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# A lattice-valued set theory

**Satoko Titani**

Chubu University, Kasugai, Aichi 487-8501, Japan. e-mail: titani@isc.chubu.ac.jp

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**Abstract.** A lattice-valued set theory is formulated by introducing the logical implication $\to$ which represents the order relation on the lattice.

## 1. Introduction

The aim of this paper is to formulate a set theory on a lattice valued universe, by introducing a natural form of implication. For a given complete lattice $\mathcal{L}$, the lattice valued universe $V^{\mathcal{L}}$ is constructed in the same way as the Boolean valued universe or Heyting valued universe.

Generally, an operator $\to_*$ on a lattice is called an *implication* if it satisfies the following conditions :

(1) $(a \to_* b) = 1$ iff $a \leqslant b$
(2) $a \wedge (a \to_* b) \leqslant b$.

For example, the operator $\to_{\mathrm{I}}$ on a complete Heyting algebra(cHa) $\Omega$, defined by

$$(a \to_{\mathrm{I}} b) = \bigvee \{c \in \Omega \mid a \wedge c \leqslant b\},$$

is an implication, which is an interpretation of intuitionistic implication. The operator $\to_{\mathcal{Q}}$ on a orthomodular lattice $\mathcal{Q}$, defined by

$$(a \to_{\mathcal{Q}} b) = a^{\perp} \vee (a \wedge b),$$

is also an implication. Implication on a lattice is not necessarily unique. In fact, every complete lattice has at least the implication $\to$ defined by

$$(a \to b) = \begin{cases} 1 & \text{if } a \leqslant b \\ 0 & \text{otherwise,} \end{cases}$$

which is the strongest implication, in the sense that

$$(a \to b) \leqslant (a \to_* b) \quad \text{for any implication } \to_*.$$

We call the strongest implication the *basic implication*. The basic implication represents the order relation $\leqslant$ on the lattice.

In a sequent $\Gamma \Longrightarrow \Delta$ of Gentzen's logical system LK or LJ, $\Longrightarrow$ is a metamathematical implication, which is interpreted as the basic implication on a lattice. That is, the metamathematical implication $\Longrightarrow$ is represented by the lattice order on the Lindenbaum algebra, and hence by the basic implication on the Lindenbaum algebra.

Now we introduce the logical operator $\to$ which corresponds to the basic implication on the Lindenbaum algebra, and call it also the *basic implication*. By introducing the logical basic implication $\to$, we can formalize the metamathematics of the theory of lattice valued sets.

Equality on $V^{\mathcal{L}}$ has a close relation to implication. For example, if $\Omega$ is a complete Heyting algebra (cHA), the equality $=_{\rm I}$ and the membership relation $\in_{\rm I}$ of the intuitionistic set theory on $V^{\Omega}$ are defined so that

$$u =_{\rm I} v \leftrightarrow_{\rm I} \forall x(x \in_{\rm I} u \leftrightarrow_{\rm I} x \in_{\rm I} v) \quad \text{and} \quad u \in_{\rm I} v \leftrightarrow_{\rm I} \exists x(u =_{\rm I} x \wedge x \in_{\rm I} v)$$

hold on $V^{\Omega}$. That is, the set theory with $\to_{\rm I}, =_{\rm I}, \in_{\rm I}$ as its implication, equality and membership relation on the Heyting valued universe $V^{\Omega}$ is an intuitionistic set theory.

In [5], we extended the intuitionistic set theory by introducing the basic implication $\to$ besides the regular intuitionistic implication $\to_{\rm I}$, and showed that the metamathematics of the set theory can be discussed in itself. In this paper, we generalize the result of [5] to lattice valued set theory.

Now we fix a universe $V$ of **ZFC** as our standpoint. That is, we assume that our metamathematics is **ZFC**. Let $\mathcal{L}$ be any complete lattice in the universe $V$, and $V^{\mathcal{L}}$ be the $\mathcal{L}$-valued universe constructed in $V$. Let $\to$ be the basic implication on $\mathcal{L}$, and $\neg$ be the complement corresponding to the basic implication :

$$\neg a = (a \to 0).$$

Our *lattice-valued set theory* ( **LZFZ** ) is a set theory on $V^{\mathcal{L}}$ with the basic implication $\to$ and the corresponding negation $\neg$. The last letter Z in **LZFZ** means Zorn's lemma.

Since the order relation on $\mathcal{L}$ is expressed by the logical operator $\to$ in **LZFZ**, we can express properties of the lattice $\mathcal{L}$ in the language of **LZFZ**.

Here we denote $(1 \to a)$ by $\Box a$, that is,

$$\Box a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1. \end{cases}$$

An element $a$ of $\mathcal{L}$ is said to be $\Box$-*closed* if $\Box a = a$.

In the same way as the case of Boolean valued universe or Heyting valued universe, the external universe $V$ can be embedded in $V^{\mathcal{L}}$ as follows: For each set $u$ in our external universe $V$ we define $\check{u} \in V^{\mathcal{L}}$ by

$$\begin{cases} \mathcal{D}\check{u} = \{\check{x} \mid x \in u\} \\ \check{u}(\check{x}) = 1. \end{cases}$$

Then for sets $u, v \in V$, $[\![\check{u} = \check{v}]\!]$ and $[\![\check{u} \in \check{v}]\!]$ are $\Box$-closed and

$$u = v \iff [\![\check{u} = \check{v}]\!] = 1 \; ; \quad u \in v \iff [\![\check{u} \in \check{v}]\!] = 1.$$

A set in $V^{\mathcal{L}}$ of the form $\check{u}$ is called a *check set*. The subclass $\check{V}$ of $V^{\mathcal{L}}$ consisting of all check sets is a copy of $V$.

The introduction of the basic implication enables us to express the sentence "u is a check set", denoted by $\mathrm{ck}(u)$, in the language of the set theory **LZFZ**:

$$\mathrm{ck}(u) \iff \forall t\, (t \in u \leftrightarrow \Box(t \in u) \wedge \mathrm{ck}(u))\, .$$

This means that we can construct a copy of the universe $V$, $\mathcal{L}$, and hence a copy of $V^{\mathcal{L}}$ in **LZFZ**.

Now we say that "a sentence holds" if the sentence is provable in our metamathematics **ZFC**. So, for a sentence $\varphi$ of **LZFZ**, we mean by "$\varphi$ is valid on lattice valued universe" that

"$[\![\varphi]\!] = 1$ on $V^{\mathcal{L}}$ for all complete lattice $\mathcal{L}$" is provable in **ZFC**.

Then we prove the "completeness" of **LZFZ** in the sense that every valid sentense of **LZFZ** is provable in **LZFZ** :

**ZFC** $\vdash$ "$[\![\varphi]\!] = 1$ on $V^{\mathcal{L}}$ for all complete lattice $\mathcal{L}$" $\Longrightarrow$ **LZFZ** $\vdash \varphi$.

That is, we have the equiprovability :

**LZFZ** $\vdash \varphi$ $\iff$ **ZFC** $\vdash$ "$[\![\varphi]\!] = 1$ on $V^{\mathcal{L}}$ for all complete lattice $\mathcal{L}$",

and we can see that **LZFZ** is a set theory with double structure : one is the structure of theory of $\mathcal{L}$-valued sets, and the other is the structure of **ZFC** on the external universe $V$. In other words, **LZFZ** is a set theory with an expression of its metamathematics in itself.

The sets of natural numbers, rational numbers, and real numbers, and also ordinals are defined in the set theory **LZFZ**, and they are all check sets.

If we introduce another implication $\rightarrow_*$ in **LZFZ**, then the corresponding equality $=_*$ and the membership relation $\in_*$ can be defined in **LZFZ** by $\in$-induction:

$$u =_* v \overset{\mathrm{def}}{\iff} \forall x (x \in u \rightarrow_* x \in_* v) \wedge \forall x (x \in v \rightarrow_* x \in_* u) \; ;$$

$$u \in_* v \overset{\mathrm{def}}{\iff} \exists x (u =_* x \wedge x \in v).$$

Then we have

$$u =_* v \iff \forall x(x \in_* u \longleftrightarrow_* x \in_* v).$$

In the set theory **LZFZ** on $V^{\Omega}$, where $\Omega$ is a cHa, we can express the sentence "$\mathcal{P}(1)$ is a cHa", which holds on $V^{\Omega}$, and define the intuitionistic implication $\to_{\mathrm{I}}$ in the language of **LZFZ** by

$$\varphi \to_{\mathrm{I}} \psi \overset{\text{def}}{\iff} 0 \in \{u \in \mathcal{P}(1) \mid \varphi \wedge (0 \in u) \to \psi\}.$$

Thus, **LZFZ**+"$\mathcal{P}(1)$ is a cHa" is an extension of intuitionistic set theory. That is, axioms of intuitionistic set theory with $\to_{\mathrm{I}}$ as its implication and with $=_{\mathrm{I}}, \in_{\mathrm{I}}$ as equality and membership relation, are provable in **LZFZ**+"$\mathcal{P}(1)$ is a cHa".

Let $\to_*$ be any implication. As for check sets, the weak equality $=_*$ and the corresponding membership relation $\in_*$ are identical with $=$ and $\in$, respectively:

$$\check{u} =_* \check{v} \iff \check{u} = \check{v}\ ; \quad \check{u} \in_* \check{v} \iff \check{u} \in \check{v}.$$

It follows that natural numbers, rational numbers which are defined in the set theory with $\to_*, =_*, \in_*$ as its implication, equality and membership relation coincide with those defined in **LZFZ** , i.e. they are check sets.

We will discuss numbers in **LZFZ** in the sequel paper.

## 2. Complete lattices

Let $\mathcal{L}$ be a complete lattice. For a subset $\{a_\alpha\}_\alpha$ of a complete lattice $\mathcal{L}$, the least upper bound of $\{a_\alpha\}_\alpha$ is denoted by $\bigvee_\alpha a_\alpha$, and the greatest lower bound of $\{a_\alpha\}_\alpha$ is denoted by $\bigwedge_\alpha a_\alpha$. The smallest element and the largest element of $\mathcal{L}$ are denoted by 0 and 1, respectively.

Define the operator $\to$ on $\mathcal{L}$ by

$$(a \to b) = \begin{cases} 1 & \text{if } a \leqslant b \\ 0 & \text{otherwise.} \end{cases}$$

$\to$ is an implication, i.e,

I1 : $(a \to b) = 1$ iff $a \leqslant b$

I2 : $a \wedge (a \to b) \leqslant b$.

We call $\to$ the *basic implication* on $\mathcal{L}$. The complement corresponding to $\to$ is defined by

$$\neg a = (a \to 0)$$

**Theorem 2.1.** *For all elements* $a, b$ *of* $\mathcal{L}$,

*N1 :* $\neg 0 = 1, \quad \neg 1 = 0$
*N2 :* $a \wedge \neg a = 0$
*N3 :* $a \leqslant \neg\neg a$
*N4 :* $\neg(a \vee b) = \neg a \wedge \neg b$

**Definition 2.1** Here we denote the formula $(1 \to a)$ by $\Box a$, that is,

$$\Box a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1. \end{cases}$$

This operator $\Box$ is called the *standard globalization*.

**Theorem 2.2.** *For all elements* $a, b, a_k, b_k, c_k$ $(k \in K)$ *of* $\mathcal{L}$,

*G1 :* $\Box a \leqslant a$
*G2 :* $\neg a = \Box \neg a$
*G3 :* $\bigwedge_k \Box a_k \leqslant \Box \bigwedge_k a_k$
*G4 : If* $\Box a \leqslant b,$ *then* $\Box a \leqslant \Box b$
*G5 :* $\Box a \wedge \bigvee_k b_k = \bigvee_k (\Box a \wedge b_k)\ ; \quad a \wedge \bigvee_k \Box b_k = \bigvee_k (a \wedge \Box b_k)\ ;$
$\Box a \vee \bigwedge_k b_k = \bigwedge_k (\Box a \vee b_k)\ ; \quad a \vee \bigwedge_k \Box b_k = \bigwedge_k (a \vee \Box b_k)$
*G6 :* $\Box a \vee \neg \Box a = 1$
*G7 : If* $a \wedge \Box c \leqslant b$, *then* $\neg b \wedge \Box c \leqslant \neg a$.
*G8 :* $(a \to b) = \bigvee \{c \in \mathcal{L} \mid c = \Box c,\ a \wedge c \leqslant b\}$

The following theorem follows from I1–I2, N1–N4 and G1–G8.

**Theorem 2.3.** *Let* $a, b \in \mathcal{L}$ *and* $\{a_k\}_{k \in K}$, $\{b_k\}_{k \in K} \subset \mathcal{L}$. *Then*

(1) *If* $a \leqslant b$ *then* $\Box a \leqslant \Box b$
(2) $\Box (\bigwedge_k a_k) = \bigwedge_k \Box a_k$
(3) $\Box a = \Box \Box a$
(4) $\bigwedge_k \Box a_k = \Box \bigwedge_k \Box a_k$
(5) $\bigvee_k \Box a_k = \Box \bigvee_k \Box a_k$
(6) $\Box (a \to b) = (a \to b)$.
(7) $(a \to b) \leqslant (\neg b \to \neg a)$
(8) *If* $\Box a \wedge b \leqslant c$ *then* $\Box a \leqslant (b \to c)$

We denote $\neg \Box \neg$ by $\Diamond$. Then we have

**Theorem 2.4.** *Let* $a, b \in \mathcal{L}$ *and* $\{a_k\}_{k \in K} \subset \mathcal{L}$.

(1) $a \leqslant \Diamond a$
(2) *If* $a \leqslant \Box b$ *then* $\Diamond a \leqslant \Box b$
(3) $\Diamond \bigvee_k a_k = \bigvee_k \Diamond a_k$
(4) $\Diamond (\Box a \wedge b) \leqslant \Box a \wedge \Diamond b$

## 3. $\mathcal{L}$-valued universe $V^{\mathcal{L}}$

Let $\mathcal{L}$ be a complete lattice with the basic implication $\rightarrow$ and the corresponding negation $\neg$. $\mathcal{L}$-valued universe $V^{\mathcal{L}}$ is constructed by induction:

$$V^{\mathcal{L}}_\alpha = \{u \mid \exists\beta<\alpha\, \exists\mathcal{D}u \subset V^{\mathcal{L}}_\beta (u : \mathcal{D}u \to \mathcal{L})\}$$
$$V^{\mathcal{L}} = \bigcup_{\alpha\in\mathrm{On}} V^{\mathcal{L}}_\alpha$$

The least $\alpha$ such that $u \in V^{\mathcal{L}}_\alpha$ is called the *rank* of $u$. For $u, v \in V^{\mathcal{L}}$, $[\![u=v]\!]$ and $[\![u\in v]\!]$ are defined by induction on the rank of $u, v$.

$$[\![u=v]\!] = \bigwedge_{x\in\mathcal{D}u} (u(x) \to [\![x\in v]\!]) \wedge \bigwedge_{x\in\mathcal{D}v} (v(x) \to [\![x\in u]\!])$$
$$[\![u\in v]\!] = \bigvee_{x\in\mathcal{D}v} [\![u=x]\!] \wedge v(x).$$

We say an element $p$ of $\mathcal{L}$ is $\Box$-*closed* if $p = \Box p$. As an immediate consequence of the definition of $[\![u = v]\!]$ we have:

**Lemma 3.1.** *For every* $u, v \in V^{\mathcal{L}}$, $[\![u=v]\!]$ *is* $\Box$-*closed.*

**Lemma 3.2.** *For* $u, v \in V^{\mathcal{L}}$ *and* $\{b_k\}_k \subset \mathcal{L}$, $[\![u=v]\!] \wedge \bigvee_k b_k = \bigvee_k [\![u = v]\!] \wedge b_k$ *; and*
*for* $u, v_k \in V^{\mathcal{L}}$ *and* $b \in \mathcal{L}$, $(\bigvee_k [\![u = v_k]\!]) \wedge b = \bigvee_k ([\![u = v_k]\!] \wedge b)$.

*Proof.* By Theorem 2.2.G5. $\Box$

**Lemma 3.3.** *Let* $u, v \in V^{\mathcal{L}}$. *Then*

(1) $[\![u=v]\!] = [\![v=u]\!]$
(2) $[\![u=u]\!] = 1$
(3) *If* $x \in \mathcal{D}u$ *then* $u(x) \leq [\![x\in u]\!]$.

*Proof.* (1) is obvious. (2) and (3) are proved by induction on the rank of $u$. Let $x \in \mathcal{D}u$. Since $[\![x=x]\!] = 1$ by induction hypothesis,

$$u(x) \leq \bigvee_{x'\in\mathcal{D}u} [\![x=x']\!] \wedge u(x') \leqslant [\![x\in u]\!],$$

and hence, $[\![u=u]\!] = 1$. $\Box$

**Theorem 3.4.** *For* $u, v, w \in V^{\mathcal{L}}$,

(1) $[\![u=v \wedge v=w]\!] \leq [\![u=w]\!]$
(2) $[\![u=v \wedge v\in w]\!] \leq [\![u\in w]\!]$
(3) $[\![u=v \wedge w\in v]\!] \leq [\![w\in u]\!]$

*Proof.* (1) We proceed by induction. Assume that $u, v, w \in V^{\mathcal{L}}_\alpha$. By Theorem 2.3.(8),

$$[\![u=v]\!] \wedge u(x) \leqslant (u(x) \to [\![x \in v]\!]) \wedge u(x) \leqslant [\![x \in v]\!]$$

for $x \in \mathcal{D}u$. Hence, by using Lemma 3.2,

$$\begin{aligned}
[\![u=v \wedge v=w]\!] \wedge u(x) &\leqslant [\![v=w]\!] \wedge \bigvee_{y \in \mathcal{D}v} [\![x=y]\!] \wedge v(y) \\
&\leqslant \bigvee_{y \in \mathcal{D}v} ([\![x=y]\!] \wedge [\![v=w]\!] \wedge v(y)) \\
&\leqslant \bigvee_{y \in \mathcal{D}v} ([\![x=y]\!] \wedge \bigvee_{z \in \mathcal{D}w} [\![y=z]\!] \wedge w(z)) \\
&\leqslant \bigvee_{y \in \mathcal{D}v} \bigvee_{z \in \mathcal{D}w} [\![x=y \wedge y=z]\!] \wedge w(z).
\end{aligned}$$

By using induction hypothesis,

$$\begin{aligned}
&\leqslant \bigvee_{z \in \mathcal{D}w} [\![x=z]\!] \wedge w(z) \\
&\leqslant [\![x \in w]\!].
\end{aligned}$$

Since $[\![u=v \wedge v=w]\!]$ is $\Box$-closed,

$$[\![u=v \wedge v=w]\!] \leqslant \bigwedge_{x \in \mathcal{D}u} (u(x) \to [\![x \in w]\!]).$$

Similarly, we have

$$[\![u=v \wedge v=w]\!] \leqslant \bigwedge_{z \in \mathcal{D}} (w(z) \to [\![z \in x]\!]).$$

Hence, $[\![u=v \wedge v=w]\!] \leqslant [\![u=w]\!]$.

(2) and (3) follows from (1) and Lemma 3.2. □

By lattice valued set theory we mean a set theory on $\mathrm{V}^{\mathcal{L}}$ whose atomic formulas are of the form $u = v$ or $u \in v$ ; and logical operations are $\wedge$, $\vee$, $\neg$, $\to$, $\forall x$, $\exists x$. We extend the definition of $[\![\varphi]\!]$ in natural way:

$[\![\neg\varphi]\!] = \neg[\![\varphi]\!]$
$[\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] \wedge [\![\varphi_2]\!]$
$[\![\varphi_1 \vee \varphi_2]\!] = [\![\varphi_1]\!] \vee [\![\varphi_2]\!]$
$[\![\varphi_1 \to \varphi_2]\!] = [\![\varphi_1]\!] \to [\![\varphi_2]\!]$
$[\![\forall x \varphi(x)]\!] = \bigwedge_{u \in V^{\mathcal{L}}} [\![\varphi(u)]\!]$
$[\![\exists x \varphi(x)]\!] = \bigvee_{u \in V^{\mathcal{L}}} [\![\varphi(u)]\!]$

We denote a formula $(\varphi \to \varphi) \to \varphi$ by $\Box\varphi$. Then $[\![\Box\varphi]\!] = \Box[\![\varphi]\!]$.

The equality axioms are valid on $V^{\mathcal{L}}$ :

**Theorem 3.5.** *For any formula $\varphi$(a) and $u, v \in V^{\mathcal{L}}$,*

$$[\![u = v \wedge \varphi(u)]\!] \leqslant [\![\varphi(v)]\!].$$

*Proof.* If $\varphi(a)$ is an atomic formula, then it is immediate from Theorem 3.3 and 3.4. Other cases follows from the fact that $[\![u = v]\!]$ is $\Box$-closed and Theorem 2.2. $\Box$

**Theorem 3.6.** *For any formula $\varphi(a)$ and $u \in V^{\mathcal{L}}$,*

(1) $[\![\forall x(x \in u \to \varphi(x))]\!] = \bigwedge_{x \in \mathcal{D}u}[\![x \in u \to \varphi(x)]\!]$

(2) $[\![\exists x(x \in u \wedge \varphi(x))]\!] = \bigvee_{x \in \mathcal{D}u}[\![x \in u \wedge \varphi(x)]\!]$

*Proof.* (1): $[\![\forall x(x \in u \to \varphi(x))]\!] \leqslant \bigwedge_{x \in \mathcal{D}u}[\![x \in u \to \varphi(x)]\!]$ is obvious. Now we show $(\geq)$. By using the fact that $[\![x \in u]\!] \leqslant \bigvee_{x' \in \mathcal{D}u}[\![x = x']\!]$, and Lemma 3.2, Theorem 3.4, we have

$$\begin{aligned}
&(\bigwedge_{x' \in \mathcal{D}u} [\![x' \in u \to \varphi(x')]\!]) \wedge [\![x \in u]\!] \\
&\quad = (\bigwedge_{x' \in \mathcal{D}u} [\![x' \in u \to \varphi(x')]\!]) \wedge [\![x \in u]\!] \wedge \bigvee_{x'' \in \mathcal{D}u} [\![x = x'']\!] \\
&\quad = \bigvee_{x'' \in \mathcal{D}u} (\bigwedge_{x' \in \mathcal{D}u} [\![x' \in u \to \varphi(x')]\!] \wedge [\![x \in u]\!] \wedge [\![x = x'']\!]) \\
&\quad \leqslant [\![\varphi(x)]\!]
\end{aligned}$$

Since $\bigwedge_{x \in \mathcal{D}u}[\![x \in u \to \varphi(x)]\!]$ is $\Box$-closed, we have

$$\bigwedge_{x \in \mathcal{D}u} [\![x \in u \to \varphi(x)]\!]) \leqslant [\![\forall x(x \in u \to \varphi(x))]\!].$$

(2): By using $[\![x \in u]\!] \leqslant \bigvee_{x \in \mathcal{D}u}[\![x = x']\!]$ again,

$$\begin{aligned}
[\![\exists x(x \in u \wedge \varphi(x))]\!] &\leqslant \bigvee_{x \in V^{\mathcal{L}}} \bigvee_{x' \in \mathcal{D}u} ([\![x = x']\!] \wedge [\![x \in u \wedge \varphi(x)]\!]) \\
&\leqslant \bigvee_{x' \in \mathcal{D}u} [\![x' \in u \wedge \varphi(x')]\!].
\end{aligned}$$ $\Box$

**Definition 3.1** *Restriction* $u \restriction p$ of $u \in V^{\mathcal{L}}$ by $p \in \mathcal{L}$ is defined by

$$\begin{cases} \mathcal{D}(u \restriction p) = \{x \restriction p \mid x \in \mathcal{D}u\} \\ (u \restriction p)(x \restriction p) = \bigvee\{u(x') \wedge p \mid x' \in \mathcal{D}u,\ x \restriction p = x' \restriction p\} \text{ for } x \in \mathcal{D}u. \end{cases}$$

If $u$ is of rank $\leqslant \alpha$ ( i.e. $u \in V_\alpha^{\mathcal{L}}$), so is $u \restriction p$, and we have

**Theorem 3.7.** *If $u, x \in V^{\mathcal{L}}$, $p, q \in \mathcal{L}$, and $p$ is $\Box$-closed (i.e, $p = \Box p$), then*

(1) $p \leqslant [\![u = u \restriction p]\!]$
(2) $[\![x \in u \restriction p]\!] = [\![x \in u]\!] \wedge p$
(3) $(u \restriction q) \restriction p = u \restriction (p \wedge q)$.

*Proof.* We proceed by induction on the rank of $u$,

(1) : For $x \in \mathcal{D}u$,

$$p \wedge u(x) \leqslant (u \restriction p)(x \restriction p) \wedge [\![x = x \restriction p]\!] \leqslant [\![x \in u \restriction p]\!]$$
$$(u \restriction p)(x \restriction p) = \bigvee_{x' \in \mathcal{D}u,\, x \restriction p = x' \restriction p} u(x') \wedge p \wedge [\![x = x' = x \restriction p]\!] \leqslant [\![x \restriction p \in u]\!].$$

Therefore, $p \leq [\![u = u \restriction p]\!]$.

(2) : By (1) and Theorem 3.5,

$$[\![x \in u]\!] \wedge p \leqslant [\![x \in u \restriction p]\!].$$

($\leqslant$) follows from the fact that $x'' \restriction p = x' \restriction p$ implies $p \leqslant [\![x'' = x']\!]$ :

$$[\![x \in u \restriction p]\!] = \bigvee_{x' \in \mathcal{D}u} [\![x = x' \restriction p]\!] \wedge \bigvee_{x'' \in \mathcal{D}u,\, x'' \restriction p = x' \restriction p} u(x'') \wedge p$$
$$\leqslant [\![x \in u]\!] \wedge p$$

(3) : $\mathcal{D}\left((u \restriction q) \restriction p\right) = \mathcal{D}\left(u \restriction (q \wedge p)\right)$, by the induction hypothesis, and

$$((u \restriction q) \restriction p)\left((x \restriction q) \restriction p\right) = \left(u \restriction (q \wedge p)\right)\left(x \restriction (q \wedge p)\right)$$

by using the fact : $(\bigvee_{x'} (u(x') \wedge q)) \wedge p = \bigvee_{x'} (u(x') \wedge q) \wedge p)$ . $\Box$

The following axioms of set theory are valid on the universe $V^{\mathcal{L}}$.

**Axiom of extensionality:** $\forall x (x \in u \leftrightarrow x \in v) \rightarrow u = v$.

*Proof.* We have $[\![\forall x (x \in u \leftrightarrow x \in v)]\!] = [\![u = v]\!]$ by Theorem 3.6 and the definition of $[\![u = v]\!]$. Hence, $[\![\forall x (x \in u \leftrightarrow x \in v) \rightarrow u = v]\!] = 1$. $\Box$

**Axiom of pair:** $\forall u, v \exists z\, \forall x (x \in z \leftrightarrow x = u \vee x = v)$.

*Proof.* For $u, v \in V^{\mathcal{L}}$ define $z$ by

$$\begin{cases} \mathcal{D}z = \{u, v\} \\ z(t) = 1 \qquad \text{for } t \in \mathcal{D}z \end{cases}$$

Then $[\![x \in z]\!] = \bigvee_{t \in \mathcal{D}z} [\![x = t]\!] \wedge z(t) = [\![x = u]\!] \vee [\![x = v]\!]$.
Therefore, $[\![\forall x (x \in z \leftrightarrow x = u \vee x = v)]\!] = 1$. $\Box$

**Axiom of union:** $\forall u \exists v \forall x (x \in v \leftrightarrow \exists y (y \in u \wedge x \in y))$.

*Proof.* For $u \in V^{\mathcal{L}}$ defined $v$ by

$$\begin{cases} \mathcal{D}v = \bigcup_{y \in \mathcal{D}u} \mathcal{D}y \\ v(x) = [\![\exists y(y \in u \wedge x \in y)]\!]. \end{cases}$$

Then, by Theorem 3.6,

$$\begin{aligned} [\![\exists y(y \in u \wedge x \in y)]\!] &= \bigvee_{y \in \mathcal{D}u} [\![y \in u]\!] \wedge [\![x \in y]\!] \\ &= \bigvee_{y \in \mathcal{D}u} [\![y \in u]\!] \wedge [\![x \in y]\!] \wedge \bigvee_{x' \in \mathcal{D}y} [\![x = x']\!] \\ &= \bigvee_{y \in \mathcal{D}u,\ x' \in \mathcal{D}y} [\![x = x']\!] \wedge [\![x' \in y \wedge y \in u]\!] \\ &= [\![x \in v]\!] \end{aligned}$$

□

**Definition 3.2** For each set $x$ we define $\check{x} \in V^{\mathcal{L}}$ by

$$\begin{cases} \mathcal{D}\check{x} = \{\check{t} \mid t \in x\} \\ \check{x}(\check{t}) = 1. \end{cases}$$

$\check{x}$ is called the *check set associated with* $x$. For check sets $\check{x}, \check{y}$, we have

$$[\![\check{x} = \check{y}]\!] = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases} \quad ; \quad [\![\check{x} \in \check{y}]\!] = \begin{cases} 1 & \text{if} \quad x \in y \\ 0 & \text{if} \quad x \notin y. \end{cases}$$

**Definition 3.3** Let

$$\mathrm{ck}(u) \overset{\mathrm{def}}{\Longleftrightarrow} \forall t(t \in u \rightarrow t \overset{\square}{\in} u \wedge \mathrm{ck}(t)).$$

Then $[\![\mathrm{ck}(\check{x})]\!] = 1$ for all $x$.

**Axiom of infinity:** $\exists u\, (\exists x(x \in u) \wedge \forall x(x \in u \rightarrow \exists y \in u(x \in y)))$ .

*Proof.* $\check{\omega}$ associated with the set $\omega$ of all natural numbers satisfies

$$[\![\exists x(x \in \check{\omega}) \wedge \forall x(x \in \check{\omega} \rightarrow \exists y \in \check{\omega}(x \in y))]\!] = 1.$$

□

**Axiom of power set:** $\forall u \exists v \forall x(x \in v \leftrightarrow x \subset u)$, where $x \subset u \overset{\mathrm{def}}{\Longleftrightarrow} \forall t(t \in x \rightarrow t \in u)$.

*Proof.* Let $u \in V_\alpha^{\mathcal{L}}$. For every $x \in V^{\mathcal{L}}$, define $x^*$ by

$$\begin{cases} \mathcal{D}x^* = \mathcal{D}u \\ x^*(t) = [\![x \subset u \wedge t \in x]\!]. \end{cases}$$

Since

$$[\![x \subset u \wedge t \in x]\!] \leqslant [\![t \in u]\!] \leqslant \bigvee_{t' \in \mathcal{D}u} [\![t = t']\!],$$

we have

$$\begin{aligned} [\![x \subset u \wedge t \in x]\!] &\leqslant \bigvee_{t' \in \mathcal{D}u} [\![t = t' \wedge x \subset u \wedge t' \in x]\!] \\ &\leqslant [\![t \in x^*]\!]. \end{aligned}$$

It follows that for every $x \in V^{\mathcal{L}}$ there exists $x^* \in V_{\alpha+1}^{\mathcal{L}}$ such that $[\![x \subset u]\!] \leq [\![x = x^*]\!]$. Now we define $v$ by

$$\begin{cases} \mathcal{D}v = \{x \in V_{\alpha+1}^{\mathcal{L}} \mid \mathcal{D}x = \mathcal{D}u\} \\ v(x) = [\![x \subset u]\!]. \end{cases}$$

Then

$$[\![\forall x (x \in v \leftrightarrow x \subset u)]\!] = 1.$$

□

**Axiom of separation:** $\forall u \exists v \, (\forall x (x \in v \leftrightarrow x \in u \wedge \varphi(x)))$.

*Proof.* For a given $u \in V^{\mathcal{L}}$ define $v$ by

$$\begin{cases} \mathcal{D}v = \mathcal{D}u \\ v(x) = [\![x \in u \wedge \varphi(x)]\!] \end{cases}$$

Then

$$[\![\forall x (x \in v \leftrightarrow x \in u \wedge \varphi(x))]\!] = 1.$$

□

We denote $\Box(a \in b)$ by $a \overset{\Box}{\in} b$.

**Axiom of collection:**

$$\forall u \left( \forall x (x \in u \to \exists y \varphi(x, y)) \to \exists v \forall x (x \in u \to \exists y \overset{\Box}{\in} v \varphi(x, y)) \right).$$

*Proof.* Let

$$p = [\![\forall x(x \in u \to \exists y \varphi(x,y))]\!] = \bigwedge_{x \in \mathcal{D}u} ([\![x \in u]\!] \to \bigvee_y [\![\varphi(x,y)]\!]).$$

It suffices to show that there exists $v$ such that

$$p \leqslant [\![\forall x(x \in u \to \exists y \overset{\Box}{\in} v \varphi(x,y))]\!].$$

Since $\mathcal{L}$ is a set, for each $x \in \mathcal{D}u$ there exists an ordinal $\alpha(x)$ such that

$$p \wedge [\![x \in u]\!] \leqslant \bigvee_{y \in V^{\mathcal{L}}_{\alpha(x)}} [\![\varphi(x,y)]\!].$$

Hence, by using the axiom of collection externally, there exists an ordinal $\alpha$ such that

$$p \wedge [\![x \in u]\!] \leqslant \bigvee_{y \in V^{\mathcal{L}}_{\alpha}} [\![\varphi(x,y)]\!] \quad \text{for all } x \in \mathcal{D}u.$$

Now we defined $v$ by

$$\begin{cases} \mathcal{D}v = V^{\mathcal{L}}_{\alpha} \\ v(y) = 1 \end{cases}$$

Then

$$p \wedge [\![x \in u]\!] \leqslant \bigvee_{y \in \mathcal{D}v} [\![y \overset{\Box}{\in} v \wedge \varphi(x,y)]\!] = [\![\exists y \overset{\Box}{\in} v \varphi(x,y)]\!] \quad \text{for all } x \in \mathcal{D}u.$$

Since $p = \Box p$, we have

$$p \leqslant [\![\forall x(x \in u \to \exists y \overset{\Box}{\in} v \varphi(x,y))]\!].$$

□

**Axiom of $\in$-induction:** $\forall x\, (\forall y(y \in x \to \varphi(y)) \to \varphi(x)) \to \forall x \varphi(x)$.

*Proof.* Let $p = [\![\forall x\, (\forall y(y \in x \to \varphi(y)) \to \varphi(x)]\!]$. We prove $p \leqslant [\![\forall x \varphi(x)]\!] = \bigwedge_{x \in V^{\mathcal{L}}} [\![\varphi(x)]\!]$ by induction on the rank of $x$. Let $x \in V^{\mathcal{L}}_{\alpha}$. Since $p \leq [\![\varphi(y)]\!]$ for all $y \in \mathcal{D}x \subset V^{\mathcal{L}}_{<\alpha}$ by induction hypothesis,

$$p \wedge [\![y \in x]\!] \leqslant [\![\varphi(y)]\!] \ \text{ for all } y \in \mathcal{D}x.$$

Hence, by using $p = \Box p$, we have

$$p \leqslant [\![\forall y(y \in x \to \varphi(y))]\!].$$

It follows that $p \leqslant [\![\forall x \varphi(x)]\!]$. $\square$

**Zorn's Lemma:** $\mathrm{Gl}(u) \wedge \forall v[\mathrm{Chain}(v,u) \to \bigcup v \in u] \to \exists z\, \mathrm{Max}(z,u)$, where

$$\mathrm{Gl}(u) \overset{\text{def}}{\Longleftrightarrow} \forall x(x \in u \to x \overset{\square}{\in} u),$$
$$\mathrm{Chain}(v,u) \overset{\text{def}}{\Longleftrightarrow} v \subset u \wedge \forall x,y(x,y \in v \to x \subset y \vee y \subset x),$$
$$\mathrm{Max}(z,u) \overset{\text{def}}{\Longleftrightarrow} z \in u \wedge \forall x(x \in u \wedge z \subset x \to z = x).$$

*Proof.* For $u \in V_\alpha^{\mathcal{L}}$, let

$$p = [\![\mathrm{Gl}(u) \wedge \forall v(\mathrm{Chain}(v,u) \to \bigcup v \in u)]\!],$$

and let U be a maximal subset of $V_\alpha^{\mathcal{L}}$ such that

$$\forall x,y \in U([\![x \in u \wedge \exists t(t \in x) \wedge y \in u \wedge \exists t(t \in y)]\!] \wedge p \leqslant [\![x \subset y \vee y \subset x]\!]).$$

$U$ is not empty. Define $v$ by

$$\begin{cases} \mathcal{D}v = U \\ v(x) = p \wedge [\![x \in u \wedge \exists t(t \in x)]\!]. \end{cases}$$

Now we prove that $p \leqslant [\![\mathrm{Max}(\bigcup v, u)]\!]$. Since $p = \square p$ and $p \wedge v(x) \leqslant [\![x \in u]\!]$ for all $x \in \mathcal{D}v$, we have $p \leqslant [\![v \subset u]\!]$. Hence, by the definition of $v$, $p \leqslant [\![\mathrm{Chain}(v,u)]\!]$. Therefore, $p \leq [\![\bigcup v \in u]\!]$. Now it suffices to show that

$$p \wedge [\![x \in u \wedge \bigcup v \subset x]\!] \leqslant [\![x \subset \bigcup v]\!] \quad \text{for} \quad x \in \mathcal{D}u.$$

Let $x \in \mathcal{D}u$ and $r = p \wedge [\![x \in u \wedge \bigcup v \subset x]\!]$. Then $r$ is $\square$-closed, and we have $r \leqslant [\![x = x \restriction r]\!]$ by Theorem 3.7. Hence $x \restriction r \in U$. In fact, for each $y \in U$, we have

$$\begin{aligned}
&[\![y \in u \wedge \exists t(t \in y) \wedge (x \restriction r) \in u \wedge \exists t(t \in x \restriction r)]\!] \wedge p \\
&\quad \leqslant [\![y \in v]\!] \wedge r \\
&\quad \leqslant [\![y \subset \bigcup v \subset x]\!] \wedge [\![x = x \restriction r]\!] \\
&\quad \leqslant [\![y \subset x \restriction r]\!] \\
&\quad \leqslant [\![y \subset x \restriction r \ \vee \ x \restriction r \subset y]\!].
\end{aligned}$$

It follows that

$$\begin{aligned}
r \wedge x(t) &\leqslant [\![x = x \restriction r \wedge x \in u \wedge t \in x]\!] \wedge p \\
&\leqslant [\![x = x \restriction r \wedge x \restriction r \in u \wedge \exists t(t \in x \restriction r)]\!] \wedge p \\
&\leqslant [\![x = x \restriction r]\!] \wedge v(x \restriction r) \\
&\leqslant [\![x \in v]\!] \leq [\![x \subset \bigcup v]\!]
\end{aligned}$$

Therefore, $r \leqslant [\![x \subset \bigcup v]\!]$. $\square$

**Definition 3.4** $\Diamond$ is the logical operation defined by $\Diamond\varphi \overset{\text{def}}{\iff} \neg\Box\neg\varphi$.

**Axiom of $\Diamond$:** $\forall u\exists v\forall x(x\in v \leftrightarrow \Diamond(x\in u))$.

*Proof.* For a given $u\in V^{\mathcal{L}}$, defined $v$ by

$$\begin{cases} \mathcal{D}v = \mathcal{D}u \\ v(x) = [\![\Diamond(x\in u)]\!]. \end{cases}$$

By using Theorem 2.4,

$$\begin{aligned} [\![\Diamond(x\in u)]\!] &= \Diamond \bigvee_{x'\in\mathcal{D}u} [\![x=x']\!] \wedge u(x') \\ &\leqslant \bigvee_{x'\in\mathcal{D}u} [\![x=x']\!] \wedge [\![\Diamond(x'\in u)]\!] = [\![x\in v]\!]. \end{aligned}$$

Hence $[\![\forall x(x\in v \leftrightarrow \Diamond(x\in u))]\!] = 1$. $\Box$

## 4. Lattice-valued set theory

Now we formulate a set theory on $V^{\mathcal{L}}$, and call it *lattice valued set theory* **LZFZ**.

Atomic symbols of **LZFZ** are:

(1) variables $x, y, z, \cdots$
(2) predicate constants $=, \in$
(3) logical symbols $\wedge,\ \vee,\ \neg,\ \to,\ \forall,\ \exists$
(4) parentheses (, ).

*Formulas* of **LZFZ** are constructed from atomic formulas of the form $x=y$ or $x\in y$ by using the logical symbols.

We denote a sentence $(\varphi \to \varphi) \to \varphi$ by $\Box\varphi$.

### 4.1. Lattice valued logic

*Lattice valued logic*, shortly **L**, is a logic on $\mathcal{L}$-valued universe $V^{\mathcal{L}}$. The rules of **L** are given by restricting **LK**. First we define $\Box$-*closed formulas* inductively by :

(1) A formula of the form $\varphi \to \psi$ or $\neg\varphi$ is $\Box$-closed.
(2) If formulas $\varphi$ and $\psi$ are $\Box$-closed, then $\varphi\wedge\psi$ and $\varphi\vee\psi$ are $\Box$-closed.
(3) If a formula $\varphi(x)$ is a $\Box$-closed formula with free variable $x$, then $\forall x\varphi(x)$ and $\exists x\varphi(x)$ are $\Box$-closed.
(4) $\Box$-closed formulas are only those obtained by (1)–(3).

$\varphi, \psi, \xi, \cdots,\ \varphi(x), \cdots$ are used to denote formulas ; $\Gamma, \Delta, \Pi, \Lambda, \cdots$ to denote finite sequences of formulas ; $\overline{\varphi}, \overline{\psi}, \cdots$ to denote $\Box$-closed formulas ; and $\overline{\Gamma}, \overline{\Delta}, \overline{\Pi}, \overline{\Lambda}, \cdots$ to denote finite sequences of $\Box$-closed formulas. A formal expression of the form $\Gamma \Longrightarrow \Delta$ is called a *sequent*.

*Logical axioms :* Axioms of **L** are sequents of the form $\varphi \Longrightarrow \varphi$.

*Structural rules:*

Thinning : $$\frac{\Gamma \Longrightarrow \Delta}{\varphi, \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \varphi}$$

Contraction : $$\frac{\varphi, \varphi, \Gamma \Longrightarrow \Delta}{\varphi, \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta, \varphi, \varphi}{\Gamma \Longrightarrow \Delta, \varphi}$$

Interchange : $$\frac{\Gamma, \varphi, \psi, \Pi \Longrightarrow \Delta}{\Gamma, \psi, \varphi, \Pi \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta, \varphi, \psi, \Lambda}{\Gamma \Longrightarrow \Delta, \psi, \varphi, \Lambda}$$

Cut : $$\frac{\Gamma \Longrightarrow \overline{\Delta}, \varphi \quad \varphi, \Pi \Longrightarrow \Lambda}{\Gamma, \Pi \Longrightarrow \overline{\Delta}, \Lambda} \qquad \frac{\Gamma \Longrightarrow \Delta, \varphi \quad \varphi, \overline{\Pi} \Longrightarrow \Lambda}{\Gamma, \overline{\Pi} \Longrightarrow \Delta, \Lambda}$$

$$\frac{\Gamma \Longrightarrow \Delta, \overline{\varphi} \quad \overline{\varphi}, \Pi \Longrightarrow \Lambda}{\Gamma, \Pi \Longrightarrow \Delta, \Lambda}$$

*Logical rules:*

$\neg$ : $$\frac{\Gamma \Longrightarrow \overline{\Delta}, \varphi}{\neg\varphi, \Gamma \Longrightarrow \overline{\Delta}} \qquad \frac{\Gamma \Longrightarrow \Delta, \overline{\varphi}}{\neg\overline{\varphi}, \Gamma \Longrightarrow \Delta} \qquad \frac{\varphi, \overline{\Gamma} \Longrightarrow \overline{\Delta}}{\overline{\Gamma} \Longrightarrow \overline{\Delta}, \neg\varphi} \qquad \frac{\overline{\varphi}, \Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg\overline{\varphi}}$$

$\wedge$ : $$\frac{\varphi, \Gamma \Longrightarrow \Delta}{\varphi \wedge \psi, \Gamma \Longrightarrow \Delta} \qquad \frac{\psi, \Gamma \Longrightarrow \Delta}{\varphi \wedge \psi, \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \overline{\Delta}, \varphi \quad \Gamma \Longrightarrow \overline{\Delta}, \psi}{\Gamma \Longrightarrow \overline{\Delta}, \varphi \wedge \psi}$$

$$\frac{\Gamma \Longrightarrow \Delta, \overline{\varphi} \quad \Gamma \Longrightarrow \Delta, \overline{\psi}}{\Gamma \Longrightarrow \Delta, \overline{\varphi} \wedge \overline{\psi}}$$

$\vee$ : $$\frac{\varphi, \overline{\Gamma} \Longrightarrow \Delta \quad \psi, \overline{\Gamma} \Longrightarrow \Delta}{\varphi \vee \psi, \overline{\Gamma} \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta, \varphi}{\Gamma \Longrightarrow \Delta, \varphi \vee \psi} \qquad \frac{\Gamma \Longrightarrow \Delta, \psi}{\Gamma \Longrightarrow \Delta, \varphi \vee \psi}$$

$$\frac{\overline{\varphi}, \Gamma \Longrightarrow \Delta \quad \overline{\psi}, \Gamma \Longrightarrow \Delta}{\overline{\varphi} \vee \overline{\psi}, \Gamma \Longrightarrow \Delta}$$

$\rightarrow$: $$\frac{\Gamma \Longrightarrow \overline{\Delta}, \varphi \quad \psi, \overline{\Pi} \Longrightarrow \Lambda}{(\varphi \rightarrow \psi), \Gamma, \overline{\Pi} \Longrightarrow \overline{\Delta}, \Lambda} \qquad \frac{\varphi, \overline{\Gamma} \Longrightarrow \overline{\Delta}, \psi}{\overline{\Gamma} \Longrightarrow \overline{\Delta}, (\varphi \rightarrow \psi)}$$

$\forall$ : $$\frac{\varphi(t), \Gamma \Longrightarrow \Delta}{\forall x \varphi(x), \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \overline{\Delta}, \varphi(a)}{\Gamma \Longrightarrow \overline{\Delta}, \forall x \varphi(x)} \qquad \frac{\Gamma \Longrightarrow \Delta, \overline{\varphi}(a)}{\Gamma \Longrightarrow \Delta, \forall x \overline{\varphi}(x)}$$

where $t$ is any term

where $a$ is a free variable which does not occur in the lower sequent.

$$\exists: \quad \frac{\varphi(a), \overline{\Gamma} \Longrightarrow \Delta}{\exists x\varphi(x), \overline{\Gamma} \Longrightarrow \Delta} \quad \frac{\overline{\varphi}(a), \Gamma \Longrightarrow \Delta}{\exists x\overline{\varphi}(x), \Gamma \Longrightarrow \Delta} \qquad \frac{\Gamma \Longrightarrow \Delta, \varphi(t)}{\Gamma \Longrightarrow \Delta, \exists x\varphi(x)}$$

where $a$ is a free variable which does not occur in the lower sequent.

where $t$ is any term

We use the following abbreviations :

$$\varphi \leftrightarrow \psi \overset{\text{def}}{\Longleftrightarrow} (\varphi \to \psi) \wedge (\psi \to \varphi).$$

**Theorem 4.1.** *The following sequents are provable in* **L***.*

(1) $\varphi \wedge (\varphi \to \psi) \Longrightarrow \psi$
(2) $\varphi \wedge \neg\varphi \Longrightarrow$
(3) $\varphi \Longrightarrow \neg\neg\varphi$
(4) $\neg(\varphi \vee \psi) \Longleftrightarrow (\neg\varphi \wedge \neg\psi)$
(5) $\Box\varphi \Longrightarrow \varphi$
(6) $\neg\varphi \Longleftrightarrow \Box\neg\varphi$
(7) $\forall x\Box\varphi(x) \Longrightarrow \Box\forall x\varphi(x)$
(8) $(\Box\varphi \to \psi) \Longleftrightarrow (\Box\varphi \to \Box\psi) \Longleftrightarrow (\neg\Box\varphi \vee \Box\psi)$
(9) $(\Box\varphi \wedge \exists x\psi(x)) \Longleftrightarrow \exists x(\Box\varphi \wedge \psi(x))$;
$(\varphi \wedge \exists x\Box\psi(x)) \Longleftrightarrow \exists x(\varphi \wedge \Box\psi(x))$
(10) $\Longrightarrow \Box\varphi \vee \neg\Box\varphi$
(11) $((\varphi \wedge \Box\xi) \to \psi) \Longrightarrow ((\neg\psi \wedge \Box\xi) \to \neg\varphi)$

### *4.2. Nonlogical axioms*

Nonlogical axioms of **LZFZ** are GA1–GA11 from the preceeding section, which are valid on lattice valued universe:

GA1. Equality $\forall u\forall v\,(u = v \wedge \varphi(u) \to \varphi(v))$.
GA2. Extensionality $\forall u, v\,(\forall x(x \in u \leftrightarrow x \in v) \to u = v)$.
GA3. Pairing $\forall u, v\exists z\,(\forall x(x \in z \leftrightarrow (x = u \vee x = v)))$.
The set $z$ satisfying $\forall x(x \in z \leftrightarrow (x = u \vee x = v))$ is denoted by $\{u, v\}$.
GA4. Union $\forall u\exists z\,(\forall x(x \in z \leftrightarrow \exists y \in u(x \in y)))$.
The set $z$ satisfying $\forall x(x \in z \leftrightarrow \exists y \in u(x \in y))$ is denoted by $\bigcup u$.
GA5. Power set $\forall u\exists z\,(\forall x(x \in z \leftrightarrow x \subset u))$, where

$$x \subset u \overset{\text{def}}{\Longleftrightarrow} \forall y(y \in x \to y \in u).$$

The set $z$ satisfying $\forall x(x \in z \leftrightarrow x \subset u)$ is denoted by $\mathcal{P}(u)$.
GA6. Infinity $\exists u\,(\exists x(x \in u) \wedge \forall x(x \in u \to \exists y \in u(x \in y)))$.
GA7. Separation $\forall u\exists v\,(\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x)))$.
The set $v$ satisfying $\forall x(x \in v \leftrightarrow x \in u \wedge \varphi(x))$ is denoted by $\{x \in u \mid \varphi(x)\}$.

GA8. Collection
$$\forall u \exists v \Big( \forall x (x \in u \to \exists y \varphi(x,y)) \to \forall x (x \in u \to \exists y \overset{\Box}{\in} v \varphi(x,y)) \Big).$$
GA9. $\in$-induction $\forall x\,(\forall y(y \in x \to \varphi(y)) \to \varphi(x)) \to \forall x \varphi(x)$.
GA10. Zorn $\mathrm{Gl}(u) \wedge \forall v\,(\mathrm{Chain}(v,u) \to \bigcup v \in u) \to \exists z\, \mathrm{Max}(z,u)$, where

$$\mathrm{Gl}(u) \overset{\text{def}}{\iff} \forall x(x \in u \to x \overset{\Box}{\in} u),$$
$$\mathrm{Chain}(v,u) \overset{\text{def}}{\iff} v \subset u \wedge \forall x,y(x,y \in v \to x \subset y \vee y \subset x),$$
$$\mathrm{Max}(z,u) \overset{\text{def}}{\iff} z \in u \wedge \forall x(x \in u \wedge z \subset x \to z = x).$$

GA11. Axiom of $\Diamond$ $\forall u \exists z \forall t(t \in z \leftrightarrow \Diamond(t \in u))$.
The set $z$ satisfying $\forall t(t \in z \leftrightarrow \Diamond(t \in u))$ is denoted by $\Diamond u$.

We say that a formula $\varphi$ is *global*, if $(\varphi \to \Box\varphi)$, and a set $u$ is *global* ($\mathrm{Gl}(u)$), if $x \in u$ is global for all $x$.

### *4.3. Well-founded relations in* **LZFZ**

Any formula with two free variables determines a binary relation. For a binary relation $A(x,y)$, we use the following abbreviations:

$$x \in \mathrm{Dom}\, A \overset{\text{def}}{\iff} \exists y A(x,y), \quad x \in \mathrm{Rge}\, A \overset{\text{def}}{\iff} \exists y A(y,x),$$

$$x \in \mathrm{Fld}\, A \overset{\text{def}}{\iff} \exists y(A(x,y) \vee A(y,x)).$$

A binary relation $\prec$ is said to be *well-founded* if the following conditions are satisfied:

WF1 $\forall x,y \neg(x \prec y \wedge y \prec x)$
WF2 $\forall x[x \in \mathrm{Fld}(\prec) \wedge \forall y(y \prec x \to \varphi(y)) \to \varphi(x)] \to \forall x(x \in \mathrm{Fld}(\prec) \to \varphi(x))$
WF3 $\forall x \exists y \forall z(z \prec x \to z \in y)$

In view of the axiom GA9 ($\in$-induction), it is clear that the relation $\in$ is itself a well-founded relation, and so is $\overset{\Box}{\in}$.

Singlton $\{x\}$ and ordered pair $\langle x,y\rangle$ are defined as usual:

$$\{x\} \overset{\text{def}}{=} \{x,x\}, \quad \langle x,y\rangle \overset{\text{def}}{=} \{\{x\},\{x,y\}\}$$

so that $x \in \{y\} \iff x = y$ and $\langle x,y\rangle = \langle x',y'\rangle \iff x = x' \wedge y = y'$ hold.

We say a binary relation $F(x,y)$ is *global*, if $\forall x,y(F(x,y) \to \Box F(x,y))$; and a global relation $F(x,y)$ is *functional*, if

$$\forall x,y,y'(F(x,y) \wedge F(x,y') \to y = y').$$

For a global functional relation $F$, we write $F(x)=y$ instead for $F(x,y)$. If $F$ is a global functional relation and $\prec$ is a well-founded relation, then $\{\langle x,y\rangle \mid F(x,y)\wedge\Diamond(x\prec u)\}$ is denoted by $F_{\prec u}$, for each set $u\in\mathrm{Fld}(\prec)$. $F_{\prec u}$ is a set by WF3, GA11($\Diamond$) and GA8(Collection).

The following theorem can be proved in the usual way, by using the fact that

$$(y\prec x\to\Box\varphi(y))\iff(\Diamond(y\prec x)\to\Box\varphi(y)).$$

**Theorem 4.2 (Recursion Principle).** *Let $\prec$ be a well-founded relation and $H$ be a global functional relation such that $\forall x\exists y H(x,y)$. Then there exists a unique global functional relation $F$ such that*

$$\mathrm{Dom}\,F=\mathrm{Fld}(\prec)\ \wedge\ \forall x\,(x\in\mathrm{Fld}(\prec)\to(F(x)=H(F_{\prec x})))\,.$$

**Definition 4.1** We define the formula $\mathrm{Ord}(\alpha)$ ("$\alpha$ is an ordinal") in **LZFZ** as follows:

$$\mathrm{Tr}(\alpha)\overset{\mathrm{def}}{\iff}\forall\beta,\gamma(\beta\in\alpha\wedge\gamma\in\beta\to\gamma\in\alpha),$$
$$\mathrm{Ord}(\alpha)\overset{\mathrm{def}}{\iff}\mathrm{Gl}(\alpha)\wedge\mathrm{Tr}(\alpha)\wedge\forall\beta(\beta\in\alpha\to\mathrm{Gl}(\beta)\wedge\mathrm{Tr}(\beta)),$$

where $\mathrm{Gl}(\alpha)\overset{\mathrm{def}}{\iff}\forall\beta(\beta\in\alpha\to\beta\overset{\Box}{\in}\alpha)$.

As an immediate consequence of the above definition, we have:

**Lemma 4.3.**

(1) $\mathrm{Ord}(\alpha)\wedge\beta\in\alpha\to\mathrm{Ord}(\beta)$
(2) $\mathrm{Gl}(X)\wedge\forall x(x\in X\to\mathrm{Ord}(x))\to\mathrm{Ord}(\bigcup X)$

**Definition 4.2** A global well-founded relation $\prec$ is called a *well-ordering* on a set $u$ if

$$(\mathrm{Fld}(\prec)=u)\wedge(\prec\text{ is transitive})\wedge(\prec\text{ is extensional}),$$

where

$$\prec\text{ is transitive}\overset{\mathrm{def}}{\iff}\forall x,y,z[(x\prec y)\wedge(y\prec z)\to(x\prec z)]$$
$$\prec\text{ is extensional}\overset{\mathrm{def}}{\iff}\forall x,y[x,y\in u\wedge\forall z(z\prec x\leftrightarrow z\prec y)\to x=y].$$

**Theorem 4.4.** *Every global set can be well-ordered, i.e. for every global set $u$, there exists a global well-ordering relation $\prec$ on $u$.*

*Proof.* Suppose $\mathrm{Gl}(u)$, and let

$$P\overset{\mathrm{def}}{=}\{\langle v,w\rangle\mid\mathrm{Gl}(v)\wedge\mathrm{Gl}(w)\wedge v\subset u\wedge(w\text{ is a well-ordering on }v)\},$$

and let $\langle v,w\rangle \prec \langle v',w'\rangle$ mean that $w = w'\lceil v$ and $v$ is an initial $w'$-section of $v'$, i.e.

$$\langle v,w\rangle \prec \langle v',w'\rangle \overset{\text{def}}{\Longleftrightarrow} (v \subset v') \wedge (w = w' \cap (v \times v)) \wedge (v \times (v' - v) \subset w').$$

If $\langle v,w\rangle \in P$, since $\mathrm{Gl}(v) \wedge \neg\neg(y \in v) \Longrightarrow y \in v$, we have

$$\langle v,w\rangle \prec \langle v',w'\rangle \wedge x \in v \wedge \langle y,x\rangle \in w' \Longrightarrow y \in v.$$

Let

$$\mathcal{I} \overset{\text{def}}{=} \{I \subset P \mid \forall p,q(p,q \in I \to p \prec q \vee p = q \vee q \prec p) \\ \wedge \forall p,q(p \in I \wedge q \prec p \to q \in I)\}.$$

Then

$$(\mathcal{I}' \subset \mathcal{I}) \wedge \forall I,I'(I,I' \in \mathcal{I}' \to I \subset I' \vee I' \subset I) \Longrightarrow \bigcup \mathcal{I}' \in \mathcal{I}.$$

By using GA10, there exists a maximal $I_0 \in \mathcal{I}$. Let

$$v_0 = \bigcup \{v \mid \langle v,w\rangle \overset{\Box}{\in} I_0\}, \quad w_0 = \bigcup \{w \mid \langle v,w\rangle \overset{\Box}{\in} I_0\}.$$

Then $\langle v_0,w_0\rangle \in P$. By maximality of $I_0$ we have $\forall x \neg(x \in u - v_0))$. $\forall x(x \in u \to x \in v_0 \vee \neg(x \in v_0))$. It follows that $u = v_0$. □

**Theorem 4.5.** *If $u$ is a global set and $\prec$ is a global well-ordering relation on $u$, then $\langle u,\prec\rangle$ is isomorphic to an ordinal $\langle \alpha,\in\rangle$, i.e. there exists $\rho$ such that*

$$(\rho : u \to \alpha) \wedge \rho(u) = \alpha \wedge \\ \forall x,y[x,y \in u \to (x \prec y \leftrightarrow \rho(x) \in \rho(y)) \wedge (x = y \leftrightarrow \rho(x) = \rho(y))].$$

*Proof.* We define by recursion in $\prec$

$$\rho(x) = \bigcup \{\rho(y) + 1 \mid y \prec x\}.$$

It is easy to see by WF2 ($\prec$-induction) that $\forall x(x \in u \to \mathrm{Ord}(\rho(x))$, and

$$\forall x[x \in u \to \forall t(t \in \rho(x) \to \exists y \prec x(t = \rho(y)))].$$

Set $\alpha = \{\rho(x) \mid x \in u\}$. Then $\mathrm{Ord}(\alpha)$, and $\langle u,\prec\rangle$ is isomorphic to $(\alpha,\in)$. □

We call $\rho(x)$ the *rank of* $x$.

*4.4. Check sets*

We define the notion of check set in **LZFZ** , by $\overset{\Box}{\in}$-recursion:

$$\mathrm{ck}(x) \overset{\mathrm{def}}{\iff} \forall t \left( t \in x \leftrightarrow t \overset{\Box}{\in} x \wedge \mathrm{ck}(t) \right).$$

That is, set

$$H(u,v) \overset{\mathrm{def}}{\iff} v = \{t \mid \langle t,t\rangle \in u\}.$$

$H$ is a global functional relation such that $\forall u \exists v H(u,v)$. Let $\prec$ be $\overset{\Box}{\in}$. $\prec$ is a global well founded relation. Since $\forall x(x \in \mathrm{Fld}(\prec))$, there exists a unique global functional relation $C(x,y)$ such that

$$\forall x\,[x \in \mathrm{Dom}(C) \wedge C(x) = H(C_{\prec x})]\,,$$

by recursion principle. If a set $x$ satisfies $C(x,x)$ then we say $x$ is a *check set* and write $\mathrm{ck}(x)$. i.e.

$$\mathrm{ck}(x) \overset{\mathrm{def}}{\iff} x = C(x).$$

The class of check sets will be denoted by $W$, i.e.

$$x \in W \overset{\mathrm{def}}{\iff} \mathrm{ck}(x).$$

**Theorem 4.6.** *The followings are provable in* **LZFZ***.*

(1) $y \in C(x) \leftrightarrow (y \overset{\Box}{\in} x) \wedge \mathrm{ck}(y)$
(2) $\mathrm{ck}(x) \leftrightarrow \forall t[t \in x \leftrightarrow (t \overset{\Box}{\in} x \wedge \mathrm{ck}(t))]$
(3) $C(x) = CC(x)$

*Proof.* (1) and (2) are immediate results of definition of $C$.

$$(3): \quad \begin{aligned} y \in CC(x) &\iff y \overset{\Box}{\in} C(x) \wedge \mathrm{ck}(y) \\ &\iff y \overset{\Box}{\in} x \wedge \mathrm{ck}(y) \\ &\iff y \in C(x) \end{aligned} \qquad \Box$$

*4.5. The model $W$ of* **ZFC** *in* **LZFZ**

An interpretation of **ZFC** in **LZFZ** is obtained by relativizing the range of quantifiers to check sets. Namely "the class $W$ of check sets is a model of **ZFC** " is provable in **LZFZ**.

We denote quantifiers relativized on check sets by $\forall^W, \exists^W$, i.e.

$$\forall^W x\varphi(x) \overset{\mathrm{def}}{\iff} \forall x(\mathrm{ck}(x) \to \varphi(x))$$

$$\exists^W x\varphi(x) \overset{\mathrm{def}}{\iff} \exists x(\mathrm{ck}(x) \wedge \varphi(x)).$$

For a formula $\varphi$ of **LZFZ** , $\varphi^W$ is the formula obtained from $\varphi$ by replacing all quantifiers $\forall x,\ \exists x,$ by $\forall^W x,\ \exists^W x$, respectively.

**Theorem 4.7.** *The following (1)–(9) are provable in* **LZFZ** *, for any formula* $\varphi$.

(1) $\forall^W x, y(x \in y \to x \overset{\Box}{\in} y)$

(2) $\forall^W x_1 \cdots x_n[\varphi^W(x_1, \cdots, x_n) \to \Box\varphi^W(x_1, \cdots, x_n)]$

(3) $\forall^W x(\forall^W y(y \in x \to \varphi^W(y)) \to \varphi^W(x)) \to \forall^W x \varphi^W(x)$

(4) $\forall\alpha[\mathrm{Ord}(\alpha) \leftrightarrow \mathrm{ck}(\alpha) \wedge \mathrm{Ord}^W(\alpha)]$

(5) $\mathrm{ck}(\emptyset)$, *where* $\emptyset$ *is the empty set.*

(6) $\forall^W x, y[\mathrm{ck}(\{x, y\}) \wedge \mathrm{ck}(\bigcup x) \wedge \mathrm{ck}(\{z \in x \mid \Box\varphi(z))\})]$

(7) *The set of natural numbers* $\omega$ *is defined as follows:*

$$\begin{aligned} \mathrm{Suc}(y) &\overset{\text{def}}{\iff} (y = \emptyset \vee \exists z(y = z + 1)), \textit{ where } z + 1 = z \cup \{z\}, \\ \mathrm{HSuc}(y) &\overset{\text{def}}{\iff} (\mathrm{Suc}(y) \wedge \forall z(z \in y \to \mathrm{Suc}(z)), \quad \textit{and} \\ \omega &\overset{\text{def}}{=} \{y : \mathrm{HSuc}(y)\}. \end{aligned}$$

*Then* $\mathrm{Ord}(\omega) \wedge \forall^W n \in \omega(n = \emptyset \vee \exists^W m \in n(n = m + 1))$.

(8) *If* $u$ *is a global set, then there exists an ordinal* $\alpha \in \mathrm{On}$ *with a bijection* $\rho : u \to \alpha$, *where* $\alpha \in \mathrm{On} \overset{\text{def}}{\iff} \mathrm{Ord}(\alpha)$, *i.e.*

$$\begin{aligned} \exists^W \alpha \in \mathrm{On}\, \exists\rho \quad [ \quad & \rho : u \to \alpha \wedge \rho(u) = \alpha \\ \wedge \quad & \forall x, y(x, y \in u \wedge \rho(x) = \rho(y) \to x = y]. \end{aligned}$$

*Proof.* (1): It follows from

$$\mathrm{ck}(x) \wedge \mathrm{ck}(y) \wedge x \in y \iff \exists t(\mathrm{ck}(x) \wedge \mathrm{ck}(y) \wedge \mathrm{ck}(t) \wedge x = y \wedge t \overset{\Box}{\in} y).$$

(2): By induction on complexity of $\varphi$. If $\varphi$ has no logical symbol, then $\varphi$ is of the form $x = y$ or $x \in y$, and hence $\varphi \to \Box\varphi$ by (1). Now we prove only the case that $\varphi$ is of the form $\exists x \psi(x, x_1, \cdots, x_n)$, since the other cases are similar. Let $\mathrm{ck}(x_1) \wedge \cdots \wedge \mathrm{ck}(x_n)$.

$$\psi^W(x, x_1, \cdots, x_n) \wedge \mathrm{ck}(x) \Longrightarrow \Box\left(\mathrm{ck}(x) \wedge \psi^W(x, x_1, \cdots, x_n)\right),$$

by using induction hypothesis. Hence,

$$\exists^W x \psi^W(x, x_1, \cdots, x_n) \Longrightarrow \Box\exists^W x \psi^W(x, x_1, \cdots, x_n).$$

(3): Let $\psi(x)$ be the formula $\mathrm{ck}(x) \to \varphi^W(x)$. Then, using $\in$-induction, we have

$$\begin{aligned}&\forall^W x[\forall^W y(y \in x \to \varphi^W(y)) \to \varphi^W(x)]\\ &\quad\Longrightarrow \forall x(\forall y(y \in x \to \psi(y)) \to \psi(x)]\\ &\quad\Longrightarrow \forall x \psi(x).\end{aligned}$$

(4): By $\in$-induction.

(5): $\mathrm{ck}(\emptyset)$ follows from:

$$\begin{aligned}x \in \emptyset &\Longrightarrow \neg(x = x)\\ &\Longrightarrow x \overset{\Box}{\in} \emptyset \wedge \mathrm{ck}(x).\end{aligned}$$

(6) $\mathrm{ck}(\{x, y\})$ : Assume $\mathrm{ck}(x) \wedge \mathrm{ck}(y)$. Then we have

$$\begin{aligned}z \in \{x, y\} &\Longleftrightarrow (z = x \vee z = y)\\ &\Longrightarrow \mathrm{ck}(z) \wedge z \overset{\Box}{\in} \{x, y\}).\end{aligned}$$

$$\begin{aligned}z \in \bigcup x &\Longrightarrow \exists t \in x(z \in t)\\ &\Longrightarrow \exists t[\mathrm{ck}(t) \wedge t \overset{\Box}{\in} x \wedge z \in t]\\ &\Longrightarrow \mathrm{ck}(z) \wedge z \overset{\Box}{\in} \bigcup x.\end{aligned}$$

$$\begin{aligned}t \in \{z \in x \mid \Box\varphi(z)\} &\Longrightarrow \mathrm{ck}(t) \wedge t \overset{\Box}{\in} x \wedge \Box\varphi(t)\\ &\Longrightarrow \mathrm{ck}(t) \wedge t \overset{\Box}{\in} \{z \in x \mid \Box\varphi(t)\}.\end{aligned}$$

(7): $\omega$ is a set by GA6 (Infinity). Let $\psi(x)$ be the formula

$$x \in \omega \to \mathrm{ck}(x) \wedge x \overset{\Box}{\in} \omega.$$

Now we prove $\forall y(y \in x \to \psi(y)) \to \psi(x)$: We have
$x \in \omega \Longrightarrow x = \emptyset \vee \exists z(x = z + 1)$, $x = \emptyset \to \psi(x)$ and

$$\begin{aligned}\forall y(y \in x \to \psi(y)) \wedge x \in \omega \wedge x = z + 1 &\Longrightarrow z \in x \wedge \mathrm{ck}(z) \wedge z \overset{\Box}{\in} \omega\\ &\Longrightarrow \mathrm{ck}(z + 1) \wedge (z + 1) \overset{\Box}{\in} \omega\\ &\Longrightarrow \mathrm{ck}(x) \wedge x \overset{\Box}{\in} \omega.\end{aligned}$$

Hence, $\mathrm{ck}(\omega)$.

It is easy to see $\forall y(y \in \omega \rightarrow \mathrm{Tr}(y) \wedge (y \subset \omega))$, by $\in$-induction, where $\mathrm{Tr}(y) \overset{\mathrm{def}}{\Longleftrightarrow} \forall s, t(s \in y \wedge t \in s \rightarrow t \in y)$. Hence $\mathrm{Tr}(\omega) \wedge \forall y(y \in \omega \rightarrow \mathrm{Tr}(y))$ and $\mathrm{Ord}(\omega)$. It is obvious that

$$\forall^W n \in \omega(n = \emptyset \vee \exists^W m \in n(n = m + 1)).$$

(8): By Theorem 4.4, there exists a global well-ordering relation $\prec$ on $u$. Define $\rho(x) = \bigcup\{\rho(y) + 1 \mid y \prec x\}$. By Theorem 4.5, $\rho$ is an isomorphism between $(u, \prec)$ and $(\alpha, \in)$, where $\alpha = \{\rho(x) \mid x \in u\}$. $\square$

**Theorem 4.8 (Interpretation of ZFC ).** *If $\varphi$ is a theorem of* **ZFC** *, then $\varphi^W$ is provable in* **LZFZ***.*

*Proof.* For a formula $\varphi(x_1, \cdots, x_n)$ of **ZFC** ,

$$\forall^W x_1, \cdots, x_n(\varphi^W \rightarrow \Box\varphi^W)$$

is provable by Theorem 4.7(2), hence,

$$\forall^W x_1, \cdots, x_n(\varphi^W \vee \neg\varphi^W)$$

is provable in **LZFZ**. Now it suffices to show that for each nonlogical axiom $A$ of **ZFC** , $A^W$ is provable in **LZFZ**.

(Equality axiom)$^W$ and (Extensionality)$^W$ are obvious.

(Pairing)$^W$: By Theorem 4.7(6),

$$\mathrm{ck}(u) \wedge \mathrm{ck}(v) \rightarrow \mathrm{ck}(\{u, v\}) \wedge \forall^W x(x \in \{u, v\} \leftrightarrow x = u \vee x = v).$$

(Union)$^W$: Similarly.

(Power set)$^W$: We have $\forall^W u, x[x \in C(\mathcal{P}(u)) \leftrightarrow \forall^W t(t \in x \rightarrow t \in u)]$.

($\in$ -induction)$^W$: By Theorem 4.7(3).

(Separation)$^W$: If $\mathrm{ck}(u)$, by Theorem 4.7(6), $\mathrm{ck}(\{x \in u \mid \varphi^W(x)\})$ and

$$\forall^W u, x\left[x \in \{x \in u \mid \varphi^W(x)\} \leftrightarrow x \in u \wedge \varphi^W(x)\right].$$

(Collection)$^W$: Suppose $\mathrm{ck}(u) \wedge \forall^W x \in u \exists^W y \varphi^W(x, y)$. By GA8(Collection),

$$\exists v \forall x \in u \exists y \overset{\Box}{\in} v(\mathrm{ck}(y) \wedge \varphi^W(z, y)).$$

Since $y \overset{\Box}{\in} v \wedge \mathrm{ck}(y) \rightarrow y \in C(v) \wedge \mathrm{ck}(C(v))$, we have

$$\exists^W v \forall^W x \in u \exists^W y \in v \varphi^W(z, y).$$

(Infinity)$^W$: By Theorem 4.7(7).

(Choice)$^W$, i.e. $\forall^W u \exists^W f \forall^W x \in u[x \neq \emptyset \rightarrow \exists!^W y \in x(\langle x, y\rangle \in f)]$, where $x \neq \emptyset$ stands for $\exists^W y(y \in x)$. By Theorem 4.6(8). there exists an ordinal $\alpha$ and a bijection $\rho : \bigcup u \rightarrow \alpha$. Define $f : u \rightarrow \bigcup u$ by

$$f(x) = \rho^{-1}(\bigcap\{\rho(t) \mid t \in x\}).$$

$\square$

*4.6. Lattice-valued model $W^{\mathcal{P}(1)}$ in $W$*

The power set $\mathcal{P}(1)$ of $1$ $(=\{\emptyset\})$ is a global set and a complete lattice with respect to the inclusion $\subset$. We write $\leqslant$ instead of $\subset$. Then $(\mathcal{P}(1), \leqslant)$ is a comlpete lattice. Let

$$(p \to q) = \{x \in 1 \mid 0 \in p \to 0 \in q\}, \quad \neg p = \{x \in 1 \mid \neg(0 \in p)\}.$$

$\to$ is the basic implication and $\neg$ is the corresponding negation on $\mathcal{P}(1)$.

For a sentence $\varphi$, let

$$|\varphi| \stackrel{\text{def}}{=} \{t \in 1 \mid \varphi\}.$$

$|\varphi|$ is an element of $\mathcal{P}(1)$, and $\varphi \iff 0 \in |\varphi|$. Thus, the complete lattice $\mathcal{P}(1)$ represents the truth value set of **LZFZ**.

The relation $\prec$ defined by

$$\alpha \prec \beta \stackrel{\text{def}}{\iff} \alpha, \beta \in \mathrm{On} \ \wedge \alpha \in \beta$$

is a well founded relation and $\mathrm{Fld}(\prec) = \mathrm{On}$. Thus, the induction on $\alpha \in \mathrm{On}$ is justified in **LZFZ**. Now we construct the $\mathcal{P}(1)$-valued sheaf model by induction on $\alpha \in \mathrm{On}$ as follows:

$$W_\alpha^{\mathcal{P}(1)} = \{u \mid \exists \beta \in \alpha \exists \mathcal{D}u \subset W_\beta^{\mathcal{P}(1)} (\mathrm{Gl}(\mathcal{D}u) \wedge u : \mathcal{D}u \to \mathcal{P}(1))\}$$
$$W^{\mathcal{P}(1)} = \bigcup_{\alpha \in \mathrm{On}} W_\alpha^{\mathcal{P}(1)}$$

On $W^{\mathcal{P}(1)}$, the atomic relation $=$ and $\in$ are interpreted as

$$[\![x = y]\!] = \bigwedge_{t \in \mathcal{D}x} (x(t) \to [\![t \in y]\!]) \wedge \bigwedge_{t \in \mathcal{D}y} (y(t) \to [\![t \in x]\!])$$
$$[\![x \in y]\!] = \bigvee_{t \in \mathcal{D}y} [\![x = t]\!] \wedge y(t).$$

Logical operations $\wedge, \vee, \to, \neg, \forall, \exists$ are interpreted as the correspondent operations on $\mathcal{P}(1)$. Then every sentence on $W^{\mathcal{P}(1)}$ has its truth value in $\mathcal{P}(1)$, and we have

**Theorem 4.9.** *For every sentence $\varphi$, "$(0 \in [\![\varphi]\!]) \longleftrightarrow \varphi$" is provable in* **LZFZ**.

*Proof.* We prove that there exists a global functional relation $F$ such that:

(i) $\mathrm{Dom}\, F = W^{\mathcal{P}(1)}$, and

(ii) for every formula $\varphi(x_1, \cdots, x_n)$ of **LZFZ** on $W^{\mathcal{P}(1)}$,

$$[\![\varphi(x_1, \cdots, x_n)]\!] = |\varphi(F(x_1), \cdots, F(x_n))|.$$

For $x \in W^{\mathcal{P}(1)}$, define $F(x)$ by

$$F(x) = \{F(t) \mid t \in \mathcal{D}x \ \wedge \ 0 \in [\![t \in x]\!]\}.$$

Then we have:

(1) $0 \in [\![x = y]\!] \Longleftrightarrow F(x) = F(y)$,
$0 \in [\![x \in y]\!] \Longleftrightarrow F(x) \in F(y)$.

(2) $\forall u \exists x (F(x) = u)$.

Proof: Let $\Psi(u) \stackrel{\text{def}}{\Longleftrightarrow} \exists x (x \in W^{\mathcal{P}(1)} \wedge u = F(x))$. Then by using GA8 (Collection) we have

$$\forall v (v \in u \to \Psi(v)) \Longrightarrow \exists \alpha [\forall v (v \in u \to \exists y \in W_\alpha^{\mathcal{P}(1)} (v = F(y))].$$

Let

$$\begin{cases} \mathcal{D}x = W_\alpha^{\mathcal{P}(1)} \\ x(y) = \{t \in 1 \mid F(y) \in u\} \end{cases}$$

Then $x \in W^{\mathcal{P}(1)}$ and $F(x) = u$. Hence, $\forall u \exists x (F(x) = u)$.

(3) $0 \in [\![\varphi(x_1, \cdots, x_n)]\!] \Longleftrightarrow \varphi(F(x_1), \cdots, F(x_n))$

Proof: We proceed by induction on the complexity of $\varphi$. If $\varphi$ is atomic, then it is (1). If $\varphi$ is of the form $\varphi_1 \vee \varphi_2,\ \varphi_1 \to \varphi_2,\ \neg \varphi_1$ or $\Box \varphi_1$, then it follows from induction hypothesis. If $\varphi(x_1, \cdots, x_n)$ is of the form $\forall x \psi(x, x_1, \cdots, x_n)$, then, by using (2),

$$\begin{aligned} 0 \in [\![\varphi]\!] &\Longleftrightarrow 0 \in \bigwedge_x [\![\psi(x, x_1, \cdots, x_n)]\!] \\ &\Longleftrightarrow \forall x (\psi(F(x), F(x_1), \cdots, F(x_n)) \\ &\Longleftrightarrow \forall z \psi(z, F(x_1), \cdots, F(x_n)). \end{aligned}$$

Similarly,

$$0 \in [\![\exists x \psi(x, x_1, \cdots, x_n)]\!] \Longleftrightarrow \exists z \psi(z, F(x_1), \cdots, F(x_n)).$$

□

### *4.7. "Completeness" of* **LZFZ**

Now we will prove in **LZFZ** that $\mathcal{P}(1)$ is lattice-isomorphic to a complete lattice $H$ which is a check set. (Theorem 4.10). As mentioned in the introduction, we mean by "a sentence $\varphi$ of **LZFZ** is valid" that

"$[\![\varphi]\!] = 1$ on $V^{\mathcal{L}}$ for all complete lattice $\mathcal{L}$" is provable in **ZFC** .

Then the "completeness" of **LZFZ** in the sense that every valid sentense of **LZFZ** is provable in **LZFZ**:

**ZFC** $\vdash$ "$[\![\varphi]\!] = 1$ on $V^{\mathcal{L}}$ for all complete lattice $\mathcal{L}$" $\Longrightarrow$ **LZFZ** $\vdash \varphi$

can be proved (Theorem 4.11).

**Theorem 4.10.** *There exists a complete lattice $H$ which is a check set and a lattice-isomorphism $\rho : \mathcal{P}(1) \to H$.*

*Proof.* Since $\mathcal{P}(1)$ is a global set, there exists a check set $H$ together with a bijection $\rho : \mathcal{P}(1) \to H$, by Theorem 4.7(8). Define operations $\bigwedge$, $\bigvee$ on $H$ as follows:

$$\bigwedge A = \rho(\bigcap_{a \in A} \rho^{-1}(a)), \quad \bigvee A = \rho(\bigcup_{a \in A} \rho^{-1}(a)),$$

$$a \to b = \begin{cases} 1, \text{ if } \rho^{-1}(a) \subset \rho^{-1}(b) \\ 0, \text{ if } \neg(\rho^{-1}(a) \subset \rho^{-1}(b)) \end{cases}$$

for $A \subset H$ such that $\mathrm{ck}(A)$, and $a, b \in H$. Then $\rho$ is a lattice-isomorphism. □

**Theorem 4.11 ("Completeness" of LZFZ).** *If a sentence $\varphi$ is valid in every lattice-valued universe, then $\varphi$ is provable in* **LZFZ***:*

$$\textbf{ZFC} \vdash \text{“}[\![\varphi]\!] = 1 \textit{ on } V^{\mathcal{L}} \textit{ for all complete lattice } \mathcal{L}\text{”} \quad \Longrightarrow \quad \textbf{LZFZ} \vdash \varphi$$

*Proof.* Suppose that a sentence $\varphi$ is valid in every lattice valued universe. This means that "$[\![\varphi]\!] = 1$ on every lattice valued universe" is provable in our external universe of **ZFC**. Since $W$ is isomorphic to $V$, ($\varphi$ is valid in every lattice valued universe)$^W$ is provable is **LZFZ**. Let $H \in W$ be a complete lattice with the basic implication which is lattice-isomorphic to $\mathcal{P}(1)$. That is, there exists a lattice-isomorphism $\rho : \mathcal{P}(1) \to H$. Construct the $H$-valued universe $W^H$ in $W$. Then $[\![\varphi]\!] = 1$ on $W^H$. It follows that $[\![\varphi]\!] = 1$ on $W^{\mathcal{P}(1)}$, and $\varphi$ is provable in **LZFZ** by Theorem 4.9. □

By Theorem 4.11, a sentence $\varphi$ holds in **LZFZ** iff $[\![\varphi]\!] = 1$, on every lattice valued universe $V^{\mathcal{L}}$. Therefore, in order to discuss **LZFZ**, it suffices to discuss the set theory on lattice valued universe.

### *4.8. Another implication*

Let $\to_*$ be any implication defined in the language of **LZFZ**. We define the corresponding $=_*$ and $\in_*$ by induction:

$$u =_* v \overset{\text{def}}{\iff} \forall x(x \in u \to_* x \in_* v) \wedge \forall x(x \in v \to_* x \in_* u)$$
$$u \in_* v \overset{\text{def}}{\iff} \exists x(x \in v \wedge u =_* x).$$

If we assume "$\mathcal{P}(1)$ is a cHa", i.e."$\mathcal{P}(1)$ is distributive", in **LZFZ**, then we have the distributive law of the logic:

$$\varphi \wedge \exists x \psi(x) \longleftrightarrow \exists x(\varphi \wedge \psi(x))$$

In fact: let $[\![\varphi]\!]$ be the truth value of $\varphi$ in $W^{\mathcal{P}(1)}$. then the following sentences are provable in **LZFZ**.

$$\begin{aligned}\varphi \wedge \exists x\psi(x) &\leftrightarrow 0 \in [\![\varphi \wedge \exists x\psi(x)]\!] \\ &\leftrightarrow 0 \in [\![\varphi]\!] \wedge \bigvee_{x\in V^{\mathcal{P}(1)}} [\![\psi(x)]\!] \\ &\leftrightarrow 0 \in \bigvee_{x\in V^{\mathcal{P}(1)}} [\![\varphi \wedge \psi(x)]\!] \\ &\leftrightarrow 0 \in [\![\exists x(\varphi \wedge \psi(x))]\!] \\ &\leftrightarrow \exists x(\varphi \wedge \psi(x))\end{aligned}$$

It follows that the intuitionistic implication $\rightarrow_{\mathrm{I}}$ can be defined by

$$(\varphi \rightarrow_{\mathrm{I}} \psi) \overset{\text{def}}{\Longleftrightarrow} 0 \in \bigcup\{u \in \mathcal{P}(1) \mid \varphi \wedge (0 \in u) \rightarrow \psi\}.$$

The corresponding $=_{\mathrm{I}}$ and $\in_{\mathrm{I}}$ are defined by induction in **LZFZ**:

$$\begin{aligned}u =_{\mathrm{I}} v &\overset{\text{def}}{\Longleftrightarrow} \forall x(x \in u \rightarrow_{\mathrm{I}} x \in_{\mathrm{I}} v) \wedge \forall x(x \in v \rightarrow_{\mathrm{I}} x \in_{\mathrm{I}} u) \\ u \in_{\mathrm{I}} v &\overset{\text{def}}{\Longleftrightarrow} \exists x(x \in v \wedge u =_{\mathrm{I}} x).\end{aligned}$$

Then we have:

**Theorem 4.12.** *It is provable in* **LZFZ** + *"$\mathcal{P}(1)$ is a cHa" that the set theory in $(\rightarrow_{\mathrm{I}}, =_{\mathrm{I}}, \in_{\mathrm{I}})$ is an intuitionistic set theory. that is,*

$$\varphi \wedge \exists x\psi(x) \longleftrightarrow \exists x(\varphi \wedge \psi(x))$$

*and axioms of intuitionistic set theory are provable in* **LZFZ**.

*Proof.* For each axiom $\varphi$ of intuitionistic set theory, $[\![\varphi]\!] = 1$ on $V^{\mathcal{P}(1)}$. cf. [5] □

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