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Coanalytic ultrafilter bases

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Abstract
We study the definability of ultrafilter bases on ω in the sense of descriptive set theory. As a main result we show that there is no coanalytic base for a Ramsey ultrafilter, while in L we can construct Π^1_1 P-point and Q-point bases. We also show that the existence of a $\mathbf{\Delta}^1_{n+1}$ ultrafilter is equivalent to that of a $\mathbf{\Pi}^1_n$ ultrafilter base, for $n \in \omega$. Moreover we introduce a Borel version of the classical ultrafilter number and make some observations.

Keywords Ultrafilter · Filter base · Definability · Coanalytic · Projective · Cardinal invariant

1 Introduction

This paper follows the line of many papers studying the definability, in the sense of descriptive set theory, of certain combinatorial subsets of the real line such as mad families [6,15,16], independent families [3,15], maximal eventually different families [18], maximal cofinitary groups [9], maximal orthogonal families of measures [7] or maximal towers and inextendible linearly ordered towers [8]. In this paper we will study the definability of ultrafilters and more specifically ultrafilter bases. Recall that a set $X \subseteq \mathcal{F}$ is a base for a filter $\mathcal{F}$, if for every $x \in \mathcal{F}$, there is $y \in X$ with $y \subseteq x$. Filters will always live on ω and contain all cofinite sets. Thus a filter is a subset of the Polish space $\mathcal{P}(\omega)$ and we can study its definability. It is well known that an ultrafilter can neither have the Baire property nor be Lebesgue measurable (see e.g. [1, Theorem 4.1.1]). This already rules out the existence of analytic ultrafilter generating sets as the generated filter will also be analytic and thus have the Baire property. But this still leaves open the possibility of a coanalytic ultrafilter base since a priori the generated set will only be $\mathbf{\Delta}^1_2$. The main purpose of this paper is to study which ultrafilters can have a coanalytic basis.

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Recall that for $x, y \in [\omega]^\omega$ we write $x \subseteq^* y$ whenever $x \setminus y$ is finite. An ultrafilter $\mathcal{U}$ is called a *P-point* if for any countable $\mathcal{F} \subseteq \mathcal{U}$, there is $x \in \mathcal{U}$ so that $\forall y \in \mathcal{F}(x \subseteq^* y)$. $\mathcal{U}$ is a *Q-point* if for any partition $\langle a_n : n \in \omega \rangle$ of ω into finite sets a_n, there is $x \in \mathcal{U}$ so that $\forall n \in \omega(|x \cap a_n| \leq 1)$. A *Ramsey* ultrafilter is an ultrafilter that is both a P- and a Q-point. A more commonly known and equivalent definition for Ramsey ultrafilters $\mathcal{U}$ is that for any coloring $c\colon [\omega]^2 \to 2$, there is $x \in \mathcal{U}$ so that c is homogeneous on x, i.e. $c \restriction [x]^2$ is constant. In fact we will show in Sect. 3 that:

Theorem 1.1 *There is a Π^1_1 base for a P-point in the constructible universe L.*

Theorem 1.2 *There is a Π^1_1 base for a Q-point in the constructible universe L.*

Section 2 will provide an introduction to the techniques employed in proving these results. In strong contrast we will show in Sect. 4 that:

Theorem 1.3 *There is no $\mathbf{\Pi}^1_1$ base for a Ramsey ultrafilter.*

Before stating the next theorem, notice that any ultrafilter that is $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ is already $\mathbf{\Delta}^1_n$. Namely suppose that φ defines an ultrafilter, then we have that $\varphi(x) \leftrightarrow \neg\varphi(\omega \setminus x)$. Moreover any base for an ultrafilter that is $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ generates a $\mathbf{\Delta}^1_n$ or respectively a $\mathbf{\Delta}^1_{n+1}$ ultrafilter.

In Sect. 5 we will compare the existence of $\mathbf{\Delta}^1_{n+1}$ ultrafilters to that of $\mathbf{\Pi}^1_n$ bases, for $n \in \omega$. As a main result we find that:

Theorem 1.4 *The following are equivalent for any $r \in 2^\omega$, $n \in \omega$.*

(1) *There is a $\Delta^1_{n+1}(r)$ ultrafilter.*
(2) *There is a $\Pi^1_n(r)$ ultrafilter base.*

Let us remark that we can use Theorem 1.4 to show that there is a Π^1_1 ultrafilter base in L by simply finding a Δ^1_2 ultrafilter. This is a much easier task than constructing a Π^1_1 base directly (see more in Sect. 2). On the other hand, Theorem 1.1 and Theorem 1.2 can not be followed in this way. As we shall see (Remark 5.2), the proof of Theorem 1.4 never yields a base for a P-point nor a Q-point. In fact, using a result of Shelah combined with Theorem 1.3, we will provide a model in which $\mathbf{\Delta}^1_2$ P-points exist but are never generated by $\mathbf{\Pi}^1_1$ bases (see Theorem 5.1).

In Sect. 6 we study the effects of adding reals to the definability of ultrafilters. Recall that a real $x \in [\omega]^\omega$ is called *splitting* over A, if for any $y \in A \cap [\omega]^\omega$, $x \cap y$ and $y \setminus x$ are infinite. We will see that classical forcing notions adding *splitting reals*, and as a consequence destroying ground model ultrafilters, namely *Cohen*, *Random* and *Silver* forcing, cannot preserve definitions for ultrafilters. This is related to a result of Brendle and Khomskii (see [4]), which shows that it is possible to destroy all ground model mad families by adding *dominating reals* while having a Π^1_1 mad family in the extension.

In Sect. 7 we introduce a new cardinal invariant that is a Borel version of the classical ultrafilter number $\mathfrak{u}$ and make some observations. Similar cardinal invariants have been defined for mad families (see [16] or [4]) and maximal independent families (see [3]).

2 An introduction to Miller's coding technique

All our constructions of coanalytic objects in L will rely on a technique that was brought to wider attention by A. Miller in his fundamental paper [15]. It is originally due to Erdős, Kunen and Mauldin [5] who used it to construct a so called *scale* with a Π^1_1 definition. For more information we also refer to [23]. The purpose of this section is to give an informal outline of the technique and to define the basic notions that we will use in our proofs. More details will appear in the proofs of Theorem 1.1 and 1.2 in the next section. We hope that this helps readers which are unfamilar with the method. Another good explanation can be found in [10, §3]

When we say that z codes the ordinal α, we mean the following. To any real $z \in 2^\omega$ we associate a relation E_z on ω defined by

$$E_z(n, m) \leftrightarrow z(2^n 3^m) = 1.$$

This relation may be a linear order and if it is a well-order and isomorphic to α we say that it codes α. Such α is unique and we define $\|z\| := \alpha$. More generally we say that z codes M if (ω, E_z) is isomorphic to $(M, \in)$. The set of $z \in 2^\omega$ coding an ordinal is denoted WO. The set WO is tightly connected to coanalytic sets. On one hand side, WO is itself Π^1_1 and on the other, for any $\boldsymbol{\Pi}^1_1$ set $X \subseteq 2^\omega$, there is a continuous function $f : 2^\omega \to 2^\omega$ so that $X = f^{-1}(\mathrm{WO})$.

There is a canonical way of defining in L various combinatorial subsets X of reals in a Δ^1_2 fashion. This goes back to Gödel who remarked in [11, p. 67] that the canonical well-order of the reals in L is in fact Δ^1_2. Typically the elements are found recursively by making adequate choices which are absolute between models of the form L_α (e.g. taking the $<_L$ least candidate which has some simple property holding with respect to the previously chosen reals).

Then $x \in X$ can be written as

$$\underbrace{\exists M}_{\exists} \,[\, \underbrace{M \text{ is well-founded}}_{\forall}, \underbrace{x \in M}_{\Delta^1_1} \text{ and } \underbrace{M \models V = L \wedge \varphi(x)}_{\Delta^1_1} \,] \tag{1}$$

or as

$$\underbrace{\forall M \models V = L, x \in M}_{\forall + \Delta^1_1}[\, \underbrace{M \text{ is not well-founded}}_{\exists} \text{ or } \underbrace{M \models \varphi(x)}_{\Delta^1_1}], \tag{2}$$

where $\varphi(x)$ is an appropriate formula in the language of set theory that expresses that there is a sequence according to the recursive construction of which x is the last member.

Quantifying over models is shorthand for quantifying over codes in 2^ω of countable models satisfying some basic set theoretic axioms. Thus e.g. (1) can be recast as "$\exists z \in 2^\omega((\omega, E_z)$ is well-founded, $x \in (\omega, E_z)$ and $(\omega, E_z) \models V = L \wedge \varphi(x))$", where $x \in (\omega, E_z)$ means that $x \in M$ for M the Mostowski collapse of (ω, E_z). It is not difficult to see that this can be expressed in a Δ^1_1 way.

As such, finding a Δ^1_2 ultrafilter base in L is very simple. The major improvement in Miller's technique is to get rid of the first existential quantifier in (1). This is done by letting x already encode a relevant well-founded model M in a Borel or even in a recursive way. Then if C is the Borel coding relation used, the definition usually looks as follows:

$$\underbrace{x \in Y}_{\forall} \text{ and } \underbrace{\forall z \in 2^\omega}_{\forall}[\underbrace{\neg C(x, z)}_{\Delta^1_1} \text{ or } \underbrace{(\omega, E_z) \models V = L \wedge \varphi(x)}_{\Delta^1_1}], \tag{3}$$

for some known coanalytic Y.

Lemma 2.1 *There is a lightface Borel set $C \subseteq (2^\omega)^3$ so that whenever z codes $\alpha < \omega_1$ and $r, y \in 2^\omega$ then $(z, r, y) \in C$ iff y codes $L_\alpha[r]$.*

Proof The claim is easy to verify by noting that an adequate E_y can be constructed by recursion on α following a rule that can be expressed in a Δ^1_1 way. Thus $(z, r, y) \in C$ can be defined by formulas $\exists (y_n)_{n\in\omega} \in (2^\omega)^\omega \left(\phi((y_n)_n, r, z) \wedge y = \bigcup_{n\in\omega} y_n\right)$ and $\forall (y_n)_{n\in\omega} \in (2^\omega)^\omega \left(\phi((y_n)_n, r, z) \to y = \bigcup_{n\in\omega} y_n\right)$, where $\phi((y_n)_n, r, z)$ is a Δ^1_1 formula expressing that $(y_n)_{n\in\omega}$ is constructed in some canonical way by recursion on the well-order (ω, E_z) according to the rules of the $L[r]$ hierarchy. □

Lemma 2.2 *There is a recursive function $(\cdot)^{+\omega} \colon 2^\omega \to 2^\omega$ so that whenever z codes α, then $(z)^{+\omega}$ codes $\alpha + \omega$.*

Proof Let $(z)^{+\omega} = y$ such that $y(2^n 3^m) = 1$ iff $\begin{cases} n \text{ even } \wedge m \text{ even } \wedge z(2^{\frac{n}{2}} 3^{\frac{m}{2}}) = 1 \\ n \text{ even } \wedge m \text{ odd} \\ n \text{ odd } \wedge m \text{ odd } \wedge n < m. \end{cases}$

□

3 Π^1_1 bases for P- and Q-points

In [8] the authors constructed, using Miller's technique, a coanalytic tower (i.e. a set $X \subseteq [\omega]^\omega$ well-ordered wrt $^* \supseteq$ and with no pseudointersection). A crucial property of the tower was that all its elements were split by the set of even natural numbers. In particular this meant that the tower could not generate an ultrafilter. We will construct in L a tower generating an ultrafilter and thus generating a P-point.

Before we start to construct the Π^1_1 P-point base, we need some ingredients.

Definition 3.1 We call $\mathcal{W}^+$ the set of $x \in [\omega]^\omega$ containing arbitrary long arithmetic progressions, i.e. $\forall k \in \omega \exists a, b \in \omega(\{a \cdot l + b : l < k\} \subseteq x)$.

The following fact follows from Van der Waerden's Theorem [21] and is well known.

Fact. *The set $\mathcal{W} = \mathcal{P}(\omega)\backslash\mathcal{W}^+$ is a proper ideal on ω. It is called the Van der Waerden ideal.*

Proof of Theorem 1.1 First, let us introduce some notation and repeat some basic facts that we are going to use in the construction. We will work in L throughout. Let $(y_\alpha)_{\alpha<\omega_1}$ enumerate $[\omega]^\omega$ via the global L well-order $<_L$. The statement "y is the α'th element according to $<_L$" is absolute between L_β's with $y \in L_\beta$ and $\alpha \in L_\beta$ (see e.g. [14, 13.19]). Let $O\colon 2^\omega \to 2^\omega$ be the following lightface Borel function: If $x \subseteq \omega$ we want to define a unique sequence $(i_n)_{n\in\omega}$ of subsets of ω so that $\max i_n < \min i_{n+1}$ and i_{n+1} is the next maximal arithmetic progression in x of length ≥ 3 above $\max i_n$ (note that any pair of natural numbers forms an arithmetic progression). Now if this sequence can be defined up to ω (in particular every i_n is finite), then we define $O(x)(n) = 1$ iff i_n has even length. Else we let $O(x)(n) = 0$. We say that an ordinal δ projects to ω, if $L_{\delta+\omega} \models$ "δ is countable". It is not hard to see that the set of ordinals projecting to ω is unbounded in ω_1 (see e.g. the proof of [22, Theorem 2.6]).

We now construct a sequence $(x_\xi, \delta_\xi)_{\xi<\omega_1}$ where $x_\xi \in [\omega]^\omega$, $\delta_\xi < \omega_1$ for every $\xi < \omega_1$, as follows. Given $(x_\xi, \delta_\xi)_{\xi<\alpha}$ we let δ_α be the least limit ordinal such that $\sup_{\xi<\alpha} \delta_\xi < \delta_\alpha$, $y_\alpha \in L_{\delta_\alpha}$ and δ_α projects to ω. We then choose $x_\alpha = x$ least in the $<_L$ well-order so that

(a) $x \subseteq^* x_\xi$ for every $\xi < \alpha$,
(b) $x \in \mathcal{W}^+$
(c) $x \subseteq y_\alpha$ or $x \subseteq \omega\backslash y_\alpha$.
(d) $O(x)$ codes δ_α.

Note that any sequence $(x_\xi)_{\xi<\omega_1}$ defined as above is a tower generating an ultrafilter. Namely, by (a), the sequence is decreasing with respect to $\subseteq^*$, and if $y \in [\omega]^\omega$ is arbitrary, say $y = y_\alpha$ for $\alpha < \omega_1$, then by (c), x_α is contained in either y or its complement.

Claim *x_α can be found in $L_{\delta_\alpha+\omega}$.*

Proof Note that the definition of $(x_\xi)_{\xi<\alpha}$ is absolute between L_β's. In particular $(x_\xi)_{\xi<\alpha}$ can be defined over L_{δ_α}. As δ_α projects to ω, there is an enumeration $(x^n)_{n\in\omega}$ of $\{x_\xi : \xi < \alpha\}$ in $L_{\delta_\alpha+\omega}$. Given y_α we have that, as $\mathcal{W}$ is an ideal, that for every $\xi < \alpha$, $y_\alpha \cap x_\xi \in \mathcal{W}^+$ or $\omega\backslash y_\alpha \cap x_\xi \in \mathcal{W}^+$. Assume wlog that for cofinally many x_ξ, $y_\alpha \cap x_\xi \in \mathcal{W}^+$ is the case. This implies that for all x_ξ this is the case as $(x_\xi)_{\xi<\alpha}$ forms a tower. Again as δ_α projects to ω, there is a real $z \in L_{\delta_\alpha+\omega} \cap 2^\omega$ coding δ_α. Now we define a sequence $(i_n)_{n\in\omega}$ of finite subsets of ω so that $\max i_n < \min i_{n+1}$, $i_n \subseteq y_\alpha \cap \bigcap_{k\leq n} x^k$, i_n consists of an arithmetic progression so that its length is $\geq n$ and it is even iff $z(n) = 1$. Moreover $\min i_n$ is chosen large enough so that $i_{n-1} \cup i_n$ cannot form an arithmetic progression. $x := \bigcup_{n\in\omega} i_n$ can be defined in $L_{\delta_\alpha+\omega}$ and satisfies (a)–(d). Thus in particular the $<_L$-least such x exists in $L_{\delta_\alpha+\omega}$. □

Claim *There is a formula $\varphi(x)$ in the language of set theory so that $\varphi(x)$ iff $\exists\xi(x = x_\xi)$ and $L_\beta \models \varphi(x)$ for some β implies that $\varphi(x)$ is true. Moreover $L_{\delta_\xi+\omega} \models \varphi(x_\xi)$ for every ξ.*

Proof $\varphi(x)$ expresses that there is an ordinal α and a sequence $(x_\xi, \delta_\xi)_{\xi \leq \alpha}$ according to the recursive definitions given above so that $x = x_\alpha$. □

Now we can check that the set $X = \{x_\xi : \xi \in \omega_1\}$ is Π^1_1. Let C and $(\cdot)^{+\omega}$ be as in Lemmas 2.1 and 2.2. Then $x \in X$ iff

$$O(x) \in \mathrm{WO} \text{ and } \forall z[\neg C(O(x)^{+\omega}, 0, z) \text{ or } (\omega, E_z) \models \varphi(x)].$$

□

We will now turn to the proof of Theorem 1.2.

Definition 3.2 The ideal Fin^2 on $\omega \times \omega$ consists of $x \in \mathcal{P}(\omega \times \omega)$ so that $\forall^\infty n \in \omega \forall^\infty m \in \omega(\langle n, m\rangle \notin x)$. Here, $\forall^\infty$ is an abreviation of "for all but finitely many".

Proof of Theorem 1.2 The ultrafilter that we construct will live on $\omega \times \omega$. Let $O\colon (\mathrm{Fin}^2)^+ \to 2^\omega$ be the following Borel function. Given $x \in (\mathrm{Fin}^2)^+$ let x_0, x_1 be the first two infinite vertical sections of x. We denote with $x_0(n)$ or $x_1(n)$ the n'th element of x_0 or x_1. Then

$$O(x)(n) = \begin{cases} 0 & \text{if } x_0(n) \geq x_1(n) \\ 1 & \text{if } x_1(n) > x_0(n). \end{cases}$$

As in the proof of Theorem 1.1 we let $(y_\alpha)_{\alpha<\omega_1}$ enumerate $[\omega \times \omega]^\omega$ and $(P_\alpha)_{\alpha<\omega_1}$ enumerate all partitions of $\omega \times \omega$ into finite sets via the well-ordering $<_L$.

Similarly to the proof of Theorem 1.1 we construct a sequence $(x_\xi, \delta_\xi)_{\xi<\omega_1}$ where $x_\xi \in (\mathrm{Fin}^2)^+$, intersections of finitely many elements in $\{x_\xi : \xi < \omega_1\}$ are in $(\mathrm{Fin}^2)^+$ and $\delta_\xi < \omega_1$ as follows.

Given $(x_\xi, \delta_\xi)_{\xi<\alpha}$ we let δ_α be the least limit ordinal such that $\sup_{\xi<\alpha} \delta_\xi < \delta_\alpha$, $y_\alpha, P_\alpha \in L_{\delta_\alpha}$ and δ_α projects to ω. $x_\alpha = x$ is then chosen least in the $<_L$ well-order so that

(a) $\{x\} \cup \{x_\xi : \xi < \alpha\}$ has all finite intersections in $(\mathrm{Fin}^2)^+$,
(b) $x \in (\mathrm{Fin}^2)^+$,
(c) $x \subseteq y_\alpha$ or $x \subseteq \omega \setminus y_\alpha$,
(d) for every $a \in P_\alpha$, $|a \cap x| \leq 1$,
(e) $O(x)$ codes δ_α.

Again we show that such an x_α exists and can be found in $L_{\delta_\alpha+\omega}$.

Claim *x_α can be found in $L_{\delta_\alpha+\omega}$.*

Proof We have that if $(x_\xi)_{\xi<\alpha}$ exists then it must be definable over L_{δ_α}. As δ_α projects to ω there is in $L_{\delta_\alpha+\omega}$ an enumeration $(x^n)_{n\in\omega}$ of all finite intersections of elements in $\{x_\xi : \xi < \alpha\}$. We are given $y_\alpha \in L_{\delta_\alpha}$. It is not hard to see that either y_α or $(\omega \times \omega) \setminus y_\alpha$ is in $(\mathrm{Fin}^2)^+$ and has $(\mathrm{Fin}^2)^+$ intersection with all x^n. Without loss of generality we assume y_α has this property. Let $P_\alpha = \{a_i : i \in \omega\}$ and $z \in 2^\omega \cap L_{\delta_\alpha+\omega}$ code δ_α. Further let $k_0 < k_1$ be first so that the k_0'th and k_1'th vertical section of y_α is infinite. Let $(p_j)_{j\in\omega}$ enumerate $\omega \times \omega$ in a way that every pair (n, m) appears infinitely often. Given $(p_j)_{j\in\omega}$ we define recursively a sequence $\langle m_i^0, m_i^1\rangle_{i\in\omega}$ and auxiliarily $(n_i)_{i\in\omega}$ as follows:

- for every i, $\langle m_i^0, m_i^1\rangle \in y_\alpha$, $\langle m_i^0, m_i^1\rangle \notin \bigcup_{j<i} a_{n_j}$ and $\langle m_i^0, m_i^1\rangle \in a_{n_i}$,
- if $i = 3j$ for $j \in \omega$, then $\langle m_i^0, m_i^1\rangle$ is in the $p_j(0)$'th infinite vertical section of $y_\alpha \cap x^{p_j(1)}$ greater than k_1,
- if $i = 1 \mod 3$ then $m_i^0 = k_0$ and $m_{i+1}^0 = k_1$ and $m_i^1 \geq m_{i+1}^1$ or $m_{i+1}^1 > m_i^1$ depending on whether $z(i) = 0$ or $z(i) = 1$.

Now the set $\{\langle m_i^0, m_i^1\rangle : i \in \omega\} \in L_{\delta_\alpha+\omega}$ satisfies (a)-(e) as can be seen from the construction. In particular $L_{\delta_\alpha+\omega}$ contains the $<_L$-least such set. □

The set $\{x_\xi : \xi < \omega_1\}$ is now a base for a Q-Point and as in the proof of Theorem 1.1 it is Π_1^1. □

4 There are no Π_1^1 Ramsey ultrafilter bases

Definition 4.1 Let $\mathcal{F}$ be a filter. Then the forcing $\mathbb{M}(\mathcal{F})$ consists of pairs $(a, F) \in [\omega]^{<\omega} \times \mathcal{F}$ such that $\max a < \min F$. A condition (b, E) extends (a, F) if b is an end-extension of a, $E \subseteq F$ and $b \backslash a \subseteq F$.

$\mathbb{M}(\mathcal{F})$ is the natural forcing to add a pseudointersection of $\mathcal{F}$. Namely, whenever G is $\mathbb{M}(\mathcal{F})$-generic over V, then $x = \bigcup\{a \in [\omega]^{<\omega} : \exists F \in \mathcal{F}((a, F) \in G)\}$ is a pseudointersection of $\mathcal{F}$ and we call x the generic real added by $\mathbb{M}(\mathcal{F})$.

Definition 4.2 Let $\mathcal{F}$ be a filter. Then we define the game $G(\mathcal{F})$ as follows:

$$\begin{array}{r|ccccc} \text{Player I} & F_0 \in \mathcal{F} & & F_1 \in \mathcal{F} & & \dots \\ \text{Player II} & & a_0 \in [F_0]^{<\omega}\backslash\{\emptyset\} & & a_0 \in [F_1]^{<\omega}\backslash\{\emptyset\} & \dots \end{array}$$

Player II wins iff $\bigcup_{n\in\omega} a_n \in \mathcal{F}$.

Lemma 4.1 *Let $\mathcal{F}$ be a filter on ω and let $\theta \geq (2^{2^{\aleph_0}})^+$. Then TFAE:*

(i) *For any countable elementary submodel $M \preccurlyeq H(\theta)$ with $\mathcal{F} \in M$, there is $x \in \mathcal{F}$, $\mathbb{M}(\mathcal{F})$ generic over M.*
(ii) *I has no winning strategy in $G(\mathcal{F})$.*

Proof (i) implies (ii): Suppose σ is a winning strategy for I in $G(\mathcal{F})$ and let $\sigma, \mathcal{F} \in M$. Wlog we assume that $\sigma(\langle\rangle) = \omega$. Thus Player II is allowed to play any a_0 as his first move and then σ carries on as if a_0 had not been played. In particular this means that any initial play a_0 of II is a legal move, i.e. $\langle a_0\rangle \in \mathrm{dom}(\sigma)$. Consider the dense sets $D_n := \{(s, F) : F \subseteq \bigcap\{\sigma(\langle s_0, \dots, s_{n-1}\rangle) : \langle s_0, \dots, s_{n-1}\rangle \in \mathrm{dom}(\sigma), \bigcup_{i<k} s_i = s\}\}$ for $n \in \omega$. $D_n \in M$ for every $n \in \omega$. By (i) there is $x \in \mathcal{F}$, $\mathbb{M}(\mathcal{F})$ generic over M. This means that for every $n \in \omega$ there is s an initial segement of x and $F \in \mathcal{F}$ so that $(s, F) \in D_n$ and $x \backslash s \subseteq F$. Now using this construct a sequence $\langle s_i\rangle_{i\in\omega}$ and $\langle F_i\rangle_{i\in\omega}$ recursively so that:

(1) $\bigcup_{i<n} s_i$ is an initial segment of x for every $n \in \omega$,
(2) $\max s_i < \min s_{i+1}$ for every $i \in \omega$,

(3) $x \setminus \bigcup_{i<n} s_i \subseteq F_n$ for every $n \in \omega$,
(4) $(\bigcup_{i<n} s_i, F_n) \in D_n$.

We find recursively that $\langle s_i \rangle_{i<n} \in \operatorname{dom}(\sigma)$, i.e. $\langle s_i \rangle_{i<n}$ is a legal move. But $\bigcup_{i \in \omega} s_i = x \in \mathcal{F}$ contradicting σ being a winning strategy for I.

(ii) implies (i): Let $M \ni \mathcal{F}$ be countable elementary and $\langle D_n \rangle_{n\in\omega}$ enumerate all dense subsets of $\mathbb{M}(\mathcal{F})$ in M. We describe a strategy for Player I: I starts by playing some F_0 so that there is $(t_0, F_0) \in D_0$. Then Player II will play $a_0 \subseteq F_0$, i.e. $(t_0 \cup a_0, F_0) \leq (t_0, F_0)$. Now I plays F_1 so that there is $(t_0 \cup a_0 \cup t_1, F_1) \in D_1$, $(t_0 \cup a_0 \cup t_1, F_1) \leq (t_0 \cup a_0, F_0)$...

By assumption there is a winning run $\langle a_i \rangle_{i\in\omega}$ for II according to this strategy. This means that $\bigcup a_i \in \mathcal{F}$ and moreover $x = \bigcup a_i \cup \bigcup t_i \in \mathcal{F}$ where t_i are as described. But x is now $\mathbb{M}(\mathcal{F})$ generic over M. □

For ultrafilters $\mathcal{U}$, I not having a winning strategy in $G(\mathcal{U})$ is equivalent to $\mathcal{U}$ being a P-point (see e.g. [19, VI.5.7]). For sake of completeness we prove a more general (in light of Lemma 4.1) version of this below. Recall that $\mathfrak{p}$ is the pseudointersection number, i.e. the least size of a set $\mathcal{B} \subseteq [\omega]^\omega$ with the finite intersection property and no pseudointersection, a set $x \in [\omega]^\omega$ such that $x \subseteq^* y$ for all $y \in \mathcal{B}$. The bounding number $\mathfrak{b}$ is the least size of a family $\mathcal{B} \subseteq \omega^\omega$ such that there is no $f \in \omega^\omega$ eventually dominating every member of $\mathcal{B}$. It is well known that $\aleph_1 \leq \mathfrak{p} \leq \mathfrak{b}$. An ultrafilter $\mathcal{U}$ is called a P_κ point if for any $\mathcal{B} \in [\mathcal{U}]^{<\kappa}$ there is a pseudointersection $x \in \mathcal{U}$ of $\mathcal{B}$. In particular a P-point is the same as a $P_{\aleph_1}$-point.

Lemma 4.2 *Assume $\kappa \leq \mathfrak{p}$, $\mathcal{U}$ is an ultrafilter and let $\theta \geq (2^{2^{\aleph_0}})^+$. Then TFAE:*

(i) *$\mathcal{U}$ is a P_κ-point.*
(i) *For every $M \preccurlyeq H(\theta)$ with $|M| < \kappa$ and $\mathcal{U} \in M$, there is $x \in \mathcal{U}$ which is $\mathbb{M}(\mathcal{U})$ generic over M.*

Proof (ii) implies (i) is trivial.

(i) implies (ii): Let $|M| < \kappa \leq \mathfrak{p}$. Then as $\mathcal{U}$ is a P_κ-point, there is $U \in \mathcal{U}$ so that $U \subseteq^* V$ for every $V \in M \cap \mathcal{U}$. Define for every $D \in M$, which is a dense open subset of $\mathbb{M}(\mathcal{U})$ and every $V \in M \cap \mathcal{U}$ a function $f_{D,V} : \omega \to \omega$ so that for $n \in \omega$:

$$\begin{aligned}\forall a \subseteq n \exists b \subseteq [n, f_{D,V}(n)) \exists V' \in M \cap \mathcal{U}((a \cup b, V') \\ \leq (a, V) \wedge (a \cup b, V') \in D \wedge U \setminus f_{D,V}(n) \subseteq V').\end{aligned}$$

The set of functions $f_{D,V}$ is smaller than $\kappa \leq \mathfrak{p} \leq \mathfrak{b}$. Thus there is one $f \in \omega^\omega$ dominating all of them. Let $i_0 = 0$, $i_{n+1} = f(i_n)$. We write $I_n = [i_n, i_{n+1})$. As $\mathcal{U}$ is an ultrafilter, either $U_0 = \bigcup_{n\in\omega} I_{2n} \cap U$ or $U_1 = \bigcup_{n\in\omega} I_{2n+1} \cap U$ is in $\mathcal{U}$. Assume wlog that $U_0 \in \mathcal{U}$.

We define a σ-centered partial order $\mathbb{P}$ as follows. $\mathbb{P}$ consists of pairs (s, F) where

(1) $s : n \to [\omega]^{<\omega}$ for some $n \in \omega$,
(2) $s(i) \subseteq I_i$ for every $i < n$,
(3) $s(i) = U \cap I_i$ when i is even,
(4) $F \in \mathcal{U} \cap M$.

A condition (t, F) extends (s, E) iff $t \supseteq s$, $F \subseteq E$ and $(t(i) \subseteq E)$ whenever $i \in \operatorname{dom} t \setminus \operatorname{dom} s$ is odd. For any $D \in M$ which is dense in $\mathbb{M}(\mathcal{U})$ we define a subset of $\mathbb{P}$, $\tilde{D}$ as follows:

$$\tilde{D} = \{(t, F) : (\bigcup_{i \in \operatorname{dom} t} t(i), F) \in D\}.$$

We claim that $\tilde{D}$ is dense in $\mathbb{P}$. Let $(s, E) \in \mathbb{P}$ be arbitrary. Then as $f_{D,E} <^* f$ there is $n \in \omega$ so that $[i_{2n+1}, f_{D,E}(i_{2n+1})) \subseteq [i_{2n+1}, i_{2n+2})$ and $2n + 1 \geq \operatorname{dom} s$. Now extend s to s_0 so that $\operatorname{dom} s_0 = 2n + 1$ and $s_0(i) = \emptyset$ for $i \in 2n + 1 \setminus \operatorname{dom} s$ odd and $s_0(i) = U \cap I_i$ for i even. By definition of $f_{D,E}$ there is $b \subseteq I_{2n+1}$ so that $\exists F \subseteq E$ with $(a \cup b, F) \in D$ where $a = \bigcup_{i<2n+1} s_0(i)$, $(a \cup b, F) \leq (a, E)$ and $U \setminus i_{2n+2} \subseteq F$. Let $t = s_0 \cup \{(2n + 1, b)\}$. Then $(t, F) \leq (s, E)$ in $\mathbb{P}$ and $(t, F) \in \tilde{D}$.

Now as $\kappa \leq \mathfrak{p}$ and by Bell's theorem (see [2]) there is a $\mathbb{P}$ generic real $g : \omega \to [\omega]^{<\omega}$ over M. But then $x := \bigcup_{i\in\omega} g(i) \in \mathcal{U}$ as $U_0 \subseteq x$ and x is $\mathbb{M}(\mathcal{U})$ generic over M. ◻

Corollary 4.3 *Suppose $\mathcal{U}$ is a P-point, $M \preccurlyeq H(\theta)$ countable and $\mathcal{U} \in M$. Then there is $x \in \mathcal{U}$, $\mathbb{M}(\mathcal{U})$ generic over M.*

Lemma 4.4 (see [12, Chapter 24]) *Assume $\mathcal{U} \in M$ is a Ramsey ultrafilter and x is $\mathbb{M}(\mathcal{U})$ generic over M. Then every $y \subseteq^* x$ is $\mathbb{M}(\mathcal{U})$ generic over M.*

Proof of Theorem 1.3 Suppose $\mathcal{U}$ is a Ramsey ultrafilter with a coanalytic base $X \subseteq [\omega]^\omega$. As X is coanalytic, there is a continuous function $f : 2^\omega \to 2^\omega$ so that

$$x \in X \leftrightarrow f(x) \in \mathrm{WO}\,.$$

Let $M \preccurlyeq H(\theta)$ be countable where $\theta \geq (2^{2^{\aleph_0}})^+$ is and $\mathcal{U}$, $f \in M$. As $\mathcal{U}$ is a P-point and by Corollary 4.3, there is $x \in \mathcal{U}$ that is $\mathbb{M}(\mathcal{U})$ generic over M. Moreover as $\mathcal{U}$ is Ramsey and by Lemma 4.4, any $y \subseteq^* x$ is also generic over M. Let N be the transitive collapse of M, $\alpha := \omega_1^N = M \cap \omega_1$ and let $y \in X$ be arbitrary such that $y \subseteq^* x$. Let $\beta = \|f(y)\|$, then $\beta \in N[y]$ and thus $\beta < \alpha = \omega_1^{N[y]}$. As y was arbitrary, we have shown that the set $X' = \{y : f(y) \in WO \wedge \|f(y)\| \leq \alpha\} \subseteq X$ contains $\{y \subseteq^* x : y \in X\}$. This means that X' also generates $\mathcal{U}$. But X' is Borel and cannot generate an ultrafilter. ◻

5 $\mathbf{\Delta}^1_2$ versus $\mathbf{\Pi}^1_1$

Using a result of Shelah we can show the following.

Theorem 5.1 *It is consistent that every P-point is $\mathbf{\Delta}^1_2$ and has no $\mathbf{\Pi}^1_1$ base.*

Proof This follows immediately by [19, Theorem XVIII.4.1] and the subsequent remark, which states that starting from L we can choose any Ramsey ultrafilter $\mathcal{U}$ and pass to an extension in which $\mathcal{U}$ generates the unique P-point up to permutation of ω. Moreover this ultrafilter will stay Ramsey.

Thus let $\mathcal{U}$ be any (definition of a) Δ^1_2 Ramsey ultrafilter in L. Now apply Shelah's theorem to this ultrafilter and pass to an extension V of L in which $\mathcal{U}^L$ generates the unique P-point and is Ramsey. In V, $\mathcal{U}^V$ will still have the finite intersection property and $\mathcal{U}^L \subseteq \mathcal{U}^V$ by Shoenfield's absolutness theorem. Thus in V, $\mathcal{U}^V$ generates the same ultrafilter as $\mathcal{U}^L$. As $\mathcal{U}^V$ is Δ^1_2, the ultrafilter it generates will be Δ^1_2 as well. We know that in V, there is for every P-point $\mathcal{V}$ a permutation f of ω so that $V \in \mathcal{V} \leftrightarrow f(V) \in \mathcal{U}$. In particular $\mathcal{V}$ has a $\Delta^1_2(f)$ definition. On the other hand, every P-point is a Ramsey ultrafilter so none of them can have a $\mathbf{\Pi}^1_1$ base by Theorem 1.3. □

Proof of Theorem 1.4 To simplify notation we assume that $r = 0$. Let $\mathcal{U}$ be a Δ^1_{n+1} ultrafilter. Let us introduce the following notation. For $y \in [\omega \times \omega]^\omega$, we let y_n be y's n'th vertical section. We let $z(y) = \{n \in \omega : y_n \neq \emptyset\}$. When $z(y)$ is infinite then we denote with y^n, the n'th nonempty vertical section of y.

The Fubini product of $\mathcal{U}$, $\mathcal{U} \otimes \mathcal{U}$, consists of all $y \in [\omega \times \omega]^\omega$ so that

$$\{n \in \omega : y_n \in \mathcal{U}\} \in \mathcal{U}.$$

$\mathcal{U} \otimes \mathcal{U}$ is again an ultrafilter. We will show that it has a Π^1_n base. Let $\varphi(x, w)$ be Π^1_1 so that

$$x \in \mathcal{U} \leftrightarrow \exists w \in 2^\omega(\varphi(x, w)).$$

Let $r : \omega \times 2^\omega \to 2^\omega$ be a recursive function such that for any sequence $\langle w_n \rangle_{n\in\omega}$ there is $w \in 2^\omega$, which is not eventually constant, so that $r(n, w) = w_n$ for every $n \in \omega$.

Let $O : [\omega \times \omega]^\omega \to 2^\omega$ be the function defined by

$$O(y)(n) = \begin{cases} 0 & \text{if } |z(y)| < \omega \\ 0 & \text{if } \min y^n \geq \min y^{n+1} \\ 1 & \text{if } \min y^n < \min y^{n+1}. \end{cases}$$

O is obviously lightface Borel. Let us define $X \subseteq [\omega \times \omega]^\omega$ as follows:

$$\begin{aligned} y \in X \leftrightarrow\ & |z(y)| = \omega \wedge \varphi(z(y), r(0, O(y))) \\ & \wedge \forall n \in \omega \exists s \in [\omega]^{<\omega}[\varphi(s \cup y^n, r(n+1, O(y)))]. \end{aligned}$$

X is obviously Π^1_n. Moreover $X \subseteq \mathcal{U} \otimes \mathcal{U}$. To see this let us decode what $y \in X$ means. The first clause in the definition of X says that y has infinitely many nonempty vertical sections. The next clause ensures that $z(y) \in \mathcal{U}$ as witnessed by $r(0, O(y))$, the 0'th real coded by $O(y)$. The last clause ensures that for every nonempty vertical section y^n of y, $s \cup y^n$ is in $\mathcal{U}$ for some finite s as witnessed by $r(n + 1, O(y))$, the $n + 1$'th real coded by $O(y)$. In particular $y^n \in \mathcal{U}$. Thus we indeed have that $y \in X \to y \in \mathcal{U} \otimes \mathcal{U}$.

Moreover we have that X is a base for $\mathcal{U} \otimes \mathcal{U}$. To see this fix $u \in \mathcal{U} \otimes \mathcal{U}$ and we show that there is $y \in X$ so that $y \subseteq u$. First let $y_0 = \bigcup\{\{n\} \times u_n : n \in \omega, u_n \in \mathcal{U}\}$, i.e. we remove from u the vertical sections that are not in $\mathcal{U}$. Then we let w_0 be such that $\varphi(z(y_0), w_0)$ holds true. Further we let w_{n+1} be such that $\varphi(y_0^n, w_{n+1})$ holds true. Let $w \in 2^\omega$ be a single real coding the sequence $\langle w_n \rangle_{n \in \omega}$ via r, i.e. $r(n, w) = w_n$ for every $n \in \omega$. Find a sequence $\langle m_n \rangle_{n \in \omega}$ so that $m_n \in y_0^n$ for every n and $w(n) = 1$ iff $m_{n+1} > m_n$. Such a sequence can be constructed recursively. Whenever $w(n) = 1$ we can simply find $m_{n+1} \in y_0^{n+1}$ large enough such that $m_{n+1} > m_n$ and if additionally $w(n+1), \dots, w(n+k)$ is a maximal block of 0s in w then we let $m_{n+1} = \cdots = m_{n+k+1} \in y^{n+1} \cap \cdots \cap y^{n+k+1}$. Finally given the sequence $\langle m_n \rangle_{n \in \omega}$ let $y = \bigcup \{\{z(y_0)(n)\} \times (y_0^n \setminus m_n) : n \in \omega\}$, where $z(y_0)(n)$ is the n'th element of $z(y_0)$. We see that $y \subseteq y_0 \subseteq u$, that $z(y) = z(y_0)$, that $y^n =^* y_0^n$ for every n and that $O(y) = w$. In particular $y \in X$ by definition of X. □

Remark 5.2 Let $\mathcal{U}$ be an ultrafilter. Then $\mathcal{U} \otimes \mathcal{U}$ is neither a P- nor a Q-point.

Proof To see that $\mathcal{U} \otimes \mathcal{U}$ is not a P-point, consider its elements of the form $[n, \omega) \times \omega$ for $n \in \omega$. Whenever y is a pseudointersection of $\{[n, \omega) \times \omega : n \in \omega\}$, then y_n, the n'th vertical section of y, is finite for every $n \in \omega$. Thus y is not a member of $\mathcal{U} \otimes \mathcal{U}$.

For Q-points, consider the partition of $\omega \times \omega$ consisting of sets of the form $a_n := \{n\} \times n$ or $b_n := (n+1) \times \{n\}$, for $n \in \omega$. Assume $y \in \mathcal{U} \otimes \mathcal{U}$ is such that $|y \cap a_n|, |y \cap b_n| \leq 1$ for every $n \in \omega$. Then there are $m < n$ so that $y_n, y_m \in \mathcal{U}$. In particular, $y_n \cap y_m$ is infinite and we find $i \in (y_n \cap y_m) \setminus n$. But then, as $i > n, m$, we have that $(m, i), (n, i) \in y \cap b_i$ and $(n, i) \neq (m, i)$ – a contradiction to $|y \cap b_i| \leq 1$. □

6 Adding reals

The purpose of the following section is to study the indestructibility of definitions for ultrafilters via forcings that add splitting reals, and as a consequence destroy all ultrafilters from a ground model (in the sense that they do not generate an ultrafilter in the extension). We will show that classical forcing notions adding splitting reals, namely *Cohen*, *Random* and *Silver* forcing, fail in preserving definitions for ultrafilters.

Let $A \subseteq V$. A set $X \in V$ is called $\mathrm{OD}(A)$ if it is definable over V from ordinals and elements of A as parameters. Recall that a poset $\mathbb{P}$ is weakly homogeneous if for any $p, q \in \mathbb{P}$, there is an automorphism $\pi : \mathbb{P} \to \mathbb{P}$ so that $\pi(p)$ is compatible to q. In this section we will denote with $\mathcal{P}_A$ the collection of weakly homogeneous $\mathrm{OD}(A)$ posets.

Lemma 6.1 *Let c be a Cohen real over V, $\mathbb{P} \in (\mathcal{P}_V)^{V[c]}$ and G a $\mathbb{P}$-generic filter over $V[c]$. Then in $V[c][G]$, c is splitting over any set of reals with the finite intersection property that is* $\mathrm{OD}(V)$.

Proof Let $X \in V[c][G]$ be an $\mathrm{OD}(V)$ set of reals with the finite intersection property, say $V[c][G] \models$ "$\dot{X} = \{x \in [\omega]^\omega : \varphi(x, a, \bar{\alpha})\}$" where $a \in V$ and $\bar{\alpha}$ is a finite sequence of ordinals. Wlog we may assume that X is a filter, since the filter generated by X is also $\mathrm{OD}(V)$. Suppose c does not split X. This means exactly that $c \in X$ or $\omega \setminus c \in X$. Thus there is $s \subseteq c$, deciding the formula and parameters defining $\mathbb{P}$, and $\dot{p}$

with $\dot{p}[c] \in G$, $(s, \dot{p}) \Vdash$ "φ defines a filter" so that either

$$(s, \dot{p}) \Vdash \dot{c} \in \dot{X}$$

or

$$(s, \dot{p}) \Vdash \omega \backslash \dot{c} \in \dot{X}.$$

But now notice that $c' = s \cup \{(n, 1-m) : (n, m) \in c, n \geq |s|\}$ is also Cohen over V with $s \subseteq c'$ (we identify c as a subset of ω with its characteristic function). Moreover $V[c] = V[c']$ and thus $\dot{\mathbb{P}}[c] = \dot{\mathbb{P}}[c']$. Let $p_0 := \dot{p}[c]$ and $p_1 := \dot{p}[c']$. Working in $V[c]$ we find that $p_0, p_1 \in \mathbb{P}$, so there is an automorphism π of $\mathbb{P}$ so that $\pi(p_1)$ is compatible to p_0. Let H be $\mathbb{P}$-generic over $V[c]$ containing p_0 and $\pi(p_1)$. In either of the above cases, $V[c][H] \models \varphi(c, a, \bar{\alpha}) \wedge \varphi(c', a, \bar{\alpha})$. This is a contradiction to $(s, \dot{p}) \Vdash$ "φ defines a filter". □

Lemma 6.2 *Let r be a random real over V, $\mathbb{P} \in (\mathcal{P}_V)^{V[r]}$ and G a $\mathbb{P}$-generic filter over $V[r]$. Then in $V[r][G]$, r is splitting over any set of reals with the finite intersection property that is* $\mathrm{OD}(V)$.

Proof Let us assume that $\mathbb{P}$ is simply the trivial forcing, since this part of the argument is essentially the same as in the last proof. As before we fix $X \in V[r]$ an $\mathrm{OD}(V)$ set with the finite intersection property and we assume that it is already a filter.

First note that any finite modification of r is still a random real. Moreover, as complementation is a measure preserving homeomorphism of 2^ω, the complement of a random real is still random. Thus any $r' =^* \omega \backslash r$ is still random.

Now similarly as in the proof for Cohen forcing we find that there is Borel set B of positive measure coded in V so that $r \in B$ and

$$B \Vdash \dot{r} \in X$$

or

$$B \Vdash \omega \backslash \dot{r} \in X.$$

Recall that for any Borel set A of positive measure, its E_0 closure $\tilde{A} = \{x \in 2^\omega : \exists y \in A(x =^* y)\}$ has full measure. To see this Let $\varepsilon > 0$ be arbitrarily small. Apply Lebesgue's density theorem to find a basic open set $[s] \subseteq 2^\omega$ so that $\frac{\mu(A \cap [s])}{\mu([s])} > 1 - \varepsilon$. Follow from this that $\mu(\tilde{A}) > 1 - \varepsilon$.

Now let $C := \{\omega \backslash x : x \in \tilde{B}\}$. C is coded in V and has full measure. Thus we have that $r \in B \cap C$. By definition of C, there is $r' \in B$ so that $r' =^* \omega \backslash r$. Moreover r' is also a random real over V by our first remark. $r, r' \in X$ and $\omega \backslash r, \omega \backslash r' \in X$ are both contradictions to X having the finite intersection property. □

Recall that Silver forcing consists of partial functions $p \colon \omega \to 2$ so that $\omega \backslash \mathrm{dom}(p)$ is infinite.

Lemma 6.3 *Let s be a Silver real over V, $\mathbb{P} \in (\mathcal{P}_V)^{V[s]}$ and G a $\mathbb{P}$-generic filter over $V[s]$. Then, in $V[s]$, there is a real splitting over any set of reals that is* $\mathrm{OD}(V)$ *in* $V[s][G]$.

Proof Again we only consider the case when $\mathbb{P}$ is trivial. Let $X \in V[s]$ be an $\mathrm{OD}(V)$ filter. Let $S_s = \{n \in \omega : |\{m < n : s(m) = 1\}|$ is even$\}$. As before assume $p \subseteq s$ is such that either

$$p \Vdash S_{\dot{s}} \in X$$

or

$$p \Vdash \omega \backslash S_{\dot{s}} \in X.$$

Let $n = \min(\omega \backslash \operatorname{dom}(p))$ and note that s' defined by $s'(i) = s(i)$ for all $i \neq n$ and $s'(n) = 1 - s(n)$ is also Silver and $p \subseteq s'$. But $S_{s'} =^* \omega \backslash S_s$. We get the same contradiction as in the last two proofs. □

Corollary 6.4 *Let $r \in 2^\omega$ and assume that there is a Cohen, a random or a Silver real over $L[r]$. Then there is no $\Delta^1_2(r)$ ultrafilter.*

In particular, the existence of a $\Delta^1_2(r)$ ultrafilter implies that $\omega_1 = \omega_1^{L[r]}$.

Proof Suppose that φ is a $\Sigma^1_2(r)$ definition for an ultrafilter and that c is a Cohen, random or Silver real over $L[r]$. In $L[r][c]$, the set defined by φ will have the finite intersection property by downwards absoluteness. Thus by Theorems 6.1, 6.2 or 6.3 respectively, $L[r][c] \models \exists x \in [\omega]^\omega \forall y \in [\omega]^\omega (\neg\varphi(y) \vee (|x \cap y| = \omega \wedge |x \cap \omega \backslash y| = \omega))$. This is a $\Sigma^1_3(x, c)$ statement, so by upwards Shoenfield absoluteness it holds true in $V \supseteq L[x][c]$. Thus φ cannot define an ultrafilter in V.

The second part follows, since whenever $\omega_1^{L[r]} < \omega_1$, there is a Cohen real in V over $L[r]$. □

Another way of seeing the above for Cohen or random forcing is to use the classical result of Judah and Shelah (see [13]), saying that the existence of a Cohen or random real over $L[r]$ is equivalent to every $\Delta^1_2(r)$ set having the Baire property or being Lebesgue measurable respectively.

Theorem 6.5 *There is no $OD(\mathbb{R})$ ultrafilter, in particular no projective one, after adding ω_1 many Cohen reals in a finite support iteration, random reals using a product of Lebesgue measure or Silver reals in a countable support iteration.*

Proof Let $\langle c_\alpha : \alpha < \omega_1 \rangle$ be Cohen reals added via a finite support iteration over a ground model V and suppose that in $V[\langle c_\alpha : \alpha < \omega_1 \rangle]$ there is an ultrafilter $\mathcal{U}$ definable from a real a and ordinals. It is well known that there is $\xi < \omega_1$ so that $a \in V[\langle c_\alpha : \alpha \in \omega_1 \backslash \{\xi\} \rangle]$. But then, by Lemma 6.1, c_ξ is splitting over $\mathcal{U}$, since $V[\langle c_\alpha : \alpha < \omega_1 \rangle] = V[\langle c_\alpha : \alpha \in \omega_1 \backslash \{\xi\} \rangle][c_\xi]$.

The argument for random reals is essentially the same.

Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \omega_1 \rangle$ be the ω_1-length countable support iteration of Silver forcing. Any real a appears in $V^{\mathbb{P}_\xi}$ for some $\xi < \omega_1$. But now note that $\mathbb{P}_{\omega_1}$ is $\mathrm{OD}(V)$

and weakly homogeneous. Moreover, $\mathbb{P}_{\omega_1} \cong \mathbb{P}_\xi * \dot{\mathbb{P}}_{\omega_1}$. Thus applying Lemma 6.3, we find that there is no ultrafilter definable from parameters in $V^{\mathbb{P}_\xi}$ over $V^{\mathbb{P}_{\omega_1}}$. In particular there is no OD($\{a\}$) ultrafilter in $V^{\mathbb{P}_{\omega_1}}$. $\square$

7 The Borel ultrafilter number

The ultrafilter number $\mathfrak{u}$ is the least size of a base for an ultrafilter. As with mad families (see [16]) and maximal independent families (see [3]) it makes sense to introduce a Borel version of the ultrafilter number that is closely related to the definability of ultrafilters.

Definition 7.1 The Borel ultrafilter number is defined as

$$\mathfrak{u}_B := \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathbf{\Delta}^1_1, \bigcup \mathcal{B} \text{ is an ultrafilter}\}.$$

Note that $\aleph_1 \leq \mathfrak{u}_B$, as a countable union of Borel sets is Borel.

Remark 7.1 Let $\mathfrak{u}'_B = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathbf{\Delta}^1_1, \bigcup \mathcal{B} \text{ is an ultrafilter base}\}$ and $\mathfrak{u}''_B = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathbf{\Delta}^1_1, \bigcup \mathcal{B} \text{ generates an ultrafilter}\}$. Then $\mathfrak{u}''_B = \mathfrak{u}'_B = \mathfrak{u}_B$.

Proof Obviously, $\mathfrak{u}''_B \leq \mathfrak{u}'_B \leq \mathfrak{u}_B$. Remember that whenever B is Borel, then the filter F_B that it generates is analytic. Thus $\mathfrak{u}''_B$ is uncountable as well. Now let $\mathcal{B}$ be a collection of Borel sets, whose union generates an ultrafilter. We may assume that $\mathcal{B}$ is closed under finite unions. For every $B \in \mathcal{B}$, let F_B be the filter generated by B. Since F_B is analytic, we can write it as an ω_1-union $F_B = \bigcup_{\alpha<\omega_1} F^\alpha_B$ of Borel sets. Now consider $\{F^\alpha_B : B \in \mathcal{B}, \alpha < \omega_1\}$. It has the same size as $\mathcal{B}$ and is a witness for $\mathfrak{u}_B$. $\square$

Any coanalytic set is an ω_1-union of Borel sets. Thus the existence of a coanalytic ultrafilter base implies that $\mathfrak{u}_B = \aleph_1$.

Theorem 7.2 $\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N}), \mathfrak{b} \leq \mathfrak{u}_B \leq \mathfrak{u}$.

Proof Let $\mathcal{B}$ be a collection of $< \mathrm{cov}(\mathcal{M})$ many Borel sets and assume that $\bigcup \mathcal{B}$ has the finite intersection property. Let $M \preccurlyeq H(\theta)$ for some large θ, so that $|M| < \mathrm{cov}(\mathcal{M})$ and $\mathcal{B} \subseteq M$. Then there is a Cohen real c over M. But then in $M[c]$, c is splitting over every $B \in \mathcal{B}$. Moreover in V it is true that c is splitting over B, by Σ_1-upwards-absoluteness. Thus c is splitting over $\bigcup \mathcal{B}$ which cannot be an ultrafilter. The argument for random forcing is exactly the same.

For $\mathfrak{b} \leq \mathfrak{u}_B$, note that any Borel filter is meager. By a classical result of Talagrand (see [20]), meager filters $\mathcal{F}$ are exactly those for which there is $f \in \omega^\omega$ so that $\forall x \in \mathcal{F} \forall^\infty n \in \omega (x \cap [n, f(n)) \neq \emptyset)$. For $\mathcal{B}$ a collection of Borel filters, we let f_B be such a function for every $B \in \mathcal{B}$. If $\mathcal{B}$ has size smaller than $\mathfrak{b}$, then there is a single function $f \in \omega^\omega$ so that $f_B <^* f$ for each $B \in \mathcal{B}$. Now note that $x_0 \cup x_1 = \omega$, where $x_0 := \bigcup_{n\in\omega}[f^{2n}(0), f^{2n+1}(0))$ and $x_1 := \bigcup_{n\in\omega}[f^{2n+1}(0), f^{2n+2}(0))$. But neither x_0 nor x_1 can be in $\bigcup \mathcal{B}$. $\square$

The following questions are answered positively in [17].

Question 7.1 Is it consistent that $\mathfrak{u}_B < \mathfrak{u}$? Is it consistent that there is a Π^1_1 ultrafilter base while $\aleph_1 < \mathfrak{u}$?

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