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Models of ZFA in which every linearly ordered set can be well ordered

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Abstract

We provide a general criterion for Fraenkel–Mostowski models of ZFA (i.e. Zermelo–Fraenkel set theory weakened to permit the existence of atoms) which implies "every linearly ordered set can be well ordered" (LW), and look at six models for ZFA which satisfy this criterion (and thus LW is true in these models) and "every Dedekind finite set is finite" (DF $=$ F) is true, and also consider various forms of choice for well-ordered families of well orderable sets in these models. In Model 1, the axiom of multiple choice for countably infinite families of countably infinite sets ($\mathsf{MC}_{\aleph_0}^{\aleph_0}$) is false. It was the open question of whether or not such a model exists (from Howard and Tachtsis "On metrizability and compactness of certain products without the Axiom of Choice") that provided the motivation for this paper. In Model 2, which is constructed by first choosing an uncountable regular cardinal in the ground model, a strong form of Dependent choice is true, while the axiom of choice for well-ordered families of finite sets ($\mathsf{AC}_{\mathrm{fin}}^{\mathrm{WO}}$) is false. Also in this model the axiom of multiple choice for well-ordered families of well orderable sets fails. Model 3 is similar to Model 2 except for the status of $\mathsf{AC}_{\mathrm{fin}}^{\mathrm{WO}}$ which is unknown. Models 4 and 5 are variations of Model 3. In Model 4 $\mathsf{AC}_{\mathrm{fin}}^{\mathrm{WO}}$ is true. The construction of Model 5 begins by choosing a regular successor cardinal in the ground model. Model 6 is the only one in which $2\mathfrak{m} = \mathfrak{m}$ for every infinite cardinal number $\mathfrak{m}$. We show that the union of a well-ordered family of well orderable sets is well orderable in Model 6 and that the axiom of multiple countable choice is false.

This paper is dedicated to the memory of James Daniel Halpern.

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1 Introduction: consequences of the axiom of choice

Definition 1 We use the following abbreviations for consequences of the axiom of choice.

1. AC (Form 1 in [5]) is the axiom of choice.
2. $\mathsf{MC}_{\aleph_0}^{\aleph_0}$ (Form 350 in [5]) is the axiom of multiple choice for countably infinite families of countably infinite sets, i.e. the statement "for every family $\mathcal{A} = \{A_n : n \in \omega\}$ with $|\mathcal{A}| = |A_n| = \aleph_0$, there is a function on $\mathcal{A}$ such that $f(A_n)$ is a non-empty finite subset of A_n for all $n \in \omega$". (The function f is called a *multiple choice function* for $\mathcal{A}$.)
3. $\mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$ (Form 330 in [5]) is the axiom of multiple choice for well orderable families of non-empty well orderable sets.
4. $\mathsf{MC}^{\aleph_0}$ (Form 126 in [5]) is the axiom of multiple choice for countably infinite families of infinite sets.
5. LW (Form 90 in [5]) is the statement "every linearly ordered set can be well ordered".
6. DF = F (Form 9 in [5]) is the statement "every Dedekind finite set is finite". (Where a set X is called *Dedekind finite* if there is no one-to-one mapping $f : \omega \to X$; otherwise, X is called *Dedekind infinite*.)
7. DC (Form 43 in [5]) is the principle of dependent choices.
8. Let κ be an infinite well-ordered cardinal number. DC_κ (Form 87(κ) in [5]) is the statement "if X is a non-empty set and R is a binary relation such that for every $\alpha < \kappa$ and every α-sequence $\mathbf{x} = (x_\xi)_{\xi<\alpha}$ of elements of X there exists $y \in X$ such that $\mathbf{x}\ R\ y$, then there is a function $f : \kappa \to X$ such that for every $\alpha < \kappa$, $(f \restriction \alpha)\ R\ f(\alpha)$". Note that $\mathsf{DC}_{\aleph_0}$ is a reformulation of the principle of dependent choices.
9. $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$ (Form 122 in [5]) is the statement "every well-ordered family of non-empty finite sets has a choice function".
10. $\mathsf{AC}_{\mathsf{WO}}^{\mathsf{WO}}$ (Form 165 in [5]) is the statement "every well-ordered family of non-empty well orderable sets has a choice function".
11. $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ (Form 3 in [5]) is the statement "for every infinite cardinal $\mathfrak{m}$, $2 \cdot \mathfrak{m} = \mathfrak{m}$". (Form 3 is equivalent to "for every infinite set X, $|2 \times X| = |X|$", i.e. for every infinite set X, there is a bijection $f : 2 \times X \to X$, where $2 = \{0, 1\}$.)

We recall that LW is equivalent to AC in ZF (i.e. Zermelo–Fraenkel set theory minus the AC), but is not equivalent to AC in ZFA (see Jech [7, Theorems 9.1 and 9.2]). We also recall that $\forall\kappa(\mathsf{DC}_\kappa)$ (where the parameter κ runs through the infinite well-ordered cardinal numbers) is equivalent to AC in ZFA; see [7, Theorem 8.1(c)]. Furthermore,

Table 1 Known relationships

	$\forall\mathfrak{m}, 2\mathfrak{m}=\mathfrak{m}$	DF = F	DC	LW	$\mathsf{AC}^{\mathsf{WO}}_{\mathsf{fin}}$	$\mathsf{AC}^{\mathsf{WO}}_{\mathsf{WO}}$	$\mathsf{MC}^{\mathsf{WO}}_{\mathsf{WO}}$	$\mathsf{MC}^{\aleph_0}_{\aleph_0}$	$\mathsf{MC}^{\aleph_0}$
$\forall\mathfrak{m}, 2\mathfrak{m}=\mathfrak{m}$	$\rightarrow$	$\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	?	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$
DF = F	$\not\rightarrow$	$\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$
DC	$\not\rightarrow$	$\rightarrow$	$\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\rightarrow$	$\rightarrow$
LW	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	?	?	$\not\rightarrow$
$\mathsf{AC}^{\mathsf{WO}}_{\mathsf{fin}}$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$
$\mathsf{AC}^{\mathsf{WO}}_{\mathsf{WO}}$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\not\rightarrow$
$\mathsf{MC}^{\mathsf{WO}}_{\mathsf{WO}}$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\rightarrow$	$\rightarrow$	$\not\rightarrow$
$\mathsf{MC}^{\aleph_0}_{\aleph_0}$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\rightarrow$	$\not\rightarrow$
$\mathsf{MC}^{\aleph_0}$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\not\rightarrow$	$\rightarrow$	$\rightarrow$

DF = F is strictly weaker than $\forall\mathfrak{m}, 2\mathfrak{m}=\mathfrak{m}$ in ZF, and $\forall\mathfrak{m}, 2\mathfrak{m}=\mathfrak{m}$ does not imply AC in ZF (see [5, Forms 3 and 9, the ZF-model $\mathcal{M}6$ and the ZFA-model $\mathcal{N}9$]).

Our original motivation for the research presented in this paper was to provide an answer to the open question "Does LW together with DF = F imply $\mathsf{MC}^{\aleph_0}_{\aleph_0}$ in the theory ZFA?", which was stated in Howard and Tachtsis [6]. The status of the above implication is also mentioned as unknown in Howard and Rubin [5].

We will consider six models of ZFA (Models 1 through 6 as described in the abstract). Of these models 4 and 5 are new. Model 1 was constructed in [10], Model 2 in [7] and Model 3 in [5]. We will show that in all six models both LW and DF = F are true. We will also show that $\mathsf{MC}^{\aleph_0}_{\aleph_0}$ is false in Models 1 and 4, but true in the other four models. We also consider the truth or falsity of the statements listed above in these models.

Besides resolving certain open problems on the relationship between the above weak choice principles (and conjunctions of those principles), it is also a central goal of this paper to explore and develop the required machinery, both set-theoretic and group-theoretic, in order to establish the relative independence results in ZFA set theory. In this direction, our aim is to provide as much information as possible on certain independence proofs and their techniques in the permutation models studied in this paper.

Table 1 summarizes the known relationships in ZFA between (most of) the statements listed in Definition 1. The symbol $\rightarrow$ in the body of the table indicates that the row heading form implies the column heading form in ZFA. The symbol $\not\rightarrow$ means that the implication does not hold. The fact that $\forall\mathfrak{m}, 2\mathfrak{m}=\mathfrak{m}$ does not imply $\mathsf{MC}^{\aleph_0}_{\aleph_0}$ is a consequence of a result from Sageev [9] where a ZF-model is constructed in which $\forall\mathfrak{m}, 2\mathfrak{m}=\mathfrak{m}$ is true and there is a countable set of countable sets of reals without a choice function (and thus without a multiple choice function—the usual order on $\mathbb{R}$ is a linear order). References for all of the other entries in the table can be found in Howard and Rubin [5].

2 A model-theoretic criterion which implies LW, and certain types of FM models satisfying this criterion

Definition 2 We will use the following notation assuming that Z is a set and S is a subset of Z.

1. $\mathrm{Sym}(Z)$ is the set of all permutations of Z.
2. If H is a subgroup of $\mathrm{Sym}(Z)$, then $\mathrm{Sym}_H(S) = \{\phi \in H : \phi(S) = S\}$ and $\mathrm{fix}_H(S) = \{\phi \in H : \forall x \in S(\phi(x) = x)\}$.
3. $\mathrm{FSym}(Z)$ is the set of all finitary permutations of Z.

Our first result, Theorem 1 below, provides a general condition under which a permutation model of ZFA satisfies LW. All of the ZFA-models in this paper satisfy this condition, and thus LW holds in all those models.

Theorem 1 *Let $\mathcal{N}$ be a permutation model which is determined by a group G of permutations of the set A of atoms, and a normal filter $\mathscr{F}$ of subgroups of G which is generated by some filter base $\mathscr{B}$ (of subgroups of G). If $\mathcal{N}$ satisfies the following condition:*

(*) for every $x \in \mathcal{N}$ and for every $B \in \mathscr{B}$ which does not support x (i.e. $B \setminus \mathrm{Sym}_G(x) \neq \emptyset$), there exists $\gamma \in B \setminus \mathrm{Sym}_G(x)$ of finite order,

then LW *is true in $\mathcal{N}$.*

Proof Let $(X, \leq)$ be a linearly ordered set in $\mathcal{N}$. Let $K \in \mathscr{F}$ be such that $K \subseteq \mathrm{Sym}_G((X, \leq))$. Since $\mathscr{B}$ is a filter base for $\mathscr{F}$, there exists $B \in \mathscr{B}$ with $B \subseteq K$, and thus $B \subseteq \mathrm{Sym}_G((X, \leq))$.

By way of contradiction, we assume that X is not well orderable in $\mathcal{N}$. Then there exists $x \in X$ such that $B \setminus \mathrm{Sym}_G(x) \neq \emptyset$. (Otherwise, if $B \subseteq \mathrm{fix}_G(X)$, then $\mathrm{fix}_G(X) \in \mathscr{F}$, and hence X is well orderable in $\mathcal{N}$—see Jech [7, Equation (4.2), p. 47]—which is a contradiction.)

By $(*)$, there exist $\gamma \in B \setminus \mathrm{Sym}_G(x)$ and an integer $n \geq 2$ such that $\gamma^n = \epsilon$, where ϵ is the identity permutation on A. (Note that for an element ϕ of G we tacitly use the same notation for the unique $\in$-automorphism of $(\mathcal{N}, \in)$ which extends ϕ.)

Since $B \subseteq \mathrm{Sym}_G((X, \leq))$, $\gamma(x) \in X$ and $\gamma(\leq) = \leq$. Furthermore, since $\leq$ is a linear order on X (in $\mathcal{N}$), either $\gamma(x) < x$ or $x < \gamma(x)$. If the first possibility occurs, then

$$x = \gamma^n(x) < \gamma^{n-1}(x) < \cdots < \gamma^2(x) < \gamma(x) < x,$$

and we thus arrived at a contradiction. In a similar manner, the second possibility also leads to a contradiction.

Thus X is well orderable in $\mathcal{N}$ as required. □

Corollary 1 *Let $\mathcal{N}$, A, G, $\mathscr{F}$, and $\mathscr{B}$, be as in the hypotheses of Theorem 1. If every element of G has finite order, or if G is a subgroup of* $\mathrm{FSym}(A)$, *then* $\mathcal{N} \models$ LW.

Lemma 1 below will be a key result for the verification of condition (*) (of Theorem 1) in the majority of our models, except for Models 2 and 4 (see Sects. 4 and 6).

Lemma 1 *Let Z be any infinite set and also let $\eta \in \mathrm{Sym}(Z)$. Then there exists $\tau \in \mathrm{Sym}(Z)$ such that*

1. $\{z \in Z : \tau(z) \neq z\} \subseteq \{z \in Z : \eta(z) \neq z\}$ *(that is, the support of τ is contained in the support of η);*
2. $\tau^2 = \epsilon$ *(where ϵ is the identity element of* $\mathrm{Sym}(Z)$*);*
3. $(\eta\tau)^2 = \epsilon$.

Proof We first consider the case where

$$\eta = (a_1, a_2, \ldots, a_n, a_{n+1}, a_{n+2} \ldots a_{2n}, a_{2n+1})$$

is a cycle of odd length. In this case, we let τ be the product of transpositions

$$\tau = (a_1, a_{2n+1})(a_2, a_{2n})(a_3, a_{2n-1}) \cdots (a_n, a_{n+2}) = \prod_{i=1}^{n} (a_i, a_{2n+2-i}).$$

Then, since τ is a product of disjoint transpositions, $\tau^2 = \epsilon$. Also,

$$\eta\tau = (a_2, a_{2n+1})(a_3, a_{2n}) \cdots (a_{n+1}, a_{n+2}) = \prod_{i=2}^{n+1} (a_i, a_{2n+3-i})$$

is a product of disjoint transpositions. Therefore, $(\eta\tau)^2 = \epsilon$.

Secondly, we assume that

$$\eta = (a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{2n})$$

is a cycle of even length. Let

$$\tau = (a_1, a_{2n})(a_2, a_{2n-1}) \cdots (a_n, a_{n+1}) = \prod_{i=1}^{n} (a_i, a_{2n+1-i}).$$

As in the previous case, $\tau^2 = \epsilon$ and

$$\eta\tau = (a_2, a_{2n})(a_3, a_{2n-1}) \cdots (a_n, a_{n+2}) = \prod_{i=2}^{n} (a_i, a_{2n+2-i})$$

which is a product of disjoint transpositions. Hence, $(\eta\tau)^2 = \epsilon$.

Our third case is the case where

$$\eta = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$$

is an infinite cycle. Let

$$\tau = (a_1, a_{-1})(a_2, a_{-2})(a_3, a_{-3}) \cdots = \prod_{i=1}^{\infty}(a_i, a_{-i}).$$

Then $\tau^2 = \epsilon$ and

$$\eta\tau = (a_1, a_0)(a_2, a_{-1})(a_3, a_{-2}) \cdots = \prod_{i=1}^{\infty}(a_i, a_{1-i}),$$

so $(\eta\tau)^2 = \epsilon$.

Now we consider the general case where η is any permutation of Z. Since any permutation can be written as a product of disjoint cycles, we assume that $\eta = \prod_{j\in J} \eta_j$ where the η_j's are pairwise disjoint cycles and J is an (infinite or finite) index set. Using cases 1, 2 and 3, we know that for each $j \in J$ there exists $\tau_j \in \mathrm{Sym}(Z)$ satisfying the three conditions of the lemma with η and τ replaced by η_j and τ_j, respectively. By condition 1 (since the η_j's are pairwise disjoint), the τ_j's are pairwise disjoint and for $j_1 \neq j_2$, τ_{j_1} is disjoint from η_{j_2}. Let $\tau = \prod_{j\in J} \tau_j$. Then $\tau^2 = \prod_{j\in J} \tau_j^2 = \epsilon$ (since disjoint cycles commute). Further,

$$(\eta\tau)^2 = \left(\prod_{j\in J} \eta_j \prod_{j\in J} \tau_j\right)^2 = \left(\prod_{j\in J} \eta_j\tau_j\right)^2 = \prod_{j\in J}(\eta_j\tau_j)^2 = \epsilon$$

(using the fact that disjoint cycles commute). This finishes the proof of the lemma. □

2.1 Types of permutation models satisfying condition (*) of Theorem 1

The majority of the FM models of our paper—in particular, all models except for Model 6 of Sect. 8—are constructed as follows: Let μ be an infinite, well-ordered, regular, cardinal number, and also let $2 \leq \lambda \leq \mu$ be a cardinal number. We start with a model M of ZFA + AC with a set A of atoms which is partitioned into a μ-sized collection of sets each having cardinality λ. Say $A = \bigcup\{A_\alpha : \alpha \in \mu\}$, where for every $\alpha \in \mu$, $|A_\alpha| = \lambda$ and for $\alpha_1 \neq \alpha_2$, $A_{\alpha_1} \cap A_{\alpha_2} = \emptyset$.

For every $\alpha \in \mu$, let $\mathscr{G}_\alpha$ be a group of permutations of A_α. (Usually, for every two distinct ordinals α, β in μ, $\mathscr{G}_\alpha$ is isomorphic to $\mathscr{G}_\beta$.) Let

$$G = \{\phi \in \mathrm{Sym}(A) : \forall\alpha \in \mu(\phi \restriction A_\alpha \in \mathscr{G}_\alpha)\}$$

G is isomorphic to the unrestricted direct product of the $\mathscr{G}_\alpha$'s and therefore in subsequent sections we shall refer to G as the *unrestricted direct product of the* $\mathscr{G}_\alpha$*'s*.

Remark 1 For use in the forthcoming Sect. 11, let us note that, in a similar way, the *weak direct product of the* $\mathscr{G}_\alpha$*'s* will mean the group

$$\{\phi \in \mathrm{Sym}(A) : (\forall \alpha \in \mu(\phi \restriction A_\alpha \in \mathscr{G}_\alpha)) \wedge \{\alpha \in \mu : \phi \restriction A_\alpha \neq \epsilon_{A_\alpha}\} \text{ is finite}\},$$

where ϵ_{A_α} denotes the identity function on A_α.

It is clear that for all $\alpha \in \mu$ and for all $\phi \in G$, $\phi(A_\alpha) = A_\alpha$. Furthermore, for every $\alpha \in \mu$, $\mathscr{G}_\alpha$ is isomorphic to the subgroup of G,

$$G_\alpha = \{\phi \in G : \forall a \notin A_\alpha(\phi(a) = a)\}. \tag{1}$$

– If $\lambda = \mu$, then the ideal I of supports is defined by

$$I = \{S : \exists E \in [\mu]^{<\omega}(S \subseteq \bigcup\{A_\alpha : \alpha \in E\})\}.$$

– If $\lambda < \mu$, then the ideal I of supports is defined by

$$I = \{S : \exists E \in [\mu]^{<\mu}(S \subseteq \bigcup\{A_\alpha : \alpha \in E\})\}.$$

In each of the above two cases, I is a normal ideal. The normal filter $\mathscr{F}$ of subgroups of G is the filter generated by the filter base $\{\mathrm{fix}_G(S) : S \in I\}$ (see [7, Chapter 4] for the definition of the terms "normal ideal" and "normal filter"). Note that, for $W \in \{[\mu]^{<\omega}, [\mu]^{<\mu}\}$, $\mathscr{F}$ is equal to the filter of subgroups of G generated by the filter base $\{\mathrm{fix}_G(\bigcup\{A_\alpha : \alpha \in E\}) : E \in W\}$ (the easy argument uses the fact that $S \subseteq S'$ implies $\mathrm{fix}_G(S') \subseteq \mathrm{fix}_G(S)$). It is the latter filter base that we use in order to check condition (*) of Theorem 1 for Models 1, 3, 4 and 5 (see the proof of the forthcoming Theorem 2).

Let $\mathcal{N}$ be the permutation model determined by M, G, and I, or equivalently, by M, G, and $\mathscr{F}$. (A set x in M is in $\mathcal{N}$ if and only if x and all elements in its transitive closure, $\mathrm{TC}(x)$, are supported by some element of I, that is, if and only if for every $y \in \{x\} \cup \mathrm{TC}(x)$ there exists $S \in I$ such that for all $\phi \in \mathrm{fix}_G(S)$, $\phi(y) = y$—equivalently, if and only if for every $y \in \{x\} \cup \mathrm{TC}(x)$, $\mathrm{Sym}_G(y) \in \mathscr{F}$.)

Note that if $x \in \mathcal{N}$ and S is a support of x of the form $\bigcup\{A_\alpha : \alpha \in E\}$, then by the fact that for all $\phi \in G$ and for all $\alpha \in \mu$, $\phi(A_\alpha) = A_\alpha$, it follows that

$$\forall \phi \in G(\phi(S) = S), \text{ so } S \text{ is a support of } \phi(x). \tag{2}$$

Another lemma which will be useful for the proofs of LW and DF = F in our models, is the following one.

Lemma 2 *Let* $\mathcal{N}$ *be a permutation model determined by* M*,* G*, and* I *as in the previous paragraph. Assume*

1. $S = \bigcup\{A_\alpha : \alpha \in E\} \in I$,
2. $\eta \in \mathrm{fix}_G(S)$ *and*
3. x *is an element of* $\mathcal{N}$ *for which* $\eta(x) \neq x$ *(and hence* x *is not supported by* S*).*

Then there exists $E' \subset \mu$ which is disjoint from E, and for each $\alpha \in E'$ there exists $\eta_\alpha \in G_\alpha$ (where G_α is given by (1)) such that for $\eta' = \prod_{\alpha \in E'} \eta_\alpha$ (and hence $\eta' \in \mathrm{fix}_G(S)$), we have $\eta'(x) \neq x$.

In particular, if $\lambda = \mu$ (so that $E \in [\mu]^{<\omega}$), then for some $\alpha \in E'$ there is $\eta_\alpha \in G_\alpha$ (and hence $\eta_\alpha \in \mathrm{fix}_G(S)$) such that $\eta_\alpha(x) \neq x$.

Proof Let $S \cup S'$ be a support of x where $S' = \bigcup\{A_\alpha : \alpha \in E'\}$ for some $E' \subset \mu$ with $E' \cap E = \emptyset$. We let η' be the permutation of A which agrees with η on S' and is the identity outside of S'. Hence, $\eta' \in \mathrm{fix}_G(A \setminus S')$, and so $\eta' \in \mathrm{fix}_G(S)$. By the definition of the group G, it follows that for each $\alpha \in E'$, there exists $\eta_\alpha \in G_\alpha$ such that $\eta' = \prod_{\alpha \in E'} \eta_\alpha$. Since η and η' agree on the support $S \cup S'$ of x, we have $\eta'(x) = \eta(x)$, and since η does not fix x, neither does η'.

The second assertion of the lemma follows from the proof of the first one and the facts that E' is finite and $\eta'(x) \neq x$ (and note that since E' is finite, $\eta' \restriction A_\alpha = \epsilon_{A_\alpha}$ for all but finitely many $\alpha \in \mu$). ◻

The following theorem essentially establishes the validity of LW in the forthcoming Models 1, 3, 4, and 5 (of Sects. 3, 5, 6, and 7, respectively).

Theorem 2 *Let $\mathcal{N}$ be a permutation model determined by M, G, and I as in the opening paragraph of Sect. 2.1. We assume that $\omega \leq \lambda \leq \mu$.*

- *(i) If for every $\alpha \in \mu$, $\mathscr{G}_\alpha$ is the group of even permutations of A_α (i.e. $\mathscr{G}_\alpha$ consists of all elements γ of $\mathrm{FSym}(A_\alpha)$ which are an even permutation of their (finite) support $\{a \in A_\alpha : \gamma(a) \neq a\}$), then $\mathcal{N} \models$ LW.*
- *(ii) If for every $\alpha \in \mu$, $\mathscr{G}_\alpha = \mathrm{Sym}(A_\alpha)$, then $\mathcal{N} \models$ LW.*

Proof We first consider the case where $\lambda < \mu$, so that

$$I = \{S : \exists E \in [\mu]^{<\mu}(S \subseteq \bigcup\{A_\alpha : \alpha \in E\})\}.$$

(i) By Theorem 1, it suffices to show that $\mathcal{N}$ satisfies condition (*). To this end, let $x \in \mathcal{N}$ and also let $S = \bigcup\{A_\alpha : \alpha \in E\}$ (for some $E \in [\mu]^{<\mu}$) which does not support x, i.e. there exists $\eta \in \mathrm{fix}_G(S)$ such that $\eta(x) \neq x$. By Lemma 2, there exists $E' \subset \mu$ which is disjoint from E, and for each $\alpha \in E'$ there exists $\eta_\alpha \in G_\alpha$ such that $\eta'(x) \neq x$, where $\eta' = \prod_{\alpha \in E'} \eta_\alpha$. Hence, $\eta'_\alpha = \eta_\alpha \restriction A_\alpha \in \mathscr{G}_\alpha$.

For each $\alpha \in E'$, we apply Lemma 1 to η'_α to obtain a permutation τ'_α on A_α with the following properties:

1. $(\tau'_\alpha)^2 = \epsilon$ and $(\eta'_\alpha \tau'_\alpha)^2 = \epsilon$.
2. τ_α, η_α and $\eta_\alpha \tau_\alpha$ (i.e. the elements of G_α which extend τ'_α, η'_α and $\eta'_\alpha \tau'_\alpha$, respectively) are all in $\mathrm{fix}_G(S)$ (since $\alpha \notin E$).
3. $\tau'_\alpha \in \mathrm{FSym}(A_\alpha)$ (by condition 1 of Lemma 1).

We may also assume that $\tau'_\alpha \in \mathscr{G}_\alpha$, i.e. that τ'_α is an even permutation of A_α. If not, then we choose two elements a and a' of A_α which are fixed by η'_α (and therefore fixed by τ'_α) and replace τ'_α by the product $\tau'_\alpha(a, a')$ of τ'_α and the transposition (a, a').

Let $\tau = \prod_{\alpha \in E'} \tau_\alpha$. Then,

$$\tau^2 = \left(\prod_{\alpha \in E'} \tau_\alpha \right)^2 = \prod_{\alpha \in E'} \tau_\alpha^2 = \epsilon; \tag{3}$$

$$(\eta' \tau)^2 = \left(\prod_{\alpha \in E'} \eta_\alpha \prod_{\alpha \in E'} \tau_\alpha \right)^2 = \prod_{\alpha \in E'} (\eta_\alpha \tau_\alpha)^2 = \epsilon. \tag{4}$$

Formulas (3) and (4) use the fact that for $\alpha_1 \neq \alpha_2$, η_{α_1} and τ_{α_1} both commute with η_{α_2} and τ_{α_2}.

Since $\eta'(x) \neq x$, it follows that either $\tau(x) \neq x$ or $\eta'\tau(x) \neq x$. Since τ and $\eta'\tau$ are both elements of $\text{fix}_G(S)$, and $\tau^2 = (\eta'\tau)^2 = \epsilon$, we conclude that condition (*) is satisfied.

Part (ii) (for the case where $\lambda < \mu$) can be proved in much the same way as (i), and so we leave it to the reader.

Now we assume that $\lambda = \mu$, so that

$$I = \{S : \exists E \in [\mu]^{<\omega} (S \subseteq \bigcup \{A_\alpha : \alpha \in E\})\}.$$

(i) Again, it suffices to show that $\mathcal{N}$ satisfies condition (*) of Theorem 1. Let $x \in \mathcal{N}$ and also let $S = \bigcup \{A_\alpha : \alpha \in E\}$ (for some $E \in [\mu]^{<\omega}$) which does not support x. By (the second assertion of) Lemma 2, there exist $E' \in [\mu]^{<\omega}$ which is disjoint from E, and $\alpha \in E'$ such that for some $\eta_\alpha \in G_\alpha$ (and hence $\eta_\alpha \in \text{fix}_G(S)$), $\eta_\alpha(x) \neq x$. Since η_α moves only finitely many atoms, η_α has finite order. Thus (*) is satisfied, finishing the proof.

Part (ii) can be proved in a similar manner, using the second assertion of Lemma 2, and Lemma 1. We thus take the liberty to leave the details to the interested reader. □

3 Model 1: $\mathcal{M}$

3.1 Motivation

We use Model 1 to establish that $\mathsf{LW} + \mathsf{DF} = \mathsf{F}$ does not imply $\mathsf{MC}_{\aleph_0}^{\aleph_0}$ in ZFA. This answers in the negative the relative open question in Howard and Tachtsis [6], and also fills the gap in information in Howard and Rubin [5].

We note that this model has been considered in Tachtsis [10, proof of Theorem 4(iv)], where it was shown that $\mathsf{DF} = \mathsf{F}$ is true in the model. In the interest of making our paper self-contained, we will provide our own proof of $\mathsf{DF} = \mathsf{F}$ in the model.

3.2 The description of $\mathcal{M}$

We construct a model $\mathcal{M}$ of ZFA starting with a model $\mathcal{M}'$ of ZFA + AC with a countably infinite set of atoms A which is partitioned into a countably infinite collection of countably infinite sets. Say $A = \bigcup\{A_k : k \in \omega\}$ where for every $k \in \omega$, $|A_k| = \aleph_0$ and for $k_1 \neq k_2$, $A_{k_1} \cap A_{k_2} = \emptyset$. Let G be the unrestricted direct product of $\mathscr{G}_n = \mathrm{Sym}(A_n)$ $(n \in \omega)$. The ideal of supports is defined (according to Subsect. 2.1) by

$$I = \{C : \exists E \in [\omega]^{<\omega}(C \subseteq \bigcup\{A_k : k \in E\})\}.$$

$\mathcal{M}$ is the permutation model determined by $\mathcal{M}'$, G, and I.

3.3 Versions of AC in $\mathcal{M}$

We first give a (known) group-theoretic result which will be useful for the verification of $\mathsf{DF} = \mathsf{F}$ and $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$ in $\mathcal{M}$.

Definition 3 Let Z be any set. If H is a subgroup of $\mathrm{Sym}(Z)$, then $|\mathrm{Sym}(Z) : H|$ denotes the index of H in $\mathrm{Sym}(Z)$.

Theorem 3 below, is due to Dixon, Neumann, and Thomas (see [2, Theorem 1]).

Theorem 3 *Let Z be countably infinite and let K be a subgroup of* $\mathrm{Sym}(Z)$ *for which* $|\mathrm{Sym}(Z) : K| < 2^{\aleph_0}$. *Then there is a finite subset Δ of Z such that* $\mathrm{fix}_{\mathrm{Sym}(Z)}(\Delta) \leq K \leq \mathrm{fix}_{\mathrm{Sym}(Z)}(\{\Delta\})$.

The subsequent Lemma 3 was originally proved by Onofri [8], and is also a consequence of Theorem 3. For the reader's convenience, we include the short proof of the lemma (using Theorem 3).

Lemma 3 *Let Z be a countably infinite set and let K be a proper subgroup of* $\mathrm{Sym}(Z)$. *Then* $|\mathrm{Sym}(Z) : K|$ *is infinite.*

Proof Let $\phi_0 \in \mathrm{Sym}(Z) \setminus K$. Toward a proof by contradiction, we assume that $|\mathrm{Sym}(Z) : K|$ is finite, so let $\mathrm{Sym}(Z)/K = \{K, \phi_0 K, \dots, \phi_n K\}$ for some $n \in \omega$ and $\phi_i \in \mathrm{Sym}(Z) \setminus K$ $(i \leq n)$. Then, by Theorem 3, there exists a finite subset $\Delta \subset Z$ such that $\mathrm{fix}_{\mathrm{Sym}(Z)}(\Delta) \leq K \leq \mathrm{fix}_{\mathrm{Sym}(Z)}(\{\Delta\})$. Since $\phi_0 \notin K$, there is $\delta \in \Delta$ such that $\phi_0(\delta) \neq \delta$. As Δ is finite, we have $|Z'| = \aleph_0$, where $Z' = Z \setminus (\Delta \cup (\bigcup\{\phi_i[\Delta] : i \leq n\}))$, and thus we may let $\psi \in \mathrm{Sym}(Z)$ such that $\psi(\delta) \in Z'$ (for example, let ψ be the transposition (δ, z') for any $z' \in Z'$). Then $\psi \notin \mathrm{fix}_{\mathrm{Sym}(Z)}(\{\Delta\})$ and $\phi_i^{-1}\psi \notin \mathrm{fix}_{\mathrm{Sym}(Z)}(\{\Delta\})$ for all $i \leq n$, and hence $\psi \notin K$ and $\phi_i^{-1}\psi \notin K$ for all $i \leq n$. Therefore, $\psi K \notin \{K, \phi_0 K, \dots, \phi_n K\} = \mathrm{Sym}(Z)/K$, a contradiction. □

Theorem 4 *In $\mathcal{M}$,* LW, $\mathsf{DF} = \mathsf{F}$, *and* $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$ *are true, but* $\mathsf{MC}_{\aleph_0}^{\aleph_0}$ *and* $\forall \mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ *are false.*

Proof By Theorem 2(ii), we immediately have that LW is true in $\mathcal{M}$.

Now, it is reasonably clear that the set $\mathscr{A} = \{A_k : k \in \omega\}$ is a countably infinite set of countably infinite sets in $\mathcal{M}$, which has no multiple choice function in $\mathcal{M}$. (If $f : \mathscr{A} \to \mathcal{P}(A)$ is a multiple choice function for $\mathscr{A}$ with support $S = \bigcup\{A_k : k \in E\}$ (for some finite $E \subset \omega$), then choose an integer $k_0 \notin E$; then $f(A_{k_0})$ is a finite subset of A_{k_0}. Then it is possible to choose $\phi \in G$ such that $\phi \in \mathrm{fix}_G(S)$ and $\phi(f(A_{k_0})) \neq f(A_{k_0})$. But, since ϕ fixes both f and A_{k_0}, $\phi(f(A_{k_0})) = f(A_{k_0})$. This gives a contradiction, so $\mathsf{MC}^{\aleph_0}_{\aleph_0}$ is false in $\mathcal{M}$.)

Claim $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ is false in $\mathcal{M}$.

Proof Indeed, there is no one-to-one mapping $f : 2 \times A \to A$ in $\mathcal{M}$. Assuming the contrary, let f be such a function in $\mathcal{M}$ with support $S = \bigcup\{A_i : i \in E\}$ for some finite $E \subset \omega$.

Now let $k \in \omega \setminus E$, and also let $x \in A_k$. Since $f((0, x)) \neq f((1, x))$, there exists $i \in 2$ such that $f((i, x)) = y$ with $y \neq x$ (and note that similarly to the following argument, y is necessarily an element of A_k). Let $z \in A_k \setminus \{x, y\}$ (recall that A_k is (countably) infinite) and also let $\psi = (x, z)$ (i.e. ψ transposes x and z but fixes all the other atoms of A). Then $\psi \in \mathrm{fix}_G(S)$, so $\psi(f) = f$. However,

$$((i, x), y) \in f \Rightarrow (\psi((i, x)), \psi(y)) \in \psi(f) \Rightarrow ((i, z), y) \in f,$$

so that $(i, x) \neq (i, z)$ and $f((i, x)) = f((i, z))$, contrary to the fact that f is one-to-one. Hence, $|2 \times A| \neq |A|$ in $\mathcal{M}$. □

Now we prove that $\mathsf{DF} = \mathsf{F}$ is true in $\mathcal{M}$. The following lemma will be useful for the proof.

Lemma 4 *Assume $x \in \mathcal{M}$ and $m \in \omega$. Let H be the subgroup $H = \{\phi \in G_m : \phi(x) = x\}$ of G_m (where G_m is given by (1) in Sect. 2.1). Then for all ϕ_1 and ϕ_2 in G_m, $\phi_1(x) = \phi_2(x)$ if and only if $\phi_1 H = \phi_2 H$.*

Proof The proof uses the standard properties of right cosets. Indeed, we have $\phi_1(x) = \phi_2(x)$, if and only if, $\phi_2^{-1}\phi_1(x) = x$, if and only if, $\phi_2^{-1}\phi_1 \in H$, if and only if, $\phi_2^{-1}\phi_1 H = H$, if and only if, $\phi_1 H = \phi_2 H$. □

Claim $\mathsf{DF} = \mathsf{F}$ is true in $\mathcal{M}$.

Proof Assume that Y is an infinite, non-well-orderable set in $\mathcal{M}$ with support $S = \bigcup\{A_i : i \in E\}$, where $E \subset \omega$ is finite. Then for some $x \in Y$, S is not a support of x. Let $S \cup S'$ be a suppport of x, where $S' = \bigcup\{A_i : i \in E'\}$ with $E \cap E' = \emptyset$. By (the second assertion of) Lemma 2, we obtain an $m \in E'$ and a $\beta \in G_m$ such that $\beta(x) \neq x$.

It follows that the set

$$H = \{\phi \in G_m : \phi(x) = x\}$$

is a proper subgroup of G_m. Since G_m is isomorphic to $\mathrm{Sym}(\omega)$ (for $G_m \simeq \mathrm{Sym}(A_m) \simeq \mathrm{Sym}(\omega)$), we may apply Lemma 3 to conclude that the set of left cosets $\{\phi H : \phi \in G_m\}$ is infinite. Let

$$W = \{\phi(x) : \phi \in G_m\}.$$

Then we know the following about W:

1. By Lemma 4 and the fact that the set of left cosets of H in G is infinite, W is infinite.
2. $W \subseteq Y$ since for all $\phi \in G_m$, $\phi \in \text{fix}_G(S)$.
3. Every element of W has support $S \cup S'$, by (2) of Sect. 2.1.

So W is an infinite subset of Y which can be well ordered in $\mathcal{M}$. Therefore, Y has a countably infinite subset in $\mathcal{M}$. □

Claim $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$ is true in $\mathcal{M}$.

Proof Let $\mathscr{X} = \{X_\alpha : \alpha \in \kappa\}$ be an infinite well-ordered set in $\mathcal{M}$ (κ is an infinite well-ordered cardinal and the mapping $\alpha \mapsto X_\alpha$ is a bijection) such that X_α is non-empty and finite for all $\alpha \in \kappa$. Let $S = \bigcup\{A_i : i \in E\}$ (for some finite $E \subset \omega$) be a support of X_α for all $\alpha \in \kappa$. We will show that S supports every element of $\bigcup \mathscr{X}$; hence, $\bigcup \mathscr{X}$ will be well orderable in the model.

Assume on the contrary that there exist $\alpha \in \kappa$ and $x \in X_\alpha$ such that S is not a support of x. By (the second assertion of) Lemma 2, there exist $m \in \omega \setminus E$ and $\eta \in G_m$ such that $\eta(x) \neq x$. Let

$$Z = \{\phi(x) : \phi \in G_m\}.$$

Then $Z \subseteq X_\alpha$ (since $x \in X_\alpha$, S is a support of X_α, and $G_m \subseteq \text{fix}_G(S)$). Hence, Z is finite (and has at least two elements). Furthermore, since $\eta \in G_m$ and $\eta(x) \neq x$, the group

$$H = \{\phi \in G_m : \phi(x) = x\}$$

is a proper subgroup of G_m. Since Z is finite, the index $|G_m : H|$ of H in G_m is finite. But this contradicts Lemma 3, since G_m is isomorphic to $\text{Sym}(\omega)$ and H is a proper subgroup of G_m. □

The above arguments complete the proof of the theorem. □
By Theorem 4, we obtain that $\mathsf{LW} + \mathsf{DF} = \mathsf{F} \not\Rightarrow \mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$ in ZFA.

4 Model 2: $\mathscr{V}$

4.1 Motivation

We recall that DC implies the axiom of countable choice (i.e. "Every countably infinite family of non-empty sets has a choice function"), which in turn implies both $\mathsf{DF} = \mathsf{F}$ and $\mathsf{MC}_{\aleph_0}^{\aleph_0}$. Furthermore, DC does not imply $\mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$ in ZF; in the Brunner/Howard permutation model $\mathcal{N}15$ in [5], DC is true but $\mathsf{MC}_{\aleph_0}^{\mathsf{WO}}$ (the axiom of multiple choice for well-ordered families of countably infinite sets) is false (see Sect. 5). The result can be transferred to ZF using Pincus' transfer theorems (see [5, Note 103, third theorem, p. 286]), and notice that $\neg\mathsf{MC}_{\aleph_0}^{\mathsf{WO}}$ is a boundable, and hence injectively boundable, statement (see [5, Note 103] for the definitions of those terms).

So in view of Theorem 4 and the above discussion, the natural question that emerges here is whether LW + DC implies $\mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$ in ZFA (recall that LW is equivalent to AC in ZF, so the above implication is vacuously true in ZF). The status of this implication is mentioned as unknown in [5]. Therein, it is also mentioned as unknown whether LW+DC implies $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$. In the model $\mathscr{V}$ of this section, we address these open questions and prove that their respective answers are in the negative. In fact, we prove a much stronger result than "LW + DC implies neither $\mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$ nor $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$ in ZFA", as Theorem 5 below clarifies.

4.2 The description of $\mathscr{V}$

We use a permutation model constructed in the proof of Jech's Theorem 8.3 in [7] (see also [5, Model $\mathcal{N}2(\aleph_\alpha)$, p. 180]). The construction starts with a ground model M of ZFA + AC which has a set A of atoms of cardinality $\aleph_\alpha$, where $\aleph_\alpha$ is an uncountable regular cardinal in M. We partition A into a disjoint union of $\aleph_\alpha$ pairs, so that $A = \bigcup\{P_\xi : \xi < \aleph_\alpha\}$ ($|P_\xi| = 2$ for all $\xi < \aleph_\alpha$, and for $\xi \neq \xi'$, $P_\xi \cap P_{\xi'} = \emptyset$). Let G be the group of all permutations of A which fix P_ξ for all $\xi < \aleph_\alpha$. (Note that G is essentially the unrestricted direct product of $\mathrm{Sym}(P_\xi)$ ($\xi < \aleph_\alpha$), and that every element of G has order 2.) Let $\mathscr{F}$ be the filter on G generated by the groups $\mathrm{fix}_G(E)$, where $E \subset A$, $|E| < \aleph_\alpha$. Let $\mathscr{V}$ be the permutation model determined by M, G and $\mathscr{F}$.

4.3 Versions of AC in $\mathscr{V}$

Theorem 5 *In $\mathscr{V}$,* LW *and* DC_ξ *for all infinite cardinals $\xi < \aleph_\alpha$ are true but* $\mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$, $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$ *and* $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ *are false.*

Proof By Corollary 1, we have $\mathscr{V} \models$ LW. Furthermore, in [7, Lemma 8.4, p. 123], it is shown that for every $\xi < \aleph_\alpha$, DC_ξ is true in $\mathscr{V}$, and that the family $\mathscr{A} = \{P_\xi : \xi < \aleph_\alpha\}$ (which is in $\mathscr{V}$ and has cardinality $\aleph_\alpha$ in $\mathscr{V}$) has no choice function in the model; hence, $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$ is false in $\mathscr{V}$.

Claim $\mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$ is false in $\mathscr{V}$[1].

Proof Fix an infinite cardinal number $\kappa < \aleph_\alpha$. Let

$$\mathscr{U} = \{U_\xi : \xi < \aleph_\alpha\}$$

be a partition of $\aleph_\alpha$ ($\xi \mapsto U_\xi$ is a bijection) into sets each of which has cardinality κ. (And note that $\mathscr{U}$ gives rise to an $\aleph_\alpha$-sized partition of $\mathscr{A} = \{P_\xi : \xi < \aleph_\alpha\}$ into κ-sized sets, namely $\{\{P_\gamma : \gamma \in U_\xi\} : \xi < \aleph_\alpha\}$, which clearly has a choice function in the model.) For each $\xi < \aleph_\alpha$, we let

$$W_\xi = \prod_{\gamma \in U_\xi} P_\gamma.$$

[1] Notice that $\mathscr{V} \models \mathsf{MC}_{\aleph_0}^{\aleph_0}$, since $\mathscr{V} \models$ DC and DC $\Rightarrow \mathsf{MC}_{\aleph_0}^{\aleph_0}$.

Then for every $\xi < \aleph_\alpha$, $W_\xi \in \mathcal{V}$ (any permutation of A in G fixes W_ξ), and furthermore, the subset $E = \bigcup\{P_\gamma : \gamma \in U_\xi\}$ of A (which has cardinality $\kappa < \aleph_\alpha$) is a support of every element of W_ξ. Thus the infinite sets W_ξ ($\xi < \aleph_\alpha$) are well orderable in $\mathcal{V}$.

Now we let

$$\mathcal{W} = \{W_\xi : \xi < \aleph_\alpha\}.$$

Then $\mathcal{W} \in \mathcal{V}$ and has cardinality $\aleph_\alpha$ in $\mathcal{V}$, so $\mathcal{W}$ is well orderable in $\mathcal{V}$. However, $\mathcal{W}$ has no multiple choice function in $\mathcal{V}$. Assume the contrary and let F be a multiple choice function for $\mathcal{W}$, which is in $\mathcal{V}$. Let $E \subset A$, $|E| < \aleph_\alpha$, be a support of F. Since $\mathcal{U}$ is an $\aleph_\alpha$-sized partition of $\aleph_\alpha$ and $|E| < \aleph_\alpha$, it follows that for some $\xi_0 < \aleph_\alpha$,

$$E \cap \left(\bigcup\{P_\gamma : \gamma \in U_{\xi_0}\}\right) = \emptyset. \tag{5}$$

Let $f \in F(W_{\xi_0})$ ($F(W_{\xi_0}) \neq \emptyset$, since F is a multiple choice function for $\mathcal{W}$), and also let $g \in W(\xi_0) \setminus F(W_{\xi_0})$ (W_{ξ_0} is infinite, whereas $F(W_{\xi_0})$ is a finite subset of W_{ξ_0}). Then $g \neq f$, so the set $J = \{\gamma \in U_{\xi_0} : g(\gamma) \neq f(\gamma)\}$ is nonempty. For each $\gamma \in J$, let ψ_γ be the permutation of A which interchanges the two elements of P_γ but fixes every atom in $A \setminus P_\gamma$. Let

$$\pi = \prod_{\gamma \in J} \psi_\gamma.$$

By the definition of π and (5), we have $\pi \in \mathrm{fix}_G(E)$, so $\pi(F) = F$, and since $\pi(W_{\xi_0}) = W_{\xi_0}$ and F is a function, we also have $\pi(F(W_{\xi_0})) = F(W_{\xi_0})$. Furthermore, it is clear that $\pi(f) = g$. Thus we have

$$f \in F(W_{\xi_0}) \Rightarrow \pi(f) \in \pi(F(W_{\xi_0})) \Rightarrow g \in F(W_{\xi_0}),$$

contradicting the fact that $g \notin F(W_{\xi_0})$. Therefore, $\mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$ is false in $\mathcal{V}$ as required. □

In [5], it is mentioned that $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ is false in $\mathcal{V}$. The argument is similar to the one given for the proof of Claim 3.3. (Assuming that there is a one-to-one mapping $f : 2 \times A \to A$ which is in $\mathcal{V}$, let E be a support of f and $\gamma < \aleph_\alpha$ such that $E \cap P_\gamma = \emptyset$. It is easy to see that $f[2 \times P_\gamma] \subseteq P_\gamma$, which is a contradiction since f is one-to-one and $|P_\gamma| < |2 \times P_\gamma|$.)

The above arguments complete the proof of the theorem. □

By Theorem 5, we immediately obtain the following corollary.

Corollary 2 $\mathsf{LW} + \mathsf{DC}$ *implies neither* $\mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$ *nor* $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$ *in* ZFA.

Clearly the above result readily yields that $\mathsf{LW} + \mathsf{DF} = \mathsf{F}$ does not imply $\mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$ in ZFA (and neither does it imply $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$). Therefore, Theorem 4 is an essential strengthening of the latter non-implication in ZFA (and note again that the model of the proof of Theorem 5 (or the model $\mathcal{N}15$ in [5]) satisfies $\mathsf{MC}_{\aleph_0}^{\aleph_0} \wedge \neg\mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$).

5 Model 3: $\mathcal{N}15$

5.1 Motivation

As already mentioned in Sect. 4, this model satisfies $\mathsf{DC} \wedge \neg \mathsf{MC}^{\mathsf{WO}}_{\aleph_0}$ (see the forthcoming Theorem 6). Therefore, the next natural question that comes up is whether LW is true in $\mathcal{N}15$. We note that the status of LW in $\mathcal{N}15$ is not specified in [5].

The answer to this open question is in the affirmative; thus filling the gap in information in [5] and providing further insight to the reader.

5.2 The description of $\mathcal{N}15$

The set A of atoms has cardinality $\aleph_1$, and is written as a union of an $\aleph_1$-sized family of pairwise disjoint countably infinite sets,

$$A = \bigcup\{B_\alpha : \alpha < \aleph_1\}, \text{ where } B_\alpha = \{a_{i,\alpha} : i \in \omega\}.$$

For each $\alpha < \aleph_1$, let $\mathcal{G}_\alpha$ be the group of even permutations on B_α. Let G be the unrestricted direct product of $\mathcal{G}_\alpha$ ($\alpha < \aleph_1$).

Let I be the ideal of all countable subsets of A. Note that I is equal to the ideal generated by all sets of the form $\bigcup\{B_\alpha : \alpha \in E\}$, where E is a countable subset of $\aleph_1$. $\mathcal{N}15$ is the permutation model determined by A, G and I.[2]

5.3 Versions of AC in $\mathcal{N}15$

Theorem 6 *In $\mathcal{N}15$, LW and DC are true, but $\mathsf{MC}^{\mathsf{WO}}_{\aleph_0}$ is false.*

Proof By Theorem 2(i), we have $\mathcal{N}15 \models \mathsf{LW}$.

Furthermore, DC is true in $\mathcal{N}15$ (for the normal ideal I comprises all countable subsets of A, and $\aleph_1$ is a regular cardinal—the argument in the proof of [7, Lemma 8.4, p. 123], and in the paragraph following this lemma, can be adapted in our case by making the obvious minor changes).

It is also easy to see that $\mathsf{MC}^{\mathsf{WO}}_{\aleph_0}$ is false in $\mathcal{N}15$. Indeed, let $\mathcal{B} = \{B_\alpha : \alpha < \aleph_1\}$. Clearly $|\mathcal{B}| = \aleph_1$ in $\mathcal{N}15$ (every permutation of A in G fixes $\mathcal{B}$ pointwise and $|\mathcal{B}| = \aleph_1$ in the ground model) and for every $\alpha < \aleph_1$, $|B_\alpha| = \aleph_0$ in $\mathcal{N}15$ (B_α is a support of each of its elements, and is countable in the ground model). Now, $\mathcal{B}$ has no multiple choice function in $\mathcal{N}15$. Assuming the contrary, let f be such a function in $\mathcal{N}15$. Let $S = \bigcup\{B_\alpha : \alpha \in E\}$, where $E \subset \aleph_1$ is countable, be a support of f. Let $\alpha_0 \in \aleph_1 \setminus E$ and u be any element of $f(B_{\alpha_0})$. Let $Z = \{z_1, z_2, z_3\}$ be a 3-element subset of B_{α_0} which is disjoint from $f(B_{\alpha_0})$, and also let $\pi = (u, z_1)(z_2, z_3)$. Then $\pi \in \mathrm{fix}_G(S)$,

[2] The model $\mathcal{N}15$ in [5] is actually a variant of a model constructed by Brunner and Howard [1]. In particular, this FM model of [1] is determined by the same set A of atoms, the same normal ideal I (of the countable subsets of A), but by the weak direct product of the $\mathcal{G}_\alpha$'s instead of the their unrestricted product. In this model, LW, DC, and $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{fin}}$ are all true (but $\mathsf{MC}^{\mathsf{WO}}_{\aleph_0}$ is false), whereas it is *unknown* whether $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{fin}}$ is valid in $\mathcal{N}15$.

and thus $\pi(f) = f$. However, $z_1 \in \pi(f(B_{\alpha_0})) \setminus f(B_{\alpha_0})$ (so $\pi(f(B_{\alpha_0})) \neq f(B_{\alpha_0})$), contradicting f's being supported by S. Hence, $\mathcal{B}$ has no multiple choice function in $\mathcal{N}15$. □

Remark 2 We note that, similarly to the proof of Claim 3.3 (of the proof of Theorem 4), $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ is false in $\mathcal{N}15$.

6 Model 4: $\mathcal{U}$, a variant of $\mathcal{N}15$

6.1 Motivation

In view of the preceding study of the model $\mathcal{N}15$, it is natural to consider a variation of this model which witnesses "$\mathsf{LW} + \mathsf{DF} = \mathsf{F} + \mathsf{AC}^{\mathsf{WO}}_{\mathsf{fin}} \not\Rightarrow \mathsf{MC}^{\aleph_0}_{\aleph_0}$" in ZFA. Indeed, our ZFA-model $\mathcal{U}$ of this section appeals to this consideration. We note that $\mathcal{U}$ does not appear in either of [1] and [5].

6.2 The description of $\mathcal{U}$

The set A of atoms is countably infinite, and is written as a union of a countably infinite family of pairwise disjoint countably infinite sets,

$$A = \bigcup\{B_n : n \in \omega\}, \text{ where } B_n = \{a_{i,n} : i \in \omega\}.$$

For every $n \in \omega$, let $\mathcal{G}_n$ be the group of even permutations of B_n. Let G be the unrestricted direct product of the $\mathcal{G}_n$'s. Let I be the ideal of subsets of A which is generated by all finite unions of B_n ($n \in \omega$). Let $\mathcal{U}$ be the Fraenkel–Mostowski model determined by A, G, and I.

Let us point out here that $\mathcal{U}$ can be generalized. Indeed, for any infinite regular cardinal number κ, we may similarly construct a permutation model $\mathcal{U}_\kappa$: The set of atoms, $A = \bigcup\{B_\alpha : \alpha < \kappa\}$ (where each of the B_α's has cardinality λ, where $\omega \leq \lambda \leq \kappa$, and $\{B_\alpha : \alpha < \kappa\}$ is disjoint), $\mathcal{G}_\alpha$ is the group of even permutations of B_α, $G = \prod_{\alpha<\kappa} \mathcal{G}_\alpha$ (the unrestricted direct product of the $\mathcal{G}_\alpha$'s), and I is the (normal) ideal generated by $\{\bigcup\{B_\alpha : \alpha \in E\} : E \in [\kappa]^{<\omega}\}$.

6.3 Versions of AC in $\mathcal{U}$

Theorem 7 *In $\mathcal{U}$,* LW, $\mathsf{DF} = \mathsf{F}$, *and* $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{fin}}$ *are true, but* $\mathsf{MC}^{\aleph_0}_{\aleph_0}$ *is false.*

Proof By Theorem 2(i), we have $\mathcal{U} \models \mathsf{LW}$.

Furthermore, the proof that $\mathsf{MC}^{\aleph_0}_{\aleph_0}$ is false in $\mathcal{U}$ is almost identical to the proof that $\mathsf{MC}^{\mathsf{WO}}_{\aleph_0}$ is false in $\mathcal{N}15$ (see the proof of Theorem 6), and we thus skip it.

Claim $\mathsf{DF} = \mathsf{F}$ is true in $\mathcal{U}$.

Proof Assume that Y is an infinite, non-well-orderable set in $\mathscr{U}$ with support $S = \bigcup\{B_i : i \in E\}$ for some finite $E \subset \omega$. Then for some $x \in Y$, S is not a support of x. Let $S \cup S'$ be a suppport of x, where $S' = \bigcup\{B_i : i \in E'\}$ with $E' \cap E = \emptyset$. By (the second assertion of) Lemma 2, we obtain an $m \in E'$ and a $\beta \in G_m$ (where G_m is given by (1) of Sect. 2.1) such that $\beta(x) \neq x$.

Consider the G_m-orbit of x, i.e. the set

$$\mathrm{Orb}_{G_m}(x) = \{\pi(x) : \pi \in G_m\}.$$

Since S is a support of Y, $x \in Y$, and for all $\pi \in G_m$, $\pi \in \mathrm{fix}_G(S)$, we conclude that $\mathrm{Orb}_{G_m}(x) \subseteq Y$. Furthermore, $\mathrm{Orb}_{G_m}(x)$ is well orderable in the model since $S \cup S'$ is a support of every element of $\mathrm{Orb}_{G_m}(x)$.

We assert that $\mathrm{Orb}_{G_m}(x)$ is infinite. If not, then $\mathrm{Sym}(\mathrm{Orb}_{G_m}(x))$ is also finite. Let $\phi : G_m \to \mathrm{Sym}(\mathrm{Orb}_{G_m}(x))$ be defined by

$$\phi(\eta)(y) = \eta(y) \ (y \in \mathrm{Orb}_{G_m}(x)).$$

Then ϕ is a homomorphism, and hence $\ker(\phi)$ is a normal subgroup of G_m and the quotient group $G_m/\ker(\phi)$ embeds into $\mathrm{Sym}(\mathrm{Orb}_{G_m}(x))$. However, $\ker(\phi)$ is a proper subgroup of G_m (for $\beta \in G_m \setminus \ker(\phi)$), and since G_m is a simple group (for $G_m \simeq \mathscr{G}_m$), we obtain that $\ker(\phi) = \{\epsilon\}$.

Thus, $G_m/\ker(\phi)$ is isomorphic to G_m, and so $\mathrm{Sym}(\mathrm{Orb}_{G_m}(x))$ contains a copy of G_m. This is a contradiction, since $\mathrm{Sym}(\mathrm{Orb}_{G_m}(x))$ is finite and G_m is infinite. Therefore, $\mathrm{Orb}_{G_m}(x)$ is infinite, and thus Y is Dedekind infinite. □

Claim $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{fin}}$ is true in $\mathscr{U}$.

Proof Letting $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ be an infinite well-ordered family (κ is an infinite well-ordered cardinal number) of non-empty finite sets in $\mathscr{U}$ and $S = \bigcup\{B_i : i \in E\}$ (where $E \in [\omega]^{<\omega}$) be a support of every V_α, we may work similarly to the proof of Claim 6.3 in order to show that every element of $\bigcup \mathcal{V}$ is supported by S, so that $\bigcup \mathcal{V}$ is well orderable in $\mathscr{U}$. Thus $\mathcal{V}$ has a choice function in $\mathscr{U}$. □

The above arguments complete the proof of the theorem. □

Remark 3 As with the model $\mathcal{N}15$, $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ is false in $\mathscr{U}$.

7 Model 5: $\mathcal{U}$

7.1 The description of $\mathcal{U}$

Suitable adjustments to the construction of the model $\mathcal{N}15$ yield the result of Theorem 5, modulo the assertion about $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{fin}}$. Indeed, fix any regular cardinal number $\aleph_{\alpha+1}$. We start with a model M of ZFA+AC with a set A of atoms which has cardinality $\aleph_{\alpha+1}$ and is written as a disjoint union,

$$A = \bigcup\{B_\beta : \beta < \aleph_{\alpha+1}\}, \text{ where } B_\beta = \{a_{\mu,\beta} : \mu < \aleph_\alpha\}.$$

For each $\beta < \aleph_{\alpha+1}$, let $\mathscr{G}_\beta$ be the group of even permutations of B_β. Let G be the unrestricted direct product of the $\mathscr{G}_\beta$'s. Let I be the ideal of all subsets of A having cardinality less than $\aleph_{\alpha+1}$. Let $\mathcal{U}$ be the permutation model determined by A, G and I.

7.2 Versions of AC in $\mathcal{U}$

Theorem 8 *In $\mathcal{U}$,* LW *and* DC_ξ *for all infinite cardinals $\xi < \aleph_{\alpha+1}$ are true, but* $\mathsf{MC}^{\mathsf{WO}}_{\mathsf{WO}}$ *is false.*

Proof By Theorem 2, $\mathcal{U} \models \mathsf{LW}$. Furthermore, the well orderable family $\mathcal{B} = \{B_\beta : \beta < \aleph_{\alpha+1}\}$, which is in $\mathcal{U}$ (any permutation of A in $\mathscr{H}$ fixes $\mathcal{B}$ pointwise), and which comprises sets that are well orderable in $\mathcal{U}$ (for every $\beta < \aleph_{\alpha+1}$, B_β is a support of each of its elements), has no multiple choice function in $\mathcal{U}$, and thus $\mathsf{MC}^{\mathsf{WO}}_{\mathsf{WO}}$ is false in $\mathcal{U}$. The fact that DC_ξ is true in $\mathcal{U}$ for all infinite cardinals $\xi < \aleph_{\alpha+1}$ can be established as in the proof of Jech's Lemma 8.4 (p. 123) in [7]. □

Remark 4 As with the model $\mathcal{N}15$, $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ is false in $\mathcal{U}$.

8 Model 6: $\mathcal{N}9$

8.1 Motivation

As mentioned in Sect. 1, $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ does not imply $\mathsf{MC}^{\aleph_0}_{\aleph_0}$ in ZF; in Sageev's ZF-model $\mathcal{M}6$ of [5], $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ is true, but there is a countably infinite family of countably infinite sets of reals without a choice function in the model (see [9]). Thus $\mathsf{MC}^{\aleph_0}_{\aleph_0}$ is false in $\mathcal{M}6$. Furthermore, LW is also false in $\mathcal{M}6$, since LW is equivalent to AC in ZF.

It is an intriguing open problem whether $\mathsf{LW} + \forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ (which is stronger than $\mathsf{LW} + \mathsf{DF} = \mathsf{F}$ in ZFA) implies $\mathsf{MC}^{\aleph_0}_{\aleph_0}$ in ZFA. In this direction, note that the Halpern/Howard permutation model $\mathcal{N}9$ in [5] satisfies $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ (and thus satisfies $\mathsf{DF} = \mathsf{F}$, see [3, Theorem 2.3]); however, the status of LW, $\mathsf{MC}^{\aleph_0}_{\aleph_0}$, and $\mathsf{MC}^{\aleph_0}$ in $\mathcal{N}9$, were open problems until now. We settle these problems by showing next that LW and $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{WO}}$ are both true in $\mathcal{N}9$, whereas $\mathsf{MC}^{\aleph_0}$ is false in $\mathcal{N}9$.

8.2 The description and basic properties of $\mathcal{N}9$

We start with a model M of ZFA + AC with a set A of atoms which has the structure of the set

$$\omega^{(\omega)} = \{s : s : \omega \to \omega \wedge (\exists n \in \omega)(\forall j > n)(s_j = 0)\}.$$

We identify A with the latter set to simplify the description of the group G.

For $s \in A$, the pseudo length of s is the least natural number k such that for all $\ell \geq k$, $s_\ell = 0$. A subset of A is called *bounded* there is an upper bound for the pseudo lengths of the elements of A. G is the group of all permutations ϕ of A such that the support of ϕ, $\{a \in A : \phi(a) \neq a\}$, is bounded.

For every $s \in A$ and every $n \in \omega$, let

$$A_s^n = \{t \in A : \forall j \geq n(t_j = s_j)\}.$$

Definition 4 (Mostly from [4]) Assume $s \in A$ and $n \in \omega$; then

1. A_s^n is called *the n-block containing s*.
2. For any $t \in A_s^n$, the *n-block code of t* is the sequence

$$(t_n, t_{n+1}, t_{n+2}, \ldots) = (s_n, s_{n+1}, s_{n+2}, \ldots).$$

   The *n-block code of A_s^n* is the n-block code of any of its elements. We will denote the n-block code of an element $t \in A$ or an n-block B by $\mathrm{bc}^n(t)$ or $\mathrm{bc}^n(B)$, respectively.
3. For any $t \in A_s^n$, the finite sequence $(t_0, t_1, t_2, \ldots, t_{n-1}) = t \restriction n$ is called the *n-location of t (in A_s^n)*.

Note the following

1. A_0^n is the set of all elements of A with pseudo length less than or equal to n. (In the expression A_0^n, 0 denotes the constant sequence all of whose terms are 0.)
2. For $s \in A$ and $n, m \in \omega$ with $n \leq m$, $A_s^n \subseteq A_s^m$.
3. If $n \leq m$, B is an n-block, B' is an m-block and $B \cap B' \neq \emptyset$ then $B \subseteq B'$. (This follows from the previous item.)
4. Any $t \in A$ is the concatenation $(t \restriction n)^\frown \mathrm{bc}^n(t)$ of the n-location of t and the n-block code t.

For each $n \in \omega$, G_n is the subgroup of G consisting of all permutations $\phi \in G$ such that

1. ϕ fixes A_0^n pointwise,
2. ϕ fixes the set of n-blocks, that is, $A_s^n = A_t^n$ if and only if $A_{\phi(s)}^n = A_{\phi(t)}^n$,
3. for each $s \in A$, the n-location of $\phi(s)$ is the same as the n-location of s.

(Note that if $n \leq m$, then $G_m \subseteq G_n$.) J is the filter of subgroups of G generated by the groups G_n, $n \in \omega$. That is, $H \in J$ if and only if H is a subgroup of G and there exists $n \in \omega$ such that $G_n \subseteq H$. It is shown in [4] that J is a normal filter, that is, closed under conjugation. $\mathcal{N}9$ is the Fraenkel–Mostowski model of ZFA which is determined by M, G, and J.

Lemma 5 *Assume that f is a one-to-one function from a subset of A into A and n is a natural number such that*

1. *A_0^n is a subset of the domain of f and f fixes A_0^n pointwise.*
2. *The domain of f is the union of n-blocks and $f(A_s^n)$ is an n-block for any n-block A_s^n contained in the domain of f.*
3. *f fixes n-locations.*

4. *The domain and range of f are bounded. That is, there is an upper bound for the pseudo-lengths of the elements of* $\mathrm{dom}(f) \cup \mathrm{ran}(f)$.

Then there is a $\phi \in G_n$ that extends f.

Proof By assumption (4) there is an $m \in \omega$ such that $\mathrm{dom}(f) \cup \mathrm{ran}(f) \subseteq A_0^m$. If we let

$$\begin{aligned}
\mathcal{B}_0 &= \{B : B \text{ is an } n\text{-block and } B \subseteq A_0^m\} \\
\mathcal{B}_1 &= \{B : B \text{ is an } n\text{-block and } B \subseteq A_0^{m+1}\} \\
\mathcal{B}_2 &= \{B : B \text{ is an } n\text{-block and } B \subseteq \mathrm{dom}(f)\} \\
\mathcal{B}_3 &= \{B : B \text{ is an } n\text{-block and } B \subseteq \mathrm{ran}(f)\}
\end{aligned}$$

then, since $\mathcal{B}_2 \subseteq \mathcal{B}_0$ and $\mathcal{B}_1 \setminus \mathcal{B}_0$ is countably infinite, we have $\mathcal{B}_1 \setminus \mathcal{B}_2$ is countably infinite. Similarly, $\mathcal{B}_1 \setminus \mathcal{B}_3$ is countably infinite. Let G be a one-to-one function from $\mathcal{B}_1 \setminus \mathcal{B}_2$ onto $\mathcal{B}_1 \setminus \mathcal{B}_3$ and define $F : A_0^{m+1} \to A_0^{m+1}$ by

$$F(s) = \begin{cases} f(s) & \text{if } s \in \mathrm{dom}(f); \\ \text{the element of } G(A_s^n) \text{ with} & \\ \quad \text{the same } n\text{-location as } s & \text{otherwise.} \end{cases}$$

Then F is a permutation of A_0^{m+1} which extends f and satisfies conditions (1), (2) and (3) of the lemma. Therefore the function ϕ defined by

$$\phi(s) = \begin{cases} F(s) & \text{if } s \in A_0^{m+1}; \\ s & \text{if } s \in A \setminus A_0^{m+1}. \end{cases}$$

extends f and is in G_n. (ϕ is bounded because the support of ϕ is a subset of A_0^{m+1}.) □

Theorem 9 *If $(X, \leq)$ is a well ordered set of non-empty well orderable sets in $\mathcal{N}9$ and G_n is a support of $(X, \leq)$ where $n > 0$, then for every $y \in X$, G_{n+1} fixes y pointwise.*

Proof Assume that $(X, \leq)$ is a well ordered set with support G_n and that $y \in X$ and $x \in y$. We prove the theorem by arguing by contradiction that for all $\beta \in G_{n+1}$, $\beta(x) = x$. Assume $\beta \in G_{n+1}$ and $\beta(x) \neq x$. G_n fixes y since G_n fixes X pointwise and since y is well orderable in $\mathcal{N}9$ there is $k > n$ such that G_k fixes y pointwise. Therefore for any $\rho \in G_n$, $\rho(x) \in y$, and hence for all $\alpha \in G_k$, $\alpha(\rho(x)) = \rho(x)$. This contradicts the following lemma.

Lemma 6 *Assume $x \in \mathcal{N}9$, n is a positive integer, and there exists $\beta \in G_{n+1}$ such that $\beta(x) \neq x$. Then for all $k \geq n$, there are $\rho \in G_n$ and $\alpha \in G_k$ such that $\alpha(\rho(x)) \neq \rho(x)$.*

Proof Assuming the hypotheses, then for $k = n$ or for $k = n + 1$ we can take $\rho = \epsilon$, the identity permutation on A and $\alpha = \beta$. We will prove the lemma for $k = n + 2$. The lemma will then follow by mathematical induction.

Since $\beta \in G$ there is an integer j such that the support of β is a subset of A_0^j. We have assumed that $\beta \in G_{n+1}$ so β fixes A_0^{n+1} pointwise. Therefore there is an atom $s \notin A_0^{n+1}$ moved by β. From this we conclude that $j > n+1$.

The plan of the proof is to get an element ρ of G_n which takes each $n+1$-block contained in A_0^j to an $n+2$-block. We will also make sure that if s and s' have the same $n+1$-location then $\rho(s)$ and $\rho(s')$ have the same $n+2$-location. Then α will be defined so that it acts on $n+2$ blocks contained in A_0^{j+1} by mirroring the action of β on $n+1$-blocks contained in A_0^j. That is, α will be $\rho\beta\rho^{-1}$.

Fix a bijection $i \mapsto (u_1(i), u_2(i))$ from ω onto $\omega \times \omega$ so that

$$u_1(0) = u_2(0) = 0. \tag{6}$$

Define $f : A_0^j \to A_0^{j+1}$ by

$$\begin{aligned} &f(s_0, s_1, \ldots, s_{n-1}, s_n, s_{n+1}, \ldots, s_{j-1}, 0, 0, \ldots) \\ &\quad = (s_0, s_1, \ldots, s_{n-1}, u_1(s_n), u_2(s_n), s_{n+1}, \ldots, s_{j-1}, 0, 0, \ldots) \end{aligned}$$

That is,

$$(f(s))_i = \begin{cases} s_i & \text{if } 0 \leq i \leq n-1; \\ u_1(s_n) & \text{if } i = n; \\ u_2(s_n) & \text{if } i = n+1; \\ s_{i-1} & \text{if } i > n+1. \end{cases}$$

(But the first form is easier to work with.)

Using the definition, we see that f has the following properties.

1. f is a bijection from A_0^j onto A_0^{j+1} (because the function $i \mapsto (u_1(i), u_2(i))$ is a bijection from ω onto $\omega \times \omega$).
2. f fixes A_0^n pointwise (using Eq. 6).
3. If s and t are in A and $\mathrm{bc}^n(s) = \mathrm{bc}^n(t)$ then $\mathrm{bc}^n(f(s)) = \mathrm{bc}^n(f(t))$.
4. For $s \in \mathrm{dom}(f) = A_0^j$, the n-location of s is

$$(s_0, s_1, \ldots, s_{n-1})$$

   which is the same as the n-location of $f(s)$.
5. For $s \in \mathrm{dom}(f)$, $\mathrm{bc}^{n+2}(f(s)) = (s_{n+1}, s_{n+2}, \ldots, s_{j-1}, 0, 0, \ldots) = \mathrm{bc}^{n+1}(s)$.
6. For $s \in \mathrm{dom}(f)$, the $n+2$-location of $f(s)$ is

$$(s_0, s_1, \ldots, s_{n-1}, u_1(s_n), u_2(s_n)).$$

7. $\mathrm{bc}^{n+2}(f(A_0^{n+1})) = \mathrm{bc}^{n+1}(A_0^{n+1}) = (0, 0, 0, \ldots)$ (by Eq. (6).

Using items (2), (3) and (4) above we see that f satisfies conditions (1), (2) and (3) of the hypotheses of Lemma 5. Further, condition (4) is satisfied since $\mathrm{dom}(f) \cup \mathrm{ran}(f) \subseteq A_0^{j+1}$. Applying the lemma we obtain a $\rho \in G_n$ that extends f.

Let $\alpha = \rho\beta\rho^{-1}$. To complete the proof, we need to argue that $\alpha(\rho(x)) \neq \rho(x)$ and that $\alpha \in G_{n+2}$. For the first of these we note that $\alpha(\rho(x)) = \rho(\beta(\rho^{-1}(\rho(x)))) = \rho(\beta(x))$. If this is equal to $\rho(x)$ we conclude that $\beta(x) = x$ which contradicts our assumptions that $\beta(x) \neq x$.

For the proof that $\alpha \in G_{n+2}$ we will need the following sublemma.

Sublemma 1 *Assume $s \in A$. Then,*

1. *If $A_s^{n+1} \subseteq A_0^j$ then $f(A_s^{n+1}) = A_s^{n+2}$.*
2. *If $s \notin A_0^{j+1}$ then $\alpha(s) = s$.*

Proof For part (1) assuming that $A_s^{n+1} \subseteq A_0^j$. It follows from (5) in the list of properties of f that $f(A_s^{n+1}) \subseteq A_{f(s)}^{n+2}$ From this we conclude that $A_{f(s)}^{n+2} \cap A_0^{j+1} \neq \emptyset$ (since $\mathrm{ran}(f) = A_0^{j+1}$). Since $n+2 < j+1$ we apply item (3) in the list following Definition 4 to conclude that $A_{f(s)}^{n+2} \subseteq A_0^{j+1} = \mathrm{ran}(f)$. To show that every element of $A_{f(s)}^{n+2}$ is in $f(A_s^{n+1})$ assume $t \in A_{f(s)}^{n+2}$. By the previous remark, $t \in \mathrm{ran}(f)$ so $t = f(s')$ for some $s' \in A_0^j$. Since t and $f(s)$ are in the same $n+2$-block, $\mathrm{bc}^{n+2}(f(s')) = \mathrm{bc}^{n+2}(t) = \mathrm{bc}^{n+2}(f(s))$. By item (5) in the list of properties of f, we have

$$\mathrm{bc}^{n+1}(s') = \mathrm{bc}^{n+2}(f(s')) = bc^{n+2}(f(s)) = bc^{n+1}(s)$$

so s' and s are in the same $n+1$-block, namely A_s^{n+1}. Hence, $t = f(s') \in f(A_s^{n+1})$.

For part (2) we assume $s \notin A_0^{j+1}$. Since, $\mathrm{ran}(f) = A_0^{j+1}$ and ρ extends f (and is a permutation of A), $\rho^{-1}(s) \notin A_0^j$, and hence $\beta(\rho^{-1}(s)) = \rho^{-1}(s)$. Therefore

$$\alpha(s) = \rho(\beta(\rho^{-1}(s))) = \rho(\rho^{-1}(s)) = s.$$

This completes the proof of the sublemma. □

To prove $\alpha \in G_{n+2}$, we argue that conditions (1), (2) and (3) in the definition of G_n are true (with n replaced by $n+2$).

- Condition (1) is the requirement that α fixes A_0^{n+2} pointwise. If $s \in A_0^{n+2}$ then $s \in A_0^{j+1}$ so $s \in \mathrm{ran}(f)$. Therefore $\rho^{-1}(s) = f^{-1}(s) \in A_0^{n+1}$ (using the sublemma, item (1)). Therefore, since $\beta \in G_{n+1}$, $\beta(\rho^{-1}(s)) = \rho^{-1}(s)$. We conclude that $\alpha(s) = \rho(\beta(\rho^{-1}(s))) = s$.
- For condition (2) we must show that for any $n+2$-block $B = A_s^{n+2}$, $\alpha(B)$ is an $n+2$-block. Since $j+1 > n+2$ (see the remark in the second paragraph of the proof of the lemma.), every $n+2$-block is either contained in $A \setminus A_0^{j+1}$ or contained in A_0^{j+1}. In the first case part (2) of the sublemma gives us $\alpha(B) = B$. In the second case $B \subseteq \mathrm{ran}(f)$ so $\rho^{-1}(B) = f^{-1}(B)$ which by the sublemma part (1) is an $n+1$-block contained in A_0^j. Since the support of β is a subset of A_0^j and $\beta \in G_{n+1}$, $\beta(\rho^{-1}(B))$ is an $n+1$ block contained in A_0^j. Applying part (1) of the sublemma again we conclude that $f(\beta(\rho^{-1}(B)))$ is an $n+2$-block. Therefore, since ρ extends f, $\rho(\beta(\rho^{-1}(B)))$ is an $n+2$-block. So $\alpha(B)$ is an $n+2$ block.

– To prove condition (3) we assume $t \in A$ and argue that the $n+2$-location of $\alpha(t)$ is the same as the $n+2$-location of t. If $t \notin A_0^{j+1}$ then the conclusion follows from part (2) of the sublemma. If $t \in A_0^{j+1}$ then $t \in \mathrm{ran}(f)$ so $\rho^{-1}(t) = f^{-1}(t) = s$ for some $s \in A_0^j = \mathrm{dom}(f)$. By item (6) in the list of properties of f, the $n+2$-location of t is $(s_0, s_1, \ldots, s_{n-1}, u_1(s_n), u_2(s_n))$ and the $n+1$-location of $\rho^{-1}(t) = f^{-1}(t)$ is $(s_0, s_1, \ldots, s_n)$. Since β fixes $n+1$-locations, the $n+1$ location of $\beta(\rho^{-1}(t))$ is $(s_0, s_1, \ldots, s_n)$. By item (6) in the list of properties of f, the $n+2$-location of $f(\beta(\rho^{-1}(t))) = \rho(\beta(\rho^{-1}(t))) = \alpha(t)$ is $(s_0, s_1, \ldots, u_1(s_n), u_2(s_n))$.

This completes the proof of the lemma. □

The lemma gives a contradiction and therefore the proof of the theorem is complete. □

8.3 Versions of AC in $\mathcal{N}9$

Theorem 10 *In $\mathcal{N}9$, the union of a well-ordered collection of well orderable sets can be well ordered and $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{WO}}$, LW and $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ are true, but the axiom of choice for families of two-element sets and $\mathsf{MC}^{\aleph_0}$ are false.*

Proof In [4], $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ was shown to be true in $\mathcal{N}9$ and the axiom of choice for families of two-elements sets was shown to be false.

The fact that the union of a well-ordered collection of well orderable sets can be well ordered follows from Theorem 9 and $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{WO}}$ follows from this. (In [4], $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{fin}}$ was shown to be true in $\mathcal{N}9$.)

Claim LW is true in $\mathcal{N}9$.

Proof We will show that $\mathcal{N}9$ satisfies condition (*) of Theorem 1. To this end, let $x \in \mathcal{N}9$ and also let $n \in \omega$ such that G_n does not support x. (Recall that $\{G_n : n \in \omega\}$ is a filter base for the filter J used to construct $\mathcal{N}9$, see Definition 4.) Then there exists $\eta \in G_n$ such that $\eta(x) \neq x$.

Since $\eta \in G_n$, the set $\eta' = \{(A^n_s, A^n_{\eta(s)}) : s \in A\}$ is a permutation of the set of n-blocks. (See item 2 in the definition of G_n.) Since η also fixes n-locations, for any $s \in A$,

$$\eta(s) = (s \restriction n)^\frown \mathrm{bc}(\eta'(A^n_s)).$$

By Lemma 1, there is a permutation τ' of the set of n-blocks such that

1. $\{B : \tau'(B) \neq B\} \subseteq \{B : \eta'(B) \neq B\}$,
2. $(\tau')^2 = \epsilon$, and
3. $(\eta'\tau')^2 = \epsilon$.

τ' determines a permutation τ of A defined by

$$\tau(s) = (s \restriction n)^\frown \mathrm{bc}(\tau'(A^n_s)).$$

Then τ has the following properties:

1. $\tau \in G_n$
2. $\tau^2 = \epsilon$ and
3. $(\eta\tau)^2 = \epsilon$

Since $\eta(x) \neq x$, we have that either $\tau(x) \neq x$ or $\eta\tau(x) \neq x$, and since both τ and $\eta\tau$ are in G_n and have finite order, we conclude that (*) is satisfied. Hence, by Theorem 1, LW is true in $\mathcal{N}9$.

Claim $\mathsf{MC}^{\aleph_0}$ is false in $\mathcal{N}9$.

Proof For each $n \in \omega$, let $\mathcal{C}_n$ be the set of n-blocks and let $W_n = \{\phi(\mathcal{C}_n) : \phi \in G\}$. Then each W_n is supported by G and therefore $\mathcal{W} = \{W_n : n \in \omega\}$ is a countably infinite set in $\mathcal{N}9$. We will show by contradiction that $\mathcal{W}$ has no multiple choice function in $\mathcal{N}9$.

Therefore assume that f is such a function which is in $\mathcal{N}9$ with support G_n. By our assumptions $f(W_{n+1})$ is a finite non-empty subset of W_{n+1} with support G_n. Choose $x \in f(W_{n+1})$ then $x = \gamma(\mathcal{C}_{n+1})$ for some $\gamma \in G$. Since the support of γ is bounded, there is a $k \in \omega$ such that the support of γ, $\{a \in A : \gamma(a) \neq a\}$, is a subset of A_0^k and we assume without loss of generality that $k \geq n+1$. It follows that for any $n+1$-block B, either B is disjoint from A_0^k or $B \subseteq A_0^k$. Therefore $\mathcal{C}_{n+1}$ is the disjoint union

$$\mathcal{C}_{n+1} = \{B \in \mathcal{C}_{n+1} : B \cap A_0^k = \emptyset\} \cup \{B \in \mathcal{C}_{n+1} : B \subseteq A_0^k\}.$$

For B in the first of these two sets $\gamma(B) = B$. So $x = \gamma(\mathcal{C}_{n+1})$ is the disjoint union

$$x = \{B \in \mathcal{C}_{n+1} : B \cap A_0^k = \emptyset\} \tag{7}$$
$$\cup \{\gamma(B) : B \in \mathcal{C}_{n+1} \text{ and } B \subseteq A_0^k\}. \tag{8}$$

The first of the two sets above ($\{B \in \mathcal{C}_{n+1} : B \cap A_0^k = \emptyset\}$) is infinite so we choose a countably infinite subset B_i, $i \in \omega$ where $B_i \neq B_j$ if $i \neq j$. (It is possible to choose $\{B_i : i \in \omega\}$ so that this set is countable in the $\mathcal{N}9$ but for our purposes this is not required.) We now choose for each $i \in \omega$ an n-block D_i which is a subset of the $n+1$-block B_i. For each $i \in \omega$ with $i \geq 1$, we let ψ_i be the element of G_n which interchanges the two n-blocks D_0 and D_i and fixes all other atoms. (That is, ψ_i is the product of transpositions $\prod_{s \in D_0}(s, s_i)$ where for each $s \in D_0$, s_i is the element of D_i with the same n-location as s.)

We note two things about ψ_i:

- Since $f(W_{n+1})$ is supported by G_n, all of the sets $\psi_i(x)$ $(i \geq 1)$ are in $f(W_{n+1})$.
- ψ_i fixes A_0^k pointwise and also fixes every element of $\{B \in \mathcal{C}_{n+1} : B \cap A_0^k = \emptyset\} \setminus \{B_0, B_i\}$ pointwise. Therefore, using Eq. (7), $\psi_i(x)$ is the disjoint union

$$\begin{aligned}\psi_i(x) = &\{B : B \in \mathcal{C}_{n+1} \text{ and } B \cap A_0^k\} \setminus \{B_0, B_i\}\\ &\cup \{\psi_i(B_0), \psi_i(B_i)\}\\ &\cup \{\gamma(B) : B \in \mathcal{C}_{n+1} \text{ and } B \subseteq A_0^k\}.\end{aligned}$$

Table 2 Forms of AC in our models

	$\mathcal{M}$	$\mathcal{V}$	$\mathcal{N}15$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{N}9$
$\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$	F	F	F	F	F	T
LW	T	T	T	T	T	T
DF = F	T	T	T	T	T	T
DC	F	T	T	F	T	F
$\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$	T	F	?	T	?	T
$\mathsf{AC}_{\mathsf{WO}}^{\mathsf{WO}}$	F	F	F	F	F	T
$\mathsf{MC}_{\mathsf{WO}}^{\mathsf{WO}}$	F	F	F	F	F	T
$\mathsf{MC}_{\aleph_0}^{\aleph_0}$	F	T	T	F	T	T
$\mathsf{MC}^{\aleph_0}$	F	T	T	F	T	F

$\psi_i(B_0)$ is the $n+1$-block B_0 with the sub-n-block D_0 replaced by the n-block D_i and $\psi_i(B_i)$ is the $n+1$-block B_i with the sub-n-block D_i replaced by the n-block D_0. Therefore neither B_0 nor B_i are in $\psi_i(x)$.

Assume that $k, j \in \omega$ are both greater than zero and that $k \neq j$. Then (among other differences) $B_k \in \psi_j(x) \setminus \psi_k(x)$ so that $\psi_k(x) \neq \psi_j(x)$. Since all of the sets $\psi_i(x)$ $(i > 0)$ are in $f(W_{n+1})$ and $f(W_{n+1})$ is finite, we have a contradiction. This completes the proof of the claim. □

The above arguments complete the proof of the theorem. □

9 Summary

Table 2 summarizes what is known (and unknown) about our six models.

10 Open questions

1. Does LW + $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ imply $\mathsf{MC}_{\aleph_0}^{\aleph_0}$ in ZFA?
2. Does $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m}$ imply $\mathsf{AC}_{\mathsf{fin}}^{\mathsf{WO}}$? (Recall that $\forall\mathfrak{m}, 2\mathfrak{m} = \mathfrak{m} \Rightarrow \mathsf{DF} = \mathsf{F} \Rightarrow \mathsf{AC}_{\mathsf{fin}}^{\aleph_0}$, where $\mathsf{AC}_{\mathsf{fin}}^{\aleph_0}$ is the axiom of choice for countably infinite families of non-empty finite sets.)

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Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

11 Appendix

The following theorem provides general information about certain types of permutation models, and thus it is interesting in its own right. Furthermore, the theorem shows that Models 1 and 4 (of Sects. 3 and 6) are respectively equal to the models determined (by the same set of atoms and the same normal ideal, and) by weak direct products of groups (see Remark 1 of Subsect. 2.1) rather than unrestricted direct products. Obvious adjustments to this theorem can be made so that to obtain analogous results for Model 5 (of Sect. 7) and for generalizations of Models 1 and 4.

Theorem 11 *Assume that the set A of atoms of the ground model M (of* ZFA + AC*) is a union of a disjoint, denumerable family $\{A_n : n \in \omega\}$, where each A_n is denumerable. For each $n \in \omega$, let $\mathscr{G}_n$ be a group of permutations of A_n, and also let G be the weak direct product of the $\mathscr{G}_n$'s. Let I be the ideal which is generated by all unions $\bigcup\{A_n : n \in E\}$, $E \in [\omega]^{<\omega}$. Let $\mathcal{M}$ be the permutation model determined by M, G, and I.*

Let $\mathscr{G}$ be the unrestricted direct product of $\mathscr{G}_n$ ($n \in \omega$), and also let $\mathcal{N}$ be the permutation model determined by M, $\mathscr{G}$, and I. Then $\mathcal{N} = \mathcal{M}$.

Proof We prove by $\in$-induction that for every $x \in M$, $\Phi(x)$ is true, where

$$\Phi(x) : \ x \in \mathcal{M} \Longleftrightarrow x \in \mathcal{N}.$$

Clearly $\Phi(x)$ is true, if $x = \emptyset$, or if $x \in A$. Assume that $y \in M$ and that for all $x \in y$, $\Phi(x)$ is true. We will show that $\Phi(y)$ is true. Assume that $y \in \mathcal{M}$. Then the following hold:

(1) y has a (countable) support $E \subset A$ relative to the group G (i.e., for every $\psi \in \text{fix}_G(E)$, $\psi(y) = y$);
(2) for every $x \in y$, $x \in \mathcal{M}$ ($\mathcal{M}$ is a transitive class);
(3) for every $x \in y$, $x \in \mathcal{N}$ (by (2) and the induction hypothesis).

We assert that E is a support of y relative to the group $\mathscr{G}$. It suffices to show that for all $\phi \in \text{fix}_{\mathscr{G}}(E)$ and for all $x \in y$, $\phi(x) \in y$ (since then $\phi(y) = y$ follows from "$\phi(y) \subseteq y$ and $\phi^{-1}(y) \subseteq y$").

To this end, let $\phi \in \text{fix}_{\mathscr{G}}(E)$ and let $x \in y$. By (3), x has a (countable) support $E' \subset A$ relative to $\mathscr{G}$. The permutation ϕ may not be in G, but we construct a permutation $\phi' \in \text{fix}_G(E)$ which agrees with ϕ on E' as follows: For each $a \in E'$, the set $\{\phi^n(a) : n \in \mathbb{Z}\}$ is countable. Therefore, since E' is countable, so is $D = \bigcup\{\{\phi^n(a) : n \in \mathbb{Z}\} : a \in E'\}$. Furthermore, D contains E' and is closed under ϕ.

We define a mapping $\phi' : A \to A$ by

$$\phi'(a) = \begin{cases} \phi(a), & \text{if } a \in D; \\ a, & \text{otherwise.} \end{cases}$$

Then the following hold:

(4) $\phi' \in G$;

(5) ϕ' fixes E pointwise (since ϕ fixes E pointwise); and
(6) ϕ' agrees with ϕ on E'.

By (4) and (5), $\phi' \in \text{fix}_G(E)$ so $\phi'(y) = y$. It follows that $\phi'(x) \in y$. Now, (6) gives $\phi'(x) = \phi(x)$, and hence $\phi(x) \in y$.

Conversely, assume that $y \in \mathcal{N}$ and that y has a support E' relative to $\mathscr{G}$. Then E' is a support of y relative to G since $G \subset \mathscr{G}$. By the induction hypothesis, every element of y is in $\mathcal{M}$, and so $y \in \mathcal{M}$. □

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