Archive for Mathematical Logic (2024) 63:921–940
https://doi.org/10.1007/s00153-024-00928-3

Mathematical Logic

Positive indiscernibles

Mark Kamsma[1]

Received: 27 June 2023 / Accepted: 16 April 2024 / Published online: 10 May 2024

Abstract

We generalise various theorems for finding indiscernible trees and arrays to positive logic: based on an existing modelling theorem for s-trees, we prove modelling theorems for str-trees, str_0-trees (the reduct of str-trees that forgets the length comparison relation) and arrays. In doing so, we prove stronger versions for basing—rather than locally basing or EM-basing—str-trees on s-trees and str_0-trees on str-trees. As an application we show that a thick positive theory has k-TP_2 iff it has 2-TP_2

Keywords Positive logic · Generalised indiscernibles · Modelling theorem · Indiscernible tree · Indiscernible array · Tree property

Mathematics Subject Classification 03C45 (Primary) · 03C95 · 03B20 (Secondary)

Contents

1 Introduction

As model theory in positive logic is maturing [1, 2, 5–7, 10] the need for the development of tools available to us in full first-order logic becomes more and more necessary. An important notion in model-theoretic arguments is that of indiscernibles. The most popular occurrence of this is an indiscernible sequence: a sequence where any two finite subsequences have the same type. The notion of indiscernibility can be generalised by replacing the linear order that indexes an indiscernible sequence by another

✉ Mark Kamsma
mark@markkamsma.nl

1 Department of Mathematics, Imperial College London, London, UK

indexing structure. In this paper we consider various kinds of trees and an array as indexing structure. The main results are modelling theorems for positive logic, which allow us to find indiscernibles indexed by trees or arrays based on an arbitrary set of parameters indexed by the same structure, while inheriting certain local structure.

Tree indiscernibles have already found deep applications in model theory. For example, in the development of Kim-independence for NSOP_1 theories [7, 13]. Their original motivation stems from the proof of [17, Theorem III.7.11], which is the celebrated theorem that a theory is simple iff it is NTP_1 and NTP_2. In that proof the existence of tree indiscernibles is claimed, and the details are filled in [12].

As any full first-order theory can be seen as a (thick) positive theory (see Remark 2.2), our main results are a direct generalisation of existing results for full first-order logic [12, 15, 18]. The main difficulty in generalising the proofs is that arguments using Ramsey's theorem break down. In these arguments one can write down a partial type that expresses indiscernibility formula by formula, using formulas of the form $\forall x_1 x_2(\varphi(x_1) \leftrightarrow \varphi(x_2))$. By compactness one can then reduce to finding parameters that are indiscernible with respect to a finite set of formulas. This can be done using Ramsey's theorem by colouring using types restricted to this finite set of formulas, of which there are finitely many. In positive logic, under the assumption of thickness, we can still write down a partial type expressing indiscernibility. However, this is no longer necessarily done formula by formula. So instead we have to use all possible types as colours, of which there are infinitely many. The solution is to replace the use of Ramsey's theorem by the Erdős-Rado theorem. This makes arguments more complicated, but we also get stronger statements (see Theorems 4.8 and 4.10).

Main results. Our main results are the following three modelling theorems. The thickness assumption is the mild assumption that being an indiscernible sequence is type-definable (Definition 2.6). Further justification and reason for this assumption is given in the discussion after Definitions 2.6 and 4.4. For the other definitions involved, we refer the reader to Sect. 3.

Theorem 1.1 *(str-modelling) Let T be a thick theory. Let $(a_\eta)_{\eta \in \omega^{<\omega}}$ be a tree of tuples of the same length and let C be any set of parameters, then there is a tree $(b_\eta)_{\eta \in \omega^{<\omega}}$ that is str-indiscernible over C and $\mathrm{EM}_{\mathrm{str}}$-based on $(a_\eta)_{\eta \in \omega^{<\omega}}$ over C.*

In the process of proving the above theorem we prove Theorem 4.8, which states that given a tall enough s-indiscernible tree we can base an str-indiscernible tree on it. This statement is interesting on its own because more information is carried over when basing one tree on another, instead of EM-basing trees on each other. Similarly, in proving the following theorem we prove that str_0-trees can be based on str-trees, see Theorem 4.10.

Theorem 1.2 *(str_0-modelling) Let T be a thick theory. Let $(a_\eta)_{\eta \in \omega^{<\omega}}$ be a tree of tuples of the same length and let C be any set of parameters, then there is a tree $(b_\eta)_{\eta \in \omega^{<\omega}}$ that is str_0-indiscernible over C and $\mathrm{EM}_{\mathrm{str}_0}$-based on $(a_\eta)_{\eta \in \omega^{<\omega}}$ over C.*

Theorem 1.3 *(array modelling) Let T be a thick theory. Let $(a_{i,j})_{i,j<\omega}$ be an array of tuples of the same length and let C be any set of parameters, then there is an array $(b_{i,j})_{i,j<\omega}$ that is array-indiscernible over C and $\mathrm{EM}_{\mathrm{ar}}$-based on $(a_{i,j})_{i,j<\omega}$ over C.*

As an application of the array modelling theorem we generalise the following from full first-order logic, completing a task from [6, Remark 7.3].

Theorem 1.4 *Let T be a thick theory. If $\varphi(x, y)$ has k-TP_2 for some $k \geq 2$ then some conjunction $\bigwedge_{i=1}^{n} \varphi(x, y_i)$ has 2-TP_2. Hence, T has k-TP_2 for some $k \geq 2$ iff T has* 2-TP_2.

2 Preliminaries

We only recall the definitions and facts about positive logic that we need, for a more extensive treatment and discussion see [1, 5] and for a more survey-like overview see [7, Section 2].

Definition 2.1 A *positive formula* in a fixed language is one that is obtained from combining atomic formulas using $\wedge$, $\vee$, $\top$, $\bot$ and $\exists$. An *h-inductive sentence* is a sentence of the form $\forall x(\varphi(x) \to \psi(x))$, where $\varphi(x)$ and $\psi(x)$ are positive formulas. A *positive theory* is a set of h-inductive sentences.

Remark 2.2 Full first-order logic can be studied as a special case of positive logic. This is done through *Morleyisation*. For this we add a relation symbol $R_\varphi(x)$ to our language for every full first-order formula $\varphi(x)$, and have our theory (inductively) express that $R_\varphi(x)$ and $\varphi(x)$ are equivalent. This way every first-order formula is equivalent to a relation symbol, and thus in particular to a positive formula.

We are interested in *positively closed* models. There are various characterisations (see e.g. [7, Definition 2.5], where they are called "existentially closed"), but we only need one.

Definition 2.3 We call a model M of a positive theory T *positively closed* or *p.c.* if for every positive formula $\varphi(x)$, whenever $M \not\models \varphi(a)$ then there is a positive formula $\psi(x)$ that implies $\neg\varphi(x)$ modulo T, i.e. $T \models \neg\exists x(\varphi(x) \wedge \psi(x))$, with $M \models \psi(a)$

A *positive type* will be a set of positive formulas, over some parameter set B, satisfied by some tuple a in some p.c. model M:

$$\mathrm{tp}(a/B) = \{\varphi(x, b) : \varphi \text{ is a positive formula, } M \models \varphi(a, b) \text{ and } b \in B\}.$$

Given a positive theory T, we will work in a monster model $\mathfrak{M}$ of that theory. For this we need to assume the *joint continuation property* or JCP for T. This means that for any two models M_1 and M_2 of T there is a third model N with homomorphisms $M_1 \to N \leftarrow M_2$. This is the positive version of working in a complete theory, and we can always extend a positive theory T to one with JCP by taking the set of all h-inductive sentences that are true in some p.c. model of T.

The monster model $\mathfrak{M}$ of T can then be constructed as usual. We let the reader fix their favourite notion of smallness (e.g., fix a big enough cardinal κ, and let "small" mean $< \kappa$). We recall the properties of a monster model $\mathfrak{M}$:

- *positively closed*, $\mathfrak{M}$ is a p.c. model of T;

- *very homogeneous*, for any small a, b, C we have that $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$ iff there is $f \in \mathrm{Aut}(\mathfrak{M}/C)$ with $f(a) = b$ (we will also write $a \equiv_C b$);
- *very saturated*, any finitely satisfiable small set of positive formulas Σ over $\mathfrak{M}$ is satisfiable in $\mathfrak{M}$.

As usual, we will omit the monster model from notation. For example, we write $\models \varphi(a)$ instead of $\mathfrak{M} \models \varphi(a)$.

We stress that the monster is only saturated for sets of *positive* formulas, and so we can only apply compactness to such sets. This is where the main challenges in generalising arguments from the full first-order setting stem from.

Definition 2.4 A sequence $(a_i)_{i \in I}$ (for some linear order I) is C*-indiscernible* if for any $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$ in I we have that $a_{i_1} \ldots a_{i_n} \equiv_C a_{j_1} \ldots a_{j_n}$.

Definition 2.5 We write $\mathrm{d}_C(a, a') \leq n$ if there are $a = a_0, a_1, \ldots, a_n = a'$ such that a_i and a_{i+1} are on a C-indiscernible sequence for all $0 \leq i < n$, and we say that a and a' have *Lascar distance* at most n.

Definition 2.6 ([3, Proposition 1.5]) A positive theory T is called *thick* if the following equivalent conditions hold:

(i) being an indiscernible sequence is type-definable, i.e. there is a partial type $\Theta((x_i)_{i<\omega})$ such that $\models \Theta((a_i)_{i<\omega})$ iff $(a_i)_{i<\omega}$ is an indiscernible sequence;
(ii) the property $\mathrm{d}_z(x, y) \leq n$ is type-definable for all $n < \omega$, i.e. for all $n < \omega$ there is a partial type $\Sigma_n(x, y, z)$ such that $\models \Sigma_n(a, a', C)$ iff $\mathrm{d}_C(a, a') \leq n$.

Note that it is essential in Definition 2.6 that we evaluate the partial types in a p.c. model, which we indeed do by working in a monster model.

Thickness is a mild assumption, as it is satisfied by many classes of positive theories.

- Any full first-order theory, seen as a positive theory (Remark 2.2), is thick.
- Any continuous theory in the sense of [4] is Hausdorff, and therefore thick.
- Jonsson theories, and even the positive Jonsson theories from [14], by definition have the property that any span $M_1 \leftarrow M_0 \rightarrow M_2$ of homomorphisms between models can be amalgamated. This implies that these theories are Hausdorff by [5, Theoreme 20]), and so in particular they are thick.
- Adding hyperimaginaries, such as the T^{heq} construction, preserves thickness, see [7, Theorem 10.17].
- Specific examples include the positive NIP theories in [8, Section 6.3] and the positive theory of exponential fields from [10]. These are both in the class of positive Jonsson theories. Another specific example, which is not in any of the classes mentioned before, is the positive theory of bilinear spaces over a fixed infinite field from [11], which is semi-Hausdorff and thus thick (see Proposition 4.14 there).

Furthermore, neostability theory, where results like the ones in this paper often find their applications, works best under the thickness assumption. For example, without thickness a stable theory may not be simple in the sense that local character fails for dividing [2, Example 4.3]. In [7] thickness is crucial for the existence of Lascar-invariant types, which are the basis for Kim-dividing in positive NSOP_1 theories.

Conventions.

- Whenever we say "formula", "type" or "theory" we will mean "positive formula", "positive type" and "positive theory" respectively, unless explicitly stated otherwise. This also means that every formula, type and theory we consider will be implicitly assumed to be positive.
- Let I be some indexing set and suppose that we have an I-indexed set of variables $(x_i)_{i \in I}$ or parameters $(a_i)_{i \in I}$. Then for any tuple $\bar{\eta} = (\eta_1, \ldots, \eta_n)$ in I we write $x_{\bar{\eta}}$ and $a_{\bar{\eta}}$ for the tuples $(x_{\eta_1}, \ldots, x_{\eta_n})$ and $(a_{\eta_1}, \ldots, a_{\eta_n})$ respectively.

3 Generalised indiscernibles

The following idea stems from [17, Definition VII.2.4].

Definition 3.1 Let $\mathcal{L}$ be a language (which we always assume to include the equality symbol), I an $\mathcal{L}$-structure, and $(a_i)_{i \in I}$ an I-indexed set of parameters. We will refer to I as the *indexing structure*. Let C be any parameter set. We say that $(a_i)_{i \in I}$ is *I-indiscernible over C* if for any two tuples $\bar{\eta}$ and $\bar{\nu}$ in I we have that

$$\mathrm{qftp}_{\mathcal{L}}(\bar{\eta}) = \mathrm{qftp}_{\mathcal{L}}(\bar{\nu}) \implies a_{\bar{\eta}} \equiv_C a_{\bar{\nu}},$$

where $\mathrm{qftp}_{\mathcal{L}}(\bar{\eta})$ is the quantifier-free $\mathcal{L}$-type of $\bar{\eta}$.

At the end of Sect. 2 we made a convention about every formula and type being positive. For the quantifier-free $\mathcal{L}$-types in the indexing structures this has no real effect. This is because we will only be interested in whether or not such quantifier-free $\mathcal{L}$-types are equal, and two tuples have the same quantifier-free $\mathcal{L}$-type in the full first-order sense if and only if they satisfy the same atomic $\mathcal{L}$-formulas. In other words, for the quantifier-free $\mathcal{L}$-formulas, where $\mathcal{L}$ is the language of some indexing structure, we may as well allow the $\neg$ symbol.

Example 3.2 We let $\mathcal{L}_< = \{<\}$ be the language with a single ordering symbol and consider ω as an $\mathcal{L}_<$-structure with the usual ordering. Then $(a_i)_{i<\omega}$ being ω-indiscernible over C means precisely that $(a_i)_{i<\omega}$ is a C-indiscernible sequence.

Definition 3.3 Let $\mathcal{L}$ be some language and let $(a_i)_{i \in I}$ and $(b_i)_{i \in I'}$ be two sets of parameters indexed by $\mathcal{L}$-structures I and I' respectively. Let furthermore C be any parameter set. We say that *$(b_i)_{i \in I'}$ is $\mathcal{L}$-based on $(a_i)_{i \in I}$ over C* if for any finite tuple $\bar{\eta}$ in I' there is a tuple $\bar{\nu}$ in I such that $\mathrm{qftp}_{\mathcal{L}}(\bar{\nu}) = \mathrm{qftp}_{\mathcal{L}}(\bar{\eta})$ and $b_{\bar{\eta}} \equiv_C a_{\bar{\nu}}$.

We say that *$(b_i)_{i \in I'}$ is locally $\mathcal{L}$-based on $(a_i)_{i \in I}$ over C* if for any finite tuple $\bar{\eta}$ in I' and any formula $\varphi(x_{\bar{\eta}})$ over C such that $\models \varphi(b_{\bar{\eta}})$ there is a tuple $\bar{\nu}$ in I such that $\mathrm{qftp}_{\mathcal{L}}(\bar{\nu}) = \mathrm{qftp}_{\mathcal{L}}(\bar{\eta})$ and $\models \varphi(a_{\bar{\nu}})$.

We note the difference in terminology from [12, Definition 3.8], where "based on" is used for what we call "locally based on", a distinction that is further promoted in [15, Definition 2.5]. The difference is important, see Example 3.11. Another difference is that we call it "(locally) $\mathcal{L}$-based on" instead of "(locally) I-based on", this is because we wish to compare sets of parameters indexed by different structures in the same language.

Remark 3.4 An alternative formulation to $(b_i)_{i\in I'}$ being locally $\mathcal{L}$-based on $(a_i)_{i\in I}$ over C is the following (see e.g. [15, Definition 2.5]). For any finite set of formulas Φ with parameters in C denote by $\text{tp}_\Phi(a/C)$ the restriction of $\text{tp}(a/C)$ to the formulas in Φ. Then $(b_i)_{i\in I'}$ is locally $\mathcal{L}$-based on $(a_i)_{i\in I}$ over C if for every finite set Φ of formulas over C and any finite tuple $\bar{\eta}$ in I' there is a finite tuple $\bar{\nu}$ in I with the same quantifier-free $\mathcal{L}$-type such that $\text{tp}_\Phi(b_{\bar{\eta}}/C) = \text{tp}_\Phi(a_{\bar{\nu}}/C)$. This is indeed an equivalent formulation, even in positive logic. It clearly implies the formulation in Definition 3.3, so we prove the converse.

Let $\varphi \in \Phi$ be some formula and let $\bar{\eta}' \subseteq \bar{\eta}$ be a tuple whose length matches the number of free variables in φ. Define $\psi_{\varphi,\bar{\eta}'}(x_{\bar{\eta}'})$ as follows: if $\models \varphi(b_{\bar{\eta}'})$ then take $\psi_{\varphi,\bar{\eta}'}(x_{\bar{\eta}'})$ to be $\varphi(x_{\bar{\eta}'})$, otherwise there is $\chi(x_{\bar{\eta}'})$ that implies $\neg\varphi(x_{\bar{\eta}'})$ modulo T such that $\models \chi(b_{\bar{\eta}'})$ and we take $\psi_{\varphi,\bar{\eta}'}(x_{\bar{\eta}'})$ to be $\chi(x_{\bar{\eta}'})$. Now let

$$\psi(x_{\bar{\eta}}) = \bigwedge\{\psi_{\varphi,\bar{\eta}'}(x_{\bar{\eta}'}) : \varphi \in \Phi \text{ and } \bar{\eta}' \subseteq \bar{\eta} \text{ matching the free variables in } \varphi\}.$$

By construction $\models \psi(b_{\bar{\eta}})$ and so there is $\bar{\nu}$ in I such that $\models \psi(a_{\bar{\nu}})$, from which it follows that $\text{tp}_\Phi(b_{\bar{\eta}}/C) = \text{tp}_\Phi(a_{\bar{\nu}}/C)$.

Example 3.5 We recall the following fact for finding indiscernible sequences [2, Lemma 1.2]. Let C be any parameter set and let κ be any cardinal, and set $\lambda = \beth_{(2^{|T|+|C|+\kappa})^+}$. Then for any sequence $(a_i)_{i<\lambda}$ of κ-tuples there is a C-indiscernible sequence $(b_i)_{i<\omega}$ such that for all $n < \omega$ there are $i_1 < \ldots < i_n < \lambda$ with $b_1 \ldots b_n \equiv_C a_{i_1} \ldots a_{i_n}$.

With C, κ and λ as above, this is exactly saying that for any sequence $(a_i)_{i<\lambda}$ there is C-indiscernible $(b_i)_{i<\omega}$ that is $\mathcal{L}_<$-based on $(a_i)_{i<\lambda}$ over C.

Definition 3.6 Let I be a structure in some language $\mathcal{L}$, let $(a_i)_{i\in I}$ be an I-indexed set of parameters and let C be some set of parameters. We write

$$\begin{aligned}\text{EM}_{\mathcal{L}}((a_i)_{i\in I}/C) = \{\varphi(x_{\bar{\eta}}) \ &: \varphi(x_{\bar{\eta}}) \text{ is a formula with parameters in } C \text{ and} \\ &\models \varphi(a_{\bar{\nu}}) \text{ for all } \bar{\nu} \text{ with } \text{qftp}_{\mathcal{L}}(\bar{\nu}) = \text{qftp}_{\mathcal{L}}(\bar{\eta})\}.\end{aligned}$$

for the $\text{EM}_{\mathcal{L}}$*-type of* $(a_i)_{i\in I}$ *over* C, which is a partial type in variables $(x_i)_{i\in I}$. For $\varphi(x_{\bar{\eta}}) \in \text{EM}_{\mathcal{L}}((a_i)_{i\in I}/C)$ we call $\text{qftp}_{\mathcal{L}}(\bar{\eta})$ the *associated quantifier-free* $\mathcal{L}$*-type*.

Let I' be a second $\mathcal{L}$-structure. We say that $(b_i)_{i\in I'}$ *is* $\text{EM}_{\mathcal{L}}$*-based on* $(a_i)_{i\in I}$ *over* C if the following holds: for any $\varphi \in \text{EM}_{\mathcal{L}}((a_i)_{i\in I}/C)$ with associated quantifier-free $\mathcal{L}$-type Δ, and any tuple $\bar{\eta}$ in I' realising Δ we have that $\models \varphi(b_{\bar{\eta}})$.

The name "EM-type" is short for "Ehrenfeucht-Mostowski type" and the above is a straightforward generalisation from the traditional case where the indexing structure is a linear order (see e.g. [19, Definition 5.1.2]).

It could be the case that $(b_i)_{i\in I'}$ is (locally) $\mathcal{L}$-based or $\text{EM}_{\mathcal{L}}$-based on $(a_i)_{i\in I}$ because I does not realise a quantifier-free $\mathcal{L}$-type that is realised in I', or vice versa.

For example, for any sequence $(a_i)_{i<\omega}$ we have that a_0 (as a singleton indexed set) is $\mathcal{L}_<$-based on $(a_i)_{i<\omega}$. To circumvent this issue we introduce the following definition, which will be satisfied by all our indexing structures of interest.

Definition 3.7 Two $\mathcal{L}$-structures I and I' are *qftp-comparable* if I' realises every quantifier-free $\mathcal{L}$-type in finitely many variables that is realised in I, and vice versa.

Remark 3.8 Let I, I' and I'' be pairwise qftp-comparable $\mathcal{L}$-structures. Suppose that $(d_i)_{i\in I''}$ is $\mathcal{L}$-based on $(b_i)_{i\in I'}$ over some parameter set C and $(b_i)_{i\in I'}$ is $\mathcal{L}$-based on $(a_i)_{i\in I}$ over C. Then $(d_i)_{i\in I''}$ is $\mathcal{L}$-based on $(a_i)_{i\in I}$ over C. We call this "transitivity of $\mathcal{L}$-basing", and similarly for locally $\mathcal{L}$-basing and $\mathrm{EM}_{\mathcal{L}}$-basing.

Remark 3.9 Let I and I' be two qftp-comparable $\mathcal{L}$-structures. Then we can equivalently be phrase the definition of being $\mathrm{EM}_{\mathcal{L}}$-based as follows. Define

$$\begin{aligned}\Sigma((x_i)_{i\in I'}) = \{\varphi(x_{\bar{\eta}}) \ :\ & \bar{\eta} \text{ is a finite tuple in } I' \text{ and}\\ & \text{for some tuple } \bar{\nu} \text{ in } I \text{ with } \mathrm{qftp}_{\mathcal{L}}(\bar{\nu}) = \mathrm{qftp}_{\mathcal{L}}(\bar{\eta})\\ & \text{we have that } \varphi(x_{\bar{\nu}}) \in \mathrm{EM}_{\mathcal{L}}((a_i)_{i\in I}/C)\}.\end{aligned}$$

Then $(b_i)_{i\in I'}$ is $\mathrm{EM}_{\mathcal{L}}$-based on $(a_i)_{i\in I}$ over C iff $\Sigma((x_i)_{i\in I'}) \subseteq \mathrm{EM}_{\mathcal{L}}((b_i)_{i\in I'}/C)$ iff $\models \Sigma((b_i)_{i\in I'})$. Note that when $I = I'$ then $\Sigma((x_i)_{i\in I}) = \mathrm{EM}_{\mathcal{L}}((a_i)_{i\in I}/C)$, so in that case we get that $(b_i)_{i\in I}$ is $\mathrm{EM}_{\mathcal{L}}$-based on $(a_i)_{i\in I}$ over C iff $\mathrm{EM}_{\mathcal{L}}((a_i)_{i\in I}/C) \subseteq \mathrm{EM}_{\mathcal{L}}((b_i)_{i\in I}/C)$ iff $(b_i)_{i\in I} \models \mathrm{EM}_{\mathcal{L}}((a_i)_{i\in I}/C)$.

Proposition 3.10 *Let $(a_i)_{i\in I}$ and $(b_i)_{i\in I'}$ be sets of parameters indexed by qftp-comparable $\mathcal{L}$-structures I and I' respectively. For any parameter set C we have that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) as below:*

(i) $(b_i)_{i\in I'}$ *is $\mathcal{L}$-based on* $(a_i)_{i\in I}$ *over C,*
(ii) $(b_i)_{i\in I'}$ *is locally $\mathcal{L}$-based on* $(a_i)_{i\in I}$ *over C,*
(iii) $(b_i)_{i\in I'}$ *is $\mathrm{EM}_{\mathcal{L}}$-based on* $(a_i)_{i\in I}$ *over C.*

Proof The implication (i) $\implies$ (ii) is clear (even without the assumption of being qftp-comparable), we prove (ii) $\implies$ (iii). So let $\varphi \in \mathrm{EM}_{\mathcal{L}}((a_i)_{i\in I}/C)$ and let Δ be the associated quantifier-free $\mathcal{L}$-type. Let $\bar{\eta}$ be a tuple in I' satisfying Δ, which exists by qftp-comparability. Suppose for a contradiction that $\not\models \varphi(b_{\bar{\eta}})$. Then there is $\psi(\bar{x})$ that implies $\neg\varphi(\bar{x})$ modulo T such that $\models \psi(b_{\bar{\eta}})$. As $(b_i)_{i\in I'}$ is locally $\mathcal{L}$-based on $(a_i)_{i\in I}$ over C there must then be $\bar{\nu}$ with $\mathrm{qftp}_{\mathcal{L}}(\bar{\nu}) = \mathrm{qftp}_{\mathcal{L}}(\bar{\eta})$ such that $\models \psi(a_{\bar{\nu}})$. However, that means that $\not\models \varphi(a_{\bar{\nu}})$, contradicting $\varphi \in \mathrm{EM}_{\mathcal{L}}((a_i)_{i\in I}/C)$. □

Example 3.11 None of the implications in Proposition 3.10 are reversible, not even assuming thickness. As an example we will consider a language with countably many constants $(c_i)_{i<\omega}$ and the structure M consisting of only the constants, which are all interpreted as distinct elements.

First we consider the full first-order theory of M and work in a monster model of this theory (in the full first-order sense). This theory has quantifier elimination, because every type is determined by its quantifier-free part. Let $(a_i)_{i<\omega}$ be a sequence of distinct elements in the monster that are not equal to any of the constant symbols. Then $(a_i)_{i<\omega}$

is locally $\mathcal{L}_<$-based on the sequence $(c_i)_{i<\omega}$ that enumerates the constants, but it is not $\mathcal{L}_<$-based on this sequence (in both cases over $\emptyset$).

Next we consider the positive theory T of M (i.e. all h-inductive sentences true in M). So T just expresses that all the constants are distinct, and M is the only p.c. model of T (up to isomorphism) and is thus the monster. Consider the sequence $(b_i)_{i<\omega}$ with constant value c_0, so $b_i = c_0$ for all $i < \omega$. Then $(b_i)_{i<\omega}$ is $\mathrm{EM}_{\mathcal{L}_<}$-based on $(c_i)_{i<\omega}$ but it is not locally $\mathcal{L}_<$-based on $(c_i)_{i<\omega}$ (in both cases over $\emptyset$).

Note that in the last case we really needed to consider an example in positive logic. The key is that $x_0 \neq x_1$ is not a positive formula and is thus not in the $\mathrm{EM}_{\mathcal{L}_<}$-type. In fact, as is well known, in full first-order logic we do have that (ii) and (iii) from proposition 3.10 are equivalent (for qftp-comparable structures). We include a proof for completeness' sake, and to point out the usage of negation.

Let $\bar{\eta}$ be any finite tuple in I' and let $\varphi(x_{\bar{\eta}})$ be any formula over C such that $\models \varphi(b_{\bar{\eta}})$. Suppose for a contradiction that there is no tuple $\bar{\nu}$ in I with $\mathrm{qftp}_{\mathcal{L}}(\bar{\nu}) = \mathrm{qftp}_{\mathcal{L}}(\bar{\eta})$ such that $\models \varphi(a_{\bar{\nu}})$. Then $\models \neg\varphi(a_{\bar{\nu}})$ for all such $\bar{\nu}$. Now picking one such $\bar{\nu}$, which exists by qftp-comparability, we get $\neg\varphi(x_{\bar{\nu}}) \in \mathrm{EM}_{\mathcal{L}}((a_i)_{i\in I}/C)$. This is a contradiction, because then $\models \neg\varphi(b_{\bar{\eta}})$ as $(b_i)_{i\in I'}$ is $\mathrm{EM}_{\mathcal{L}}$-based on $(a_i)_{i\in I}$ over C.

Remark 3.12 Versions of the proof of Proposition 3.10 and the argument at the end of Example 3.11 reveal an equivalent formulation of being $\mathrm{EM}_{\mathcal{L}}$-based, which is a negative version of being locally $\mathcal{L}$-based. Namely, $(b_i)_{i\in I'}$ is $\mathrm{EM}_{\mathcal{L}}$-based on $(a_i)_{i\in I}$ over C if for any finite tuple $\bar{\eta}$ in I' and any formula $\varphi(x_{\bar{\eta}})$ over C such that $\not\models \varphi(b_{\bar{\eta}})$ there is a tuple $\bar{\nu}$ in I such that $\mathrm{qftp}_{\mathcal{L}}(\bar{\nu}) = \mathrm{qftp}_{\mathcal{L}}(\bar{\eta})$ and $\not\models \varphi(a_{\bar{\nu}})$.

We will continue working with the EM-type perspective, because that makes it immediately clear that type-definable behaviour is captured and thus carried over.

Remark 3.13 In Example 3.5 we saw that we can base an indiscernible sequence on a sufficiently long sequence. In Example 3.11 we saw examples of indiscernible sequences that are $\mathrm{EM}_{\mathcal{L}_<}$-based on sequences of length ω (the sequences $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ there are indiscernible). This can always be done and illustrates the use of EM-types.

That is, for any sequence $(a_i)_{i<\omega}$ of tuples of the same length and any parameter set C there is a C-indiscernible sequence $(b_i)_{i<\omega}$ that is $\mathrm{EM}_{\mathcal{L}_<}$-based on $(a_i)_{i<\omega}$ over C. To see this we let λ be the cardinal from Example 3.5. Then we define

$$\begin{aligned}\Sigma((x_i)_{i<\lambda}) = \{\varphi(x_{i_1}, \ldots, x_{i_n}) \ : \ & \varphi(x_1, \ldots, x_n) \in \mathrm{EM}_{\mathcal{L}_<}((a_i)_{i<\omega}/C) \\ & \text{and } i_1 < \ldots < i_n < \lambda\},\end{aligned}$$

note that this is the same construction of Σ as in Remark 3.9, just specialised to $\mathcal{L}_<$. Let $(a'_i)_{i<\lambda}$ be any realisation of Σ, which exists by compactness because every finite part of Σ is realised by $(a_i)_{i<\omega}$. Then use Example 3.5 to $\mathcal{L}_<$-base a C-indiscernible sequence $(b_i)_{i<\omega}$ on $(a'_i)_{i<\lambda}$. By Proposition 3.10 this means that $(b_i)_{i<\omega}$ is in particular $\mathrm{EM}_{\mathcal{L}_<}$-based on $(a'_i)_{i<\lambda}$ over C, which is in turn $\mathrm{EM}_{\mathcal{L}_<}$-based on $(a_i)_{i<\omega}$ over C by construction. Since EM-basing is transitive, we conclude that $(b_i)_{i<\omega}$ is as required.

In the remainder of this section we provide tools to deal with carrying over indiscernibility and EM-types between indexing structures in different languages. We refer to Example 4.3 for examples.

Definition 3.14 ([16, Definition 3.2]) Let $\mathcal{L}$ and $\mathcal{L}'$ be languages and I and I' be structures in those respective languages. We call a function $f : I \to I'$ *qftp-respecting* if for any two finite tuples $\bar{\eta}$ and $\bar{\nu}$ in I we have that $\text{qftp}_{\mathcal{L}}(\bar{\eta}) = \text{qftp}_{\mathcal{L}}(\bar{\nu})$ implies $\text{qftp}_{\mathcal{L}'}(f(\bar{\eta})) = \text{qftp}_{\mathcal{L}'}(f(\bar{\nu}))$.

Lemma 3.15 *(Re-indexing lemma) Let $f : I \to I'$ be a qftp-respecting function between structures I and I' in languages $\mathcal{L}$ and $\mathcal{L}'$ respectively. Let C be any parameter set.*

(i) Let $(a'_i)_{i \in I'}$ and $(a_i)_{i \in I}$ be such that $a_i = a'_{f(i)}$ for all $i \in I$. If $(a'_i)_{i \in I'}$ is I'-indiscernible over C then $(a_i)_{i \in I}$ is I-indiscernible over C.

(ii) Let $(a'_i)_{i \in I'}$, $(b'_i)_{i \in I'}$, $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ be such that $a_i = a'_{f(i)}$ and $b_i = b'_{f(i)}$ for all $i \in I$. If f is surjective and $(b_i)_{i \in I}$ is $\text{EM}_{\mathcal{L}}$-based on $(a_i)_{i \in I}$ over C then $(b'_i)_{i \in I'}$ is $\text{EM}_{\mathcal{L}'}$-based on $(a'_i)_{i \in I'}$ over C.

(iii) Suppose that there is $g : J \to I$, where J is an $\mathcal{L}'$-structure, such that fg is an $\mathcal{L}'$-embedding. Let $(a'_i)_{i \in I'}$, $(b_i)_{i \in I}$, $(a_i)_{i \in I}$ and $(b'_j)_{j \in J}$ be such that $a_i = a'_{f(i)}$ and $b'_j = b_{g(j)}$ for all $i \in I$ and $j \in J$. If $(b_i)_{i \in I}$ is $\text{EM}_{\mathcal{L}}$-based on $(a_i)_{i \in I}$ over C then $(b'_j)_{j \in J}$ is $\text{EM}_{\mathcal{L}'}$-based on $(a'_i)_{i \in I'}$ over C.

Proof

(i) Let $\bar{\eta}$ and $\bar{\nu}$ be tuples in I such that $\text{qftp}_{\mathcal{L}}(\bar{\eta}) = \text{qftp}_{\mathcal{L}}(\bar{\nu})$ then $\text{qftp}_{\mathcal{L}'}(f(\bar{\eta})) = \text{qftp}_{\mathcal{L}'}(f(\bar{\nu}))$ because f is qftp-respecting. Hence, by I'-indiscernibility of $(a'_i)_{i \in I'}$ we have that $a_{\bar{\eta}} = a'_{f(\bar{\eta})} \equiv_C a'_{f(\bar{\nu})} = a_{\bar{\nu}}$.

(ii) Let $\varphi \in \text{EM}_{\mathcal{L}'}((a'_i)_{i \in I'}/C)$ and let Δ' be the associated quantifier-free $\mathcal{L}'$-type. Let $\bar{\eta}'$ be any tuple in I' satisfying Δ'. By surjectivity there is a tuple $\bar{\eta}$ in I such that $f(\bar{\eta}) = \bar{\eta}'$. For any tuple $\bar{\nu}$ in I such that $\text{qftp}_{\mathcal{L}}(\bar{\nu}) = \text{qftp}_{\mathcal{L}}(\bar{\eta})$ we have that $\text{qftp}_{\mathcal{L}'}(f(\bar{\nu})) = \text{qftp}_{\mathcal{L}'}(f(\bar{\eta})) = \Delta'$ because f is qftp-respecting. So $\models \varphi(a'_{f(\bar{\nu})})$, and hence $\models \varphi(a_{\bar{\nu}})$. As $\bar{\nu}$ as arbitrary satisfying $\text{qftp}_{\mathcal{L}}(\bar{\eta})$, we see that $\varphi \in \text{EM}_{\mathcal{L}}((a_i)_{i \in I}/C)$ with $\text{qftp}_{\mathcal{L}}(\bar{\eta})$ as the associated quantifier-free $\mathcal{L}$-type. Since $(b_i)_{i \in I}$ is $\text{EM}_{\mathcal{L}}$-based (over C) on $(a_i)_{i \in I}$, we get that $\models \varphi(b_{\bar{\eta}})$ and hence $\models \varphi(b'_{\bar{\eta}'})$. As $\bar{\eta}'$ was arbitrary satisfying Δ' we conclude that $\varphi \in \text{EM}_{\mathcal{L}'}((b'_i)_{i \in I'}/C)$, as required.

(iii) Let $\varphi \in \text{EM}_{\mathcal{L}'}((a'_i)_{i \in I'}/C)$ and let Δ' be the associated quantifier-free $\mathcal{L}'$-type. Let $\bar{\eta}'$ be any tuple in J satisfying Δ'. Let $\bar{\nu}$ be any tuple in I satisfying $\text{qftp}_{\mathcal{L}}(g(\bar{\eta}'))$. Then $\text{qftp}_{\mathcal{L}'}(f(\bar{\nu})) = \text{qftp}_{\mathcal{L}'}(fg(\bar{\eta}')) = \Delta'$ because f is qftp-respecting and because of our assumption on fg. We thus have that $\models \varphi(a'_{f(\bar{\nu})})$ and so $\models \varphi(a_{\bar{\nu}})$. As $\bar{\nu}$ was arbitrary satisfying $\text{qftp}_{\mathcal{L}}(g(\bar{\eta}'))$ and such $\bar{\nu}$ exists (take $\bar{\nu} = g(\bar{\eta}')$) we get that $\varphi \in \text{EM}_{\mathcal{L}}((a_i)_{i \in I}/C)$ with $\text{qftp}_{\mathcal{L}}(g(\bar{\eta}'))$ as the associated quantifier-free $\mathcal{L}$-type. Since $(b_i)_{i \in I}$ is $\text{EM}_{\mathcal{L}}$-based on $(a_i)_{i \in I}$ over C, we get that $\models \varphi(b_{g(\bar{\eta}')})$ and hence $\models \varphi(b'_{\bar{\eta}'})$. As $\bar{\eta}'$ was arbitrary satisfying Δ' we conclude that $\varphi \in \text{EM}_{\mathcal{L}'}((b'_j)_{j \in J}/C)$, as required.

□

4 The tree modelling theorems

Definition 4.1 ([12, Definition 2.1]) For any ordinals α and β we view the set $\alpha^{<\beta}$ of functions $\eta : \gamma \to \alpha$ with $\gamma < \beta$ as a tree with the usual partial ordering: $\eta \trianglelefteq \nu$ iff ν is an extension of η. We will simultaneously view a function $\eta : \gamma \to \alpha$ as a sequence of elements in α of length γ. We put further structure on $\alpha^{<\beta}$ as follows, where $\eta, \nu \in \alpha^{<\beta}$:

- we write $\eta \wedge \nu$ for the *meet* of η and ν with respect to $\trianglelefteq$, i.e. the largest initial segment shared by η and ν;
- we write $\eta <_{\text{lex}} \nu$ for the *lexicographical ordering*, i.e. either $\eta \triangleleft \nu$ or η and ν are incomparable with respect to $\trianglelefteq$ and for the least ordinal γ such that $\eta(\gamma) \neq \nu(\gamma)$ we have that $\eta(\gamma) < \nu(\gamma)$;
- we write $\ell(\eta)$ for the *length* or *level* of η, i.e. $\ell(\eta)$ is the domain of η;
- we write $\eta <_{\text{len}} \nu$ iff $\ell(\eta) < \ell(\nu)$;
- for $\gamma < \beta$ we let $P_\gamma = \{\eta \in \alpha^{<\beta} : \ell(\eta) = \gamma\}$.

Based on the above we put different structures on $\alpha^{<\beta}$ using the following languages.

- the *Shelah language* $\mathcal{L}_{\text{s}} = \{\trianglelefteq, \wedge, <_{\text{lex}}, (P_\gamma)_{\gamma<\beta}\}$,
- the *strong Shelah language* $\mathcal{L}_{\text{str}} = \{\trianglelefteq, \wedge, <_{\text{lex}}, <_{\text{len}}\}$,
- the language $\mathcal{L}_{\text{str}_0} = \{\trianglelefteq, \wedge, <_{\text{lex}}\}$.

To abbreviate notation we will write "EM_{s}" and "s-based" for "$\text{EM}_{\mathcal{L}_{\text{s}}}$" and "$\mathcal{L}_{\text{s}}$-based". Whenever we consider a tree $\alpha^{<\beta}$ as an $\mathcal{L}_{\text{s}}$-structure we will call it an *s-tree* and write "s-indiscernible" instead of "$\alpha^{<\beta}$-indiscernible". For any two tuples $\bar{\eta}$ and $\bar{\nu}$ in $\alpha^{<\beta}$ we will write $\bar{\eta} \equiv^{\text{s-qf}} \bar{\nu}$ for $\text{qftp}_{\mathcal{L}_{\text{s}}}(\bar{\eta}) = \text{qftp}_{\mathcal{L}_{\text{s}}}(\bar{\nu})$.

We abbreviate notation involving $\mathcal{L}_{\text{str}}$ and $\mathcal{L}_{\text{str}_0}$ in a similar way, replacing the "s" in the above by "str" or "str_0" respectively.

Definition 4.2 For $\gamma_0, \ldots, \gamma_{n-1} \in \alpha$ we write $\langle \gamma_0, \ldots, \gamma_{n-1} \rangle$ for the function $i \mapsto \gamma_i$, which is an element of $\alpha^{<\beta}$. For $\eta, \nu \in \alpha^{<\beta}$ we write $\eta^\frown\nu$ for the concatenation of η and ν. Formally:

$$(\eta^\frown\nu)(i) = \begin{cases} \eta(i) & \text{if } i < \ell(\eta) \\ \nu(j) & \text{if } i = \ell(\eta) + j \end{cases}$$

For $n < \omega$ we will write η^n for the concatenation of η with itself n times:

$$\eta^n = \underbrace{\eta^\frown\eta^\frown \ldots ^\frown\eta}_{n \text{ times}}$$

So $\langle 0 \rangle^n$ is the sequence of n zeroes, or formally: the constant function $n \to \{0\}$.

Example 4.3 We apply the re-indexing lemma (Lemma 3.15) to the different tree structures.

(i) Consider the identity function $f : \omega^{<\omega} \to \omega^{<\omega}$, where the domain carries the structure of an str-tree and the codomain that of an str_0-tree. Then f is just a reduct of structures, and hence qftp-respecting. By Lemma 3.15(i) we thus see that if a tree $(a_\eta)_{\eta\in\omega^{<\omega}}$ is str_0-indiscernible over C then it is also str-indiscernible over C. Similarly, we could consider the domain and codomain of f to be an s-tree and str-tree respectively instead. Now f is not simply a reduct, but it is still qftp-respecting as the predicates $(P_n)_{n<\omega}$ determine the relation $<_{\text{len}}$. So if $(a_\eta)_{\eta\in\omega^{<\omega}}$ is str-indiscernible over C then it is s-indiscernible over C.

(ii) Like in the previous point, we consider the identity function $f : \omega^{<\omega} \to \omega^{<\omega}$ as a qftp-respecting function between an str-structure and str_0-structure, or between an s-structure and an str-structure. We get the following from Lemma 3.15(ii): if $(b_\eta)_{\eta\in\omega^{<\omega}}$ is EM_s-based on $(a_\eta)_{\eta\in\omega^{<\omega}}$ over C then it is also EM_str-based on $(a_\eta)_{\eta\in\omega^{<\omega}}$ over C, and if $(b_\eta)_{\eta\in\omega^{<\omega}}$ is EM_str-based on $(a_\eta)_{\eta\in\omega^{<\omega}}$ over C then it is also EM_{str_0}-based on $(a_\eta)_{\eta\in\omega^{<\omega}}$ over C.

(iii) Let $L \subseteq \omega$ be an infinite set, and enumerate L in order as $\ell_0 < \ell_1 < \dots$. For $\eta \in \omega^{<\omega}$ of length n we define $\nu_\eta \in \omega^{<\omega}$ of length ℓ_n as:

$$\nu_\eta(k) = \begin{cases} \eta(i) & \text{if } k = \ell_{i+1} - 1 \\ 0 & \text{otherwise} \end{cases}$$

Define $f_L : \omega^{<\omega} \to \omega^{<\omega}$ as $f_L(\eta) = \nu_\eta$. The picture to keep in mind here is that of including one tree into another by only having the levels in L taking non-zero values. If we consider the domain and codomain of f_L as s-trees then f_L is qftp-respecting. We write $\omega^{<\omega}\restriction_L$ for the image of f_L with the induced s-structure, and call this *the restriction of $\omega^{<\omega}$ to levels L*. If $(a_\eta)_{\eta\in\omega^{<\omega}}$ is s-indiscernible over C then so is $(a_\eta)_{\eta\in\omega^{<\omega}\restriction_L}$. This construction works in the exact same way for str-trees and str_0-trees.

(iv) Fix some $\eta \in \omega^{<\omega}$ and consider $f_\eta : \omega \to \omega^{<\omega}, i \mapsto \eta^\frown\langle i\rangle$. So f_η injects the linear order ω into the tree by sending it to the immediate successors of η. We consider ω as the usual $\mathcal{L}_<$-structure and $\omega^{<\omega}$ as an s-structure. Then f_η is qftp-respecting. So if $(a_\eta)_{\eta\in\omega^{<\omega}}$ is s-indiscernible over C then $(a'_i)_{i<\omega}$ defined by $a'_i = a_{f_\eta(i)}$ is a C-indiscernible sequence, by Lemma 3.15(i).

The following notation will be useful in proofs. We define it for finite tuples, but it would make sense for infinite tuples.

Definition 4.4 Let $\bar{\eta} = (\eta_1, \dots, \eta_n)$ be a tuple in a tree $\alpha^{<\beta}$. We define:

- $\ell(\bar{\eta}) = \{\ell(\eta_i) : 1 \le i \le n\}$ to be the set of levels of the elements in $\bar{\eta}$,
- $\text{cl}_\wedge(\bar{\eta})$ to be the closure of $\bar{\eta}$ under meets.

The thickness assumption in the main theorems is needed to make the various forms of indiscernibility type-definable, which we make precise below in Proposition 4.5. We also rely on a result from [7] (see Theorem 4.6), whose proof heavily relies on s-indiscernibility being type-definable. While we do not exclude the possibility of achieving the same results without the thickness assumption, it seems unlikely. Meanwhile, the discussion after Definition 2.6 shows the relevance of our results, even with the thickness assumption.

Proposition 4.5 *Let T be a thick theory. The properties of being s-indiscernible, being str-indiscernible and being str$_0$-indiscernible are type-definable. This is done by taking the partial type that states that any two tuples of variables, whose tuples of indices have the same quantifier-free type (in the relevant language), have Lascar distance at most* 2.

More precisely, define the partial type $\pi_{\mathrm{s}}((x_\eta)_{\eta\in\omega^{<\omega}}, y)$ to be

$$\bigcup\{d_y(x_{\bar{\eta}}, x_{\bar{\nu}}) \leq 2 : \bar{\eta}, \bar{\nu} \text{ are finite tuples in } \omega^{<\omega} \textit{ with } \bar{\eta} \equiv^{\text{s-qf}} \bar{\nu}\},$$

define $\pi_{\mathrm{str}}((x_\eta)_{\eta\in\omega^{<\omega}}, y)$ to be

$$\bigcup\{d_y(x_{\bar{\eta}}, x_{\bar{\nu}}) \leq 2 : \bar{\eta}, \bar{\nu} \text{ are finite tuples in } \omega^{<\omega} \textit{ with } \bar{\eta} \equiv^{\text{str-qf}} \bar{\nu}\},$$

and define $\pi_{\mathrm{str}_0}((x_\eta)_{\eta\in\omega^{<\omega}}, y)$ to be

$$\bigcup\{\mathrm{d}_y(x_{\bar{\eta}}, x_{\bar{\nu}}) \leq 2 : \bar{\eta}, \bar{\nu} \text{ are finite tuples in } \omega^{<\omega} \textit{ with } \bar{\eta} \equiv^{\text{str}_0\text{-qf}} \bar{\nu}\}.$$

Then for any parameter set C we have that $\models \pi_{\mathrm{s}}((a_\eta)_{\eta\in\omega^{<\omega}}, C)$ iff $(a_\eta)_{\eta\in\omega^{<\omega}}$ is s-indiscernible over C, and similarly for π_{str} and π_{str_0} and being str-indiscernible and being str$_0$-indiscernible over C.

Proof For π_{s} this is [7, Corollary 5.7]. We prove the case for π_{str} using a similar argument, and the case for π_{str_0} uses the exact same argument as for π_{str}.

If $\models \pi_{\mathrm{str}}((a_\eta)_{\eta\in\omega^{<\omega}}, C)$ then for any finite tuples $\bar{\eta}, \bar{\nu}$ in $\omega^{<\omega}$ with $\bar{\eta} \equiv^{\text{str-qf}} \bar{\nu}$ we have that $\mathrm{d}_C(a_{\bar{\eta}}, a_{\bar{\nu}}) \leq 2$, and so in particular $a_{\bar{\eta}} \equiv_C a_{\bar{\nu}}$.

Conversely, suppose that $(a_\eta)_{\eta\in\omega^{<\omega}}$ is str-indiscernible over C. Let $\bar{\eta}, \bar{\nu}$ be finite tuples in $\omega^{<\omega}$ such that $\bar{\eta} \equiv^{\text{str-qf}} \bar{\nu}$. We may assume that both $\bar{\eta}$ and $\bar{\nu}$ are closed under meets. As $\bar{\eta}$ and $\bar{\nu}$ are both finite, there is some $k < \omega$ such that they are both contained in $k^{<k}$. Write $\bar{\eta} = (\eta_1, \ldots, \eta_n)$. For $i < \omega$ and $1 \leq j \leq n$ we define $\chi_i^j = \langle k \rangle^{(i+1)k} \frown \eta_j$, and write $\bar{\chi}_i = (\chi_i^1, \ldots, \chi_i^n)$. One straightforwardly verifies that $\bar{\eta}, \bar{\chi}_0, \bar{\chi}_1, \ldots$ and $\bar{\nu}, \bar{\chi}_0, \bar{\chi}_1, \ldots$ are indiscernible sequences with respect to quantifier-free $\mathcal{L}_{\mathrm{str}}$-formulas. Then $\mathrm{d}_C(a_{\bar{\eta}}, a_{\bar{\nu}}) \leq 2$ follows from str-indiscernibility of $(a_\eta)_{\eta\in\omega^{<\omega}}$. □

Theorem 4.6 *(s-modelling, [7, Proposition 5.8]) Let T be a thick theory. Let $(a_\eta)_{\eta\in\omega^{<\omega}}$ be a tree of tuples and let C be any set of parameters, then there is a tree $(b_\eta)_{\eta\in\omega^{<\omega}}$ that is s-indiscernible over C and* EM$_{\mathrm{s}}$*-based on $(a_\eta)_{\eta\in\omega^{<\omega}}$ over C.*

Proof The cited result [7, Proposition 5.8] only states the above theorem for trees of finite height (i.e., trees indexed by $\omega^{\leq k}$ for some $k < \omega$). However, type-definability of s-indiscernibility in thick theories is also established there [7, Corollary 5.7]. So the statement here is really just an easy application of compactness, and so we still attribute it to [7]. □

Using the $<_{\mathrm{len}}$ relation in $\mathcal{L}_{\mathrm{str}}$ we get the following fact.

Fact 4.7 *Let $\bar{\eta}$ and $\bar{\nu}$ be finite meet-closed tuples in $\alpha^{<\beta}$, then $\bar{\eta} \equiv^{\text{s-qf}} \bar{\nu}$ iff $\bar{\eta} \equiv^{\text{str-qf}} \bar{\nu}$ and $\ell(\bar{\eta}) = \ell(\bar{\nu})$.*

Theorem 4.8 *Let C be any parameter set, κ be any cardinal, and let $\lambda = \beth_{(2^{|T|+|C|+\kappa})^+}$. Given any tree $(a_\eta)_{\eta\in\omega^{<\lambda}}$ of κ-tuples that is s-indiscernible over C, there is a tree $(b_\eta)_{\eta\in\omega^{<\omega}}$ that is str-indiscernible over C str-based on $(a_\eta)_{\eta\in\omega^{<\lambda}}$ over C.*

The proof of this theorem relies on the Erdős-Rado theorem, which we will recall for the reader's convenience. For $n < \omega$ and cardinals κ, λ, μ the notation $\kappa \to (\lambda)^n_\mu$ means that for any function $f : [\kappa]^n \to \mu$ we can find $X \subseteq \kappa$ with $|X| = \lambda$ such that f is constant on $[X]^n$, where $[\kappa]^n$ and $[X]^n$ are the sets of subsets of size n of κ and X respectively.

Fact 4.9 *(Erdős-Rado) For any $n < \omega$ and any infinite cardinal μ we have that*

$$\beth_n^+(\mu) \to (\mu^+)^{n+1}_\mu.$$

Proof Let S be the set of types over C in $\max(\kappa, \omega)$ many variables. Then λ has the following properties:

(i) λ is a limit cardinal with $\mathrm{cf}(\lambda) > |S|$,
(ii) for all $\mu < \lambda$ and $n < \omega$ there is $\mu' < \lambda$ such that $\mu' \to (\mu)^n_{|S|}$.

Property (i) is immediate, and (ii) follows from Erdős-Rado (Fact 4.9).

For any tuple $\bar{\eta}$ in $\omega^{<\omega}$ and any type $p(x_{\bar{\eta}})$, we write

$$\Sigma_{p,\bar{\eta}}((x_\nu)_{\nu\in\omega^{<\omega}}) = \bigcup\{p(x_{\bar{\nu}}) : \bar{\nu} \equiv^{\text{str-qf}} \bar{\eta}\}$$

for the partial type expressing that in the tree $(x_\nu)_{\nu\in\omega^{<\omega}}$ any tuple indexed by something with the same str-quantifier-free type as $\bar{\eta}$ has type p.

There are countably many quantifier-free $\mathcal{L}_{\text{str}}$-types in finitely many variables. Enumerate the ones that are the type of a meet-closed tuple in $\omega^{<\omega}$ as $(\Delta_i)_{i<\omega}$. For $i < \omega$ we let $\bar{\eta}_i$ be a tuple in $\omega^{<\omega}$ satisfying Δ_i. By induction we will construct types $(p_i(x_{\bar{\eta}_i}))_{i<\omega}$ over C such that:

(1) $\bigcup_{j\leq i} \Sigma_{p_j,\bar{\eta}_j}$ is consistent,
(2) for every $\mu < \lambda$ there is $I \subseteq \lambda$ with $|I| = \mu$ such that for all $j \leq i$ we have that whenever $\bar{\nu} \equiv^{\text{str-qf}} \bar{\eta}_j$ with $\ell(\bar{\nu}) \subseteq I$ then $\models p_j(a_{\bar{\nu}})$.

Suppose that $(p_j)_{j<i}$ has been constructed, we construct p_i. Set $n = |\ell(\bar{\eta}_i)|$. We define $f : [\lambda]^n \to S$ as follows. For $E \in [\lambda]^n$ we let $\bar{\eta}_E$ be some tuple in $\omega^{<\lambda}$ such that $\bar{\eta}_E \equiv^{\text{str-qf}} \bar{\eta}_i$ and $\ell(\bar{\eta}_E) = E$. Then we set

$$f(E) = \mathrm{tp}(a_{\bar{\eta}_E}/C).$$

Let now $\mu < \lambda$ be arbitrary. By (ii) there is $\mu' < \lambda$ such that $\mu' \to (\mu)^n_{|S|}$. By (2) there is $I \subseteq \lambda$ with $|I| = \mu'$ such that for all $j < i$ we have that whenever $\bar{\nu} \equiv^{\text{str-qf}} \bar{\eta}_j$ with $\ell(\bar{\nu}) \subseteq I$ then $\models p_j(a_{\bar{\nu}})$ (in case $i = 0$ we just take $I = \mu'$). We apply $\mu' \to (\mu)^n_{|S|}$

to the restriction of f to I to find $I_\mu \subseteq I$ with $|I_\mu| = \mu$ such that f is constant on $[I_\mu]^n$. We write q_μ for the constant value of f on $[I_\mu]^n$, that is $q_\mu = f(E)$, where $E \in [I_\mu]^n$.

As $\mu < \lambda$ was arbitrary, there is such an I_μ and q_μ for every $\mu < \lambda$. By (i) we then find a cofinal subset $J \subseteq \lambda$ of cardinals such that $q_\mu = q_{\mu'}$ for any $\mu, \mu' \in J$. Set $p_i = q_\mu$, where $\mu \in J$. For part (2) of the induction hypothesis we note that for every $\mu < \lambda$ there is $\mu' \in J$ with $\mu < \mu'$. We verify (2) for $I_{\mu'}$ (technically we would need to take a μ-sized subset, but that we can clearly do). For $j < i$ the required property follows by construction of $I_{\mu'}$, in particular because it is a subset of what we called I in its construction. For $j = i$ the required property follows because f is constant on $[I_{\mu'}]^n$, together with s-indiscernibility of $(a_\eta)_{\eta \in \omega^{<\lambda}}$ and Fact 4.7. Then part (1) follows from part (2): let $\mu \in J$ be any infinite cardinal and let L be the first ω elements of I_μ, then $\bigcup_{j \leq i} \Sigma_{p_j, \bar{\eta}_j}$ is realised by the restriction $(a_\eta)_{\eta \in \omega^{<\lambda}\restriction_L}$ to levels L (see Example 4.3(iii)).

This finishes the inductive construction of $(p_i(x_{\bar{\eta}_i}))_{i<\omega}$. Set

$$\Sigma((x_\eta)_{\eta \in \omega^{<\omega}}) = \bigcup_{i<\omega} \Sigma_{p_i, \bar{\eta}_i}.$$

By (1) this is consistent, so we let $(b_\eta)_{\eta \in \omega^{<\omega}}$ be a realisation, which has the following properties:

(a) for any finite tuple $\bar{\eta}$ in $\omega^{<\omega}$ there is some i such that $\mathrm{cl}_\wedge(\bar{\eta}) \equiv^{\text{str-qf}} \bar{\eta}_i$,
(b) if $\mathrm{cl}_\wedge(\bar{\eta}) \equiv^{\text{str-qf}} \bar{\eta}_i$ then $b_{\mathrm{cl}_\wedge(\bar{\eta})} \models p_i$.

It follows that $(b_\eta)_{\eta \in \omega^{<\omega}}$ is str-indiscernible over C. Let $\bar{\eta} \equiv^{\text{str-qf}} \bar{\nu}$ be finite tuples in $\omega^{<\omega}$. Then $\mathrm{cl}_\wedge(\bar{\eta}) \equiv^{\text{str-qf}} \mathrm{cl}_\wedge(\bar{\nu})$. By (a) there is $i < \omega$ such that $\mathrm{cl}_\wedge(\bar{\eta}) \equiv^{\text{str-qf}} \bar{\eta}_i$. Then by (b) we get that $b_{\mathrm{cl}_\wedge(\bar{\eta})} \models p_i$ and $b_{\mathrm{cl}_\wedge(\bar{\nu})} \models p_i$, so since p_i is a type over C we get $b_{\bar{\eta}} \equiv_C b_{\bar{\nu}}$ after restricting the types. Finally, $(b_\eta)_{\eta \in \omega^{<\omega}}$ is str-based on $(a_\eta)_{\eta \in \omega^{<\lambda}}$: for any finite tuple $\bar{\eta}$ in $\omega^{<\omega}$ we have by (a) that there is $i < \omega$ with $\mathrm{cl}_\wedge(\bar{\eta}) = \bar{\eta}_i$. By construction p_i is realised by $a_{\bar{\nu}'}$ for some $\bar{\nu}'$ in $\omega^{<\lambda}$ with $\bar{\nu}' \equiv^{\text{str-qf}} \bar{\eta}_i$. Using the fact that $\bar{\nu}' \equiv^{\text{str-qf}} \mathrm{cl}_\wedge(\bar{\eta})$ we find $\bar{\nu} \subseteq \bar{\nu}'$ such that $\bar{\nu} \equiv^{\text{str-qf}} \bar{\eta}$. By (b) we have that $\mathrm{tp}(b_{\mathrm{cl}_\wedge(\bar{\eta})}/C) = p_i$, so restricting types yields $b_{\bar{\eta}} \equiv_C a_{\bar{\nu}}$, as required. □

Theorem 1.1 ***repeated.*** *Let T be a thick theory. Let $(a_\eta)_{\eta \in \omega^{<\omega}}$ be a tree of tuples of the same length and let C be any set of parameters, then there is a tree $(b_\eta)_{\eta \in \omega^{<\omega}}$ that is str-indiscernible over C and* $\mathrm{EM}_{\mathrm{str}}$*-based on $(a_\eta)_{\eta \in \omega^{<\omega}}$ over C.*

Proof By Theorem 4.6 we find $(a'_\eta)_{\eta \in \omega^{<\omega}}$ that is s-indiscernible over C and is EM_{s}-based on $(a_\eta)_{\eta \in \omega^{<\omega}}$ over C. Let λ be the cardinal in Theorem 4.8. Using Proposition 4.5 we can write down a partial type Σ for a tree $(a''_\eta)_{\eta \in \omega^{<\lambda}}$ that is s-indiscernible over C and $\mathrm{EM}_{\mathrm{str}}$-based on $(a'_\eta)_{\eta \in \omega^{<\omega}}$ over C. Using $(a'_\eta)_{\eta \in \omega^{<\omega}}$, and renaming levels whenever needed, we see that Σ is finitely satisfiable. So we find our tree $(a''_\eta)_{\eta \in \omega^{<\lambda}}$ that is s-indiscernible over C and $\mathrm{EM}_{\mathrm{str}}$-based on $(a'_\eta)_{\eta \in \omega^{<\omega}}$ over C. We apply Theorem 4.8 to $(a''_\eta)_{\eta \in \omega^{<\lambda}}$ to find $(b_\eta)_{\eta \in \omega^{<\omega}}$ that is str-indiscernible over C and str-based on $(a''_\eta)_{\eta \in \omega^{<\lambda}}$ over C. Being str-based and EM_{s}-based both imply being $\mathrm{EM}_{\mathrm{str}}$-based, and being $\mathrm{EM}_{\mathrm{str}}$-based is transitive, so $(b_\eta)_{\eta \in \omega^{<\omega}}$ is $\mathrm{EM}_{\mathrm{str}}$-based on $(a_\eta)_{\eta \in \omega^{<\omega}}$ over C. □

The proof strategies in Theorems 4.8 and 1.1 are very similar to the case for indiscernible sequences. The use of the Erdős-Rado theorem in Theorem 4.8 is very similar how one constructs an indiscernible sequence based on a very long sequence (see Example 3.5 and the reference there). Then Theorem 1.1 is similar to Remark 3.13: we use compactness to stretch the input and then apply the previous result that uses the Erdős-Rado theorem. One key difference though is that Theorem 4.8 requires the input to already be s-indiscernible. This is why we assume thickness in Theorem 1.1. That way we can use type-definability of s-indiscernibility to guarantee that the stretched input remains s-indiscernible.

Theorem 4.10 *Let C be any parameter set. Given any tree $(a_\eta)_{\eta\in\omega^{<\omega}}$ that is str-indiscernible over C, there is a tree $(b_\eta)_{\eta\in\omega^{<\omega}}$ that is str$_0$-indiscernible over C and str$_0$-based on $(a_\eta)_{\eta\in\omega^{<\omega}}$ over C.*

Proof Fix $1 \leq k < \omega$. By induction on $m < \omega$ we define $f_k^m : k^{\leq m} \to \omega^{<\omega}$ and $l_k^m < \omega$ as in [15, Claim A.7] (which in turn is based on [9, page 142]):

$$
\begin{aligned}
f_k^m(\emptyset) &= \emptyset && \text{for all } m < \omega,\\
l_k^m &= \max\{\ell(f_k^m(\eta)) + 1 : \eta \in k^{\leq m}\},\\
f_k^{m+1}(\langle i\rangle^\frown \eta) &= \langle i\rangle^\frown \langle 0\rangle^{(i+1)l_k^m\frown} f_k^m(\eta).
\end{aligned}
$$

Then for all $k, m < \omega$:

(*) f_k^m is an $\mathcal{L}_{\text{str}_0}$-embedding such that $\eta <_{\text{lex}} \nu$ implies $f_k^m(\eta) <_{\text{len}} f_k^m(\nu)$.

Let $\bar\eta$ be a finite tuple in $\omega^{<\omega}$ and let $k, m < \omega$ be such that $\bar\eta$ is contained in $k^{\leq m}$. Then we assign the type $p_{\bar\eta}(x_{\bar\eta}) = \text{tp}(a_{f_m^k(\bar\eta)}/C)$ to $\bar\eta$. By (*) and str-indiscernibility, the type $p_{\bar\eta}$ does not depend on k or m and whenever $\bar\eta \equiv^{\text{str}_0\text{-qf}} \bar\nu$ then $p_{\bar\eta} = p_{\bar\nu}$ (after renaming variables). Define

$$
\Sigma((x_\eta)_{\eta\in\omega^{<\omega}}) = \bigcup\{p_{\bar\eta} : \bar\eta \text{ is a finite tuple in } \omega^{<\omega}\}.
$$

Let Σ_0 be any finite part of Σ, and let $k, m < \omega$ be such that the variables appearing in Σ_0 are contained in $(x_\eta)_{\eta\in k^{\leq m}}$. Then Σ_0 is realised by $(a'_\eta)_{\eta\in k^{\leq m}}$ where $a'_\eta = a_{f_k^m(\eta)}$ for all $\eta \in k^{\leq m}$. So by compactness we find a realisation $(b_\eta)_{\eta\in\omega^{<\omega}}$ of Σ, which is the tree we needed to construct. Indeed, let $\bar\eta$ be any finite tuple in $\omega^{<\omega}$ and let $k, m < \omega$ be such that $\bar\eta$ is contained in $k^{\leq m}$. Then $\text{tp}(b_{\bar\eta}/C) = p_{\bar\eta} = \text{tp}(a_{f_k^m(\bar\eta)}/C)$, and so because $f_k^m(\bar\eta) \equiv^{\text{str}_0\text{-qf}} \bar\eta$ this shows that $(b_\eta)_{\eta\in\omega^{<\omega}}$ is str$_0$-based on $(a_\eta)_{\eta\in\omega^{<\omega}}$ over C. Let now $\bar\nu$ be such that $\bar\nu \equiv^{\text{str}_0\text{-qf}} \bar\eta$, then $\text{tp}(b_{\bar\eta}/C) = p_{\bar\eta} = p_{\bar\nu} = \text{tp}(b_{\bar\nu}/C)$ and we have established str$_0$-indiscernibility over C. □

Theorem 1.2 ***repeated.*** *Let T be a thick theory. Let $(a_\eta)_{\eta\in\omega^{<\omega}}$ be a tree of tuples of the same length and let C be any set of parameters, then there is a tree $(b_\eta)_{\eta\in\omega^{<\omega}}$ that is str$_0$-indiscernible over C and* EM_{str_0}*-based on $(a_\eta)_{\eta\in\omega^{<\omega}}$ over C.*

Proof By Theorem 1.1 there is a tree $(a'_\eta)_{\eta\in\omega^{<\omega}}$ that is str-indiscernible over C and EM_{str}-based on $(a_\eta)_{\eta\in\omega^{<\omega}}$ over C. By Theorem 4.10 there is then a tree $(b_\eta)_{\eta\in\omega^{<\omega}}$

that is str_0-indiscernible over C and str_0-based on $(a'_\eta)_{\eta\in\omega^{<\omega}}$ over C. Being EM_{str}-based and being str_0-based both imply being EM_{str_0}-based, and being EM_{str_0}-based is transitive, so we conclude that $(b_\eta)_{\eta\in\omega^{<\omega}}$ is the required tree. □

5 The array modelling theorem

Definition 5.1 ([12, Definition 5.4]) We define the following structure on $\omega \times \omega$, where $(i, j), (s, t) \in \omega \times \omega$:

- $(i, j) <_1 (s, t)$ iff $i < s$,
- $(i, j) <_2 (s, t)$ iff $i = s$ and $j < t$.

We define the *array language* to be $\mathcal{L}_{\text{ar}} = \{<_1, <_2\}$ and we call the structure $\omega \times \omega$ an *array*. We abbreviate notation involving $\mathcal{L}_{\text{ar}}$ in a similar way as described at the end of Definition 4.1, replacing the "s" there by "ar" or "array".

Theorem 1.3 ***repeated.*** *Let T be a thick theory. Let $(a_{i,j})_{i,j<\omega}$ be an array of tuples of the same length and let C be any set of parameters, then there is an array $(b_{i,j})_{i,j<\omega}$ that is array-indiscernible over C and* EM_{ar}*-based on $(a_{i,j})_{i,j<\omega}$ over C.*

Proof Let J be the $<_{\text{lex}}$-order type of $\omega^{<\omega}$ and, using compactness, let $(a'_{i,j})_{i<\omega, j\in J}$ be an array that is EM_{ar}-based on $(a_{i,j})_{i,j<\omega}$ over C. Here $\omega \times J$ carries the expected $\mathcal{L}_{\text{ar}}$-structure: $(i, j) <_1 (s, t)$ iff $i < s$, and $(i, j) <_2 (s, t)$ iff $i = s$ and $j < t$ in the order on J. Let $f' : \omega^{<\omega} \to J$ be the $<_{\text{lex}}$-order isomorphism and define $f : \omega^{<\omega} \to \omega \times J$ as $f(\eta) = (\ell(\eta), f'(\eta))$. Then f is an injection such that for any $\eta, \nu \in \omega^{<\omega}$:

(i) $\eta <_{\text{len}} \nu$ implies $f(\eta) <_1 f(\nu)$,
(ii) $\ell(\eta) = \ell(\nu)$ and $\eta <_{\text{lex}} \nu$ implies $f(\eta) <_2 f(\nu)$.

So in particular, f is qftp-respecting, where $\omega^{<\omega}$ is considered as an str-tree. Define a tree $(a^*_\eta)_{\eta\in\omega^{<\omega}}$ by $a^*_\eta = a'_{f(\eta)}$. By Theorem 1.1 we find a tree $(b^*_\eta)_{\eta\in\omega^{<\omega}}$ that is str-indiscernible over C and EM_{str}-based on $(a^*_\eta)_{\eta\in\omega^{<\omega}}$. Define $g : \omega \times \omega \to \omega^{<\omega}$ by $g(i, j) = \langle 0\rangle^{2i} {}^\frown \langle j + 1\rangle$ and define an array $(b_{i,j})_{i,j<\omega}$ by $b_{i,j} = b^*_{g(i,j)}$. We claim that $(b_{i,j})_{i,j<\omega}$ is the required array.

First, we note that g is qftp-respecting. Indeed, for any $(i, j), (s, t) \in \omega \times \omega$ we have that:

- $(i, j) <_1 (s, t)$ implies $g((i, j)) <_{\text{len}} g((s, t))$ and $g((s, t)) <_{\text{lex}} g((i, j))$,
- $(i, j) <_2 (s, t)$ implies $g((i, j)) <_{\text{lex}} g((s, t))$,
- $g((i, j)) \trianglelefteq g((s, t))$ iff $(i, j) = (s, t)$.

That leaves the relations between any meets in the image of g to be checked, but this is also straightforward using the fact that the meet of a finite number of nodes in $\omega^{<\omega}$ can be written as the meet of two of those nodes, together with the fact that $g((i, j)) \wedge g((s, t)) = \langle 0\rangle^{2\min(i,s)}$ (unless $(i, j) = (s, t)$, of course). We can thus apply the re-indexing lemma, Lemma 3.15(i), and get that $(b_{i,j})_{i,j<\omega}$ is array-indiscernible over C because $(b^*_\eta)_{\eta\in\omega^{<\omega}}$ is str-indiscernible over C.

We will again use the re-indexing lemma, Lemma 3.15(iii), to conclude that $(b_{i,j})_{i,j<\omega}$ is $\mathrm{EM}_{\mathrm{ar}}$-based on $(a'_{i,j})_{i<\omega, j\in J}$ over C. From this the result then follows because $(a'_{i,j})_{i<\omega, j\in J}$ is $\mathrm{EM}_{\mathrm{ar}}$-based on $(a_{i,j})_{i,j<\omega}$ over C. So we only need to verify that for any finite tuple $\bar{\eta}$ in $\omega \times \omega$ we have that $\bar{\eta} \equiv^{\text{ar-qf}} fg(\bar{\eta})$. Indeed, let $(i, j), (s, t) \in \omega \times \omega$, then

- f and g are both injective functions, so equality is preserved and reflected;
- if $(i, j) <_1 (s, t)$ then $g((i, j)) <_{\text{len}} g((s, t))$, and so $fg((i, j)) <_1 fg((s, t))$;
- if $fg((i, j)) <_1 fg((s, t))$ then $2i + 1 = \ell(g((i, j))) < \ell(g((s, t))) = 2s + 1$ and so $(i, j) <_1 (s, t)$;
- if $(i, j) <_2 (s, t)$ then $\ell(g((i, j))) = \ell(g((s, t)))$ and $g((i, j)) <_{\text{lex}} g((s, t))$, so $fg((i, j)) <_2 fg((s, t))$;
- if $fg((i, j)) <_2 fg((s, t))$ then $\ell(g((i, j))) = \ell(g((s, t)))$ and $f'(g((i, j))) < f'(g((s, t)))$, the latter means that $g((i, j))) <_{\text{lex}} g((s, t))$ and since their lengths are the same, and so $i = s$, we must have $j + 1 < t + 1$ and thus $(i, j) <_2 (s, t)$.

□

Remark 5.2 The proof of Theorem 1.3 is based on that of [12, Theorem 5.5]. However, there the existence of a non-existing embedding is claimed. In more detail, we view $\omega^{<\omega}$ as an $\mathcal{L}_{\text{ar}}$-structure by interpreting $<_1$ as $<_{\text{len}}$ and setting $\eta <_2 \nu$ iff $\ell(\eta) = \ell(\nu)$ and $\eta <_{\text{lex}} \nu$. The claim is then that there is an $\mathcal{L}_{\text{ar}}$-embedding $f : \omega^{<\omega} \to \omega \times \omega$. Such an f cannot exist. Our proof fixes this by stretching the original array, so that our f can play the role of this embedding, and the third author of [12] has a similar fix in [16, Corollary 3.8].

Suppose that an f as above exists. Let $\eta, \nu \in \omega^{<\omega}$ be such that $\ell(\eta) = \ell(\nu)$, and let $(i, j) = f(\eta)$ and $(s, t) = f(\nu)$. Then we must have $i = s$, because $<_1$ is interpreted as $<_{\text{len}}$ in the tree and $<_1$ is preserved and reflected by f. Let g be the restriction of f to the nodes of length two. Then the image of g is contained in $\{n\} \times \omega$ for some $n < \omega$. So $g(\langle 1, 0\rangle) = (n, k)$ for some $k < \omega$. However, $\langle 0, i\rangle <_{\text{lex}} \langle 1, 0\rangle$ for all $i < \omega$. We thus get that the second coordinate of $g(\langle 0, i\rangle)$ is strictly less than k for all $i < \omega$, contradicting injectivity of g. So f cannot exist.

Though we have no use for it, the following may be useful in future applications.

Proposition 5.3 *Let T be a thick theory. The property of being array-indiscernible is type-definable. That is, let $\pi_{\text{ar}}((x_{i,j})_{i,j<\omega}, y)$ be the union of the following partial types:*

- *a partial type that expresses that $((x_{i,j})_{j<\omega})_{i<\omega}$ is y-indiscernible;*
- *for each $i < \omega$, a partial type that expresses that $(x_{i,j})_{j<\omega}$ is indiscernible over $(y, (x_{k,j})_{k,j<\omega,k\neq i})$.*

Then for any parameter set C we have that $\models \pi_{\text{ar}}((a_{i,j})_{i,j<\omega}, C)$ iff $(a_{i,j})_{i,j<\omega}$ is array-indiscernible over C.

Proof If $(a_{i,j})_{i,j<\omega}$ is array-indiscernible over C then we clearly have that:

- $((a_{i,j})_{j<\omega})_{i<\omega}$ is C-indiscernible;
- for each $i < \omega$, $(a_{i,j})_{j<\omega}$ is indiscernible over $(C, (a_{k,j})_{k,j<\omega,k\neq i})$.

So $\models \pi_{\text{ar}}((a_{i,j})_{i,j<\omega}, C)$.

For the converse, we assume $\models \pi_{\text{ar}}((a_{i,j})_{i,j<\omega}, C)$ and let $(q_1, r_1), \ldots, (q_n, r_n) \in \omega \times \omega$ and $(s_1, t_1), \ldots, (s_n, t_n) \in \omega \times \omega$ be such that $(q_1, r_1) \ldots (q_n, r_n) \equiv^{\text{ar-qf}} (s_1, t_1) \ldots (s_n, t_n)$. For notational convenience we may assume that $(q_1, r_1) \leq_1 (q_2, r_2) \leq_1 \ldots \leq_1 (q_n, r_n)$ and (necessarily) the same for the (s_i, t_i). Then because $((a_{i,j})_{j<\omega})_{i<\omega}$ is C-indiscernible we have that

$$a_{q_1,r_1} \ldots a_{q_n,r_n} \equiv_C a_{s_1,r_1} \ldots a_{s_n,r_n}.$$

Now let $1 \leq m \leq n$ be maximal such that $s_1 = s_m$ (so for all $1 \leq m' \leq m$ we have that $s_1 = s_{m'}$). Then because $(a_{s_1,j})_{j<\omega}$ is indiscernible over $(C, (a_{k,j})_{k,j<\omega,k\neq s_1})$, we have that

$$a_{s_1,r_1} \ldots a_{s_n,r_n} \equiv_C a_{s_1,t_1} \ldots a_{s_m,t_m} a_{s_{m+1},r_{m+1}} \ldots a_{s_n,r_n}.$$

Repeating this process we find

$$a_{s_1,r_1} \ldots a_{s_n,r_n} \equiv_C a_{s_1,t_1} \ldots a_{s_n,t_n},$$

and thus $a_{q_1,r_1} \ldots a_{q_n,r_n} \equiv_C a_{s_1,t_1} \ldots a_{s_n,t_n}$, as required. □

6 An application: TP_2

The definition of 2-TP_2 in positive logic first appeared in [10, Definition 6.1]. We take the version from [6], which defines k-TP_2 for $k \geq 2$.

Definition 6.1 ([6, Definition 4.5]) A formula $\varphi(x, y)$ has the *k-tree property of the second kind* (k-TP_2) for $k \geq 2$ if there are $(a_{i,j})_{i,j<\omega}$ and $\psi(y_1, \ldots, y_k)$ that implies $\neg\exists x(\varphi(x, y_1) \wedge \ldots \wedge \varphi(x, y_k))$ modulo T such that:

(i) for all $\sigma \in \omega^\omega$ the set $\{\varphi(x, a_{i,\sigma(i)}) : i < \omega\}$ is consistent,
(ii) for all $i < \omega$ and $j_1 < \ldots < j_k < \omega$ we have that $\models \psi(a_{i,j_1}, \ldots, a_{i,j_k})$.

We say that a theory T has k-TP_2 if some formula has k-TP_2.

Lemma 6.2 *Let T be a thick theory. A formula $\varphi(x, y)$ has k-TP_2 for $k \geq 2$ if and only if there is an array-indiscernible array $(a_{i,j})_{i,j<\omega}$ such that*

(i) $\{\varphi(x, a_{i,0}) : i < \omega\}$ is consistent,
(ii) $\{\varphi(x, a_{0,i}) : i < \omega\}$ is k-inconsistent.

Proof If $\varphi(x, y)$ has k-TP_2 as witnessed by $(a'_{i,j})_{i,j<\omega}$ and $\psi(y_1, \ldots, y_k)$ then we can use Theorem 1.3 to EM_{ar}-base an array-indiscernible array $(a_{i,j})_{i,j<\omega}$ on $(a'_{i,j})_{i,j<\omega}$. Then points (i) and (ii) in Definition 6.1 are captured by the EM_{ar}-type of $(a'_{i,j})_{i,j<\omega}$ and respectively yield points (i) and (ii) for $(a_{i,j})_{i,j<\omega}$ as above.

Conversely, let $(a_{i,j})_{i,j<\omega}$ be as above. Then $((i, \sigma(i)))_{i<\omega} \equiv^{\text{ar-qf}} ((i, 0))_{i<\omega}$ for any $\sigma \in \omega^\omega$, so by (i) and array-indiscernibility $\{\varphi(x, a_{i,\sigma(i)}) : i < \omega\}$ is consistent. By

(ii) we have that $\not\models \exists x(\varphi(x, a_{0,0}) \wedge \ldots \wedge \varphi(x, a_{0,k-1}))$. Let $\psi(y_1, \ldots, y_k)$ be such that it implies $\neg\exists x(\varphi(x, y_1) \wedge \ldots \wedge \varphi(x, y_k))$ modulo T and $\models \psi(a_{0,0}, \ldots, a_{0,k-1})$. Then by array-indiscernibility $\models \psi(a_{i,j_1}, \ldots, a_{i,j_k})$ for all $i < \omega$ and $j_1 < \ldots < j_k < \omega$. We conclude that $(a_{i,j})_{i,j<\omega}$ and ψ witness k-TP_2 for $\varphi(x, y)$. □

Theorem 1.4 ***repeated.*** *Let T be a thick theory. If $\varphi(x, y)$ has k-TP_2 for some $k \geq 2$ then some conjunction $\bigwedge_{i=1}^n \varphi(x, y_i)$ has 2-TP_2. Hence, T has k-TP_2 for some $k \geq 2$ iff T has 2-TP_2.*

Proof Based on [12, Proposition 5.7]. We prove this by induction on $k \geq 2$. The base case, $k = 2$, is trivial. So assume that the theorem holds for $2, \ldots, k-1$ and suppose that $\varphi(x, y)$ has k-TP_2. By Lemma 6.2 there is an array-indiscernible array $(a_{i,j})_{i,j<\omega}$ such that

(i) $\{\varphi(x, a_{i,0}) : i < \omega\}$ is consistent,
(ii) $\{\varphi(x, a_{0,i}) : i < \omega\}$ is k-inconsistent.

We finish the induction step, and thus the proof, by considering two cases.

- The case where $\{\varphi(x, a_{i,0}), \varphi(x, a_{i,1}) : i < \omega\}$ is consistent. Define an array $(b_{i,j})_{i,j<\omega}$ by $b_{i,j} = (a_{i,2j}, a_{i,2j+1})$ for all $i, j < \omega$, then $(b_{i,j})_{i,j<\omega}$ is array-indiscernible. Let $\varphi'(x, y_0, y_1)$ be $\varphi(x, y_0) \wedge \varphi(x, y_1)$. Then by assumption $\{\varphi'(x, b_{i,0}) : i < \omega\}$ is consistent, while $\{\varphi'(x, b_{0,i}) : i < \omega\}$ is $\lceil k/2 \rceil$-inconsistent. So by Lemma 6.2 we have that $\varphi'(x; y_0, y_1)$ has $\lceil k/2 \rceil$-TP_2, and we conclude this case by using the induction hypothesis.
- The case where $\{\varphi(x, a_{i,0}), \varphi(x, a_{i,1}) : i < \omega\}$ is inconsistent. By compactness there is then $n < \omega$ such that $\{\varphi(x, a_{i,0}), \varphi(x, a_{i,1}) : i < n\}$ is inconsistent. Define an array $(b_{i,j})_{i,j<\omega}$ by $b_{i,j} = (a_{ni,j}, \ldots, a_{ni+n-1,j})$, then $(b_{i,j})_{i,j<\omega}$ is array-indiscernible. Let $\varphi'(x, y_0, \ldots, y_{n-1})$ be $\varphi(x, y_0) \wedge \ldots \wedge \varphi(x, y_{n-1})$. Then $\{\varphi'(x, b_{i,0}) : i < \omega\}$ is consistent, while by assumption $\{\varphi'(x, b_{0,i}) : i < \omega\}$ is 2-inconsistent. So by Lemma 6.2 we have that $\varphi'(x; y_0, \ldots, y_{n-1})$ has 2-TP_2, completing the induction step for this case.

□

Acknowledgements We thank the anonymous referee for their suggestions for improvement, and in particular for suggesting a simpler version of Example 3.11 compared to a previous version of this paper.

Author Contributions M.K. is the sole author, hence all work on this paper is his.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

1. Ben-Yaacov, I.: Positive model theory and compact abstract theories. J. Math. Log. **03**(01), 85–118 (2003)
2. Ben-Yaacov, I.: Simplicity in compact abstract theories. J. Math. Log. **03**(02), 163–191 (2003)
3. Ben-Yaacov, I.: Thickness, and a categoric view of type-space functors. Fundam. Math. **179**, 199–224 (2003)
4. Ben-Yaacov, I., Berenstein, A., Ward Henson, C., Usvyatsov, A.: Model theory for metric structures. In: Zoé, C., Dugald, M., Anand, P., Alex, W. (eds.) Model Theory with Applications to Algebra and Analysis, vol. 2, pp. 315–427. Cambridge University Press, Cambridge (2008)
5. Ben-Yaacov, I., Poizat, B.: Fondements de la logique positive. J. Symbol. Logic **72**(4), 1141–1162 (2007)
6. Dmitrieva, A., Gallinaro, F., Kamsma, M.: Dividing lines between positive theories. J. Symbol. Logic (2023). https://doi.org/10.1017/jsl.2023.89
7. Dobrowolski, J., Kamsma, M.: Kim-independence in positive logic. Model Theory **1**(1), 55–113 (2022)
8. Dobrowolski, J., Mennuni, R.: The Amalgamation Property for automorphisms of ordered abelian groups, (2023). arXiv:2209.03944
9. Džamonja, M.: On $\triangleleft^*$-maximality. Ann. Pure Appl. Logic **125**(1), 119–158 (2004)
10. Haykazyan, L.: Existentially closed exponential fields. Israel J. Math. **241**(1), 89–117 (2021)
11. Kamsma, Mark: Bilinear spaces over a fixed field are simple unstable. Annals of Pure Appl. Logic **174**(6), 103268 (2023)
12. Kim, B., Kim, H.-J., Scow, L.: Tree indiscernibilities, revisited. Arch. Math. Logic **53**(1), 211–232 (2014)
13. Kaplan, I., Ramsey, N.: On Kim-independence. J. Eur. Math. Soc. **22**(5), 1423–1474 (2020)
14. Poizat, B., Yeshkeyev, A.: Positive Jonsson theories. Log. Univers. **12**(1), 101–127 (2018)
15. Scow, L.: Indiscernibles, EM-types, and Ramsey classes of trees. Notre Dame J. Formal Logic **56**(3), 429–447 (2015)
16. Scow, Lynn: Ramsey transfer to semi-retractions. Ann. Pure Appl. Logic **172**(3), 102891 (2021)
17. Shelah, S.: Classification Theory and the Number of Nonisomorphic Models, 2nd edn. North-Holland Publishing, Amsterdam (1990)
18. Takeuchi, K., Tsuboi, A.: On the existence of indiscernible trees. Ann. Pure Appl. Logic **163**(12), 1891–1902 (2012)
19. Tent, K., Ziegler, M.: A Course in Model Theory. Cambridge University Press, Cambridge (2012)