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End extensions of models of fragments of *PA*

C. Dimitracopoulos[1] · V. Paschalis[1]

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Abstract

In this paper, we prove results concerning the existence of proper end extensions of arbitrary models of fragments of Peano arithmetic (PA). In particular, we give alternative proofs that concern (a) a result of Clote (Fundam Math 127(2):163–170, 1986); (Fundam Math 158(3):301–302, 1998), on the end extendability of arbitrary models of Σ_n-induction, for $n \geq 2$, and (b) the fact that every model of Σ_1-induction has a proper end extension satisfying Δ_0-induction; although this fact was not explicitly stated before, it follows by earlier results of Enayat and Wong (Ann Pure Appl Log 168:1247–1252, 2017) and Wong (Proc Am Math Soc 144:4021–4024, 2016).

Keywords Arithmetized completeness theorem · Fragments of Peano arithmetic · End extensions

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1 Introduction

We work with subsystems of Peano arithmetic (PA) in the usual first-order language of arithmetic LA. As usual, for $n \in \mathbb{N}$, $I\Sigma_n$ (respectively, $I\Pi_n$) denotes the induction schema for Σ_n (respectively, Π_n) formulas (plus the well-known base theory PA^-), $B\Sigma_n$ denotes $I\Delta_0$ plus the collection schema for Σ_n formulas and exp denotes the axiom expressing "exponentiation is total" (recall that there is a Δ_0 formula representing the graph of the function 2^x). As usual, we will identify formulas, proofs etc. with their Gödel numbers. For details, the reader can consult [11] or [12].

✉ C. Dimitracopoulos
cdimitr@phs.uoa.gr

V. Paschalis
pasva@yahoo.com

1 Department of History and Philosophy of Science, National and Kapodistrian University of Athens, 15771 Zografou, Greece

Since the mid 1970's, two of the main aims of research work on fragments of PA have been (1) to study relationships among the various fragments and (2) to produce miniaturizations of significant results concerning PA, such as the so-called "MacDowell–Specker theorem", i.e., the following result.

Theorem 1 [14] *Every model of PA has a proper elementary end extension.*

One of the first, ground-breaking papers devoted to these aims was [16]; the first two of the main results contained in this paper can be summarized as follows.

Theorem 2

(a) *For all $n \in \mathbb{N}$,*

$$I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n,$$

and the converse implications do not hold.

(b) *For any $n \geq 2$ and any countable structure M,*

$$M \models B\Sigma_n \Leftrightarrow M \text{ has a proper } \Sigma_n\text{-elementary end extension } K \models I\Delta_0.$$

Furthermore, for any M as above, if M has a proper Σ_1-elementary end extension satisfying $I\Delta_0$, then $M \models B\Sigma_2$ (and hence M has a proper Σ_2-elementary end extension satisfying $I\Delta_0$).

Remarks 1

(1) In fact, part (a) of Theorem 2 was more comprehensive, in the sense that it referred to relationships among many more than two fragments of PA, but we have restricted our attention to the ones that are relevant to our subsequent work.
(2) As noted in [16], the proof of implication ($\Leftarrow$) in part (b) of Theorem 2 does not rely on the countability of the model M.

In view of the fact that the MacDowell–Specker theorem holds for any model of PA, it was natural to wonder whether Theorem 2(b) could be generalized. Indeed, one reads in [3]:

> The Kirby–Paris construction used very strongly the countability of the model. In view of the cardinality-free statement of the MacDowell–Specker Theorem, we might expect the conclusion of Theorem 1 to hold for models of any cardinality. Such a possibility was first suggested by A. Wilkie.

In [3], Clote claimed he had proved such a generalization, i.e., that he had proved the implication $\Rightarrow$ of Theorem 2(b) for arbitrary models of $B\Sigma_n$, $n \geq 2$, but it turned out much later that his claim was incorrect. The correct result was stated by the same author in [4] and is as follows.

Theorem 3 *For any $n \geq 2$, every model of $I\Sigma_n$ has a proper Σ_n-elementary end extension satisfying $I\Delta_0$.*

Remark 2 The main tool used by Clote was the same as that used in [16], i.e., the construction of a restricted ultrapower of the initial model M.

Given Theorem 3, one of the questions remaining to be answered was whether or not this result holds for $n = 0, 1$. Let us consider first the case when $n = 1$, i.e. the following question.

Question 1 *Does every model of $I\Sigma_1$ have a proper Σ_1-elementary end extension satisfying $I\Delta_0$?*

It is easy to see that this problem has a negative solution. Indeed, if every model of $I\Sigma_1$ had a proper Σ_1-elementary end extension satisfying $I\Delta_0$, then, by Theorem 2(b), every model of $I\Sigma_1$ would satisfy $B\Sigma_2$, which is impossible, since, by Theorem 2(a), there exists a model of $I\Sigma_1 + \neg B\Sigma_2$.

Given this negative answer to Question 1, let us become less ambitious and ask an apparently "easier" question.

Question 2 *Does every model of $I\Sigma_1$ have a proper Δ_0-elementary end extension satisfying $I\Delta_0$?*

Recalling that, for any structures M, K for LA, if $M \subset_e K \models I\Delta_0$, then $M <_0 K$ (see Theorem 2.7, page 24, in [12]), the new problem can be equivalently stated in the following form.

Problem 1 Does every model of $I\Sigma_1$ have a proper end extension satisfying $I\Delta_0$?

Using Theorem 2(a), it is easy to see that the question whether Theorem 3 holds for $n = 0$ has a negative answer, too. Indeed, suppose that every model of $I\Delta_0$ had a proper Δ_0-elementary end extension satisfying $I\Delta_0$. Now recall that, for any structures M, K for LA, if $M \subset_e K \models I\Delta_0$, then $M \models B\Sigma_1$ (see, e.g., Theorem 1 in [18]). Hence, every model of $I\Delta_0$ would satisfy $B\Sigma_1$, which is impossible, since, by Theorem 2(a), there exists a model of $I\Delta_0 + \neg B\Sigma_1$.

In Sect. 2 of the present paper, we provide an alternative proof of Clote's result, i.e., Theorem 3. By modifying this proof, in the third section, we show that Problem 1 has a positive solution. The final section of the paper is devoted to some remarks and related problems.

Concerning the main result of Sect. 3, our attention has been recently called to the fact that it was implicit in earlier work of A. Enayat and T.L. Wong. Indeed, this result follows by either the proof of Theorem 1 in [19] or combining Corollary 4.2 in [8] and Theorem 3.1 in [10]. Nevertheless, we feel our proof, which involves definable elements, is worth presenting, as it could be of use in resolving related problems.

Before we proceed to the main part of the paper, we note that the guiding principle for the proofs that will follow in the sequel has been to use a single argument (modulo necessary modifications), whose essence can be described in short as follows:

(i) starting with a consistent theory T_0 in a suitable extension of LA, we extend it to a maximal consistent theory T, using a lemma concerning the possibility of witnessing certain bounded existential quantifiers with appropriate constants, and

(ii) taking as the required extension of the initial model M a structure K with universe an appropriate set of elements definable in a model of the theory T.

For the proof of part (i), the main tool needed is a technical lemma, which concerns the construction, via induction, of a suitable branch in an infinite tree in M (whose nodes are appropriate formulas of the extended language). The proof of part (ii) essentially relies on a combination of well-known facts concerning definable elements in fragments of PA.

2 A proof of Clote's result

Our main aim in this section is to expand our sketch in the last part of [7] to obtain a full proof of Theorem 3. Our approach is reminiscent of the one taken for the proof of the MacDowell–Specker theorem given by Gaifman in [9] (see also Section 8.2 in [12]). Let us now proceed to the proof.

Proof Let M be a model of $I\Sigma_n$, $n \geq 2$. $LA(M)$ denotes the language obtained from LA by adding a set of new constant symbols $\{c_a : a \in M\}$ and $LA(M, c)$ the language obtained from $LA(M)$ by adding the new constant symbol c. For convenience, we will often identify the structure M with the structure M^* for $LA(M)$ defined by $M^* = \langle M, \{a : a \in M\}\rangle$.

Let $(\theta_i(x, \vec{v}))_{i\in\mathbb{N}}$ be an enumeration of all Σ_{n-1} formulas. Note that we may assume each θ_i is of the form $\exists y \leq v_{k+1} \eta_i(y, x, v_1, \dots, v_{k+1})$ (indeed, we can use dummy variables to introduce an extra bounded existential quantifier at the front). We will need the next lemma, in the proof of which, for every $i \in \mathbb{N}$ and $w \in M$, we will be working with an enumeration in M, in increasing order, of all (codes of) $LA(M)$ formulas obtained from the LA formula $\theta_i(x, \vec{v})$, if we replace the variables $\vec{v}$ by constant symbols from the set $\{c_a : a \leq w\}$. Clearly, we may assume that this enumeration is made in a canonical manner, i.e., we first enumerate all formulas involving constants from the set $\{c_0\}$, then all formulas involving constants from the set $\{c_0, c_1\}$, at least one of which is c_1, ... and finally all formulas involving constants from the set $\{c_0, c_1, \dots, c_w\}$, at least one of which is c_w; the idea is that, for each $w \in M$, the enumeration of formulas involving constants from $\{c_0, \dots, c_w\}$ will be a "proper initial segment" of the enumeration of all formulas involving the constant c_{w+1}. For each $i \in \mathbb{N}$ and $w \in M$, we call the sequence of $LA(M)$ formulas enumerated as above "canonical (i, w) sequence".

To make the enumeration more intuitive, we give the following description: We construct a countable sequence of "blocks", corresponding to the formulas $(\theta_i)_{i\in\mathbb{N}}$ mentioned above. For each i, the $(i+1)$-th block will consist of a sequence of "floors", corresponding to the elements of M that may be used to replace the variables $\vec{v}$ appearing in the formula θ_i. The canonicity of the enumeration can be depicted as in Fig. 1.

In the next lemma, given a canonical (i, w) sequence of formulas $(\theta_i^j)_{j\leq lh(i,w)}$, where $lh(i, w)$ denotes the length of this sequence, we construct another sequence of formulas $(\theta_i^j)^*$, $j \leq lh(i, w)$, as follows: $(\theta_i^j)^*$ is *either* the $(j+1)$-th formula of the canonical (i, w) sequence, i.e., a formula of the form $\exists y \leq c_{a_{k+1}} \eta_i(y, x, c_{a_1}, \dots, c_{a_{k+1}})$, *or* the tautology $0=0$.

c_w	$\theta_0(x, \{c_0, \ldots, c_w\})$	$\theta_1(x, \{c_0, \ldots, c_w\})$	$\cdots$	$\theta_i(x, \{c_0, \ldots, c_w\})$
$\vdots$				
c_1	$\theta_0(x, \{c_0, c_1\})$	$\theta_1(x, \{c_0, c_1\})$	$\cdots$	$\theta_i(x, \{c_0, c_1\})$
c_0	$\theta_0(x, \vec{c}_0)$	$\theta_1(x, \vec{c}_0)$	$\cdots$	$\theta_i(x, \vec{c}_0)$
	$\theta_0(x, \vec{v}_0)$	$\theta_1(x, \vec{v}_1)$	$\cdots$	$\theta_i(x, \vec{v}_i)$

Fig. 1 Enumeration

Let us now proceed with the exact statement and proof of the lemma. □

Lemma 4 *For every $k \in \mathbb{N}$,*

$$M \models \forall w \exists t_0 \ldots \exists t_k \exists s_0 \ldots \exists s_k \left\{ \bigwedge_{i=0}^{k} lh(t_i) = lh(s_i) \wedge \bigwedge_{i=0}^{k} \text{“}t_i \textit{ codes the sequence of all formulas of the form } \theta_i(x, \vec{c}) \textit{ with the indices of } \vec{c} \leq w\text{''} \wedge \bigwedge_{i=0}^{k} \forall p < lh(s_i) \left[[\forall z \exists x > z Sat_{n-1}(S_{i-1} \wedge \bigwedge_{j=0}^{p-1} (s_i)_j \wedge \theta_i^p(x, \vec{c}), x) \wedge (s_i)_p = \theta_i^p(x, \vec{c})] \vee [\neg \forall z \exists x > z Sat_{n-1}(S_{i-1} \wedge \bigwedge_{j=0}^{p-1} (s_i)_j \wedge \theta_i^p(x, \vec{c}), x) \wedge (s_i)_p = (0=0)] \right] \right\}$$

where (i) Sat_{n-1} *denotes a complete* Σ_{n-1} *formula and* $(s)_p$ *denotes the* p*-th term of* (*the sequence coded by*) s *and* (ii) S_{i-1} *denotes* $\bigwedge_{m=0}^{i-1} \bigwedge_{j=0}^{lh(s_m)-1} (s_m)_j$*, i.e. the conjunction of the set of formulas obtained by having considered all the layers of all blocks corresponding to the canonical* (m, w) *sequences, for* $0 \leq m \leq i-1$ (*we note also that, when* $i = 0$*,* S_{i-1} *denotes any tautology*).

Proof We will use induction in the metalanguage. So, we suppose that all $m < k$ have the property and show that k also has the property; we may assume $k \neq 0$, since the argument for the case $k = 0$ is essentially similar to the one that follows. In other words, we suppose that

$$\text{for all } m < k,\ M \models \forall w \exists t_0 \ldots \exists t_m \exists s_0 \ldots \exists s_m \left\{ \bigwedge_{i=0}^{m} lh(t_i) = lh(s_i) \wedge \bigwedge_{i=0}^{m} \text{“}t_i \text{ codes the canonical } (i, w) \text{ sequence''} \wedge \bigwedge_{i=0}^{m} \forall p < lh(s_i) [[\ldots] \vee [\ldots]] \right\} \tag{1}$$

and prove that

$$M \models \forall w \exists t_0 \ldots \exists t_k \exists s_0 \ldots \exists s_k \left\{ \bigwedge_{i=0}^{k} lh(t_i)=lh(s_i) \wedge \bigwedge_{i=0}^{k} \text{“}t_i \text{ codes} \right.$$
$$\left. \text{the canonical } (i, w) \text{ sequence”} \wedge \bigwedge_{i=0}^{k} \forall p<lh(s_i)\, [[\ldots] \vee [\ldots]] \right\}. \tag{2}$$

Note that, for any $i \in \mathbb{N}$ and $w \in M$,

(a) there exists a unique element t_i of M coding, in increasing order, all formulas obtained from θ_i as we described before. In fact, this element lies below an exponential bound, depending on the Gödel number of θ_i and on w; for example, t_i can be bounded by $(\ulcorner \theta_i \urcorner)^{w^2}$.
(b) as in (a) above, a similar exponential bound can be put on s_i; for simplicity, we will use the same bound as for t_i.

It follows that (2) is equivalent to

$$M \models \forall w \forall b \Big[b = (max\{\ulcorner \theta_0 \urcorner, \ldots, \ulcorner \theta_k \urcorner\})^{w^2}$$
$$\rightarrow \exists t_0 \ldots \exists t_k{<}b \exists s_0 \ldots \exists s_k{<}b \left\{ \bigwedge_{i=0}^{k} lh(t_i) = lh(s_i) \wedge \bigwedge_{i=0}^{k} \text{“}t_i \text{ codes} \right.$$
$$\left. \text{the canonical } (i, w) \text{ sequence”} \wedge \bigwedge_{i=0}^{k} \forall p<lh(s_i)\, [[\ldots] \vee [\ldots]] \right\} \Big]. \tag{3}$$

Now notice that $\forall b\, [\ldots]$ is a $\Sigma_0(\Sigma_n)$ formula, since it has been constructed by using connectives, bounded quantification and instances of the Π_n formula $\forall z \exists x{>}z Sat_{n-1}(\ldots)$ (for the exact definition of the class $\Sigma_0(\Sigma_n)$, see, e.g., Definition 2.2, page 62, of [11]). But it is well-known that $I\Sigma_n$ implies $I\Sigma_0(\Sigma_n)$, for all $n \in \mathbb{N}$ (see, e.g. Lemma 2.14, page 65, in [11]). Therefore, we may use induction in M to prove (3) or, equivalently, (2).

Since the case $w{=}0$ is similar to the general inductive step, it suffices to show that, for any $a \in M$, if a satisfies $\forall b\, [\ldots]$, then $a{+}1$ also satisfies this formula. So, assuming that, for $b{=}(max\{\ulcorner \theta_0 \urcorner, \ldots, \ulcorner \theta_k \urcorner\})^{a^2}$,

$$M \models \exists t_0 \ldots \exists t_k{<}b \exists s_0 \ldots \exists s_k{<}b \left\{ \bigwedge_{i=0}^{k} lh(t_i) = lh(s_i) \wedge \bigwedge_{i=0}^{k} \text{“}t_i \text{ codes} \right.$$
$$\left. \text{the canonical } (i, a) \text{ sequence”} \wedge \bigwedge_{i=0}^{k} \forall p<lh(s_i)\, [[\ldots] \vee [\ldots]] \right\}, \tag{4}$$

we will show that, for $B=(max\{\ulcorner\theta_0\urcorner,\ldots,\ulcorner\theta_k\urcorner\})^{(a+1)^2}$,

$$M\models\exists t_0\ldots\exists t_k<B\exists s_0\ldots\exists s_k<B\left\{\bigwedge_{i=0}^{k}lh(t_i)=lh(s_i)\wedge\bigwedge_{i=0}^{k}\text{“}t_i\text{ codes the canonical }(i,a+1)\text{ sequence”}\wedge\bigwedge_{i=0}^{k}\forall p<lh(s_i)\,[[\ldots]\vee[\ldots]]\right\}.\tag{5}$$

In what follows, let $\Theta(j,w,t_0,\ldots,t_j,s_0,\ldots,s_j)$ denote the formula

$$\bigwedge_{i=0}^{j}lh(t_i)=lh(s_i)\wedge\bigwedge_{i=0}^{j}\text{“}t_i\text{ codes the canonical }(i,w)\text{ sequence of formulas”}\wedge\bigwedge_{i=0}^{j}\forall p<lh(s_i)[\ldots].$$

By (1), there exist unique elements $t_0^{a+1},\ldots,t_{k-1}^{a+1},s_0^{a+1},\ldots,s_{k-1}^{a+1}$ of M such that

$$M\models\Theta\left(k-1,a+1,t_0^{a+1},\ldots,t_{k-1}^{a+1},s_0^{a+1},\ldots,s_{k-1}^{a+1}\right)\tag{6}$$

and, by (4), there exist unique elements $t_0^a,\ldots,t_k^a,s_0^a,\ldots,s_k^a$ such that

$$M\models\Theta\left(k,a,t_0^a,\ldots,t_k^a,s_0^a,\ldots,s_k^a\right).\tag{7}$$

Finally, it is clear that there exists a unique element t_k^{a+1} of M coding the canonical $(k,a+1)$ sequence of formulas (obtained from θ_k).

Note that, by the canonicity of the construction of the t_i's and the induced canonicity of the construction of the s_i's, we have that

(i) for any $0\leq i\leq k$, t_i^a is an initial segment of t_i^{a+1}
(ii) for any $0\leq i\leq k-1$, s_i^a is an initial segment of s_i^{a+1}.

Now let $\theta_k^0,\ldots,\theta_k^L$ be the formulas coded by t_k^{a+1}, that is, the formulas (in increasing order) in the $(k+1)$-th block of $LA(M)$ formulas described before the Lemma. We claim that

$$M\models\forall l\leq L\exists S\Bigg\{lh(S)=l+1\wedge\forall p\leq l\left[\left[\forall z\exists x>zSat_{n-1}\left(\bigwedge_{i=0}^{k-1}s_i^{a+1}\wedge\bigwedge_{j=0}^{p-1}(S)_j\wedge\theta_k^p(x,\vec{c}),x\right)\wedge(S)_p=\theta_k^p(x,\vec{c})\right]\vee\left[\neg\forall z\exists x>zSat_{n-1}\left(\bigwedge_{i=0}^{k-1}s_i^{a+1}\wedge\bigwedge_{j=0}^{p-1}(S)_j\wedge\theta_k^p(x,\vec{c}),x\right)\wedge(S)_p=(0=0)\right]\right]\Bigg\},\tag{8}$$

where, for the sake of technical simplicity, we identify codes of sequences, e.g., s_i^{a+1}, with the corresponding sequences of terms (i.e., formulas). Note that, for each $i \leq k-1$, s_i^{a+1} is a sequence of formulas corresponding to the canonical $(i, a+1)$ sequence of formulas and S corresponds to a sequence of formulas obtained from the canonical $(k, a+1)$ sequence.

To prove (8), we use induction on l; note that the formula after $\forall l \leq L$ in (8) is (equivalent to) a $\Sigma_0(\Sigma_n)$ formula, so we may use induction in M as before. For simplicity, we will deal with the case $l = 0$ only.

First note that, by the construction of the sequences $s_0^{a+1}, \ldots, s_{k-1}^{a+1}$, we have that

$$M \models \forall z \exists x > z Sat_{n-1}\left(\bigwedge_{i=0}^{k-1} s_i^{a+1}\right). \tag{9}$$

Since, clearly,

$$M \models \forall z \exists x > z Sat_{n-1}\left(\bigwedge_{i=0}^{k-1} s_i^{a+1} \wedge \theta_k^0(x, \vec{c}), x\right)$$
$$\vee \neg \forall z \exists x > z Sat_{n-1}\left(\bigwedge_{i=0}^{k-1} s_i^{a+1} \wedge \theta_k^0(x, \vec{c}), x\right)$$

and hence that $M \models \exists S \Gamma(S)$, where $\Gamma(S)$ denotes the formula in $\{\ldots\}$ in (8).

Having proved our claim, we let s_k^{a+1} be the unique S satisfying the formula $\{\ldots\}$ in (8) for $l = L$. It is not difficult to check that s_k^a is a proper initial segment of s_k^{a+1} and that

$$M \models \Theta(k, a+1, t_0^{a+1}, \ldots, t_k^{a+1}, s_0^{a+1}, \ldots, s_k^{a+1}),$$

which finishes the proof of the Lemma. □

We continue with a lemma, which concerns a nice property of the sequence $(\theta_i^j)^*$, $j \leq lh(i, w)$, constructed according to the procedure stated in Lemma 4. This property can be named "witness property" and can be described in short as follows: if a formula of the form $\exists y \leq c_{a_{k+1}} \eta(y, x, c_{a_1}, \ldots, c_{a_{k+1}})$ becomes a member of the sequence corresponding to $\exists y \leq v_{a_{k+1}} \eta(y, x, \vec{v})$, then there exists $b \leq a_{k+1}$ such that the formula $\eta(c_b, x, c_{a_1}, \ldots, c_{a_{k+1}})$ becomes a member of the sequence corresponding to the formula $\eta(y, x, \vec{v})$. The exact statement and proof of this fact are given below.

Lemma 5 *Suppose that $\exists y \leq v_{k+1} \eta(y, x, \vec{v})$ and $\eta(y, x, \vec{v})$ appear as θ_m and θ_l, respectively, in the enumeration $(\theta_i(x, \vec{v}))_{i \in \mathbb{N}}$ of Σ_{n-1} formulas, where $m > l$. Then, for any elements $\vec{a}$ of M,*

$$M \models \forall w \geq max(\vec{a}) \forall t_0 \ldots \forall t_m \forall s_0 \ldots \forall s_m \left\{ \bigwedge_{i=0}^{m} lh(t_i) = lh(s_i) \wedge \bigwedge_{i=0}^{m} \text{“}t_i \textit{ codes the}\right.$$
$$\textit{sequence of all formulas of the form } \theta_i(x, \vec{c}) \textit{ with the indices of } \vec{c} \leq w\text{”} \wedge$$

$$\bigwedge_{i=0}^{m} \forall p{<}lh(s_i)\Bigg[\Bigg[\forall z\exists x{>}zSat_{n-1}\left(S_{i-1}\wedge \bigwedge_{j=0}^{p-1}(s_i)_j\wedge\theta_i^p(x,\vec{c}),x\right)\wedge(s_i)_p=\theta_i^p(x,\vec{c})\Bigg]$$
$$\vee\left[\neg\forall z\exists x{>}zSat_{n-1}\left(S_{i-1}\wedge \bigwedge_{j=0}^{p-1}(s_i)_j\wedge\theta_i^p(x,\vec{c}),x\right)\wedge(s_i)_p=(0=0))\right]\Bigg]$$
$$\left.\wedge\exists r{<}lh(s_m)[(s_m)_r=\exists y{\le}c_{a_{k+1}}\eta(y,x,\vec{c})]\rightarrow\exists y{\le}a_{k+1}\exists q{<}lh(s_l)[(s_l)_q{=}\eta(c_y,x,\vec{c})]\right\}.$$

(*if* $l{>}m$*, a similar statement holds, which can be proved in the same way*)

Proof Assume $b{\ge}\vec{a}, d_0,\ldots,d_m, e_0,\ldots,e_m$ and f are elements of M such that the formula

$$\bigwedge_{i=0}^{m} lh(t_i){=}lh(s_i)\wedge\ldots\wedge f{<}lh(s_m)\wedge(s_m)_f{=}\exists y{\le}c_{a_{k+1}}\eta(y,x,\vec{c})$$

is satisfied in M when we replace w by b, t_i by d_i and s_i by e_i, for all $0{\le}i{\le}m$, and r by f. It follows that

$$M\models\forall z\exists x{>}zSat_{n-1}(S_{m-1}\wedge\bigwedge_{j=0}^{f-1}(s_m)_j\wedge\exists y{\le}c_{a_{k+1}}\eta(y,x,\vec{c}),x).$$

By well-known properties of the formula Sat_{n-1}, we deduce that

$$M\models\forall z\exists x{>}z\exists y{\le}a_{k+1}Sat_{n-1}\left(S_{m-1}\wedge\bigwedge_{j=0}^{f-1}(s_m)_j\wedge\eta(c_y,x,\vec{c}),x\right).\qquad(10)$$

But this implies that

$$M\models\exists y{\le}a_{k+1}\forall z\exists x{>}zSat_{n-1}\left(S_{m-1}\wedge\bigwedge_{j=0}^{f-1}(s_m)_j\wedge\eta(c_y,x,\vec{c}),x\right).\qquad(11)$$

Indeed, if (11) failed, then we would have

$$M\models\forall y{\le}a_{k+1}\exists z\forall x{>}z\neg Sat_{n-1}\left(S_{m-1}\wedge\bigwedge_{j=0}^{f-1}(s_m)_j\wedge\eta(c_y,x,\vec{c}),x\right),$$

which, by the fact that M satisfies Σ_n collection, would give

$$M\models\exists t\forall y{\le}a_{k+1}\exists z{<}t\forall x{>}z\neg Sat_{n-1}\left(S_{m-1}\wedge\bigwedge_{j=0}^{f-1}(s_m)_j\wedge\eta(c_y,x,\vec{c}),x\right).$$

But then there exists $T{\in}M$ such that

$$M \models \forall y{\le}a_{k+1}\forall x{>}T\neg Sat_{n-1}\left(S_{m-1}\wedge\bigwedge_{j=0}^{f-1}(s_m)_j\wedge\eta(c_y,x,\vec{c}),x\right),$$

which contradicts (10).

Therefore, (11) holds and so there exists $g{\in}M$, $g{\le}a_{k+1}$ such that

$$M \models \forall z\exists x{>}zSat_{n-1}\left(S_{m-1}\wedge\bigwedge_{j=0}^{f-1}(s_m)_j\wedge\eta(c_g,x,\vec{c}),x\right). \tag{12}$$

Let now h be an element of M such that $M \models h{<}lh(t_l)\wedge(t_l)_h = \eta(x,c_g,\vec{c})$. If $M \models (s_l)_h = (0=0)$, then it would be the case that

$$M \models \neg\forall z\exists x{>}zSat_{n-1}\left(S_{l-1}\wedge\bigwedge_{j=0}^{h-1}(s_l)_j\wedge\eta(c_g,x,\vec{c}),x\right),$$

which would contradict (12). It follows that $M \models (s_l)_h = \eta(c_g,x,\vec{c})$, as required. □

Let us now consider the following theory in the language $LA(M,c)$

$$T = Th(M) + c{>}M + \Theta,$$

where $Th(M)$ denotes, as usual, the elementary diagram of M, $c{>}M$ denotes the set $\{c{>}c_a : a{\in}M\}$ and Θ denotes the set $\bigcup_{i\in\mathbb{N}}\{(\theta_i^j)^* : j \in M\}$, where $(\theta_i^j)^*$ is the formula $\theta_i(c,\vec{c})$ or 0=0, depending on the term $(s_{i,w})_j$ of (the sequence coded by) $s_{i,w}$, for some $w{\in}M$, or, equivalently, for any $w{\in}M$ such that the indices of $\vec{c}$ are below w; note that, once $(\theta_i^j)^*$ enters the enumeration at a level, it will stay there "to the end" (of the enumeration).

Observe that T is a consistent set of sentences of $LA(M,c)$; indeed, every finite subset T' of this set is satisfied in the structure for $LA(M,c)$ obtained from M by interpreting each constant c_a as the corresponding element of M and c as a suitably large element of M. Let J be a model of (a maximal consistent extension Σ of) T and let $K^{n-1}(J)$ be the substructure of J whose universe is the set of Σ_{n-1} definable elements in J. Clearly, we may identify $K^{n-1}(J)$ with the substructure K of $J \restriction LA$ (i.e., the reduct of J to LA), whose universe is the set of elements Σ_{n-1} definable in $J{\restriction}LA$ using parameters from the set $\{c_a^J : a{\in}M\}\cup\{c^J\}$ (for the precise definition, see, e.g., p. 130 in [12]). To complete the proof of Theorem 3, it suffices to prove the following result.

Lemma 6 *M is (isomorphic to) a proper Σ_n-elementary initial segment of K and $K \models I\Delta_0$.*

Proof Let $f : M \to K$ be the function defined by $f(a) = c_a^J$, for each $a{\in}M$. Clearly, f maps M isomorphically onto a substructure L of K and so it suffices to show that K is a proper Σ_n-elementary end extension of L.

First, note that $K \prec_{n-1} J{\restriction}LA$, by Theorem 10.1, p. 131, in [12]. It follows that $K^* \models \Pi_n(M)$, where $\Pi_n(M)$ denotes the set of Π_n sentences of $LA(M)$ which are true in M^*, i.e., the natural expansion of M to a structure for $LA(M)$. Therefore, $L \prec_n K$.

Since, clearly, K is a proper extension of L, it remains to show that $L \subset_e K$. So let $a \in M$ and $d \in K$ such that $K \models d < c_a^J$ or, equivalently, $J \models d < c_a^J$. By the definition of K, there exist a Σ_{n-1} formula $\varphi(y, x, \vec{v})$ and $\vec{a} \in M$ such that $\varphi(y, c, \vec{c})$ defines d in J. Then $J \models \exists y{\le}c_a^J \varphi(y, c, \vec{c})$ and hence $\exists y{\le}c_a\varphi(y, c, \vec{c}) \in T$; indeed, if $\exists y{\le}c_a\varphi(y, c, \vec{c}) \notin T$ held, we would have that $M{\models}\exists z\forall x{>}z\neg\exists y{\le}c_a\varphi(y, x, \vec{c})$, i.e., $M \models \forall x{>}c_{a^*}\forall y{\le}c_a\neg\varphi(y, x, \vec{c})$ for some $a^*{\in}M$, which, given that $c{>}c_{a^*}{\in}T$, would imply that $J \models \forall y{\le}c_a^J\neg\varphi(y, c, \vec{c})$. By Lemma 5, there exists $b{\in}M$ such that $M \models b{<}a$ and $\varphi(c_b, c, \vec{c}){\in}T$. Therefore, $J \models \varphi(c_b, c, \vec{c})$ and hence $d = c_b^J$, as required.

Finally, note that $K \models I\Delta_0$, since $K{\prec_{n-1}}J{\restriction}LA$ and $J{\restriction}LA \models I\Delta_0$. □

The proof of Theorem 3 is now complete. □

Remark 3 It is easy to see that, in Theorem 3, the model K obtained satisfies a theory strictly stronger than $I\Delta_0$. Indeed, by Theorem 0.2 in [13], for $n{\ge}2$, $I\Sigma_n \Rightarrow I\Pi_n^-$, where $I\Pi_n^-$ denotes the theory of induction for Π_n parameter free formulas. But, clearly, $I\Pi_n^-$ holds in M and $I\Pi_n^-$ is a set of Σ_{n+1} sentences (recall Theorem 0.6 in [13]). Therefore, $K \models I\Pi_n^-$. Now recall that $I\Pi_n^-$ is strictly stronger than $I\Delta_0$; indeed, by Proposition 1.10 in [13], ($B\Sigma_1$ and hence) $I\Delta_0$ does not prove $I\Pi_1^-$. It is probably known that the model K satisfies a theory strictly stronger than $I\Pi_n^-$, but we will not pursue this question further, as it lies outside our area of interest in the present paper.

3 End extending models of Σ_1-induction

In this section, we show that it is possible to modify the proof of Theorem 3 to prove that Problem 1 has a positive solution, i.e. to prove the next result.

Theorem 7 *Every model of $I\Sigma_1$ has a proper end extension satisfying $I\Delta_0$.*

The main difference between the proof of Theorem 3 in Section 2 and the modification we are about to use to prove Theorem 7 is that in the latter we have to use a consistency statement instead of the formula concerning satisfiability of formulas that was used in the former. Note that

(i) it would not be possible to work with formulas of the form $\forall z\exists x{>}z\ Sat_0(\ldots)$ here, as this would require the use of $I\Sigma_2$, while we only have that $M \models I\Sigma_1$. So we have to follow an alternative approach, namely one that employs the use of a formal consistency statement, by means of which the problematic quantifier complexity induced by the expression $\forall z\exists x{>}z$ is "absorbed" within this statement; actually, the appearance of the expression $\forall z \exists x{>}z$ can be avoided altogether, since the same role can be played, essentially, by the constant c, through including the condition $c{>}M$ in the theory whose consistency is being considered.

(ii) it is not necessary to work with a formula expressing semantic tableau consistency, as we did in [6]. Indeed, in that paper, we had to work with the restricted notion of consistency $Tabcon(\ldots)$, since the model under consideration there satisfied a theory strictly weaker than $I\Sigma_1$; here $I\Sigma_1$ is satisfied in the model and hence the usual formula $Con(\ldots)$ may be employed - recall the well-known fact that the notions of (usual) consistency and semantic tableau consistency are equivalent in models in which superexponentiation is total and hence in models satisfying $I\Sigma_1$.

Let us proceed now to proving Theorem 7.

Proof Let M be an arbitrary model of $I\Sigma_1$ and $LA(M)$, $LA(M, c)$ be the extensions of LA defined at the beginning of the proof of Theorem 3.

We first mention a result which is well-known, but we recall it for the sake of completeness. □

Lemma 8 *$M \models Con(I\Delta_0 + \Delta + c{>}M)$, where Δ denotes the Δ_0-diagram of M.*

Proof This is essentially a variant of the proof of Lemma 8.10 in [17]. Note that working with the formula $Con(\ldots)$ does not differ essentially from working with the formula $Tabcon(\ldots)$ in [17], since M satisfies $I\Sigma_1$. □

Now we proceed to a result similar to Lemma 4, in the proof of which we will be working with (a) an enumeration $(\theta_i(c, \vec{v}))_{i\in\mathbb{N}}$ of all Σ_1 formulas (as in the previous section, we may assume that each θ_i is of the form $\exists y{\le}v_{k+1}\eta_i(y, c, v_1, \ldots, v_{k+1})$, where η_i is a Σ_1 formula) and (b) an enumeration in M, in increasing order, of the set of all (codes of) $LA(M, c)$ formulas obtained from the $LA(c)$ formula $\theta_i(c, \vec{v})$, if we replace the variables $\vec{v}$ by constant symbols from the set $\{c_a : a{\le}w\}$. As before, we will assume that this enumeration is made in a "canonical manner" (see explanation before Lemma 4).

Lemma 9 *For every $k \in \mathbb{N}$,*

$$M \models \forall w \exists t_0 \ldots \exists t_k \exists s_0 \ldots \exists s_k \left\{ \bigwedge_{i=0}^{k} lh(t_i) = lh(s_i) \wedge \bigwedge_{i=0}^{k} \text{“}t_i \textit{ codes the sequence of all formulas of the form } \theta_i(c, \vec{c}) \textit{ with the indices of } \vec{c}{\le}w\text{”} \wedge \bigwedge_{i=0}^{k} \forall p{<}lh(s_i) \left[\left[Con(T_0 \wedge S_{i-1} \wedge \bigwedge_{j=0}^{p-1} (s_i)_j \wedge \theta_i^p(c, \vec{c})) \wedge (s_i)_p{=}\theta_i^p(c, \vec{c}) \right] \vee \left[\neg Con(T_0 \wedge S_{i-1} \wedge \bigwedge_{j=0}^{p-1} (s_i)_j \wedge \theta_i^p(c, \vec{c})) \wedge (s_i)_p{=}(0{=}0) \right] \right] \right\},$$

where T_0 denotes $I\Delta_0{+}\Delta{+}c{>}M$ and S_{i-1} denotes $\bigwedge_{m=0}^{i-1} \bigwedge_{j=0}^{lh(s_m)-1} (s_m)_j$ (note, as before, that S_{i-1} denotes any tautology, when $i = 0$).

Proof The proof is essentially identical with that of Lemma 4, the only difference being that the formula $Con(\ldots)$ is used instead of the formula $Sat_{n-1}(\ldots)$ to extend the initial theory T_0. □

Next comes a lemma concerning a witnessing property analogous to the one stated in Lemma 5.

Lemma 10 *Suppose that $\exists y{\le}v_{k+1}\eta(y, c, \vec{v})$ and $\eta(y, c, \vec{v})$ appear as θ_m and θ_l, respectively, in the enumeration $(\theta_i(c, \vec{v}))_{i\in\mathbb{N}}$ of Σ_1 formulas, where $m{>}l$. Then, for any elements $\vec{a}$ of M,*

$$M \models \forall w{\ge}max(\vec{a})\forall t_0\ldots\forall t_m\forall s_0\ldots\forall s_m\left\{\bigwedge_{i=0}^{m} lh(t_i)=lh(s_i)\wedge\bigwedge_{i=0}^{m}\text{“}t_i \textit{ codes the sequence of all formulas of the form } \theta_i(c,\vec{c}) \textit{ with the indices of } \vec{c}{\le}w\text{”}\wedge\right.$$
$$\bigwedge_{i=0}^{m}\forall p{<}lh(s_i)\left[\left[Con\left(T_0\wedge S_{i-1}\wedge\bigwedge_{j=0}^{p-1}(s_i)_j\wedge\theta_i^p(c,\vec{c})\right)\wedge(s_i)_p{=}\theta_i^p(c,\vec{c})\right]\right.$$
$$\left.\vee\left[\neg Con\left(T_0\wedge S_{i-1}\wedge\bigwedge_{j=0}^{p-1}(s_i)_j\wedge\theta_i^p(c,\vec{c})\right)\wedge(s_i)_p=(0{=}0)\right]\right]$$
$$\left.\wedge\exists r{<}lh(s_m)[(s_m)_r{=}\exists y{\le}c_{a_{k+1}}\eta(y,c,\vec{c})]\rightarrow\exists y{\le}a_{k+1}\exists q{<}lh(s_l)[(s_l)_q{=}\eta(c_y,c,\vec{c})]\right\}.$$

(*if $l{>}m$, a similar statement holds, which can be proved in the same way*)

Proof Assume $b{\ge}\vec{a}, d_0,\ldots,d_m, e_0,\ldots,e_m$ and f are elements of M such that the formula

$$\bigwedge_{i=0}^{m} lh(t_i)=lh(s_i)\wedge\cdots\wedge r{<}lh(s_m)\wedge(s_m)_r=\exists y{\le}c_{a_{k+1}}\eta(y,c,\vec{c})$$

is satisfied in M when we replace w by b, t_i by d_i and s_i by e_i, for all $0{\le}i{\le}m$, and r by f. It follows that

$$M\models Con\left(T_0+S_{m-1}\wedge\bigwedge_{j=0}^{f-1}(s_m)_j\wedge\exists y{\le}c_{a_{k+1}}\eta(y,c,\vec{c})\right). \tag{13}$$

We claim that

$$M\models\exists y{\le}a_{k+1}Con\left(T_0{+}S_{m-1}\wedge\bigwedge_{j=0}^{f-1}(s_m)_j\wedge\eta(c_y,c,\vec{c})\right). \tag{14}$$

Indeed, if (14) failed, then we would have

$$M \models \forall y{\le}a_{k+1}\neg Con\left(T_0 + S_{m-1} \wedge \bigwedge_{j=0}^{f-1}(s_m)_j \wedge \eta(c_y, c, \vec{c})\right),$$

that is,

$$M \models \forall y{\le}a_{k+1}\exists s\text{“}s \text{ is a proof of } \perp \text{ from } T_0 + S_{m-1} \wedge \bigwedge_{j=0}^{f-1}(s_m)_j \wedge \eta(c_y, c, \vec{c})\text{”}.$$

Recalling now that $I\Delta_0 \vdash \forall x\forall y(x{\le}y{\leftrightarrow}x{<}y \vee x{=}y)$ and that $M \models I\Sigma_1$, we can prove by induction that

$$M \models \forall z{\le}a_{k+1}\exists s\text{“}s \text{ is a proof of } \perp \text{ from } T_0 + S_{m-1} \wedge \bigwedge_{j=0}^{f-1}(s_m)_j \wedge \exists y{\le}c_z\eta(y, c, \vec{c})\text{”}.$$

Therefore, (14) holds and so there exists $g{\in}M$, $g{\le}a_{k+1}$ such that

$$M \models Con\left(T_0 \wedge S_{m-1} \wedge \bigwedge_{j=0}^{f-1}(s_m)_j \wedge \eta(c_g, c, \vec{c})\right). \tag{15}$$

Let now h be an element of M such that $M \models h{<}lh(t_l) \wedge (t_l)_h = \eta(c_g, x, \vec{c})$. If $M \models (s_l)_h = (0 = 0)$, then it would be the case that

$$M \models \neg Con\left(T_0 \wedge S_{l-1} \wedge \bigwedge_{j=0}^{h-1}(s_l)_j \wedge \eta(c_g, c, \vec{c})\right),$$

which would contradict (15). It follows that $M \models (s_l)_h = \eta(c_g, c, \vec{c})$, as required. □

We now consider the theory $T = I\Delta_0{+}\Delta{+}c{>}M{+}\Theta$ in the language $LA(M, c)$, where Θ denotes the set $\bigcup_{i\in\mathbb{N}}\{(\theta_i^j)^* : j \in M\}$, with $(\theta_i^j)^*$ denoting the formula $\theta_i(c, \vec{c})$ or 0=0, depending on the term $(s_{i,w})_j$ of (the sequence coded by) $s_{i,w}$, for any $w \in M$ large enough.

T is clearly consistent, so let J be a model of T and K denote the substructure of $J{\restriction}LA$ whose universe is the set of elements Σ_1 definable in $J{\restriction}LA$ using parameters from the set $\{c_a^J : a \in M\} \cup \{c^J\}$. To complete the proof of Theorem 7, it suffices to prove the following result.

Lemma 11 *M is (isomorphic to) a proper initial segment of K and $K \models I\Delta_0$.*

Proof First, note that $K \models I\Delta_0$, since $J \models I\Delta_0$ and $K{<_1}J \restriction LA$ (recall Theorem 10.1 in [12]).

Now let $f : M \to K$ be the function defined by $f(a) = c_a^J$, for any $a \in M$. Since T contains the diagram of M, f maps M isomorphically onto a substructure L of K and so it suffices to show that K is a proper end extension of L. Clearly, K is a proper extension of L, so it remains to show that $L \subset_e K$. But this can be proved by an easy modification of the second part of the proof of Lemma 6, using Lemma 10. □

The proof of Theorem 7 is now complete. □

Remark 4 As we have noted before, Theorem 7 follows from earlier work of A. Enayat and T. L. Wong; indeed, this result follows by *either* the proof of Theorem 1 in [19] *or* a combination of Corollary 4.2 in [8] and Theorem 3.1 in [10]. In fact, Theorem 1 in [19] leads to a positive answer of the second part of Problem 1 in [15], i.e., the following question: is every model M of $I\Sigma_1$ properly end extendable to $K \models I\Sigma_1$ such that $M \models I\Sigma_1^*$ in K? Note that $I\Sigma_1^*$ denotes the theory, in the second-order language of arithmetic, which concerns induction for Σ_1^* formulas, i.e., formulas of the form $\exists\vec{x}_1\forall\vec{x}_2\ldots\varphi(\vec{x}_1, \vec{x}_2, \ldots)$, with n alternating blocks of similar first-order quantifiers where the only quantifiers in φ are bounded first-order.

4 Concluding remarks

In view of the negative answer to Question 1, Theorem 7 is the best possible, concerning the quantifier complexity of the formulas that are satisfied in M iff they are satisfied in its end extension. Similarly, Clote's result, i.e. Theorem 3, is the best possible, in the sense that, for any $n \geq 2$, there exists a model of $I\Sigma_n$ which has no proper Σ_{n+1}-elementary end extension satisfying $I\Delta_0$. Indeed, if there were no such model, by implication $\Leftarrow$ of Theorem 2(b), all models of $I\Sigma_n$, $n\geq 2$, would satisfy $B\Sigma_{n+1}$, which is impossible, by Theorem 2(a), i.e., by the fact that $I\Sigma_n \not\Rightarrow B\Sigma_{n+1}$.

One can naturally wonder whether Theorem 3 holds if we consider the theory $B\Sigma_n$ instead of the theory $I\Sigma_n$.

Problem 2 Does every model of $B\Sigma_n$, $n\geq 2$, have a proper Σ_n-elementary end extension satisfying $I\Delta_0$?

Recalling that $B\Sigma_1 \Leftrightarrow B\Sigma_0$ and that $I\Delta_0$ is strictly weaker than $B\Sigma_1$, the following problem also arises naturally.

Problem 3 Does every model of $B\Sigma_1$ have a proper end extension satisfying $I\Delta_0$?

Both these problems remain open, with Problem 3 being considered to be of particular interest (see "Fundamental problem F" in [5]).

Concerning Problem 2, it has a positive answer, if attention is restricted to countable models (recall implication ($\Rightarrow$) of Theorem 2(b)).

It is easy to see that the answer to Problem 2 is also positive, if we demand that the end extension is Σ_{n-1}-elementary, for $n \geq 3$; indeed, if M is a model of $B\Sigma_n$, $n \geq 3$, then M satisfies $I\Sigma_{n-1}$ and hence, by Clote's result, it has a proper Σ_{n-1}-elementary end extension satisfying $I\Delta_0$. In view of this observation, it remains to see what happens in the case $n = 2$, i.e., to determine the answer to the following question.

Question 3 *Does every model of $B\Sigma_2$ have a proper Σ_1-elementary end extension satisfying $I\Delta_0$?*

A more general question worth considering, in view of Theorem 7 and the negative answer to Question 1 (discussed in the first section) is as follows.

Question 4 *Is there a theory T, with strength strictly between those of $I\Sigma_2$ and $I\Sigma_1$, such that every model of T has a proper Σ_1-elementary end extension satisfying $I\Delta_0$?*

Problem 3 was studied extensively in [18]. In fact, Wilkie and Paris proved that it has a positive answer, if (a) attention is restricted to countable models and (b) the initial model satisfies an extra condition. Among the results proved in [18], the following is perhaps the most interesting one.

Theorem 12 [18] *Every countable model of $B\Sigma_1 + exp$ has a proper end extension satisfying $I\Delta_0$.*

Remarks 5

(1) Note that Problem 3 is considered especially interesting, since it is connected to problems in computational complexity theory (for details, see, e.g., [5]).
(2) Problem 3 is known to have a negative answer, under an assumption concerning the collapse of the Δ_0 hierarchy. Indeed, in [18] it was proved that there exists a countable model of $B\Sigma_1$ which does not have a proper end extension satisfying $I\Delta_0$, *provided that* the Δ_0 hierarchy collapses provably in $I\Delta_0$, i.e., there exists a fixed n such that, for any $\theta \in \Delta_0$ there is $\eta \in \Delta_0$ in prenex normal form with at most n alternations of bounded quantifiers such that $I\Delta_0 \vdash \theta \leftrightarrow \eta$ (this hypothesis is usually denoted by $I\Delta_0 \vdash \neg\Delta_0 H$).

In view of the above results and comments, it is natural to wonder whether Theorem 7 or Theorem 12 can be improved; in particular, it seems worthwhile to study the following problem.

Problem 4 Does every model of $B\Sigma_1 + exp$ have a proper end extension satisfying $I\Delta_0$?

Among attempts made towards solving Problem 4, the one deserving special mention is due to Z. Adamowicz, who extended in [1] Theorem 12 above, by proving that every model of $B\Sigma_1 + exp$ that is *cofinal with ω* has a proper end extension satisfying $I\Delta_0$.

In connection with Problem 4, one can ask the following, apparently more difficult, question.

Problem 5 Does every countable model of $B\Sigma_1 + \Omega_1$ have a proper end extension satisfying $I\Delta_0$?

Here, as usual, Ω_1 denotes the axiom expressing that the function $x^{|x|}$ is total, where $|x|$ denotes the length of the logarithmic expansion of x – recall that the strength of Ω_1 was studied extensively in [17].

Let us end this section by noting that a very interesting study of Problem 5 was carried out by Z. Adamowic in [2]. One of the main results of this specific paper states that, under an assumption weaker than $I\Delta_0 \vdash \neg\Delta_0 H$, there exists a model of $B\Sigma_1 + \Omega_1$ which does not have a proper end extension satisfying $I\Delta_0$.

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