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A note on Spector's quantifier-free rule of extensionality

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Abstract. In this note we show that the so-called weakly extensional arithmetic in all finite types, which is based on a quantifier-free rule of extensionality due to C. Spector and which is of significance in the context of Gödel's functional interpretation, does not satisfy the deduction theorem for additional axioms. This holds already for Π^0_1-axioms. Previously, only the failure of the stronger deduction theorem for deductions from (possibly open) assumptions (with parameters kept fixed) was known.

1. Introduction

Let E-HA$^\omega$ denote the system of extensional intuitionistic arithmetic in all finite types as defined in [5]. Concerning equality, E-HA$^\omega$ only contains equality $=_0$ between numbers as a primitive predicate. For $\rho = 0\rho_k \ldots \rho_1$, $x_1 =_\rho x_2$ is defined as $\forall y_1^{\rho_1}, \ldots, y_k^{\rho_k} (x_1 y_1 \ldots y_k =_0 x_2 y_1 \ldots y_k)$. In the context of Gödel's functional ('Dialectica') interpretation, a variant WE-HA$^\omega$ (weakly extensional intuitionistic arithmetic in all finite types) of E-HA$^\omega$ is of relevance which instead of the extensionality axioms (E) for all types only has the following quantifier-free rule of extensionality

$$\text{QF-ER:}\ \ \frac{A_0 \to s =_\rho t}{A_0 \to r[s] =_\tau r[t]},$$

where A_0 is quantifier-free, $s^\rho, t^\rho, r[x^\rho]^\tau$ are arbitrary terms of the system and $\rho, \tau \in$ are arbitrary types. WE-PA$^\omega$ denotes the variant of WE-HA$^\omega$ with classical logic.

In contrast to (E), Gödel's functional interpretation trivially satisfies QF-ER which was introduced in [4] for that very reason. It has been observed in the

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literature ([5](3.5.15 and 1.6.12), see also [6] for corrections) that WE-HA$^{\omega}$ doesn't satisfy the deduction theorem for 'deductions from open assumptions' (whose free variables are treated as parameters and hence are not permitted as proper variables in the quantifier rules as formulated in [5]).[1]

The argument proceeds as follows: consider

$$f =_1 g \vdash_{\text{WE-HA}^{\omega}} f =_1 g,$$

where f, g are free function variables.

QF-ER yields

$$f =_1 g \vdash_{\text{WE-HA}^{\omega}} \forall z^2(zf =_0 zg).$$

The deduction theorem for derivations under assumptions would yield

$$\vdash_{\text{WE-HA}^{\omega}} f =_1 g \to \forall z^2(zf =_0 zg),$$

which is underivable in WE-HA$^{\omega}$ as follows from [2] and the fact that WE-HA$^{\omega}$ has a functional interpretation in (the weakly extensional version of) Gödel's T.

This, however, leaves it open whether the deduction theorem also fails for assumptions added as axioms, i.e. assumptions which implicitly are understood as universally closed.

In this note we show that the deduction theorem (both for WE-HA$^{\omega}$ as well as for WE-PA$^{\omega}$) already fails for Π^0_1-axioms.

2. Results

Theorem 1. *There exists a Π^0_1-sentence A and a quantifier-free formula B such that*

$$\text{WE-HA}^{\omega} + A \vdash B, \ \textit{but} \ \text{WE-PA}^{\omega} \nvdash A \to B.$$

Proof. Let Con_{PA} the standard consistency predicate for Peano arithmetic PA. In WE-HA$^{\omega}$, Con_{PA} can be written as $A :\equiv \forall x^0(t_{\text{PA}}x =_0 0)$ for a suitable closed term t_{PA} of WE-HA$^{\omega}$.

$$\text{WE-HA}^{\omega} + A \vdash t_{\text{PA}} =_1 0^1,$$

where $0^1 := \lambda x^0.0^0$. By QF-ER we obtain

$$\text{WE-HA}^{\omega} + A \vdash x^2(t_{\text{PA}}) =_0 x(0^1),$$

where x^2 is a free variable of type 2. Let's assume now that

$$(*)\ \text{WE-PA}^{\omega} \vdash A \to x^2(t_{\text{PA}}) =_0 x(0^1).$$

[1] In order to avoid this consequence, Troelstra uses a weaker form of QF-ER where the premise of the rule is required to be derivable without assumptions. In this paper we deal with Spector's original rule and our definition of WE-HA$^{\omega}$ thereby differs from Troelstra's definition in [5]. The deduction theorem for deductions from assumption, however, does hold – under an appropriate variable condition – for the quantifier-free fragment qf-WE-HA$^{\omega}$ of WE-HA$^{\omega}$ (see [1]).

Then a fortiori

$$\text{WE-PA}^\omega \vdash A \to \forall x \leq_2 1^2(x(t_{\text{PA}}) =_0 x(0^1))$$

and hence

$$\text{WE-PA}^\omega \vdash \forall x \leq_2 1^2 \exists y^0 \big(t_{\text{PA}} y =_0 0 \to x(t_{\text{PA}}) =_0 x(0^1)\big),$$

where $1^2 := \lambda x^1.S0$ and $x_1 \leq_2 x_2 :\equiv \forall y^1(x_1 y \leq_0 x_2 y)$. By corollary 3.4 from [3] there exists a closed term s^0 of WE-HA$^\omega$ such that

$$\text{WE-HA}^\omega \vdash \forall y \leq_0 s(t_{\text{PA}} y =_0 0) \to \forall x \leq_2 1(x(t_{\text{PA}}) =_0 x(0^1)).$$

By the computability of every fixed closed term s in WE-HA$^\omega$, there exists a number $n \in \mathbb{N}$ such that

$$\text{WE-HA}^\omega \vdash s =_0 \overline{n}.$$

Since (by Σ^0_1-completeness of WE-HA$^\omega$)

$$\text{WE-HA}^\omega \vdash \forall y \leq_0 \overline{n}(t_{\text{PA}} y =_0 0),$$

we get

$$\text{WE-HA}^\omega \vdash \forall x \leq_2 1^2(x(t_{\text{PA}}) =_0 x(0^1))$$

and therefore

$$\text{WE-HA}^\omega \vdash t_{\text{PA}} =_1 0, i.e.$$

$$\text{WE-HA}^\omega \vdash Con_{\text{PA}},$$

which contradicts Gödel's second incompleteness theorem, since WE-HA$^\omega$ is conservative over Heyting arithmetic HA (as follows by formalizing the model HEO of all hereditarily effective operations in HA, see [5]). Hence $(*)$ above is false. So the theorem holds with $B :\equiv \big(x^2(t_{\text{PA}}) =_0 x(0)\big)$ and A as above.

Corollary 1. *The deduction theorem for both* WE-PA$^\omega$ *and* WE-HA$^\omega$ *fails already for closed Π^0_1-axioms.*

Remark 1. The argument above can be applied also to stronger systems which allow a functional interpretation by majorizable functionals. Then we have to use a consistency predicate for a sufficiently strong system.

Final comments. The failure of the deduction theorem for WE-PA$^\omega$ (already for Π^0_1-axioms) might suggest that a system like Troelstra's [5] PA$^\omega$(=(HA$^\omega$)^c) which is neutral with respect to extensionality but still only contains equality for numbers as a primitive predicate, would be more favorable in the context of functional interpretation. However, we believe that for applications to mathematics and the extraction of data from given proofs it is desirable to have as much extensionality avalaible as possible. If we work in WE-PA$^\omega + A$ and want to shift A to an implicative premise of the conclusion, then we can do this provided that we restrict $+$ to $\oplus$ where WE-PA$^\omega \oplus A$ means that A must not be used in the proof of the premise of an application of QF-ER. This is a less severe restriction than to work in PA$^\omega + A$.

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