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Construction with opposition: cardinal invariants and games

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Abstract

We consider several *game* versions of the cardinal invariants $\mathfrak{t}$, $\mathfrak{u}$ and $\mathfrak{a}$. We show that the standard proof that parametrized diamond principles prove that the cardinal invariants are small actually shows that their game counterparts are small. On the other hand we show that $\mathfrak{t} < \mathfrak{t}_{Builder}$ and $\mathfrak{u} < \mathfrak{u}_{Builder}$ are both relatively consistent with ZFC, where $\mathfrak{t}_{Builder}$ and $\mathfrak{u}_{Builder}$ are the principal game versions of $\mathfrak{t}$ and $\mathfrak{u}$, respectively. The corresponding question for $\mathfrak{a}$ remains open.

Keywords Cardinal invariants of the continuum · Transfinite games · Parametrized diamond principles

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1 Introduction

The main purpose of this paper is to propose a measure of robustness of transfinite constructions. The general question is whether a transfinite recursive construction of an object A with a property φ can survive outside interference. This is formulated in terms of a transfinite game where two players, the *Builder* and the *Spoiler*, take turns in constructing the object A. The Builder tries to make sure the resulting object has property φ and the Spoiler wins if the resulting object does not satisfy the property φ. The construction envisioned by the Builder is *robust* if it produces a winning strategy in the game.

Even though the natural scope of such research is much wider, we have restricted ourselves to the case of cardinal invariants of the continuum, and constructions of length ω_1. For the vast majority of cardinal invariants such considerations are moot as the invariants are *super-robust* in the sense that the existence of a winning strategy for the Builder is equivalent to the cardinal invariant in question being $\aleph_1$. The winning strategy for the Builder would be described by simply taking a witness and playing its elements one by one independently of the moves of the Spoiler. This is the case for instance of all *Borel* cardinal invariants in the sense of [15]. There are, however, a few cardinal invariants *with structure* for which such a simplistic strategy fails, e.g. the *almost disjointness number* $\mathfrak{a}$, the *tower number* $\mathfrak{t}$, and the *ultrafilter number* $\mathfrak{u}$. In these games the Builder and the Spoiler agree that they construct an almost disjoint family (resp. decreasing chain) of infinite subsets of ω of size (length) ω_1, and hence cannot ignore each other's moves, the distinguishing property φ being maximality for $\mathfrak{a}$ and $\mathfrak{t}$, and being a *reaping*[1] family for $\mathfrak{u}$.

The starting point for our investigations is the observation that recursive constructions of length ω_1 produced by the (parametrized) $\diamondsuit$ principles tend to be robust in this sense.

We first review briefly the genesis of the relevant $\diamondsuit$-like principles. *Jensen's Diamond* principle $\diamondsuit$ [11] holds if there is a sequence of functions $\langle f_\alpha : \alpha < \omega_1 \rangle$ such that $f_\alpha \in 2^\alpha$ for every $\alpha \in \omega_1$, and such that for every $f \in 2^{\omega_1}$, the set

$$\{\alpha < \omega_1 : f \restriction_\alpha = f_\alpha\}$$

is stationary.

Devlin and Shelah's *weak diamond* principle Φ (see [8]) asserts that for every $F : 2^{<\omega_1} \to 2$, there is $g : \omega_1 \to 2$ such that for every $f : \omega_1 \to 2$, the set

$$\{\alpha < \omega_1 : F(f \restriction_\alpha) \neq g(\alpha)\}$$

is stationary.

Devlin and Shelah showed that Φ is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$, and suffices for some of the weak consequences of $\diamondsuit$. On the other hand,

[1] Recall that a family $\mathcal{R} \subseteq [\omega]^\omega$ is *reaping* if for every $A \subseteq \omega$ there is an $R \in \mathcal{R}$ such that $R \subseteq A$ or $A \cap R = \emptyset$.

Proposition 1.1 (folklore) *If $\diamondsuit$ holds and $R \subseteq A \times B$ is a relation with* $\operatorname{dom}(R) = A$*, then for every $F : 2^{<\omega_1} \to A$, there is a function $g : \omega_1 \to B$ such that for every $f \in 2^{\omega_1}$, the set*

$$\{\alpha < \omega_1 : F(f\restriction_\alpha) R g(\alpha)\}$$

is stationary.

Proof Let $\langle f_\alpha : \alpha < \omega_1\rangle$ be a diamond sequence. For $F : 2^{<\omega_1} \to A$, let $g(\alpha)$ be any $b \in B$ such that $F(f_\alpha) R b$. This is the desired g. □

Following [15], we say that a triple (A, B, R) is an *invariant* if

1. A and B are sets of cardinality at most $\mathfrak{c}$,
2. $R \subseteq A \times B$,
3. for every $a \in A$, there is $b \in B$ such that $(a, b) \in R$,
4. for every $b \in B$, there is $a \in A$ such that $(a, b) \notin R$,

and its *evaluation* $\langle A, B, R\rangle$ is given by

$$\langle A, B, R\rangle = \min\{|X| : X \subseteq B \text{ and } \forall a \in A \exists b \in X (a R b)\}.$$

Finally, an invariant (A, B, R) is *Borel* if A, B and R are Borel subsets of some Polish spaces. Given a Borel subset A of some Polish space, a map $F : 2^{<\omega_1} \to A$ is *Borel* if for every $\delta < \omega_1$, the restriction of F to 2^δ is a Borel function.

Definition 1.1 [15] Let (A, B, R) a Borel invariant. $\diamondsuit(A, B, R)$ denotes the statement: for every Borel map $F : 2^{<\omega_1} \to A$, there is $g : \omega_1 \to B$ such that for every $f : \omega_1 \to 2$, the set

$$\{\alpha \in \omega_1 : F(f\restriction_\alpha) R g(\alpha)\}$$

is stationary.

Note that $\diamondsuit$ is equivalent to $\diamondsuit(2^\omega, 2^\omega, =)$. The main point for introducing these principles is that for many standard cardinal invariants of the continuum $\mathfrak{j}$, there are Borel invariants (A, B, R) such that $\mathfrak{j} = \langle A, B, R\rangle$, and the use of $\diamondsuit$ can be measured by the *parametrized $\diamondsuit$-principles* much in the same way as the use of CH can be measured by the cardinal invariants of the continuum. When a cardinal invariant has a natural representation as an evaluation of a Borel invariant, we abuse the notation and identify the invariant with its evaluation. In particular,

- the *unbounding number* $\mathfrak{b} = \langle\omega^\omega, \omega^\omega, \not\geq^*\rangle$, where $f \geq^* g$ if $\{n \in \omega : f(n) < g(n)\}$ is finite, and
- the *reaping number* $\mathfrak{r} = \langle[\omega]^\omega, [\omega]^\omega, \mathbf{R}\rangle$, where $A\mathbf{R}B$ if $B \subseteq^* A$ or $A \cap B =^* \emptyset$[2].

In general, we write $\diamondsuit(A, R)$ instead of $\diamondsuit(A, A, R)$ and, in particular, $\diamondsuit(2, \neq)$ instead of $\diamondsuit(2, 2, \neq)$.

[2] Here $B \subseteq^* A$ means that $B \setminus A$ is finite, and $A \cap B =^* \emptyset$ says that $A \cap B$ is finite.

A sequence $\langle X_\alpha : \alpha < \delta\rangle$ of infinite subsets of ω is a *tower* if

1. $X_\alpha \subseteq^* X_\beta$ for all $\beta < \alpha < \delta$, and
2. for every $X \in [\omega]^\omega$ there is $\alpha < \delta$ such that $X \not\subseteq^* X_\alpha$.

A family $\{A_\alpha : \alpha < \delta\}$ of infinite subsets of ω is a *maximal almost disjoint (MAD) family* if

1. $A_\alpha \cap A_\beta$ is finite for all $\beta < \alpha < \delta$, and
2. for every $X \in [\omega]^\omega$ there is $\alpha < \delta$ such that $X \cap A_\alpha$ is infinite.

The first condition in both definitions defines the *structure* we mention above, while the second condition is the requirement of maximality. We denote by $\mathfrak{a}$ the minimal size of an infinite MAD family, and by $\mathfrak{t}$ the minimal length of a tower. Finally $\mathfrak{u}$ denotes the minimal character of a non-principal ultrafilter on ω. For more on cardinal invariants of the continuum see e.g. [5].

It is well known (see [15]) that:

- Assuming $\diamondsuit(2, \neq)$ there is a tower of length ω_1, i.e. $\mathfrak{t} = \omega_1$.
- Assuming $\diamondsuit(\mathfrak{b})$ there is a MAD family of size ω_1, i.e. $\mathfrak{a} = \omega_1$.
- Assuming $\diamondsuit(\mathfrak{r})$ there is an ω_1-generated ultrafilter, i.e. $\mathfrak{u} = \omega_1$.

We have already mentioned that these and similar constructions are robust in the above mentioned sense—the existence of a winning strategy for the Builder in the corresponding game, as we shall see in what follows. Then we shall consider the question of whether the cardinal invariant being ω_1 is sufficient for the existence of a winning strategy for the Builder.

We fix the following notation for the rest of the paper: Given an infinite countable ordinal δ, we fix a bijection $e_\delta : \omega \to \delta$. We denote by $\mathrm{pair}(\omega_1)$ the countable ordinals of the form $\beta + 2k$, with β limit and $k \in \omega$, and let $\mathrm{odd}(\omega_1) = \omega_1 \setminus \mathrm{pair}(\omega_1)$.

Let us make a simple yet important remark concerning the parametrized $\diamondsuit$-principles here: The definition of the function $F : 2^{<\omega_1} \to A$ almost always requires some simple coding. It has to do with the domain of the function. We shall say that the domain consists of pairs (s, X) where X is a subset of ω (usually infinite) and s is a sequence of subsets of ω of length some countable ordinal α with some structure (e.g. consisting of infinite sets which are almost disjoint or $\subseteq^*$-decreasing) which constitute an approximation to an object we want to construct. The coding can be described as follows: Given such a pair (X, s), where $s = \langle s_\xi : \xi < \alpha\rangle$ let $\sigma_{(s,X)} : \omega \cdot (1 + \alpha) \to 2$ defined by

$$\sigma_{(s,X)}(n) = 1 \text{ if and only if } n \in X, \text{ and}$$
$$\sigma_{(s,X)}(\omega \cdot (1 + \xi) + n) = 1 \text{ if and only if } n \in s_\xi.$$

For any given $\alpha < \omega_1$, the set of such $\sigma_{(s,X)}$'s is easily seen to be Borel, and as the values of F outside of this Borel set are irrelevant for our constructions, we can let F outside this set be constant. As guessing happens on a stationary set, we can also ignore the value of F at heights which are not irreducible (i.e. not of the form $\alpha = \omega \cdot \alpha$).

2 The tower number game

Consider the *tower game* $G_{\mathfrak{t}}$ of length ω_1 played as follows: Players Builder and Spoiler take turns playing a $\subseteq^*$-decreasing transfinite sequence $\langle Y_\alpha : \alpha < \omega_1 \rangle$ of infinite sets of ω, the Builder playing at even stages pair(ω_1), and the Spoiler playing at odd stages odd(ω_1).

Builder	Y_0		$\cdots$	Y_α		$\cdots$
Spoiler		Y_1	$\cdots$		$Y_{\alpha+1}$	$\cdots$

The Builder wins the match if $\langle Y_\alpha : \alpha < \omega_1 \rangle$ is a tower; otherwise, the Spoiler wins.

The first instance of the phenomenon discussed in the introduction is the following:

Proposition 2.1 *Assuming $\diamondsuit(2, \neq)$, the Builder has a winning strategy in the game $G_{\mathfrak{t}}$.*

Proof Given an infinite $\subseteq^*$-decreasing sequence $s = \{Y^s_\xi : \xi < \delta(s)\}$ with $\delta(s)$ limit, we will define a strictly increasing sequence $\{l^s_i : i \in \omega\}$ of natural numbers. Fix an increasing sequence $\{\delta_i : i \in \omega\} \subseteq \delta(s)$ converging to $\delta(s)$. Let

$$l^s_0 = \min\left(Y^s_{\delta_o}\right),$$

and

$$l^s_{i+1} = \min\left(\bigcap_{j \le i+1} Y^s_{\delta_j} \setminus (l^s_i + 1)\right).$$

We will define $F : 2^{<\omega_1} \to 2$ using the above described coding. For a decreasing $\subseteq^*$-sequence $s = \{Y^s_\xi : \xi < \delta(s)\}$ of length an infinite limit ordinal and $X \subseteq \omega$ infinite, define $F(s, X)$ as follows:

$$F(s, X) = \begin{cases} 0 & \text{if } X \subseteq^* \left\{l^s_{2i} : i \in \omega\right\}, \\ 1 & \text{otherwise.} \end{cases}$$

As the function is defined only using countable intersections and complements using only the fixed sequence $\{\delta_i : i \in \omega\} \subseteq \delta(s)$ as a parameter, and since its domain is Borel, it is Borel.

Let $g : \omega_1 \to 2$ be a $\diamondsuit(2, \neq)$-sequence for F. We are going to use g to define a winning strategy for the Builder.

Suppose $s = \{Y^s_\xi : \xi < \delta(s)\}$ is a partial play of the game with $\delta(s)$ an infinite limit ordinal. The Builder is going to choose $Y_{\delta(s)}$ as follows:

$$Y_{\delta(s)} = \begin{cases} \{l^s_{2i} : i \in \omega\} & \text{if } g(\delta(s)) = 0, \\ \{l^s_{2i+1} : i \in \omega\} & \text{otherwise.} \end{cases}$$

Let $s = \{Y^s_\xi : \xi < \omega_1\}$ be a complete match played by the Builder according to the strategy described above. Let $X \subseteq \omega$. Then if δ is an infinite limit ordinal such that $F(s\restriction_\delta, X) \neq g(\delta)$, it is straightforward to see that $X \not\subseteq^* Y_\delta = Y^s_\delta$ (note that $\delta(s\restriction_\delta) = \delta$). □

The previous Proposition has non-trivial content as we shall see next.

Theorem 2.1 *It is consistent with* ZFC *that* $\mathfrak{t} = \omega_1$ *and the Builder does not have a winning strategy in* $G_\mathfrak{t}$.

Before embarking on the proof, let us do some preparation.

Let $\mathcal{F}$ be a filter on ω. The *Laver-Prikry* forcing associated with $\mathcal{F}$, denoted by $\mathbb{L}_\mathcal{F}$ consists of subtrees $T \subseteq \omega^{<\omega}$ which have a stem $\sigma \in T$, denoted by $\text{stem}(T)$, such that for every $\tau \in T$, either $\tau \subseteq \sigma$ or $\tau \supseteq \sigma$. Besides, for every $\tau \in T$ extending σ, the set $\{n \in \omega : \tau^\frown\langle n\rangle \in T\}$ belongs to $\mathcal{F}$. The order on $\mathbb{L}_\mathcal{F}$ is given by inclusion.

Assume CH. Let $\mathcal{Y} = (Y_\alpha : \alpha < \omega_1)$ be a tower. Let $(f_\alpha : \alpha < \omega_1)$ list all partial functions from $\omega \to \omega$ with infinite range. Construct $(A_\alpha : \alpha < \omega_1)$ and $(B_\alpha : \alpha < \omega_1)$ so that for all $\alpha < \omega_1$,

- $A_\alpha \subseteq^* B_\alpha \subseteq^* A_\beta$ for $\beta < \alpha$,
- B_α is chosen according to a given rule (= a strategy that we want to "kill"), and [3]
- if $\text{ran}(f_\alpha\restriction_{B_\alpha})$ is infinite, then $\text{ran}(f_\alpha\restriction_{A_\alpha})$ is almost disjoint from some Y_{β_α}.

To choose A_α note that there is $\beta_\alpha < \omega_1$ such that $\text{ran}(f_\alpha\restriction_{B_\alpha})\setminus Y_{\beta_\alpha}$ is infinite because $\mathcal{Y}$ is a tower. Now let $A_\alpha = B_\alpha \cap f_\alpha^{-1}\left(\text{ran}\left(f_\alpha\restriction_{B_\alpha}\right)\setminus Y_{\beta_\alpha}\right)$. This is as required. Let $\mathcal{F}$ be the filter generated by the sequence $\{A_\alpha : \alpha < \omega_1\}$. Consider Laver forcing $\mathbb{L}_\mathcal{F}$ with $\mathcal{F}$.

The following lemma is based on ideas of Baumgartner and Dordal [4]. A similar argument appears elsewhere, see e.g. [6, Lemma 41].

Lemma 2.1 $\mathbb{L}_\mathcal{F}$ *preserves* $\mathcal{Y}$ *as a tower.*

Proof Let $\dot{X}$ be a name for an infinite subset of ω. Without loss of generality, we may assume its increasing enumeration (also denoted by $\dot{X}$) dominates the generic Laver real. Fix $n \in \omega$. Say that $\sigma \in \omega^{<\omega}$ *favours* $\dot{X}(n) = k$ if given any $T \in \mathbb{L}_\mathcal{F}$ with $\text{stem}(T) = \sigma$, there is $S \leq T$ such that $S \Vdash \dot{X}(n) = k$ (alternatively, no $T \in \mathbb{L}_\mathcal{F}$ with $\text{stem}(T) = \sigma$ forces $\dot{X}(n) \neq k$). Note that if σ favours $\dot{X}(n) = k$, then $|\sigma| > n$. Define the rank rk_n by recursion as follows:

- $\text{rk}_n(\sigma) = 0$ if σ favours $\dot{X}(n) = k$ for some $k \in \omega$,
- for $\alpha > 0$, $\text{rk}_n(\sigma) = \alpha$ if $\neg(\text{rk}_n(\sigma) < \alpha)$ and $\{i : \text{rk}_n(\sigma^\frown i) < \alpha\} \in \mathcal{F}^+$.

Claim 2.1 *For all* σ *and* n, $\text{rk}_n(\sigma)$ *is defined.*

Proof Suppose $\text{rk}_n(\sigma)$ is undefined. Build a tree $T \in \mathbb{L}_\mathcal{F}$ with $\text{stem}(T) = \sigma$ such that $\text{rk}_n(\tau)$ is undefined for all $\tau \in T$ with $\tau \supseteq \sigma$. Let $S \leq T$ be such that S decides $\dot{X}(n)$, say $S \Vdash \dot{X}(n) = k$. Let $\tau = \text{stem}(S)$. Then $\text{rk}_n(\tau) = 0$ because τ favours $\dot{X}(n) = k$, a contradiction. □

[3] What the given rule is, is irrelevant at this point, and will be specified in the forthcoming text.

Fix a pair n, σ such that $\mathrm{rk}_n(\sigma) = 1$. So σ does not favour $\dot{X}(n) = k$ for any k but $\{i : \sigma^\frown i$ favours $\dot{X}(n) = k$ for some $k\}$ belongs to $\mathcal{F}^+$. Define a partial function $f : \omega \to \omega$ as follows: $\mathrm{dom}(f) = \{i : \sigma^\frown i$ favours $\dot{X}(n) = k$ for some $k\}$ and, for $i \in \mathrm{dom}(f)$, let $f(i)$ be some k such that $\sigma^\frown i$ favours $\dot{X}(n) = k$. Note that since $\mathrm{rk}_n(\sigma) \neq 0$, $f^{-1}(\{k\}) \notin \mathcal{F}^+$ for all $k \in \omega$. There is $\alpha = \alpha(n, \sigma)$ such that $f = f_\alpha$. Let β be larger than all the $\beta_{\alpha(n,\sigma)}$.

Claim 2.2 $\Vdash \dot{X} \not\subseteq^* Y_\beta$.

Proof Fix $m \in \omega$ and $T \in \mathbb{L}_{\mathcal{F}}$. It suffices to find $k > m$, $k \notin Y_\beta$, and $S \leq T$ such that $S \Vdash k \in \dot{X}$. Let $\sigma = \mathrm{stem}(T)$ and $n > \max\{m, |\sigma|\}$. In particular, $\mathrm{rk}_n(\sigma) > 0$. By extending σ if necessary, we may assume $\mathrm{rk}_n(\sigma) = 1$. By construction, there is $F \in \mathcal{F}$ such that $\mathrm{ran}\left(f_{\alpha(n,\sigma)}\restriction_F\right)$ is almost disjoint from Y_β. Since $f^{-1}_{\alpha(n,\sigma)}(\{k\}) \notin \mathcal{F}^+$ for all $k \in \omega$ and $\mathrm{dom}\left(f_{\alpha(n,\sigma)}\right) \cap F \cap \mathrm{succ}_T(\sigma) \in \mathcal{F}^+$, we may find $i \in \mathrm{dom}\left(f_{\alpha(n,\sigma)}\right) \cap F \cap \mathrm{succ}_T(\sigma)$ and $k \in \omega$ such that $f_{\alpha(n,\sigma)}(i) = k$ and $k \notin Y_\beta$. Hence $\sigma^\frown i$ favours $\dot{X}(n) = k$, and there is $S \leq T$ with $\mathrm{stem}(S) \supseteq \sigma^\frown i$ such that $S \Vdash \dot{X}(n) = k$. Clearly $k \geq n > m$, and we are done. □

This finishes the proof of the lemma. □

Recall yet another version of $\diamondsuit$. For a given uncountable regular cardinal κ and a stationary set $E \subseteq \kappa$, we say that the principle $\diamondsuit_E$ holds if there is a sequence $\langle d_\gamma : \gamma \in E\rangle$ such that for every $X \subseteq \kappa$, the set $\{\gamma \in E : X \cap \gamma = d_\gamma\}$ is stationary. Now we are ready to prove the theorem:

Proof of Theorem 2.1 Assume $\diamondsuit_{E^{\omega_2}_{\omega_1}}$ and CH. Fix a tower $\mathcal{Y} = (Y_\alpha : \alpha < \omega_1)$ as above. Construct a finite support iteration $\left(\mathbb{P}_\gamma, \dot{\mathbb{Q}}_\gamma : \gamma < \omega_2\right)$, where the initial segments of the iteration have size at most $\aleph_1$. Use the diamond to guess (initial segments of) names of strategies for the Builder. This is a standard argument which has been used a lot, e.g. in [16]; for the reader's convenience we present an outline, following roughly the one in [7, proof of Theorem 8].

Think of the diamond sequence as acting on the product $\omega_2 \times \mathbb{P}_{\omega_2}$, that is, there is a sequence $\langle S_\gamma \subseteq \gamma \times \mathbb{P}_\gamma : \gamma \in E^{\omega_2}_{\omega_1}\rangle$ such that for all $T \subseteq \omega_2 \times \mathbb{P}_{\omega_2}$, the set $\{\gamma \in E^{\omega_2}_{\omega_1} : T \cap (\gamma \times \mathbb{P}_\gamma) = S_\gamma\}$ is stationary. This can be done, because the initial segments $\mathbb{P}_\gamma$ of the iteration have size ω_1, by building up a bijection $F : \omega_2 \to \mathbb{P}_{\omega_2}$ recursively along the definition of the iteration, such that $F\restriction_\gamma : \gamma \to \gamma \times \mathbb{P}_\gamma$ is a bijection for all $\gamma \in E^{\omega_2}_{\omega_1}$. Next, again along the definition of the iteration, we produce a name $\dot{f} \in V^{\mathbb{P}_{\omega_2}}$ for a bijection between ω_2 and $([\omega]^\omega)^{<\omega_1} \times [\omega]^\omega$ such that for all $\gamma \in E^{\omega_2}_{\omega_1}$, $\dot{f}\restriction_\gamma \in V^{\mathbb{P}_\gamma}$ is a name for a bijection between γ and $(([\omega]^\omega)^{<\omega_1} \times [\omega]^\omega) \cap V^{\mathbb{P}_\gamma}$. (Note that a strategy $\Sigma : ([\omega]^\omega)^{<\omega_1} \to [\omega]^\omega$ is a subset of $([\omega]^\omega)^{<\omega_1} \times [\omega]^\omega$.)

At stage $\gamma \in E^{\omega_2}_{\omega_1}$ such that S_γ is a $\mathbb{P}_\gamma$-name for a subset of γ and $\dot{f}[S_\gamma] \in V^{\mathbb{P}_\gamma}$ is a strategy for Builder, we force with $\dot{\mathbb{Q}}_\gamma = \mathbb{L}_{\dot{\mathcal{F}}}$ where $\dot{\mathcal{F}}$ is constructed from $\dot{A}_\alpha$ and $\dot{B}_\alpha$ as above and the $\dot{B}_\alpha$ are obtained from the $\dot{A}_\beta$, $\dot{B}_\beta$, $\beta < \alpha$, using the strategy $\dot{f}[S_\gamma]$. In all other cases we let $\dot{\mathbb{Q}}_\gamma$ be Cohen forcing. Force with $\mathbb{P}_{\omega_2}$.

Since towers are preserved in limit steps of finite support iterations (see e.g. [3,4,9]), the lemma implies that $\mathcal{Y}$ is still a tower in $V^{\mathbb{P}_{\omega_2}}$. In particular $\mathfrak{t} = \omega_1$.

On the other hand, for each strategy $\Sigma = \dot{f}[T]$ of the Builder in $V^{\mathbb{P}_{\omega_2}}$, where T is a $\mathbb{P}_{\omega_2}$-name for a subset of ω_2, there is $\gamma \in E^{\omega_2}_{\omega_1}$ such that $\Sigma\restriction_{V^{\mathbb{P}_\gamma}} = \dot{f}[T \cap (\gamma \times \mathbb{P}_\gamma)] = \dot{f}[S_\gamma]$ is a strategy in $V^{\mathbb{P}_\gamma}$ and was used to construct the filter $\dot{\mathcal{F}}$. Hence there is a game according to Σ which the Builder looses, as witnessed by the $\mathbb{L}_{\dot{\mathcal{F}}}$-generic set added in $V^{\mathbb{P}_{\gamma+1}}$. □

Consider the longer version of the tower game $G^\delta_{\mathfrak{t}}$ of length δ played as follows: Players Builder and Spoiler take turns playing a $\subseteq^*$-decreasing transfinite sequence $\langle Y_\alpha : \alpha < \delta\rangle$ of infinite subsets of ω, the Builder playing at even stages pair(δ), and the Spoiler playing at odd stages odd(δ).

Builder	Y_0		$\cdots$	Y_α		$\cdots$
Spoiler		Y_1	$\cdots$		$Y_{\alpha+1}$	$\cdots$

The Spoiler wins the match if $\langle Y_\alpha : \alpha < \delta\rangle$ is not a tower; otherwise, the Builder wins.

Given the previous theorem, it is natural to define $\mathfrak{t}_{Builder}$ as the least ordinal δ such that the Builder has a winning strategy in the game $G^\delta_{\mathfrak{t}}$. The previous result then says $\mathfrak{t} < \mathfrak{t}_{Builder}$ is consistent. The following is a special case of a result of Vojtáš [17, Theorem 7]; we include the short proof for the sake of completeness.

Lemma 2.2 $\mathfrak{t}_{Builder}$ *is a regular cardinal.*

Proof Let α be minimal such that the Builder has a strategy Σ that makes her win in at most α moves. Let $\{\gamma_\xi : \xi < cf(\alpha)\}$ be a club subset of α such that for even ξ, γ_ξ is also even and $\gamma_{\xi+1} = \gamma_\xi + 1$. We construct a strategy Σ' for the Builder that makes her win in at most $cf(\alpha)$ steps such that for each run $\bar{A} = \{A_\eta : \eta < \xi\}$ according to Σ' of length an even ξ, there is a run $\bar{B} = \{B_\gamma : \gamma < \delta_\xi\}$ according to Σ of length δ_ξ such that $B_{\gamma_\eta} = A_\eta$ for all $\eta < \xi$ and

$$\delta_\xi = \begin{cases} \gamma_\xi & \text{if } \xi \text{ is limit} \\ \gamma_\zeta + 1 & \text{if } \xi = \zeta + 1 \text{ is even successor.} \end{cases}$$

Suppose we are at step ξ. If ξ is a limit ordinal, then either $\bar{A}$ has no pseudointersection and the Builder already won or, since $\bar{A}$ is a cofinal subsequence of $\bar{B}$, we can let $\Sigma'(\bar{A})$ be $\Sigma(\bar{B})$. If ξ is an even successor, say $\zeta + 1$, consider the corresponding game $\bar{B}$ whose final move is $B_{\gamma_\zeta} = A_\zeta$. Notice that since there is no strategy which makes the Builder win in less than α steps, Σ cannot make the Builder win below the set A_ζ in less than α steps. In particular, there must be a game according to Σ and extending $\bar{B}$ which still has a move, with the Builder following Σ, at stage γ_ξ. Let $\bar{B}'$ be this extension of length γ_ξ and let $\Sigma'(\bar{A})$ be this move $\Sigma(\bar{B}')$.

This describes the strategy Σ'. It is clear that the Builder must win after at most $cf(\alpha)$ steps. □

We may also define $\mathfrak{t}_{Spoiler}$ as the supremum of all ordinals δ such that the Spoiler has a winning strategy in the game $G^\delta_{\mathfrak{t}}$. It is easy to see that the Spoiler has no winning strategy in $G^\delta_{\mathfrak{t}}$ for $\delta = \mathfrak{t}_{Spoiler}$ moves (for otherwise the game could be continued one further move and would still be winning for the Spoiler). Hence, $\mathfrak{t}_{Spoiler}$ can be characterized as the least δ such that the Spoiler has no winning strategy in the game $G^{\delta}_{\mathfrak{t}}$. Again we see (this is a special case of [17, Theorem 6]):

Lemma 2.3 *$\mathfrak{t}_{Spoiler}$ is a regular cardinal.*

Proof Suppose $\mathfrak{t}_{Spoiler} = \alpha$ is minimal such that no strategy of the Spoiler of the game with α moves is winning. Let Σ be a strategy of the Spoiler of the game with $cf(\alpha)$ moves. We need to see that Σ is not winning. As in the previous proof, let $\{\gamma_\xi : \xi < cf(\alpha)\}$ be a club subset of α such that for even ξ, γ_ξ is also even and $\gamma_{\xi+1} = \gamma_\xi + 1$. We shall build a strategy Σ' of the Spoiler with α moves such that for every run $\bar{B} = \{B_\gamma : \gamma < \alpha\}$ according to Σ' there is a run $\bar{A} = \{A_\eta : \eta < cf(\alpha)\}$ according to Σ such that $A_\eta = B_{\gamma_\eta}$. Since Σ' is not winning, one such run $\bar{B}$ is won by the Builder. But then the Builder also wins the corresponding run $\bar{A}$ according to Σ, as required.

As in the previous proof, let

$$\delta_\xi = \begin{cases} \gamma_\xi & \text{if } \xi \text{ is limit} \\ \gamma_\zeta + 1 & \text{if } \xi = \zeta + 1 \text{ is even successor} \end{cases}$$

for even ξ.

Now suppose ξ is even and Σ' has been constructed for a run $\bar{B} = \{B_\gamma : \gamma < \delta_\xi\}$. Let $\bar{A} = \{A_\eta : \eta < \xi\}$ be the corresponding run according to Σ. If ξ is limit, consider the move B_{γ_ξ} of the Builder. Let $A_\xi = B_{\gamma_\xi}$ be the corresponding move of the Builder in the other game. Then let $B_{\gamma_{\xi+1}} = \Sigma'(\bar{B} \cup \{B_{\gamma_\xi}\}) = \Sigma(\bar{A} \cup \{A_\xi\}) = A_{\xi+1}$, that is, the Spoiler plays in Σ' what Σ tells her to play in the other game. If $\xi = \zeta + 1$ is successor, the last move of the Spoiler was B_{γ_ζ}. Note that $\delta_\xi \leq \gamma_\xi$ are both even ordinals. So, let ϵ_ξ be such that $\delta_\xi + \epsilon_\xi = \gamma_\xi$. Since $\epsilon_\xi < \alpha$, the Spoiler has a winning strategy of length ϵ_ξ below the set B_{γ_ζ} (i.e with B_{γ_ζ} as the first move). Let Σ' in the interval $[\delta_\xi, \gamma_\xi)$ be this strategy. Let $\bar{B}' = \{B_\gamma : \gamma < \gamma_\xi\}$ be an extension of $\bar{B}$ following this strategy. Now continue as in the limit case: let B_{γ_ξ} be the next move of the Builder (such a move exists because the strategy of the Spoiler was winning so far); let $A_\xi = B_{\gamma_\xi}$ and let $B_{\gamma_{\xi+1}} = \Sigma'(\bar{B} \cup \{B_{\gamma_\xi}\}) = \Sigma(\bar{A} \cup \{A_\xi\}) = A_{\xi+1}$. Clearly this works. □

By modifying the proof of Theorem 2.1 a little we see:

Theorem 2.2 *It is consistent that $\mathfrak{t} = \mathfrak{t}_{Spoiler} = \omega_1 < \mathfrak{t}_{Builder} = \omega_2 = \mathfrak{c}$.*

Proof We first observe:

Lemma 2.4 *Assume* CH *and let Σ be a strategy of the Builder (of length ω_1). Also assume there are towers $(\mathcal{Y}^\beta : \beta < \omega_1)$. Then there is a filter $\mathcal{F}$ containing a run of the game according to Σ such that $\mathbb{L}_{\mathcal{F}}$ preserves all $\mathcal{Y}^\beta$.*

Proof To see this simply redo the construction before Lemma 2.1 by diagonalizing against ω_1 towers instead of just one. More explicitly, let $\mathcal{Y}^\beta = (Y^\beta_\alpha : \alpha < \omega_1)$ for $\beta < \omega_1$. Let $(S^\beta : \beta < \omega_1)$ partition ω_1 into sets of size ω_1. Let $\{\gamma^\beta_\alpha : \alpha < \omega_1\}$ be the enumeration of S^β. Let $(f_\alpha : \alpha < \omega_1)$ list all partial functions from ω to ω with infinite range. Then construct $(A_\alpha : \alpha < \omega_1)$ and $(B_\alpha : \alpha < \omega_1)$ recursively such that for all $\alpha, \beta < \omega_1$,

- $A_\alpha \subseteq^* B_\alpha \subseteq^* A_{\alpha'}$ for $\alpha' < \alpha$,
- $B_\alpha = \Sigma(A_{\alpha'} : \alpha' < \alpha)$ for even α, B_α is arbitrary for odd α, and
- if $\operatorname{ran}(f_\alpha \restriction_{B_{\gamma_\alpha^\beta}})$ is infinite, the $\operatorname{ran}(f_\alpha \restriction_{A_{\gamma_\alpha^\beta}})$ is almost disjoint from some $Y^\beta_{\beta_\alpha}$.

Clearly this can be done. Let $\mathcal{F}$ be the filter generated by the sequence $\{A_\alpha : \alpha < \omega_1\}$. Fix $\beta < \omega_1$. Preservation of $\mathcal{Y}^\beta$ by $\mathbb{L}_\mathcal{F}$ is immediate by Lemma 2.1. □

Now, as in the proof of Theorem 2.1, assume $\diamondsuit_{E^{\omega_2}_{\omega_1}}$ and CH. Use the diamond to guess (initial segments of) names of strategies for both the Builder and the Spoiler. Simultaneously construct a finite support iteration $\left(\mathbb{P}_\gamma, \dot{\mathbb{Q}}_\gamma : \gamma < \omega_2\right)$ and a sequence of (names of) towers $\left(\dot{\mathcal{Y}}^\beta : \beta < \omega_2\right)$ such that $\left(\dot{\mathcal{Y}}^\beta : \beta \leq \gamma\right) \in V^{\mathbb{P}_\gamma}$. At stage γ first consider the (name of the) strategy of the Spoiler handed down by $\diamondsuit_{E^{\omega_2}_{\omega_1}}$. Since CH still holds while there are 2^{ω_1} many games following the strategy, one of these games must be winning for the Builder, that is, there is a tower $\dot{\mathcal{Y}}^\gamma \in V^{\mathbb{P}_\gamma}$ that is a run according to the strategy. Now, as in the proof of Theorem 2.1, use the lemma to get a filter $\dot{\mathcal{F}}$ containing a run of the game according to the (name of the) Builder's strategy handed down by $\diamondsuit_{E^{\omega_2}_{\omega_1}}$ such that $\dot{\mathbb{Q}}_\gamma = \mathbb{L}_{\dot{\mathcal{F}}}$ preserves all $\dot{\mathcal{Y}}^\beta$, $\beta \leq \gamma$.

By the argument of Theorem 2.1, a strategy of the Builder of length ω_1 cannot be winning. Similarly, if Σ is a strategy of the Spoiler of length ω_1, there is $\gamma < \omega_2$ such that $\Sigma \restriction_{V^{\mathbb{P}_\gamma}}$ is a strategy in $V^{\mathbb{P}_\gamma}$ and was guessed by the diamond. This means that the tower $\mathcal{Y}^\gamma$ is preserved as a run of the game according to Σ which is won by the Builder. □

However we do not know:

Open question 2.1 *Is $\mathfrak{t} < \mathfrak{t}_{Spoiler}$ consistent?*

On the other hand, $\mathfrak{t}_{Builder} \leq \mathfrak{h}$, where $\mathfrak{h} = \min\{\text{height}(\mathcal{T}) : \mathcal{T} \subseteq \left([\omega]^\omega, {}^*\supseteq\right)$ is a base tree$\}$[4] is the *distributivity number* of $\mathcal{P}(\omega)/\text{fin}$.[5] To see this note that the Builder can simply make sure to play along a branch of the base tree $\mathcal{T}$ which, of course, produces a winning strategy. In particular, $\mathfrak{h} = \omega_1$ is sufficient for the existence of a winning strategy for the Builder in the game $G_\mathfrak{t}$ (of length ω_1).

This proof actually gives a little more. Note that in general the Builder has a distinct advantage over the Spoiler in that her moves appear on a closed unbounded subset

[4] A *base tree* is a set $\mathcal{T} \subseteq [\omega]^\omega$ which is a tree when ordered by $\supseteq^*$ and is such that every element of $[\omega]^\omega$ contains an element of $\mathcal{T}$. The existence of such a tree was proved by Balcar, Pelant and Simon in [2], see also [1].

[5] Remember that $\langle [\omega]^\omega, \subseteq^* \rangle$ is a preorder. Therefore, the set of its classes of equivalence, $\mathcal{P}(\omega)/\text{fin}$, defined by $X \equiv_{\text{fin}} Y$ if and only if $X \subseteq^* Y$ and $Y \subseteq^* X$, defines a partial order $\langle \mathcal{P}(\omega)/\text{fin}, \leq_{\text{fin}} \rangle$, where $[X]_{\text{fin}} \leq_{\text{fin}} [Y]_{\text{fin}}$ if and only if $X \subseteq^* Y$. Given a partial order $\langle P, \leq \rangle$, we say that a set $D \subseteq P$ is *dense* if for every $p \in P$, there is $q \in D$ such that $q \leq p$. A subset set $D \subseteq P$ is *open* if whenever $p \in D$ and $q \leq p$, then $q \in D$. As usual, we refer only to P as the partial order if the order is clear from the context. For a partial order P, we define its *distributivity number* $\mathfrak{h}(P)$ as the minimum α such that for some collection $\{D_\xi : \xi < \alpha\}$ of open dense sets, its intersection $\bigcap_{\xi < \alpha} D_\xi$ is not dense.

of ω_1 (pair$(\omega_1) \in$ Club(ω_1), while odd(ω_1) is not stationary). Let $G^*_{\mathfrak{t}}$ be the game in which the players switch places, that is, the Builder plays at odd steps while the Spoiler plays at even steps. It is obvious that a winning strategy of the Builder in $G^*_{\mathfrak{t}}$ gives her a winning strategy in $G_{\mathfrak{t}}$ as well, while the implication goes the other way round for the Spoiler. Furthermore, the winning strategy described here from $\mathfrak{h} = \omega_1$ is robust in the sense that it is irrelevant in which order the players play; that is, the latter hypothesis implies a winning strategy for the Builder even in $G^*_{\mathfrak{t}}$. We shall see below (Corollary 2.2) that $\diamondsuit(2, \neq)$ is not sufficient for this.

Define $\mathfrak{t}^*_{Builder}$ and $\mathfrak{t}^*_{Spoiler}$ similarly as the unstarred versions. The four cardinal numbers $\mathfrak{t}_{Builder}$, $\mathfrak{t}_{Spoiler}$, $\mathfrak{t}^*_{Builder}$ and $\mathfrak{t}^*_{Spoiler}$ are due to Vojtáš [17] in a more general context, where he also showed they are regular cardinals [17, Theorem 6 and Theorem 7]. Also

$$\mathfrak{h} \geq \mathfrak{t}^*_{Builder} \geq \max\{\mathfrak{t}^*_{Spoiler}, \mathfrak{t}_{Builder}\} \geq \min\{\mathfrak{t}^*_{Spoiler}, \mathfrak{t}_{Builder}\} \geq \mathfrak{t}_{Spoiler} \geq \mathfrak{t}$$

is obvious. A straightforward modification of the proof of Theorem 2.2 actually shows the consistency of $\mathfrak{t}_{Builder} > \mathfrak{t}^*_{Spoiler}$. As in Question 2.1, we do not know whether $\mathfrak{t}^*_{Spoiler} > \mathfrak{t}$ is consistent.

The following lemma is a special case of a result by Foreman [10].

Lemma 2.5 $\mathfrak{t}^*_{Builder} = \mathfrak{h}$.

Proof This is immediate using [10, Theorem on page 718] and realizing that given a cardinal λ, the Builder has a winning strategy in λ steps in the game $G^*_{\mathfrak{t}}$ if and only if I has a winning strategy in the game $G^{II}_{\lambda^+}$ played in $\mathcal{P}(\omega)$/fin described in [10]. □

By the above discussion, both $\diamondsuit(2, \neq)$ and $\mathfrak{h} = \omega_1$ imply the existence of a winning strategy for the Builder in the game $G_{\mathfrak{t}}$ in ω_1 many steps. Both are consequences of CH. The two statements are independent, however: in the Mathias model, $\diamondsuit(2, \neq)$ holds [15, Theorem 6.6] and $\mathfrak{h} > \omega_1$, while in a model of Judah and Shelah [13], $\mathfrak{h} = \omega_1$ and $\diamondsuit(2, \neq)$ fails[6]. In particular we have:

Corollary 2.1 *The Builder having a winning strategy in $G_{\mathfrak{t}}$ does not imply $\diamondsuit(2, \neq)$.*

Corollary 2.2 *It is consistent that $\diamondsuit(2, \neq)$ holds and the Builder has no winning strategy in $G^*_{\mathfrak{t}}$. In particular it is consistent that $\mathfrak{t}^*_{Builder} > \mathfrak{t}_{Builder}$.*

Proof As remarked $\diamondsuit(2, \neq)$ and $\mathfrak{h} = \mathfrak{c} = \omega_2$ hold in the Mathias model. By Proposition 2.1 and Lemma 2.5, $\mathfrak{t}_{Builder} = \omega_1$ and $\mathfrak{t}^*_{Builder} = \omega_2$ follow. □

Another classical upper bound of $\mathfrak{t}$ is the *additivity* add$(\mathcal{M})$ *of the meager ideal* $\mathcal{M}$, that is, the least κ such that there is a family of κ many meager sets whose union is not meager. Since, as observed in the Introduction, cardinals like add$(\mathcal{M})$ are equal to their game versions, one might conjecture that $\mathfrak{t}_{Builder} \leq$ add$(\mathcal{M})$ holds in ZFC. However, this is not what the proof of $\mathfrak{t} \leq$ add$(\mathcal{M})$ gives for the latter uses towers of dense sets of rationals and not just of arbitrary sets of natural numbers. And, in fact, we show the following:

[6] They prove, in fact, that it is consistent there is a Q-set of reals while the null ideal has a basis of size ω_1. The latter implies $\mathfrak{h} = \omega_1$ while by Theorem 6.16 in [15], $\diamondsuit(2, \neq)$ implies there are no Q-sets.

Theorem 2.3 $\mathfrak{t}_{Builder} = \mathfrak{c} = \omega_2 > \mathrm{add}(\mathcal{M}) = \omega_1$ *is consistent.*

Before starting with the proof we review some notions and some facts. Recall that a non-principal ultrafilter $\mathcal{U}$ on ω is *Ramsey* if for every partition $\{A_n : n \in \omega\}$ of ω such that $A_n \notin \mathcal{U}$ for all $n \in \omega$, there is $X \in \mathcal{U}$ such that $X \cap A_n$ has one element for all $n \in \omega$. Say a function $\varphi : \omega \to [\omega]^{<\omega}$ is a *slalom* if $|\varphi(n)| \leq n + 1$ for all $n \in \omega$. A forcing notion $\mathbb{P}$ has the *Laver property* if given any condition $p \in \mathbb{P}$, any function $h \in \omega^\omega$ and any $\mathbb{P}$-name $\dot{f}$ for a function bounded by h, there are $q \leq p$ and a slalom φ such that $q \Vdash \forall n\ (\dot{f}(n) \in \varphi(n))$. A forcing with the Laver property does not add Cohen reals (see Lemma 7.2.3 in [3]) and thus in particular preserves the additivity of the meager ideal, that is, if $\mathrm{add}(\mathcal{M}) = \omega_1$ holds in the ground model, it still holds in the generic extension. Like standard Mathias forcing (Lemma 7.2.2 and Corollary 7.4.7 in [3]) used in the proof of the previous theorem, Mathias forcing with a Ramsey ultrafilter $\mathcal{U}$, which is forcing equivalent to Laver forcing $\mathbb{L}_{\mathcal{U}}$ with $\mathcal{U}$ (see e.g. Theorem 1.20 in [12]), has the Laver property. Furthermore, the Laver property is preserved in countable support iterations (Theorem 6.3.34 in [3]).

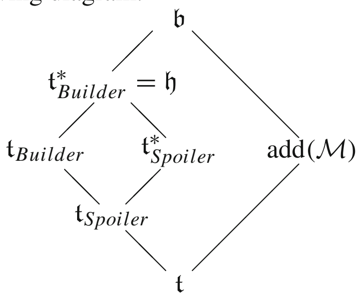
Proof of Theorem 2.3 As in the proof of Theorem 2.1 we assume $\diamondsuit_{E^{\omega_2}_{\omega_1}}$ and CH. Construct a countable support iteration $\left(\mathbb{P}_\gamma, \dot{\mathbb{Q}}_\gamma : \gamma < \omega_2\right)$. We use the diamond again to guess (initial segments of) names of strategies for the Builder. Again this can be done, exactly like in the proof of Theorem 2.1, because the initial steps $\mathbb{P}_\gamma$ of the iteration have (a dense subset of) size ω_1. At stage γ consider (the name of) Builder's strategy $\dot{\Sigma}$ handed down by $\diamondsuit_{E^{\omega_2}_{\omega_1}}$. As in the argument before Lemma 2.1, we can construct, in $V^{\mathbb{P}_\gamma}$, a run of the game according to $\dot{\Sigma}$ such that the ω_1-sequence of the sets played generates a Ramsey ultrafilter $\dot{\mathcal{U}}$. Now let $\dot{\mathbb{Q}}_\gamma = \mathbb{L}_{\dot{\mathcal{U}}}$. Force with $\mathbb{P}_{\omega_2}$.

By the discussion in the paragraph preceding the proof, the whole iteration has the Laver property, and $\mathrm{add}(\mathcal{M}) = \omega_1$ thus follows.

To see $\mathfrak{t}_{Builder} = \omega_2$, assume Σ is a strategy of Builder for a game of length ω_1. By $\diamondsuit_{E^{\omega_2}_{\omega_1}}$ there is $\gamma < \omega_2$ such that $\Sigma{\restriction}_{V^{\mathbb{P}_\gamma}}$ is a strategy and was used to construct the ultrafilter $\mathcal{U}$. Hence there is a game following Σ which the Builder looses, as witnessed by the $\mathbb{L}_{\mathcal{U}}$-generic set added in $V^{\mathbb{P}_{\gamma+1}}$. □

Note that this gives an alternative proof of Theorem 2.1. However, the original argument is more direct in that it uses less black-boxed forcing theory. Also, in Theorem 2.1, we additionally have the consistency of $\mathfrak{t} < \mathfrak{t}_{Builder} = \mathrm{add}(\mathcal{M})$.

The order relationship between the cardinals we considered in this section can be summarized in the following diagram.



3 The ultrafilter number game

Recall that a filter $\mathscr{F}$ on ω is a *P-filter* if for each countable collection $\{Y_n : n \in \omega\} \subseteq \mathscr{F}$ there is a $Y \in \mathscr{F}$ such that $Y \subseteq^* Y_n$ for every $n \in \omega$. A non-principal ultrafilter $\mathscr{F}$ on ω is called a *P-point* if it is a P-filter.

The *ultrafilter game* $G_\mathfrak{u}$ is played as before, the Builder and the Spoiler taking turns constructing a $\subseteq^*$-decreasing sequence $\langle U_\alpha : \alpha < \omega_1\rangle$ (the Builder playing at pair(ω_1)-stages, while the Spoiler plays at odd(ω_1)-stages).

Builder	U_0		$\cdots$	U_α		$\cdots$
Spoiler		U_1	$\cdots$		$U_{\alpha+1}$	$\cdots$

The difference is in how we declare a winner. The Builder now has a harder task as she wins the match if the filter generated by $\{U_\alpha : \alpha < \omega_1\}$ is an ultrafilter; otherwise, the Spoiler wins.

Again, the proof of the following result mimicks closely the proof of Theorem 7.8 in [15]. We include it for the benefit of the reader.

Proposition 3.1 $\diamondsuit(\mathfrak{r})$ *implies the Builder has a winning strategy in the game* $G_\mathfrak{u}$.

Proof For a $\subseteq^*$-decreasing infinite sequence $s = \{U^s_\xi : \xi < \delta(s)\}$, we define the strictly increasing sequence $\{k^s_i : i \in \omega\} \subseteq \bigcup_{\xi<\delta(s)} U^s_\xi$ as follows: Remember that we have fixed a bijective function $e_\delta : \omega \to \delta$ for every infinite ordinal $\delta < \omega$. Let

$$k^s_0 = \min\left(U^s_{e_{\delta(s)}(0)}\right),$$

and

$$k^s_{i+1} = \min\left(\bigcap_{j\le i+1} U^s_{e_{\delta(s)}(j)} \setminus (k^s_i + 1)\right).$$

Given $C \subseteq \omega$ and an infinite $\subseteq^*$-decreasing sequence s, we define a Borel map F as follows: $F(s, C) = \{i \in \omega : k^s_i \in C\}$ if $\{i \in \omega : k^s_i \in C\}$ is infinite, and $F(s, C) = \{i \in \omega : k^s_i \notin C\}$ otherwise.

The function F uses the coding described at the end of the introduction, and is Borel since its domain is Borel and it is defined by countable boolean operations using e_δ as a parameter.

Let g be the respective $\diamondsuit(\mathfrak{r})$-guessing function for F. We will show that g defines a winning strategy for the Builder as follows: If $s = \{U^s_\xi : \xi < \delta(s)\}$ is a partial match with $\delta(s)$ even, let $U_{\delta(s)} = \{k^s_i : i \in g(\delta(s))\}$. It is not difficult to see that any complete match $s = \{U^s_\xi : \xi < \omega_1\}$ according to the strategy defined by g is a $\subseteq^*$-decreasing sequence. It is also straightforward to show that the set $\mathscr{F}_s = \{X \in [\omega]^\omega : \exists\delta < \omega_1(U^s_\delta \subseteq^* X)\}$ is a filter. We are done if $\mathscr{F}_s$ is an ultrafilter.

Let $C \subseteq \omega$. Since g is a $\diamondsuit(\mathfrak{r})$-sequence, we can find $\delta < \omega_1$ such that either $|g(\delta) \cap F(s{\restriction}_\delta, C)| < \aleph_0$ or $|g(\delta)\setminus F(s{\restriction}_\delta, C)| < \aleph_0$.

We will show that either $U_\delta \subseteq^* C$ or $U_\delta \subseteq^* \omega\setminus C$ where $U_\delta = U^s_\delta$ (note that $\delta(s{\restriction}_\delta) = \delta$).

Case 1: $|g(\delta) \cap F(s\restriction_\delta, C)| < \aleph_0$. Let $j \in \omega$ such that $g(\delta) \cap F(s\restriction_\delta, C) \subseteq j$. Then $U_\delta \setminus k_j^{s\restriction_\delta} \subseteq C$ if $\{i \in \omega : k_i^{s\restriction_\delta} \in C\}$ is finite, and $U_\delta \setminus k_j^{s\restriction_\delta} \subseteq \omega \setminus C$ otherwise.

Case 2: $|g(\delta) \setminus F(s\restriction_\delta, C)| < \aleph_0$. Let $j \in \omega$ such that $g(\delta) \setminus j \subseteq F(s\restriction_\delta, C)$. Then $U_\delta \setminus k_j^{s\restriction_\delta} \subseteq C$ if $\{i \in \omega : k_i^{s\restriction_\delta} \in C\}$ is infinite, and $U_\delta \setminus k_j^{s\restriction_\delta} \subseteq \omega \setminus C$ otherwise. □

Note that it was enough that the set of guesses of the diamond sequence was just non-empty. It is a simple exercise left to the reader to show that

Lemma 3.1 *CH implies that the Builder has a winning strategy in $G_\mathfrak{u}$.*

In fact, the stronger statement that the Builder has a winning strategy also in the game $G_\mathfrak{u}^*$ where the Builder and the Spoiler switch places easily follows from CH. Since CH does not imply $\diamondsuit(\mathfrak{r})$ by Proposition 8.2 and Theorem 8.3 in [15], we have the following:

Corollary 3.1 *The Builder having a winning strategy in $G_\mathfrak{u}$ does not imply $\diamondsuit(\mathfrak{r})$.*

Again, we will show that all of this is not gratuitous.

Theorem 3.1 *$\mathfrak{u} = \omega_1$ does not imply that the Builder has a winning strategy in the game $G_\mathfrak{u}$.*

Rather than constructing an *ad hoc* forcing model for this, we show that this holds in a model constructed by Shelah in [16, Chapter XVIII, Section 4]. We shall review some standard facts about ultrafilters first. Given two ultrafilters $\mathcal{U}, \mathcal{V}$ on ω, we recall the *Rudin-Keisler* order $\leq_{RK}$ defined as follows: $\mathcal{U} \leq_{RK} \mathcal{V}$ if and only if there is a function $f : \omega \to \omega$ such that $\mathcal{U} = \{X \in \omega : \exists Y \in \mathcal{V}(f[Y] \subseteq X)\}$, and they are *RK-equivalent*, denoted by $\mathcal{U} \equiv_{RK} \mathcal{V}$ if such f exists which is, moreover, bijective. We recall the following fact, which shows that Ramsey ultrafilters are $\leq_{RK}$-minimal:

Fact 3.1 *Let $\mathcal{U}$ and $\mathcal{U}'$ be two ultrafilters with $\mathcal{U}$ Ramsey and $\mathcal{U}' \leq_{RK} \mathcal{U}$. Then $\mathcal{U}' \equiv_{RK} \mathcal{U}$.*

Proof of Theorem 3.1 Let $V \models \mathsf{CH} + 2^{\omega_1} = \omega_2$, let $\mathbb{P}^{\omega_2}$ be the countable support iteration used by Shelah to construct a model with a unique P-point ([16, Chapter XVIII, Theorem 4.1]), and let G be $\mathbb{P}^{\omega_2}$-generic.

We shall show that $V[G]$ is the model we need. We will be able to deduce this from the following three facts which hold there:

1. In V there is a Ramsey ultrafilter $\mathcal{U}_0$ such that $\mathcal{U}_0$ still generates a Ramsey ultrafilter in $V[G]$, and thus $V[G] \models \mathfrak{u} = \omega_1$ (see [16, Chapter XVIII, Remark 4.1A]).
2. Every P-point of $V[G]$ is RK-equivalent to $\mathcal{U}_0$.
3. The forcing $\mathbb{P}^{\omega_2}$ is ω^ω-bounding.

We shall show that in $V[G]$, the Builder does not have a winning strategy. Suppose that Σ is a winning strategy for the Builder in $V[G]$. Then by a standard reflection argument, there is $\alpha < \omega_2$ such that $\Sigma_0 = \Sigma \cap V[G_\alpha]$ is a winning strategy in $V[G_\alpha]$.

Now as $V[G_\alpha] \models \mathsf{CH}$, we may list all strictly increasing functions in ω^ω of $V[G_\alpha]$ as $\{F_\xi : \xi < \omega_1\}$. Next, for $\xi < \omega_1$, define a function $G_\xi \in \omega^\omega$ such that whenever

$F_\xi(m) \le n < F_\xi(m+1)$, then $G_\xi(n) = m$. Note that we have $G_\xi(n) \le n \le F_\xi(n)$ for every n. Since $\mathcal{U}_0$ generates a Ramsey ultrafilter in $V[G_\alpha]$ by (1), for each ξ there is $A_\xi \in \mathcal{U}_0$ such that the intervals $[G_\xi(n), F_\xi(n))$ for $n \in A_\xi$ are pairwise disjoint. (Indeed, define an interval partition $(I_k : k \in \omega)$ of ω such that whenever $G_\xi(n) \in I_k$ then $F_\xi(n) \in I_k \cup I_{k+1}$. Since $\mathcal{U}_0$ is Ramsey, there is $A \in \mathcal{U}_0$ such that $|A \cap I_k| \le 1$ for all $k \in \omega$. By further pruning A to A_ξ, we may assume it intersects either only intervals I_k such that $k \equiv 0 \mod 3$ or $k \equiv 1 \mod 3$ or $k \equiv 2 \mod 3$. It is easy to see A_ξ is as required.)

Now, still in $V[G_\alpha]$, play a game $(U_\xi : \xi < \omega_1)$ in which Builder follows the strategy Σ_0 while Spoiler plays sets $U_{2\cdot\xi+1}$ such that

$$\{n \in \omega : [G_\xi(n), F_\xi(n)) \cap U_{2\cdot\xi+1} \neq \emptyset\} \notin \mathcal{U}_0. \qquad (\star)$$

To see this is possible fix $\xi < \omega_1$. If $U_{2\cdot\xi}$ meets only finitely many of the intervals $[G_\xi(n), F_\xi(n))$ where $n \in A_\xi$, we may let $U_{2\cdot\xi+1} = U_{2\cdot\xi}$. So suppose $\{n \in A_\xi : [G_\xi(n), F_\xi(n)) \cap U_{2\cdot\xi} \neq \emptyset\}$ is infinite. Since $\mathcal{U}_0$ is an ultrafilter, there is $B \in \mathcal{U}_0$ with $B \subseteq A_\xi$ such that

$$\{n \in A_\xi \setminus B : [G_\xi(n), F_\xi(n)) \cap U_{2\cdot\xi} \neq \emptyset\}$$

is still infinite. We then let

$$U_{2\cdot\xi+1} = \bigcup\{[G_\xi(n), F_\xi(n)) \cap U_{2\cdot\xi} : n \in A_\xi \setminus B\}.$$

This completes the construction.

As the strategy Σ_0 is winning in $V[G_\alpha]$, the sequence of U_ξ's produces a P-point $\mathcal{V}$ in $V[G_\alpha]$. It suffices to show that $\mathcal{V}$ does no longer generate a P-point in $V[G]$.[7] For if this is the case, the sequence of U_ξ's will remain a Σ-legal play in $V[G]$, but Spoiler will win the game, a contradiction.

By (2), all P-points are RK-equivalent to $\mathcal{U}_0$ in $V[G]$. Hence it suffices to show that $\mathcal{V}$ cannot be RK-equivalent to $\mathcal{U}_0$ in $V[G]$. Suppose it were, and $f : \omega \to \omega$ is the bijection witnessing this. Since, by (3), the extension $V[G]$ is ω^ω-bounding over $V[G_\alpha]$, there is ξ such that both f and f^{-1} are everywhere dominated by F_ξ, more explicitly, $f(n) < F_\xi(n)$ and $f^{-1}(n) < F_\xi(n)$ for all n. Note that the latter implies that $f(n) \ge G_\xi(n)$ for all n, that is, $f(n) \in [G_\xi(n), F_\xi(n))$ for all n. It then follows by $(\star)$ that $f^{-1}[U_{2\cdot\xi+1}] = \{n \in \omega : f(n) \in U_{2\cdot\xi+1}\}$ does not belong to $\mathcal{U}_0$, and f cannot witness RK-equivalence, the final contradiction. □

Let us state the following here explicitly:

Open question 3.1 *Does the Builder have a winning strategy in the game $G_\mathfrak{u}$ if and only if she has a winning strategy in the game $G^*_\mathfrak{u}$?*

It would be tempting to define now cardinals $\mathfrak{u}_{Builder}$ and $\mathfrak{u}_{Spoiler}$ as we did in Section 2 for the generalized tower game. This, however, is problematic, for the following reason. Consider the Cohen model, that is, the model obtained by adding at

[7] Actually, it would be even enough to show that $\mathcal{V}$ does not generate an ultrafilter, since an ultrafilter witnessing a victory must be a P-point.

least ω_2 Cohen reals over a model of CH. In this model, all $\subseteq^*$-decreasing sequences have length some ordinal $\gamma < \omega_2$ while on the other hand $\mathfrak{u} = \mathfrak{c} \geq \omega_2$. This means that the game $G_\mathfrak{u}$ is always won by the Spoiler, no matter what its length is. The reason for this problem is that a win of the Builder in $G_\mathfrak{u}$ produces a P-point generated by a decreasing chain and not just an arbitrary ultrafilter.

So let us consider the *modified ultrafilter game* $G'_\mathfrak{u}$ in which the Builder and the Spoiler take turns in building a filter base $\{U_\alpha : \alpha < \omega_1\}$, with the Builder playing at even steps. The Builder wins again if the filter generated by $\{U_\alpha : \alpha < \omega_1\}$ is an ultrafilter; otherwise the Spoiler wins. $G'^*_\mathfrak{u}$ is defined similarly, with the players switching places. It turns out that for plays of length ω_1 these games are equivalent to the original ones, in the following sense.

Lemma 3.2 1. *The Builder has a winning strategy in $G_\mathfrak{u}$ if and only if she has a winning strategy in $G'_\mathfrak{u}$.*
2. *The Builder has a winning strategy in $G^*_\mathfrak{u}$ if and only if she has a winning strategy in $G'^*_\mathfrak{u}$.*

Proof 1. First assume Σ is a winning strategy of the Builder in $G_\mathfrak{u}$. We construct a strategy Σ' of the Builder in $G'_\mathfrak{u}$ by associating with each game $\bar{A} = \{A_\xi : \xi < \omega_1\}$ according to Σ' a game $\bar{C} = \{C_\xi : \xi < \omega_1\}$ according to Σ with $A_\xi = C_\xi$ for even ξ. This means that if the Builder wins $\bar{C}$ then she also wins $\bar{A}$ and, thus, Σ' is a winning strategy.
If $\xi = \zeta + 1$ is odd, we let $C_\xi := A_\xi \cap C_\zeta$ and note that this set must be infinite because $C_\zeta = A_\zeta$ and the players build a filter base in $G'_\mathfrak{u}$. Also C_ξ is a legal move of the Spoiler in $G_\mathfrak{u}$. For even ξ, simply let $A_\xi = \Sigma'(\bar{A}) := \Sigma(\bar{C}) = C_\xi$. Again this is clearly a legal move of the Builder in $G'_\mathfrak{u}$.
Now assume Σ' is a winning strategy of the Builder in $G'_\mathfrak{u}$. Construct a strategy Σ of the Builder in $G_\mathfrak{u}$ by associating with each run $\bar{C} = \{C_\xi : \xi < \omega_1\}$ according to Σ a run $\bar{A} = \{A_\xi : \xi < \omega_1\}$ according to Σ' with $A_\xi = C_\xi$ for odd ξ.
If ξ is odd, let $A_\xi := C_\xi$ and note this is a legal move for the Spoiler in $G'_\mathfrak{u}$. For even ξ let $C_\xi = \Sigma(\bar{C})$ be a pseudointersection of the C_ζ for $\zeta < \xi$ and $A_\xi = \Sigma'(\bar{A})$. Such a pseudointersection exists because these sets form a countable filter base. Clearly, if the Builder wins $\bar{A}$, she also wins $\bar{C}$.
2. Similar. □

This lemma should be thought of as saying that producing an ω_1-generated ultrafilter by a game is equally difficult as producing a P-point generated by a $\subseteq^*$-decreasing ω_1-chain. It is unknown, however, whether $\mathfrak{u} = \omega_1$ implies the existence of an ω_1-generated P-point.[8]

Now consider the game $G'_\mathfrak{u}$ of arbitrary length and define $\mathfrak{u}_{Builder}$ and $\mathfrak{u}_{Spoiler}$ as in the previous section: the former is the least ordinal α such that the Builder has a strategy

[8] One may also consider the same games in the context of the previous section: declare the Builder the winner if the sequence $\{U_\alpha : \alpha < \omega_1\}$ has no pseudointersection. Since these games are naturally related to the *pseudointersection number* $\mathfrak{p}$, denote them by $G_\mathfrak{p}$ and $G^*_\mathfrak{p}$. The analogue of Lemma 3.2 obviously holds: the Builder has a winning strategy in $G_\mathfrak{p}$ iff she has a winning strategy in $G_\mathfrak{t}$, and similarly for the starred games. This can be seen as the game-theoretic version of the classical result stating that $\mathfrak{p} = \omega_1$ iff $\mathfrak{t} = \omega_1$ (see e.g. Theorem 6.25 in [5]). The much deeper $\mathfrak{p} = \mathfrak{t}$ was proved by Malliaris and Shelah [14].

that makes her win in $G'_{\mathfrak{u}}$ in at most α many steps, while the latter is the supremum of all ordinals α such that the Spoiler has a winning strategy in the game $G'_{\mathfrak{u}}$ with α moves. Clearly $\mathfrak{u} \leq \mathfrak{u}_{Spoiler} \leq \mathfrak{u}_{Builder}$ and Theorem 3.1 says that $\mathfrak{u} < \mathfrak{u}_{Builder}$ is consistent. Apart from that we know little:

Open question 3.2 1. *Is* $\mathfrak{u} < \mathfrak{u}_{Spoiler}$ *consistent? Is* $\mathfrak{u}_{Spoiler} < \mathfrak{u}_{Builder}$ *consistent?*
2. *Are* $\mathfrak{u}_{Builder}$ *and* $\mathfrak{u}_{Spoiler}$ *cardinals?*

Finally note that if we also consider $G'^{*}_{\mathfrak{u}}$ of arbitrary length and the corresponding ordinals, we still have:

Fact 3.2 $\mathfrak{u}_{Builder} \leq \mathfrak{u}^{*}_{Builder}$ *and* $\mathfrak{u}_{Spoiler} \leq \mathfrak{u}^{*}_{Spoiler}$.

Proof To see for example the former, let Σ be a winning strategy of the Builder of length $\alpha = \mathfrak{u}^{*}_{Builder}$ in $G'^{*}_{\mathfrak{u}}$. We produce a winning strategy Σ' of the same length in $G'_{\mathfrak{u}}$ such that whenever $\{A_\gamma : \gamma < \alpha\}$ is a run according to Σ' then $\{B_\gamma : \gamma < \alpha\}$ is a run according to Σ with $B_{\gamma+2} = A_{\gamma+1}$ for all γ and $B_{\gamma+1} \cap B_\gamma = A_\gamma$ for limit γ. Clearly this works. □

4 The maximal almost disjoint number game

The last example we consider here is the *maximal almost disjoint game* $G_{\mathfrak{a}}$, which is played as follows. To avoid trivialities, it starts by fixing a partition $\{A_n : n \in \omega\}$ of ω into infinite pieces, and then the Builder and the Spoiler take turns extending it to an AD family $\{A_\alpha : \alpha \leq \beta\}$ (the Builder playing at stages in pair(ω_1), while the Spoiler plays at ordinals in odd(ω_1)).

Builder	A_0		$\cdots$	A_α		$\cdots$
Spoiler		A_1	$\cdots$		$A_{\alpha+1}$	$\cdots$

The Builder wins the match if the family $\{A_\alpha : \alpha < \omega_1\}$ is a maximal almost disjoint family; otherwise, the Spoiler wins.

We could also consider the game $G^{*}_{\mathfrak{a}}$ played according to the same rules but the Spoiler playing at pair(ω_1), while the Builder plays at odd(ω_1). However, in this case it is easy to see that the two games are equivalent:

Lemma 4.1 *The Builder has a winning strategy in the game* $G_{\mathfrak{a}}$ *if and only if she has a winning strategy in the game* $G^{*}_{\mathfrak{a}}$.

Proof First assume Σ is a winning strategy of the Builder in $G_{\mathfrak{a}}$. We construct a strategy Σ' of the Builder in $G^{*}_{\mathfrak{a}}$ by associating with each game $\bar{A} = \{A_\xi : \xi < \omega_1\}$ according to Σ' a game $\bar{B} = \{B_\xi : \xi < \omega_1\}$ according to Σ such that $A_\xi \cup A_{\xi+1} = B_\xi \cup B_{\xi+1}$ for all even ordinals ξ. Thus, since the Builder wins $\bar{B}$, she must also win $\bar{A}$, and the strategy Σ' is winning.

At even ξ, let $B_\xi = \Sigma(\bar{B}\restriction_\xi)$ be the move of the Builder according to Σ. Let A_ξ be an arbitrary move of the Spoiler in $G^{*}_{\mathfrak{a}}$. Next choose $A_{\xi+1}$ almost disjoint from $\bar{A}\restriction_{\xi+1}$ such that $B_\xi \subseteq A_\xi \cup A_{\xi+1}$ and $(A_\xi \cup A_{\xi+1}) \setminus B_\xi$ is infinite. This is clearly possible by the inductive assumption on the sequences $\bar{A}\restriction_\xi$ and $\bar{B}\restriction_\xi$. Let $\Sigma'(\bar{A}\restriction_{\xi+1}) = A_{\xi+1}$

and put $B_{\xi+1} = (A_\xi \cup A_{\xi+1}) \setminus B_\xi$. Note that this is a legal move of the Spoiler in $G_\mathfrak{a}$. This clearly works.

Now assume Σ is winning for the Builder in $G^*_\mathfrak{a}$ and construct Σ' winning for her in $G_\mathfrak{a}$. This is almost the same except that this time, when associating with the Σ'-game $\bar{A} = \{A_\xi : \xi < \omega_1\}$ the Σ-game $\bar{B} = \{B_\xi : \xi < \omega_1\}$, we guarantee that $A_\xi \cup A_{\xi+1} = B_\xi \cup B_{\xi+1}$ for all odd ordinals ξ and $A_\xi = B_\xi$ for all limit ordinals ξ. Details are left to the reader. □

Proposition 4.1 $\diamondsuit(\mathfrak{b})$ *implies the Builder has a winning strategy in* $G_\mathfrak{a}$.

Proof Let F be the Borel function into ω^ω defined in Theorem 7.2 in [15] which we reproduce here. For every infinite countable ordinal, consider the bijective function $e_\delta : \omega \to \delta$. The domain of F is the set of all pairs (s, B) such that:

1. $s = \{A^s_\xi : \xi < \delta(s)\}$ with $\delta = \delta(s)$ an infinite countable ordinal,
2. the collection $s \cup \{B\}$ is an almost disjoint family of infinite subsets of ω,
3. the set $I(s, B) = \left\{ i \in \omega : B \cap A^s_{e_\delta(i)} \setminus \bigcup_{j<i} A^s_{e_\delta(j)} \neq \emptyset \right\}$ is infinite.

Choose an increasing enumeration $I(s, B) = \{i^{s,B}_k : k \in \omega\}$ and define F as follows:

$$F(s, B)(k) = \min \left(B \cap A^s_{e_\delta\left(i^{s,B}_k\right)} \setminus \bigcup_{j<i^{s,B}_k} A^s_{e_\delta(j)} \right).$$

Let $g : \omega_1 \to \omega^\omega$ be a $\diamondsuit(\mathfrak{b})$-sequence for F. Without loss of generality, $g(\delta)$ is a strictly increasing function for every $\delta < \omega_1$.

We show that g allows us to construct a winning strategy for the Builder as follows: Let $s = \{A^s_\xi : \xi < \delta(s)\}$ be a partial match of the game $G_\mathfrak{a}$ with $\delta = \delta(s) \in \mathrm{pair}(\omega_1)$. The Builder plays A^s_δ as follows: if

$$A = \omega \setminus \bigcup_{i\in\omega} \left(A^s_{e_\delta(i)} \setminus \left(\bigcup_{j<i} A^s_{e_\delta(j)} \cup g(\delta)(i) \right) \right)$$

is infinite, we let $A^s_\delta = A$. Otherwise A^s_δ is an arbitrary infinite set almost disjoint from the members of s.

We will see that $\{A^s_\xi : \xi \leq \delta\}$ is an almost disjoint family. Observe first that the set

$$A^s_{e_\delta(i)} \cap \left(g(\delta)(i) \cup \bigcup_{j<i} A^s_{e_\delta(j)} \right) = \left(A^s_{e_\delta(i)} \cap g(\delta)(i) \right) \cup \left(A^s_{e_\delta(i)} \cap \bigcup_{j<i} A^s_{e_\delta(j)} \right)$$

is finite for every $i \in \omega$. Therefore for $i \in \omega$, the intersection $A^s_{e_\delta(i)} \cap A \subseteq A^s_{e_\delta(i)} \cap \left(g(\delta)(i) \cup \bigcup_{j<i} A^s_{e_\delta(j)} \right)$ is finite.

We show that this is a winning strategy. Let $s = \{A^s_\xi : \xi < \omega_1\}$ be a complete match where the Builder played according to the strategy defined by g. We show that s is maximal. Let $B \in [\omega]^\omega$. Consider $f \in 2^{\omega_1}$ coding (B, s), i.e. $f(n) = 1$ iff $n \in B$, and $f(\omega \cdot (1 + \xi) + n) = 1$ iff $n \in A^s_\xi$.

We should find $\delta < \omega_1$ such that $B \cap A^s_\delta$ is infinite.

Aiming towards a contradiction assume that it is not the case, that is $\{B\} \cup \{A^s_\xi : \xi < \omega_1\}$ is an AD family, and for every indecomposable ordinal δ (1)–(3) are satisfied. Let δ be an indecomposable ordinal where $g(\delta)$ guesses f, so in particular, $F(s, B) \not\geq^* g(\delta)$.

Let $\{i_k = i_k^{s\restriction\delta, B} : k \in \omega\}$ be the increasing enumeration of $I(s, B)$. For $k \in \omega$, let $l_k = F(s, B)(k)$, i.e.

$$l_k = \min\left(B \cap A^s_{e_\delta(i_k)} \setminus \bigcup_{j<i_k} A^s_{e_\delta(j)}\right).$$

Observe that the family $\{A^s_{e_\delta(i)} \setminus \bigcup_{j<i} A^s_{e_\delta(j)} : i \in \omega\}$ is disjoint, so the application $k \mapsto l_k$ is injective. Since we have $F(s, B) \not\geq^* g(\delta)$, the set

$$X = \{l_k : g(\delta)(k) > F(s, B)(k)\}$$

is infinite. It is enough to show $X \subseteq A^s_\delta$. Indeed let $l_k \in X$. Then $l_k < g(\delta)(k) \leq g(\delta)(i_k)$ and so

$$l_k \notin A^s_{e_\delta(i_k)} \setminus \left(\bigcup_{j<i_k} A^s_{e_\delta(j)} \cup g(\delta)(i_k)\right).$$

Since $g(\delta)$ is increasing we see that for all $i \geq i_k$,

$$l_k \notin A^s_{e_\delta(i)} \setminus \left(\bigcup_{j<i} A^s_{e_\delta(j)} \cup g(\delta)(i)\right).$$

This implies that $l_k \in A$. In particular, A is infinite and $A^s_\delta = A$. Hence $X \subseteq A^s_\delta$ follows. □

An even simpler task is to show that

Lemma 4.2 *If CH holds, then the Builder has a winning strategy in $G_\mathfrak{a}$.*

Proof Let $\{X_\alpha : \alpha \in \mathrm{odd}(\omega_1)\}$ be an enumeration of $[\omega]^\omega$.

Fact 4.1 *Any infinite countable almost disjoint sequence can be extended.*

If $\langle A_\xi : \xi < \alpha\rangle$ is a partial match for α an infinite limit ordinal, using Fact 4.1 let the Builder play any infinite set A_α extending the sequence.

Let $\langle A_\xi : \xi \leq \alpha \rangle$ be a partial match of infinite length, where the Spoiler has played A_α with $\alpha \in \mathrm{odd}(\omega_1)$. If there is $\xi \leq \alpha$ such that $A_\xi \cap X_\alpha$ is infinite, then let the Builder play any $A_{\alpha+1}$ almost disjoint from the previous ones using again Fact 4.1. Otherwise, let $A_{\alpha+1} = X_\alpha$. It is clear now that any complete match $\langle A_\xi : \xi < \omega_1 \rangle$ defines a maximal almost disjoint family. □

Since CH does not imply $\diamondsuit(\mathfrak{b})$ by Proposition 8.2 and Theorem 8.3 in [15], we have the following:

Corollary 4.1 *The Builder having a winning strategy in $G_\mathfrak{a}$ does not imply $\diamondsuit(\mathfrak{b})$.*

We have still the following open question:

Open question 4.1 *Does $\mathfrak{a} = \omega_1$ imply the Builder has a winning strategy in $G_\mathfrak{a}$?*

As in the preceding sections, we may now consider longer games and the corresponding ordinals $\mathfrak{a}_{Builder}$ and $\mathfrak{a}_{Spoiler}$. Obviously $\mathfrak{a} \leq \mathfrak{a}_{Spoiler} \leq \mathfrak{a}_{Builder}$, and a more general version of the preceding question asks whether these three numbers are equal. As for $\mathfrak{u}$, we even do not know whether $\mathfrak{a}_{Builder}$ and $\mathfrak{a}_{Spoiler}$ necessarily are cardinals.

Also, if we define $\mathfrak{t}_{NoSpoiler}$ as the minimum ordinal where the Spoiler does not have a winning strategy in the game $G_\mathfrak{t}$ of length α, we have mentioned in Section 2 that $\mathfrak{t}_{NoSpoiler} = \mathfrak{t}_{Spoiler}$. With similar definitions, we do not know whether $\mathfrak{u}_{NoSpoiler} = \mathfrak{u}_{Spoiler}$ or $\mathfrak{a}_{NoSpoiler} = \mathfrak{a}_{Spoiler}$ hold.

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