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Definable Tietze extension property in o-minimal expansions of ordered groups

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Abstract
The following two assertions are equivalent for an o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, 0, \ldots)$. There exists a definable bijection between a bounded interval and an unbounded interval. Any definable continuous function $f : A \to M$ defined on a definable closed subset of M^n has a definable continuous extension $F : M^n \to M$.

Keywords o-Minimal theory · Definable Tietze extension property

Mathematics Subject Classification Primary 03C64

1 Introduction

In this paper, we investigate the Tietze extension property for functions definable in an o-minimal expansion of an ordered Abelian group. For the basics of o-minimality, good references are [2, 6, 7]. In this context, the Tietze extension property is defined as follows:

Definition 1.1 Consider an expansion $\mathcal{M} = (M, <, \ldots)$ of a dense linear order without endpoints. The structure $\mathcal{M}$ enjoys the *definable Tietze extension property* if, for any positive integer n, any definable closed subset A of M^n and any continuous definable function $f : A \to M$, there exists a definable continuous extension $F : M^n \to M$ of f.

The definable Tietze extension property is a convenient tool for the geometric study of o-minimal structures. We prove the following theorem in this paper.

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Theorem 1.2 *Consider an o-minimal expansion $\mathcal{M} = (M, <, +, 0, \dots)$ of an ordered group. The following are equivalent:*

1. *There exists a definable bijection between a bounded interval and an unbounded interval.*
2. *The structure $\mathcal{M}$ enjoys the definable Tietze extension property.*

We make a comment on the theorem. Miller and Starchenko studied the asymptotic behavior of o-minimal expansions of an ordered group $\mathcal{M} = (M, <, +, \dots)$ in [4]. They introduced the notion of linear boundedness. An o-minimal structure is called *linearly bounded* if, for any definable function $f : M \to M$, there exists a definable automorphism $\lambda : M \to M$ with $|f(x)| \leq \lambda(x)$ for all sufficiently large $x \in M$. Their main theorem is that there exists a definable binary operation $\cdot$ such that $(M, <, +, \cdot)$ is an ordered real closed field when the structure is not linearly bounded.

Peterzil and Edmundo studied the subclass of linearly bounded o-minimal expansions of ordered groups [1, 5]. An o-minimal structure $\mathcal{M}$ is *semi-bounded* if any set definable in $\mathcal{M}$ is already definable in the o-minimal structure generated by the collection of all bounded sets definable in $\mathcal{M}$. Edmundo gave equivalent conditions for an o-minimal expansion of an ordered group to be semi-bounded in [1, Fact 1.6]. The condition (1) in our theorem is the negation of one of them. Theorem 1.2 gives a new equivalent condition. An o-minimal expansion of an ordered group is semi-bounded if and only if it does not have the definable Tietze extension property. In our proof, we use the following facts:

- In an o-minimal expansion $\mathcal{M} = (M, <, +, 0, \dots)$ of an ordered group which is not semi-bounded, we can define a real closed field whose universe is an unbounded subinterval of M and whose ordering agrees with $<$ [1, Fact 1.6].
- An o-minimal expansion of an ordered field enjoys the definable Tietze extension property [7, Chapter 8, Corollary 3.10].

We introduce the terms and notations used in this paper. The term 'definable' means 'definable in the given structure with parameters' in this paper. For a linearly ordered structure $\mathcal{M} = (M, <, \dots)$, an interval is a nonempty definable set of the form $\{x \in M \mid a * x *' b\}$, where $a, b \in M \cup \{\pm\infty\}$ and $*, *' \in \{<, \leq\}$. The interval is denoted by $]a, b[$ when both $*$ and $*'$ are the symbol $<$. It is denoted by $[a, b]$ when both $*$ and $*'$ are $\leq$. We define $[a, b[$ and $]a, b]$, similarly. An interval is called bounded if both a and b belong to M. It is called unbounded otherwise. We consider the order topology on M and its product topology on the Cartesian product M^n in the paper. The notation $M_{>r}$ denotes the set $\{x \in M \mid x > r\}$ for any $r \in M$.

2 Proof

We now begin to prove Theorem 1.2. An o-minimal structure is always definably complete. We use this fact without notice. We first prove two lemmas.

Lemma 2.1 *Consider an o-minimal structure. The structure has a definable bijection between a bounded interval and an unbounded interval if and only if it has a definable homeomorphism between a bounded interval and an unbounded interval.*

Proof It is immediate from the monotonicity theorem [7, Chapter 3, Theorem 1.2]. □

Lemma 2.2 *Consider a definably complete expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, \ldots)$. If the structure $\mathcal{M}$ has a strictly monotone definable homeomorphism between a bounded open interval and an unbounded open interval, any two open intervals are definably homeomorphic and there exists a definable strictly increasing homeomorphism between them.*

Proof By the assumption, there exists a strictly monotone definable homeomorphism $\varphi : I \to J$, where I is a bounded open interval and J is an unbounded open interval. We may assume that $I =]0, u[$ for some $u > 0$. In fact, an open interval $]u_1, u_2[$ is obviously definably homeomorphic to $]0, u_2 - u_1[$. We may further assume that φ is strictly increasing because the map $\tau :]0, u[\to]0, u[$ defined by $\tau(t) = u - t$ is a definable homeomorphism.

We next reduce to the case in which $J =]0, \infty[$. We have only three possibilities; that is $J =]v, +\infty[$, $J =] - \infty, v[$ and $J = M$ for some $v \in M$. In the first and second cases, we may assume that $J =]0, \infty[$ because $J =]v, +\infty[$ and $J =]-\infty, v[$ are obviously definably homeomorphic to $]0, \infty[$. In the last case, set $u' = \varphi^{-1}(0)$. Then the restriction of φ to the open interval $]0, u'[$ is a definable homeomorphism between $]0, u'[$ and $] - \infty, 0[$. Hence, we can reduce to the second case. We have constructed a strictly increasing definable homeomorphism $\varphi :]0, u[\to]0, \infty[$. We fix such a homeomorphism.

We next construct a definable strictly increasing homeomorphism between an arbitrary bounded open interval and $]0, \infty[$. We may assume that the bounded interval is of the form $]0, v[$. We have nothing to do when $v = u$. When $v < u$, the map defined by $\varphi(t + u - v) - \varphi(u - v)$ for all $t \in]0, v[$ is a definable homeomorphism between $]0, v[$ and $]0, \infty[$. When $v > u$, consider the map $\psi :]0, v[\to]0, \infty[$ given by $\psi(t) = t$ for all $t \leq v - u$ and $\psi(t) = \varphi(t + u - v) + v - u$ for the other case. The map ψ is the desired definable homeomorphism. We have constructed a definable homeomorphism between $]0, u[$ and all open intervals other than M.

The remaining task is to construct a definable homeomorphism between $]0, u[$ and M. There exists a strictly increasing definable homeomorphisms $\psi_1 :]0, u/2[\to] - \infty, 0[$ and $\psi_2 :]u/2, u[\to]0, \infty[$. The definable map $\psi :]0, u[\to M$ given by $\psi(t) = \psi_1(t)$ for $t < u/2$, $\psi(t) = 0$ for $t = u/2$ and $\psi(t) = \psi_2(t)$ for $t > u/2$ is a definable homeomorphism. The function ψ is well defined because $(M, +)$ is a divisible group by [3, Proposition 2.2]. □

The following proposition is a part of [1, Fact 1.6].

Proposition 2.3 *Consider an o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, 0, \ldots)$. The followings are equivalent:*

1. *There exists a definable bijection between a bounded interval and an unbounded interval.*
2. *In $\mathcal{M}$, we can define a real closed field whose universe is an unbounded subinterval of M and whose ordering agrees with $<$.*

We now begin to prove Theorem 1.2.

Proof of Theorem 1.2 We first show that the condition (1) implies the condition (2).

There exist an unbounded subinterval I of M, two elements 0^* and 1^* in I, and definable functions $\oplus, \otimes : I \times I \to I$ such that $(I, 0^*, 1^*, \oplus, \otimes)$ is a real closed field with the ordering $<$ by Proposition 2.3. The subinterval I is obviously an open interval.

If $I = M$, the assertion (2) directly follows from the original definable Tietze extension theorem [7, Chapter 8, Corollary 3.10].

We next consider the other case. Consider a definable continuous function $f : A \to M$ defined on a definable closed subset A of M^n. We construct a definable continuous extension $F : M^n \to M$ of the function f. There exists a definable homeomorphism $\sigma : M \to I$ by Lemmas 2.1 and 2.2. The notation σ_n denotes the homeomorphism from M^n onto I^n induced by σ. The definable set $\sigma_n(A)$ is contained I^n. Consider the definable continuous function $f_\sigma : \sigma_n(A) \to I$ defined by $f_\sigma(x) = (\sigma \circ f \circ \sigma_n^{-1})(x)$. Its graph is obviously contained in I^{n+1}.

We consider a new structure $\mathcal{I}$ whose universe is I. Let $\mathfrak{S}_n$ be the set of all subset of I^n definable in $\mathcal{M}$. Set $\mathfrak{S} = \bigcup_{n \geq 0} \mathfrak{S}_n$. For any $S \in \mathfrak{S}$, we introduce new predicate symbol R_S and we define $\mathcal{I} \models R_S(x)$ by $x \in S$. The structure $\mathcal{I} = (I, <, \{R_S\}_{S \in \mathfrak{S}})$ is obviously an o-minimal structure. Since the operators $\oplus$ and $\otimes$ are definable in $\mathcal{I}$, the structure $\mathcal{I}$ is an o-minimal expansion of an ordered field. The $\mathcal{M}$-definable set $\sigma_n(A)$ and the $\mathcal{M}$-definable function f_σ are also definable in the structure $\mathcal{I}$. Note that the function f_σ is also continuous under the topology induced by the ordering of the real closed field $(I, 0^*, 1^*, \oplus, \otimes)$ because the two structures $\mathcal{I}$ and $\mathcal{M}$ share the same order $<$. There exists a continuous extension $F_\sigma : I^n \to I$ of f_σ definable in $\mathcal{I}$ by the original definable Tietze extension theorem [7, Chapter 8, Corollary 3.10]. The function F_σ is also definable in $\mathcal{M}$ by the definition of the structure $\mathcal{I}$. The function $F = \sigma^{-1} \circ F_\sigma \circ \sigma_n$ is the desired definable continuous extension of f definable in $\mathcal{M}$.

We next show that the condition (2) implies the condition (1). We construct a definable bijection between a bounded interval and an unbounded interval. Take a positive element c in M. Consider the definable closed set $A = \{(x, y) \in M^2 \mid x \leq 0 \text{ or } x \geq c\}$ and the definable continuous function $f : A \to M$ given by $f(x, y) = y$ if $x \geq c$ and $f(x, y) = 0$ otherwise. By the condition (2), there exists a definable continuous extension $F : M^2 \to M$ of f. The notation g denotes the restriction of F to $[0, c] \times M$.

Consider the sets $S_{t,y} = \{x \in [0, c] \mid g(x, y) = t\}$ for all $t \geq 0$ and $y \geq 0$. The definable sets $S_{t,y}$ is not empty for $y > t$ by the intermediate value theorem [3, Corollary 1.5]. The definable function $\varphi_t : M_{>t} \to [0, c]$ is given by $\varphi_t(y) = \sup S_{t,y}$. For any $t > 0$, there exists a nonnegative u_t such that the restriction $\varphi_t|_{M_{>u_t}}$ of the function $\varphi_t(y)$ to $M_{>u_t} = \{y \in M \mid y > u_t\}$ is continuous and strictly monotone or constant for $y > u_t$ by the monotonicity theorem.

We consider the following two cases separately.

(a) The restriction $\varphi_t|_{M_{>u_t}}$ is continuous and strictly monotone for some $t > 0$.
(b) The restriction $\varphi_t|_{M_{>u_t}}$ is constant for any $t > 0$.

In the case (a), the restriction $\varphi_t|_{M_{>u_t}}$ gives a bijection between a bounded interval and an unbounded interval. We have finished the proof in this case. We next consider the case (b). By the definition of the function φ_t, the following assertion holds true:

For any $t > 0$, there exist a point $x_t \in [0, c]$ and a nonnegative u_t such that $g(x_t, y) = t$ for all $y > u_t$.

In fact, we have only to take $y' > u_t$ and set $x_t = \varphi_t(y')$. Consider the definable map $\psi : [0, \infty[\to [0, c]$ given by $\psi(t) = x_t$, where x_t is the point defined above. Since $\varphi_t|_{M_{>u_t}}$ is constant, the point x_t is independent of the choice of y'. It means that ψ is well defined. The map ψ is injective. In fact, if $\psi(t) = \psi(t')$, we have $t = g(\psi(t), y') = g(\psi(t'), y') = t'$ for a sufficiently large y'. By the monotonicity theorem, there exists $c > 0$ such that the restriction $\psi|_{]c,\infty[}$ of ψ to $]c, \infty[$ is continuous and monotone. The restriction $\psi|_{]c,\infty[}$ is strictly monotone because ψ is injective. Therefore, it gives a definable bijection between a bounded interval and an unbounded interval. □

Corollary 2.4 *An o-minimal expansion of an ordered group is semi-bounded if and only if it does not have the definable Tietze extension property.*

Proof The corollary follows from Theorem 1.2 and [1, Fact 1.6]. □

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