}$. Dividing equation 60 by q_i, we obtain a definition of $\frac{1}{q_i}$ in terms of $\frac{q_i}{q_i} = 1$ and $\frac{p_i}{q_i}$.

It remains to show that we can express $\frac{1}{l}$, since then all other truth values of $Ł_{l+1}$ can be expressed by addition $\dot{+}$. (Recall from Section 2.1, that addition $\dot{+}$ is definable in Ł.)

Consider $\frac{1}{q_1}$ and $\frac{1}{q_2}$, and let $l = \mathrm{lcm}(q_1, q_2)$. Then

$$\frac{1}{q_1} \dot{+} \frac{1}{q_2} = \frac{q_1 + q_2}{q_1 q_2} = \frac{\frac{q_1+q_2}{\mathrm{gcd}(q_1,q_2)}}{\frac{q_1 q_2}{\mathrm{gcd}(q_1,q_2)}} = \frac{\frac{q_1+q_2}{\mathrm{gcd}(q_1,q_2)}}{l} \tag{61}$$

Since $\mathrm{gcd}(\frac{q_1+q_2}{\mathrm{gcd}(q_1,q_2)}, l) = 1$, Euclid's algorithm gives us a formula for $\frac{1}{l}$ like in formula 60. This proves the lemma for the case of $n = 2$. Since $\mathrm{lcm}(q_1, \ldots, q_n) = \mathrm{lcm}(q_n, \ldots \mathrm{lcm}(q_3, \mathrm{lcm}(q_1, q_2)) \ldots)$, an easy induction yields the result for arbitrary n. □

Theorem 14. *Let* S *be a (possibly empty) subset of the truth values for* $Ł_k$. *Then the following are equivalent*

1. $Ł_k + S$ *has interpolation.*
2. *All truth values of* $Ł_k$ *are explicitly definable in* $Ł_k + S$.
3. 0^+ *is explicitly definable in* $Ł_k + S$.
4. $gr(S) = 1$.
5. $Ł_k + S$ *has uniform interpolation.*

Proof. $1 \Longrightarrow 2$: Let $v = \frac{p}{q}$ be some $Ł_k$ truth value, then we consider the functions $\max(px, 1-(q-p)x)$ and $\max(0, \min(py, 1-(q-p)y))$. By McNaughton's theorem, there are $Ł$ formulas $\phi(x)$ and $\psi(y)$ which realize those functions. A short calculation shows that $\phi(x) \longrightarrow \psi(y)$ is a tautology with unique interpolant v in $Ł$, and therefore, in $Ł_k + S$. Since by assumption $Ł_k + S$ has interpolation, we conclude that v is expressible in $Ł_k + S$.

$2 \Longrightarrow 5$: Consider $\exists x.A(x, \mathbf{y})$, and let $t_1, \ldots, t_k$ be the terms for the truth values of $Ł_k$. Since $\max$ is definable in every $Ł_k$, the supremum can be computed by $\bigvee_{1 \leq i \leq n} A(t_i, \mathbf{y})$.

$5 \Longrightarrow 1$: Trivial.

$3 \Longrightarrow 2$: Trivial, because all truth values can be defined as iterated addition of 0^+.

$4 \Longrightarrow 3$: If $gr(S) = 1$ then we can apply Euclid's algorithm to obtain a sequence of subtractions which defines $\frac{1}{k-1} = 0^+$.

$2 \Longrightarrow 4$: Suppose that for each truth value v there is a formula t_v which defines v explicitly, yet $gr(S) \neq 1$. By elementary properties of $\gcd$ it follows that there is no linear combination of the elements in S which yields 0^+.

Now consider t_{0^+}. Using the definitions of the truth functions, t_{0^+} can be written as an arithmetical expression using just subtraction and addition, because all occurrences of $\max, \min$ can be eliminated for constant arguments. Therefore, t_{0^+} describes a linear combination for 0^+. Contradiction. □

Corollary 12. *The results of Theorem 14 remain true, if the logic contains* $\triangle$.

Proof. To see this, we observe that the addition of $\triangle$ does not change the set of explicitly definable truth values. Then, the other equivalences follow in anology to the proof of Theorem 14. □

Corollary 13 ([11]). $Ł_k, k \geq 3$ *does not have interpolation.*

5. The number of intermediate logics with interpolation

We know from the preceding sections that no finite Gödel logic but G_3, and no finite Łukasiewicz logic has interpolation. On the other hand, if we add truth functions for all truth values, then the logic has interpolation because maximum and minimum can be expressed.

The surprising fact is that the number of intermediate logics is closely connected to Fibonacci numbers.

Let S be a set of additional truth constants, i.e., for $n+1-$valued logics, $S \subseteq \{\frac{1}{n}, \ldots, \frac{n-1}{n}\}$, and let $\overline{S}$ denote the gaps between the truth values, i.e.

$$\overline{S} = \{x - y \mid x \in S \cup \{0\}, x = \min\{z \mid z > y\}\}$$

For a set S of rationals, and an integer n, let nS denote $\{ns \mid s \in S\}$.

For a finite-valued logic $\mathcal{L}$, let $\hat{\mathcal{L}}$ denote the logic with additional truth constants for all truth values.

Theorem 15. *1. There are exactly F_{n+1} intermediate logics between G_{n+1} and $\hat{G}_{n+1}$, where F_{n+1} is the $n+1$st Fibonacci number.*

2. There are exactly $\binom{|S|+1}{n-|S|-1}$ intermediate logics $G_{n+1}+S$ between G_{n+1} and $\hat{G}_{n+1}$.

Proof.

1. The result is obtained by the following bijection of truth values to finite strings: A logic $G_{n+1} + S$ is represented by a string $w \in \{0,1\}^{n+1}$ such that every position in the string indicates if the corresponding truth constant is in the language. Let us call w the *characteristic string* of $G_{n+1} + S$.
   Thus, for example, $G_5 + \{\frac{1}{4}\}$ is represented by 11001. By our characterization from Theorem 11 it follows that a logic $G_{n+1}+S$ has interpolation if and only if its characteristic string is in the language

   $$I = \{10, 1\}^*1$$

   because a 0 symbol can appear only immediately after a 1 symbol, since otherwise 00 would violate the condition on the size of the gaps. Therefore, the number of intermediate logics is equal to

   $$f_n = |\{0,1\}^{n+1} \cap I| = |\{0,1\}^n \cap \underbrace{\{10,1\}^*}_{I'} \tag{62}$$

   The last equation follows from the fact that the symbol 1 in the end is fixed, and cannot contribute to the overall number.
   The crucial observation is that every string in I' can be written as the concatenation of $A = 10$ and $B = 1$ in a *unique way*. This immediately follows from the fact that the occurrences of 0 fix the positions of the A strings, and hence all other positions are occupied by B strings:

   $$1\;1\;\underbrace{1\;0}_{A}\underbrace{1\;0}_{A}\;1\;\underbrace{1\;0}_{A}$$

   Obviously, $f_1 = 1$, and $f_2 = 2$. Let $n \geq 3$. A string of size n can either start with A or B. In the first case, there are f_{n-2} possibilities for the rest of the string, in the other case there are f_{n-1} possibilities.

Thus, we obtain the recursive definition

$$f_1 = 1, f_2 = 2, \ f_{n+2} = f_{n+1} + f_n \tag{63}$$

Therefore, f_n is equal to F_{n+1}, the Fibonacci sequence.

2. Let $|S| = l-1$. To obtain the result for $l-1$ additional truth constants, we again consider the characteristic string of $G_{n+1} + S$, with the additional restriction, that it contains $l-1+2 = l+1$ occurrences of 1; the additional 2 truth values are for truth and falsity. Like above, it is sufficient to consider the string w composed of the first n symbols of the characteristic string. Like above, w can be written as the concatenation of $A = 10$ and $B = 1$ in a unique way. Let $\#A$ be the number of As in this concatenation, and $\#B$ the number of Bs. Then there are $\binom{\#A + \#B}{\#A}$ possible strings, and it remains to calculate $\#A$ and $\#B$ to obtain the result.
   Let $k = n - l$ be the number of 0 symbols in the string. Since every occurrence of 0 is invoked by an A, it follows that $\#A = k = n - l$, $\#B = n - |A|k = n - 2k = 2l - n$, and therefore $\#A + \#B = l$. This concludes the proof. □

Remark. From the proof of the Theorem, we immediately obtain that

$$|S| \leq n \leq 2|S|$$

is a necessary condition for the interpolation of $G_{n+1} + S$.

Corollary 14. $F_{n+1} = \sum_{\lceil \frac{n}{2} \rceil \leq l \leq n} \binom{l}{n-l}$

Theorem 16. *There are* $R_k(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \binom{d-1}{k-1}$ *intermediate logics between* $Ł_{n+1}$ *and* $\hat{Ł}_{n+1}$ *with exactly* k *additional truth constants.*

Proof. From Theorem 14 we know that

$$\gcd(nS \cup \{n\}) = 1$$

characterizes the logics $Ł_i + S$ with the interpolation property. Evidently, this condition is equivalent to

$$\gcd(n\overline{S}) = 1$$

and thus we obtain very similar characterizations of interpolation in Gödel and Łukasiewicz logic. It is shown in [5,4] that there are $R_k(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \binom{d-1}{k-1}$ sets S with this property, and thus the result follows. □

Corollary 15. *There are* $\sum_{d|n} \mu\left(\frac{n}{d}\right) 2^{d-1}$ *intermediate logics between* $Ł_{n+1}$ *and* $\hat{Ł}_{n+1}$.

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