Arch. Math. Logic (2000) 39: 145–154

Archive for
Mathematical
Logic

On relative enumerability of Turing degrees

Shamil Ishmukhametov

Department of Mathematics, 42 Tolstoy st., Ulyanovsk State University, 432700, Ulyanovsk, Russia. e-mail: ishm@mci.ulsu.ru

Received: 17 January 1996

Abstract. Let **d** be a Turing degree, R[**d**] and Q[**d**] denote respectively classes of recursively enumerable (r.e.) and all degrees in which **d** is relatively enumerable. We proved in Ishmukhametov [1999] that there is a degree **d** containing differences of r.e.sets (briefly, d.r.e.degree) such that R[**d**] possess a least element **m**>**0**. Now we show the existence of a d.r.e. **d** such that R[**d**] has no a least element. We prove also that for any REA-degree **d** below **0**′ the class Q[**d**] cannot have a least element and more generally is not bounded below by a non-zero degree, except in the trivial cases.

Introduction

The recursively enumerable (r.e.) degrees play an important role in Mathematical Logic, since they are exactly the degrees in which axiomatizable theories lie. Relativising this class to some oracle A we obtain so called A-REA (or **a**-REA, we replace A by its degree) degrees. The hierarchy of **a**-REA degrees was defined in Jockusch and Shore [1984]. Clearly, if a degree **d** is **a**-REA, then it is **b**-REA for every **b**, **a**<**b**, so the problem arises to find, given a degree **d**, a least degree **a** such that **d** is **a**-REA. If such **a** exists it plays for the degree **d** the same role as **0** for the r.e.degrees, namely, any set A ∈**a** contains a minimum of information necessary to enumerate some set from **d**.

We say, a degree **d** is non-trivially REA (or simply REA), if **d**>**0** and there is a degree **b**<**d** in which **d** is relatively enumerable.

The distribution of REA-degrees is not studied completely. It follows from the Friedberg Inversion Theorem that any degree greater than or equal to **0**′ is REA. By a result of Posner [1972] each high degree below **0**′ is enumerable in a low degree, so all degrees, located sufficiently high, are REA. On the other hand, since the r.e. degrees are REA, such degrees exist in all jump classes. The most important latest result concerning REA degrees is Cooper's characterization of the relation of relative enumerability in terms of the relation $\leq$ (Cooper [1990]):

A degree **d** is relatively enumerable in **c**, if and only if for any degrees **a** and **b** greater than **c**, **d**$\cup$**a** is splittable over **a** avoiding **b**.

In Ishmukhametov [1999] we studied the class R[**d**] for a d.r.e. degree **d**>**0**. It is known that d.r.e. degrees form a proper subclass of 2-REA degrees (see Jockusch and Shore [1984]). We proved that for some **d** R[**d**] is an upper cone with an r.e. base **a**>**0**. We proved also, uniting our method with the Cooper and Yi construction of isolated d.r.e. degree [1995] that there is a d.r.e. degree **d** which is enumerable just in one r.e. degree **a** below **d**.

We conjectured that this **a** bounds not only r.e. but all degrees in which **d** is recursively enumerable. Our conjecture was based on a proposition (see Ishmukhametov [1999) that, given a d.r.e. set D, there exists a least degree **a** in which D is enumerable (in the class of all degrees).

Now we refute our conjecture and show that there is no a REA-degree **d** below **0**$'$ such that the class Q[**d**] possess a least element greater than **0**. Moreover, for no such **d** Q[**d**] is bounded below by a non-zero degree. Namely, we show that, given a REA degree **d** and **a**, **0**<**a**<**d**, there exists a degree **c** in which **d** is enumerable and **a**$\not\leq$**c**. By the relativized Sacks Splitting Theorem (see Soare [1987], p.124), **d** is splittable over **c** avoiding **a** (into some $\mathbf{d}_0$ and $\mathbf{d}_1$). At least one of $\mathbf{d}_i$ is incomparable with **a**. So we obtain as a corollary to our theorem that, given a REA-degree **d**$\leq$**0**$'$, for any **a**, **0**<**a**<**d**, there exists a degree incomparable with **a**.

For other results concerning this subject see Cooper and Yi [1995], Arslanov, Lempp and Shore [1996a], [1996b], La Forte [1995], [1996], Arslanov, La Forte, and Slaman [1998]. We follow notations and terminology of Soare [1987].

1. Studying class R[d]

Let D be a d.r.e. set, and a recursive approximation of D denoted by $\{D^s\}$ is given. $D^s(x)$ is a current (at stage s) guess of $D(x)$. We assume that for any x $D^0(x) = 0$, and there are no more than two different s such that $D^s(x) \neq D^{s+1}(x)$. Assume also that for any s there is no more than one x such that $D^s(x) \neq D^{s+1}(x)$. Denote by $[D]$ the set of all numbers such that there is $s,$ $D^s(x) = 1$ and by $s(x)$ a stage at which x enters $[D]$.

Definition. $B[D] = \{s(x) : x \in [D] - D\}$ is the associated set for D.

Clearly, $B[D]$ is r.e. and recursive in D. It is known also (see Ishmukhametov [1999]) that the degree of the set B is a least degree in which D is r.e. Below we show the latter cannot be said about all d.r.e.sets from the degree of D.

Theorem 1. *There is a d.r.e.degree* **d**, *for which R[***d***] has no a minimum element.*

Proof. We shall build d.r.e.sets D and $\{D_e\}_{e\in\omega}$ such that if D is T-equivalent to a 2-REA set $A\oplus W_k^A$ then $D_e \equiv_T D$ and $A \not\leq_T B_e$, where B_e is the associated set for D_e. We set the following list of requirements:

$$N_{e,j}:\ (D=\Phi_m^{A\oplus W_k^A}\wedge A\oplus W_k^A=\Phi_n^D)\rightarrow(D_e\equiv_T D\ \wedge\ A\neq\Phi_j^{B_e}),$$

where $e=<m,n,k,i>,\ A=W_i,\ \{\Phi_j,W_j\}_{j\in\omega}$ is a recursive enumeration of all partial recursive functionals and r.e.sets.

Define length functions as follows (we drop everywhere an index s):

$$\begin{aligned} l(e,s)&=\max\{x:(\forall y<x)D(y)=\Phi_m^{(A\oplus W_k^A)\restriction\tau_y}(y)\\ &\qquad\wedge(A\oplus W_k^A)\restriction\tau_y^*=\Phi_n^D\restriction\tau_y^*\},\\ \text{where } \tau_y^*&=\max\{\tau_y,use\{W_k^A\restriction\tau_y\}\},\\ l(e,j,s)&=\max\{x:(\forall y<x)\,A(y)=\Phi_j^{B_e}(y)\},\\ \text{where } e&=<m,n,k,i>,\ A=W_i. \end{aligned}$$

Assume that all parameters of computations defined at a stage s do not exceed s. We begin with a description of the basic module for a requirement $N_{e,j}$ in isolation.

Basic module

(1) Choose a pair $x,x+1$ of unused witnesses.
(2) Wait for a stage $s',\ l(e,s')>x+1$, then enumerate x into both D and D_e. Restrain D up to s'.
(3) Wait for a stage $s''>s'$, $l(e,s'')>x+1$, then restrain D up to s''.
(4) Wait for a stage s''', $l(e,j,s''')>s''$. Remove x from D, enumerate $x+1$ into D_e , and restrain B_e up to s'''.

Define $D_e(y)$ for all y which are not witnesses of $N_{e,j}$- requirements, $j\in\omega$, equal to $D(y)$.

Clearly, each $N_{e,j}$ acts finitely often and restrains a finite interval so the cooperation of different $N_{e,j}$ can be organized in the usual way.

Now we verify the Basic Module.

Lemma 1.1. *Each* $N_{e,j}$ *is met.*

Proof. Fix $k =< e, j >$ and assume the Lemma for all $k' < k$. Assume, the left part of $N_{e,j}$ holds, and let s_0 be a least stage after which no requirement with higher priority acts.

Consider witnesses x and $x + 1$ of $N_{e,j}$ defined after stage s_0. Since, $\liminf l(e, s) = \infty$, each step 1,2, and 3 of the Basic Module occurs. If $N_{e,j}$ fails then $\liminf l(e, j, s) = \infty$, and step 4 also occurs.

We shall prove, that $A^{s'''} \upharpoonright s'' \neq A \upharpoonright s''$. $(*)$

Since $D(x) = \Phi_m^{A \oplus W_k^A}(x)$ at stages s', s'' and s''', then when we put x into D, and then remove it from there, $\Phi_m^{A \oplus W_k^A}(x)$ subsequently takes values 0,1,0. This means, $A \oplus W_k^A \upharpoonright \tau_x$ changes between stages s' and s'', (τ_x is the use of computation $\Phi_m^{A \oplus W_k^A}(x)$ at stage s') and after stage s'' returns its old value because $\Phi_m^D \upharpoonright \tau_x^*$ returns the old value after removing x from D. So there is some $z < \tau_x$, $A \oplus W_k^A(z)$ changes between stages s' and s'', and then returns. Since, A is r.e., then $W_k^A(z')$ changes for $z' = (z - 1)/2$. If the first value of $W_k^A(z')$ was 1, then the change of $W_k^A(z')$ is possible only if $A \upharpoonright \tau_x^*$ changes. But $A \upharpoonright \tau_x^*$ is equal to $\Phi_m^D \upharpoonright \tau_x^*$ at stages s' and s''' and cannot change between stages s' and s'''.

So the first value $W_k^A(z')$ was 0, and z was enumerated in W_k^A between stages s' and s''. In order to return $W_k^A(z')$ to 0 it is necessary to change the oracle $A \upharpoonright s''$, and $(*)$ holds.

Since $A \upharpoonright s'' = \Phi_j^{B_e^*} \upharpoonright s''$ at stage s''', after $A \upharpoonright s''$ change, we get the disagreement $A \upharpoonright s'' \neq \Phi_j^{B_e^*} \upharpoonright s''$, and $N_{e,j}$ is met.

This finishes the proof.

Lemma 1.2. *If* $\liminf l(e, s) = \infty$, *then* $D_e \equiv_T D$.

Proof. If x is not a witness of some $N_{e,j}$-requirement, then $x \in D \leftrightarrow x \in D_e$.

Assume, x and $x + 1$ are witnesses of a requirement $N_{e,j}$. Wait for a stage s', at which $l(e, s)$ exceeds $x+1$. At stage s', either x is put into D and D_e or it is restrained by a higher priority requirement. In the latter case, both x and $x + 1$ are not ever put into D and D_e. Assume the first case occurs. By the construction, if x is ever removed from D, then $x + 1$ is enumerated into D_e. Therefore, $x \notin D \leftrightarrow x + 1 \in D_e$.

The proof of the Theorem follows immediately from the lemmas.

2. Studying class Q[d]

Now we study, for a given Δ_2^0 degree **d**>**0**, the class Q[**d**] of all degrees, less **d**, in which **d** is r.e. Our main result concerning such classes we divide into two parts. First we show that for any proper d.r.e degree **d** and any r.e.

a<**d** there is an ω-r.e. **c** such that **d**=**c**-REA, and **a**$\not\leq$**c**. Then we generalize this result to all REA degrees below **0**$'$

Let D be a d.r.e. set. Fix a recursive enumeration of D, and let $B[D]$ be the set associated with D, and $s(x)$ a stage, at which x enters $[D]$.

Definition 2.1. *Number $x \in D$ is called minimal (under the given enumeration of D), if*

$$(\forall y < x) y \in D \to y \in D^{s(x)}.$$

Denote by $M[D]$ the set of all minimal elements of D. By definition, $M[D] \subseteq D$.

Lemma 2.1. *For any set E such that $M[D] \subseteq E \subseteq [D]$,* $D \leq_T E \oplus B[D]$.

Proof. To compute $D(x)$ for $x \in \omega$, find a least s_0 such that for any $y \leq y$ $y \in E \to s(y) \leq s_0$. If $x \notin D^{s_0}$, then $x \notin D$. Assume $x \in D^{s_0}$. Then

$$x \in D \leftrightarrow s(x) \notin B[D].$$

This finishes the proof.

Theorem 2. *Given a properly d.r.e.* **d** *and a r.e.* **a** *such that* **0**<**a**< **d**, *there exists an ω-r.e.* **c**<**d** *such that* **d** *is r.e. in* **c** *and* **a**$\not\leq$**c**.

Proof. Let A be a r.e.set from **a**, D a d.r.e.set from **d**. Fix recursive enumerations of sets A and D. Assume, the enumeration of D satisfies requirements under which lemma 2.1 holds. B is the associated set for D, **b** is its degree. We assume that **a**$\leq$**b**, otherwise, the theorem is established. The latter allows us to assume that there is a recursive function h such that each time when a number n is enumerated in A $h(n)$ enters B.

We construct an ω-r.e.set $E \subset [D]$ and define $C = \{s(y) : y \notin E\}$ (where as before $s(x)$ is a stage when x enters $[D]$). Then, E is C-REA. To guarantee $D \leq_T E \oplus B$ we build E satisfying the condition $Min(D) \subseteq E$.

To ensure the backwards reducibility, both sets E and C are permitted by numbers entered D. This also ensures that E and C are ω-r.e.

To meet $A \not\leq_T C$ we set an infinite list of requirements:

$$N_j : A \neq \Phi_j^C,\ j \in \omega.$$

Define the length function as follows:

$$l(j,s) = \max\{x : (\forall y < x)\, A(y) = \Phi_j^C(y) \text{ at stage s}\}$$

Our general strategy for satisfying requirements $N_j, j \in \omega$, is based on the next lemma:

Lemma 2.2. *Let $A = \Phi_j^C$ and $\{C^s\}_{s\in\omega}$ be a recursive approximation for C. For any natural n there is a triple $\langle x, s', s''\rangle$ such that $C^{s'} \upharpoonright n = C^{s''} \upharpoonright n = C \upharpoonright n$, and $\Phi_j^{C^{s'}}(x)\downarrow \neq \Phi_j^{C^{s''}}(x)\downarrow$.*

Proof. If the lemma fails, then there is a stage s_0 such that any value $\Phi_{j,s}^{C^s}(x)$, computed after this stage, is valid and A is recursive contrary to assumptions of the theorem.

Below we consider a module for a requirement $N_j,\ j \in \omega$, in isolation.

Basic module

The basic module consists of a list of procedures. Procedure $\langle 0\rangle$ starts first. Each next procedure is started by the previous one. Procedure $\langle n\rangle$ defines a value $\Sigma^B(n)$ of a recursive functional Σ^B in such a way that if the procedure fails to satisfy N_j then $D(n) = \Sigma^B(n)$. Fix a number n and consider instructions for procedure $\langle n\rangle$:

(1) If $D(n)$ changed twice till this stage, then it cannot change more. In this case define $\Sigma^B(n) = 0$ with use $\sigma(n) = 0$. Start procedure $\langle n+1\rangle$.

If the current value $D(n)$ is 1, then define $\Sigma^B(n) = 1$ with use $\sigma(n) = s(n)+1$. Restrain $C(s(n))$. Start procedure $\langle n+1\rangle$.

If later $D(n)$ takes value 0, then $s(n)$ enters B and $\Sigma^B(n)$ is destroyed. Redefine it then equal to 0.

If $n \notin D^{s'}, s' \leq s$, go to the next step.

(2) Wait for $D(n)$ to change or for a stage s'', at which there is a triple $\langle x_n, s', s''\rangle$ such that:

2.1. $l(j, s'') > x_n$,

2.2. $\Phi_{j,s}^{C^{s'}}(x_n) \neq \Phi_{j,s}^{C^{s''}}(x_n)$,

2.3. Numbers $x_0 = \mu x[s(x) < \phi^{s'}(x_n) \wedge C^{s'}(s(x)) \neq C^{s''}(s(x))]$ and $z_0 = \mu z[C^{s'}(z) \neq C^{s''}(z)]$ exceed all parameters of the previous procedures.

If there are several triples satisfying 1-3 choose among them triple with least x_n and $c_n = \max\{\phi^{s'}(x_n), \phi^{s''}(x_n)\}$.

If $D(n)$ changes first return to step 1, otherwise go to the next step.

(3) Let u be a least stage such that $C^t \upharpoonright c_n = C^s \upharpoonright c_n$ for $u \leq t \leq s$. Define $\Sigma^B(n) = 0$ with use $\sigma(n) = \max\{u,\ h(n)+1\}$. Restrain $C \upharpoonright \sigma(n)$ and start procedure $\langle n+1\rangle$.

(4) Wait for $D(n)$ or $B \upharpoonright \sigma(n)$ to change. If the latter occurs first, drop restraint on $C \upharpoonright \sigma(n)$, stop procedures $> n$ and return to step 2. Otherwise go to the next step.

(5) Simultaneously return all $C(y),\ y < c_n$, to their values at stage s' or s'' forcing $\Phi_j^C(x_n)$ to take the value different from $A^s(x_n)$. This action includes

also the change of appropriate $E(x)$, $s(x) = y$. Stop procedures $> n$. Go to the next step.

6. Wait for $B \upharpoonright \sigma(n)$ to change, then return to 1.

This establishes the module. Notice that due to the module we change some values $C(y)$ and $E(x)$ such that x is not an element of $Min(D^s)$. But if later $D(n)$ returns to 0, condition $Min(D^s) \subseteq E^s$ can become invalid. In order to prevent it each time when some n is removed from D we add to E all elements of $Min(D^s)$. This causes an infinte injury to the basic module. Nevertheless the following lemma ensures a satisfaction of the requirement N_j:

Lemma 2.3. *If for every $k \leq n$ $\Phi_j^C(k) = A(k)$ then $\Sigma^B(n) \downarrow = D(n)$.*

Proof. Fix some $n \in \omega$. Assume the lemma for all $k < n$. If n was enumerated in D before the stage when procedure $\langle n \rangle$ is started the proof is clear. Assume n was not in D before a stage s when a triple $\langle x_n, s', s'' \rangle$ satisfying conditions 2.1-2.3 of the basic module appears. The restraint on $C \upharpoonright \sigma(n)$ posed at stage s can be destroyed in the following cases:

1. Some requirement of higher priority acts. By inductive assumptions we can assume that this case does not occur after some stage s_0.

2. Restoring condition $Min(D) \subseteq E$ we return some number x to E and remove $s(x)$ from C. This happens if some m is removed from D. Let u be such as in the Basic Module. Clearly, this m was put into D before stage u and $s(m) < u \leq \sigma(n)$. Therefore, when the restraint on $C \upharpoonright z_0$ is destroyed $\Sigma^B(n)$ becomes undefined, and procedure $\langle n \rangle$ can be initialized.

Procedure $\langle n \rangle$ eventually finds a triple $\langle x_n, s', s'' \rangle$ such that $C^s \upharpoonright c_n = C \upharpoonright c_n$ (this is ensured by lemma 2.2 and the way of selection of such triples). At this stage we define $\Sigma^B(n) = D(n)$. If later n is enumerated in D then we get the disagreement $A(x_n) \neq \Phi_j^C(x_n)$. Since the restraint on $C \upharpoonright c_n$ is not injured, $A(n)$ must be changed and become equal to $\Phi_j^C(x_n)$. Then $h(n)$ is enumerated in B and $\Sigma^B(n)$ becomes undefined. Again, $D(n) = \Sigma^B(n)$. This finishes the proof.

Since $D \not\leq_T B$, there is a (least) n such that either $\Sigma^B(n) \uparrow$, or $\Phi_j^C(x_n) \downarrow \neq A(x_n)$.

In the full construction we use a linear ordering of requirements. Each requirement can have two possible outcomes: finite and infinite due to defined or undefined is the value $\Sigma^B(n)$ where n is a least number such that $D(n) \neq \Sigma^B(n)$. The finite case is clear. In the case $\Sigma^B(n) \uparrow$ we arrange a construction in such a way that all values $\Sigma^B(k)$, $k \geq n$, becomes undefined simultaneosly at B-true stages (i.e such stages s that $B^s \upharpoonright b_s = B \upharpoonright b_s$, b_s is a number enumerated at stage s in B). This causes a delay of $C(y)$-changes relating to $D(n)$-change but again we are able to prove $E \leq_T D$

since $C(y)$-changes happen no later than at the nearest B-true stage after $D(n)$ has changed.

We leave details to the reader. This establishes the theorem.

Theorem 3. *For any Δ_2^0 degrees* **a,d,** **a**<**d**, *if* **d** *is REA, then there is a degree* **c** *such that* **d**=**c**-*REA and* **a**$\not\leq$**c**.

Proof. Assume, **d**=**b**-REA,**b**<**d** and **a**≤**b**, otherwise the theorem is established.

Let $A \in$**a**, $B \in$**b**, $D \in$**d**, and $D = W^B$ is r.e. in B.

We shall construct sets C and $E = C - REA$ such that $E \equiv_T D$ and $A \not\leq_T C$. A list of priority requirements will be the same as in the previous theorem:

$$N_j : A \neq \Phi_j^C,\ j \in \omega.$$

Since D is r.e. in B, there is a total 1-1 function $f = \Phi^B$, with $range(f) = D$. Define a recursive function $f(s,x) = \Phi_s^{B_s}(x)$. Clearly, $f(x) = \lim f(s,x)$. Define a new set (a kind of the associated set for D) as follows:

$\hat{B} = \{\langle s{+}1, y\rangle : \exists x < s,\ f(s,x) \neq f(s{+}1,x) = y \wedge D^s(y) \neq D(y)\}$.

Clearly, $\hat{B} \leq_T B$, and D is $\hat{B}$-REA.

If $A \not\leq_T \hat{B}$, then the theorem is established. Assume, $A \leq_T \hat{B}$.

We construct C as a subset of a set $P = \{\langle s+1, y\rangle : \exists x < s,\ f(s,x) \neq f(s+1,x) = y\}$. Define for $x = \langle s, y\rangle \in P$ the value $s(y) = x$, and let $E = \{y :\ s(y) \notin C\}$.

In our construction we need to know in advance a recursive approximation for set C. Since $C \in \Delta_2^0$, some recursive approximation exists. To find it we use the Fixed Point Theorem applied to a recursive enumeration W_0^K, W_1^K, $W_2^K \ldots$, K is a creative set, of all Σ_2^0-sets. More exactly, given a natural e, we construct a Δ_2^0-set C_e. Then computing a fixed point e_0 of the function h, $(\forall e)\ W_{h(e)}^K = C_e$, we find a required $C = W_{f(e_0)}^K$. Fix a number e. Denote $C_e = W_e^K$, C is the constructed set, and C^s is its approximation defined at the end of stage s.

Define the length function as follows:

$$l(j,s) = \max\{x : (\forall y < x)\, A(y) = \Phi_{j,s}^{C^s}(y)$$

Notice that since our construction now is recursive in B, we use in definition of l real values $A(y)$ (not approximations). Let D^s be a finite subset of D enumerated for s stages (using oracle B).

Our main strategy for constructing E and C is same as earlier. We enumerate D and put into E all enumerated numbers.

Any other change of a $E(x)$ is possible only simultaneously with appropriate change of $C(y), y = s(x)$, and is permitted by a number $z < x$ entered D.

Let $K = \{k_o, k_1, k_2, ...\}$ be a recursive approximation of the creative set K. Define $C_{e,s} = W_{e,s}^{K^s}$. We assume that if $x \in W_{e,s}^{K^s}(x)$, then $\phi(x) < k_s$. We say, $C_{e,s},\ s \in \omega$, is a Σ_2^0- approximation for C_e.

Our module for requirements $N_j,\ j \in \omega$, is based on the next lemma:

Lemma 2.4. *Let $C_s,\ s \in \omega$, is a Σ_2^0- approximation for a set C, and Φ_j^C is total and non-recursive. Then for any natural n there is a triple $\langle x, s, s'\rangle'$ such that*

$$\Phi_{j,s'}^{C_{s'}}(x) \neq \Phi_{j,s''}^{C_{s''}}(x), \text{ and } C_{s'} \upharpoonright n = C_{s''} \upharpoonright n = C \upharpoonright n. \qquad (**)$$

Proof. Notice that for any x there are infinitely many s such that $C_s \upharpoonright x = C \upharpoonright x$. So for any x there are infinitely many s such that $\Phi_{j,s}^{C_s}(x) = \Phi_j^C(x)$. If the lemma is false then there is a n_0 such that for each triple $\langle x, s', s''\rangle$ satisfying $(**)$ there exists $n < n_0$ such that either $C_{s'}(n)$ or $C_{s''}(n)$ is incorrect. Then any computation $\Phi_{j,s}^{C_s}(x)$ using correct values $C(n)$ for $n < n_0$ is correct, and Φ_j^C is recursive contrary to our assumptions.

The basic module for an isolated requirement N_j consists of a list of procedures. Procedure $\langle 0\rangle$ starts first. Each next procedure is started by the previous one. Procedure $\langle n\rangle$ defines a value $\Sigma(n)$ of a functional Σ a recursive in A in such a way that if the procedure fails to satisfy N_j then $D(n) = \Sigma(n)$. All computations (except approximation for C_e) are made using oracle B. Consider instructions for a procedure $\langle n\rangle$:

(1) If $n \in D^s$, then define $\Sigma(n) = 1$ with use $\sigma(n) = 0$. Start procedure $\langle n+1\rangle$. Otherwise go to the next stage.

(2) Wait for n to be enumerated in D or for a stage s, at which there is a pair $\langle x_n, s'\rangle$, $s' < s$, such that $l(j,s) > x_n$, $\Phi_{j,s'}^{C_{e,s'}}(x_n) \downarrow \neq \Phi_{j,s}^{C^s}(x_n)$ and $z_n = \mu z[z < s \wedge s(z) \downarrow = y \ \rightarrow\ C_{e,s'}(z) \neq C^s(z)]$ exceeds n.

Additionally we assume that $u_n = \mu u[u < \phi_j^{C_{e,s'}}(x_n) \wedge C^s(u) \neq C_{e,s'}(u)]$ exceeds all parameters of the previous procedures. If $D(n)$ is enumerated first return to step 1, otherwise go to the next step.

(3) Define $\Sigma(n) = 0$. Restrain $C \upharpoonright s$ and start procedure $\langle n+1\rangle$.

(4) Wait for $D(n)$ to change, then change all $C(y), y < \phi_{j,s'}^{C_e}(x_n)$, to compat them with $C_{e,s'}(y)$. Change simultaneously $E(x)$ such that $C(s(x))$ has been changed. Notice that for any changed $E(x)$ x exceeds n. This establishes the module.

Since by the construction a change of $E(x)$ is permitted by a change of $D(n)$, $n \leq x$, then E is recursive in D. Besides, for any x, if x enters D at a stage s and $D^s \upharpoonright x = D \upharpoonright x$, then by the construction $x \in E$, and D is recursive in E.

As in the previous theorem, if procedure $\langle n \rangle$ fails to satisfy N_j then $D(n) = \Sigma(n)$ and if all procedures fail then D is recursive in B contrary to assumptions of the theorem. Since now using oracle B we avoid the infinite case there is a least n such that procedure $\langle n+1 \rangle$ is never started and N_j is satisfied in a finite number of stages. The satisfaction of all requirements can be proved by induction for $e = e_0$ being a fixed point of function h.

This establishes the theorem.

We proved that in non-trivial cases the class $Q[\mathbf{d}]$ has no a least element. We don't know, whether there is a REA-degree **d** (in particular, a d.r.e.degree **d**) such that $Q[\mathbf{d}]$ possess minimal elements.

References

1. M.M. Arslanov, S. Lempp, R.A. Shore [1996], On isolating r.e. and isolated d.r.e. degrees, in: S.B. Cooper, T. Slaman and S. Wainer, eds, Compurability, Enumerability, and Unsolvability; Directions in Recursion Theory. London Math. Soc. Lect. Note Series **224**, Cambridge, Univ. Press, 1996, pp. 61–80.
2. M.M. Arslanov, S. Lempp, R.A. Shore [1996] Interpolating d.r.e. and REA degrees between r.e. degrees, Ann. Pure and Applied Logic, **78**, 1996, pp. 29–56.
3. M.M. Arslanov, G.L. La Forte, T.A. Slaman [1998] Relative enumerability in the difference hierarchy, J. Symb. Logic v. **68** no. **2**, 1998, pp. 411–420.
4. S.B. Cooper [1990] The jump is definable in the structure of the degrees of unsolbability, Preprint series, Leeds, No.1, England.
5. S.B. Cooper and X. Yi [1995] Isolated d.r.e. degrees, Math. Log. Quart.
6. R. Epstein, R. Haass, R. Kramer [1981] Hierarchies of sets and degrees below $\mathbf{0}'$, Logic Year 1979–1980 (M. Lerman, J. Schmerl and R. Soare, editors), Lecture Notes in Mathematics, vol. **859**, Springer-Verlag, Berlin, pp.32–48.
7. C. Jockusch, R. Shore [1984] Pseudojump operators II: transfinite iterations, hierarchies, and minimal covers, J. Symbolic Logic **49**, pp. 1205-1236.
8. G. LaForte [1995] Phenomena in the n-r.e. and n-REA degrees. Ph.D. thesis, University of Michigan, Ann Arbor, 1995.
9. G. LaForte [1996] The isolated d.r.e. dgrees are dense in the r.e. degrees Math. Log. Quarterly, vol.**42**, no. **2**.
10. Sh.T. Ishmukhametov [1999]. On the r.e. predecessors of d.r.e.degrees, Archive for Math. Logic **38**, 373–386.
11. D.B. Posner [1977] High degrees, Ph.D. Dissertation, University of California, Berkeley.
12. R.I. Soare [1987] Recursively Enumerable Sets and Degrees, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, xviii+437 pp. 1987.