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# Complete and atomic Tarski algebras

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**Abstract**
Tarski algebras, also known as implication algebras or semi-boolean algebras, are the $\{\rightarrow\}$-subreducts of Boolean algebras. In this paper we shall introduce and study the complete and atomic Tarski algebras. We shall prove a duality between the complete and atomic Tarski algebras and the class of covering Tarski sets, i.e., structures $\langle X, \mathcal{K}\rangle$, where $X$ is a non-empty set and $\mathcal{K}$ is non-empty family of subsets of $X$ such that $\bigcup \mathcal{K} = X$. This duality is a generalization of the known duality between sets and complete and atomic Boolean algebras. We shall also analize the case of complete and atomic Tarski algebras endowed with a complete modal operator, and we will prove a duality for these algebras.

**Keywords** Tarski algebras · Tarski sets · Representation theorem · Complete and atomic Tarski algebras · Modal operator

**Mathematics Subject Classification** 03B45 · 03G25

## 1 Introduction

The variety of Tarski algebras, also know as implication algebras or semi-boolean algebras, was introduced by Abbott [2] (see also [3,13]). These algebras are the algebraic counterpart of the $\{\rightarrow\}$-fragment of the propositional classical calculus. It is known that these algebras are also join semilattices with a last element. With the aim of developing a representation for the finite Tarski algebras, the notion of the Tarski set was introduced in [6]. A Tarski set is a pair $\langle X, \mathcal{K}\rangle$, where $X$ is a non-empty set and $\mathcal{K}$ is non-empty subfamily of the power algebra $\mathcal{P}(X)$. As was shown in [6], every finite Tarski algebra $A$ can be represented as a Tarski algebra of sets $T_{\mathcal{K}}(X) \subset \mathcal{P}(X)$, for some finite Tarski set $\langle X, \mathcal{K}\rangle$. This representation is a generalization of the known

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Birkhoff's representation theorem for finite Boolean algebras that assert that a finite Boolean algebra $A$ can be represented as the power set of its set of atoms $\mathrm{At}(A)$; each element of the Boolean algebra corresponds to the set of atoms below it (the join of which is the element). This power set representation can be constructed more generally for any complete atomic Boolean algebra (see for instance [10]). The main objective of this paper is to introduce the class of complete and atomic Tarski algebras. We prove that for any Tarski set $\langle X, \mathcal{K}\rangle$ such that $\bigcup \mathcal{K} = X$ (called a covering Tarski set) the Tarski algebra of sets $T_{\mathcal{K}}(X)$ is complete and atomic. Moreover, if $A$ is a complete and atomic Tarski algebra, then there exists a covering Tarski set $\langle X, \mathcal{K}\rangle$ such that $A \cong T_{\mathcal{K}}(X)$. This bijection between covering Tarski sets and complete and atomic Tarski algebras is indeed a categorical duality.

It is well known that each Kripke frame $\langle X, R\rangle$ has naturally associated an complete and atomic Boolean algebra endowed with a *completely additive* operator. It was proved by Thomason [14] (see also [12]) that the category of all completely additive complete atomic modal algebras (CAMA) is dually equivalent to the category of all generalized approximation spaces. In this paper we shall generalize this duality to the setting of complete and atomic Tarski algebras endowed with a complete modal operator.

The present paper is organized as follows. In Sect. 2 we shall provide all the needed information on Tarski algebras, implicative filters and completely implicative filters, to make the paper self-contained. In Sect. 3 we will introduce the notion of complete and atomic Tarski algebras. In this section we will prove that the set of completely implicative filters is isomorphic to the set of dual atoms of a Tarski algebra. In Sect. 4 we will prove the categorical duality between the category **CTA** whose objects are the complete and atomic Tarski algebras and whose morphisms are the complete Tarski homomorphisms, and the category **CTS** whose objects are the covering Tarski sets and whose morphism are the functions between sets satisfying a continuity condition. In Sect. 5 we shall extended the above result to complete and atomic Tarski algebras endowed with a complete modal operator. The results of this section extend those given by Thomasson [14].

## 2 Tarski algebras

We will recall the definitions and some basic properties of Tarski algebras. For detailed proofs of the results of this section see [2–4], and [13].

**Definition 1** An algebra $\langle A, \to, 1\rangle$ of type $(2, 0)$ is a Tarski algebra if it satisfies the following identities:

*T1.* $1 \to a \approx a$,
*T2.* $a \to a \approx 1$,
*T3.* $a \to (b \to c) \approx (a \to b) \to (a \to c)$,
*T4.* $(a \to b) \to b \approx (b \to a) \to a$.

We note that the conditions T1 to T3 are an axiomatization of the variety of Hilbert algebras (see [8]). We denote by $\mathcal{T}$ the variety of Tarski algebras and by $\mathcal{B}$ the variety of

Boolean algebras. Every Boolean algebra $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Tarski algebra where the implication $\rightarrow$ is defined by $a \rightarrow b = \neg a \vee b$.

In a Tarski algebra $A$ we can define an order relation $\leq$ by setting $a \leq b$ if and only if $a \rightarrow b = 1$. It is well known that $\langle A, \leq\rangle$ is an ordered set and that $A$ is a join-semilattice where the supremum of two elements $a, b \in A$ is defined by $a \vee b = (a \rightarrow b) \rightarrow b$ (see [3]). In [2,3] it was proved that there exists a bijective correspondence between Tarski algebras and join-semilattices where each principal filter is a Boolean algebra. If $A$ is a Tarski algebra, then $\langle [a), \vee, \wedge_a, \neg_a, a, 1\rangle$ is a Boolean algebra, where $[a) = \{b \in A : a \leq b\}$, the negation of an element $b \in [a)$ is defined as $\neg_a b = b \rightarrow a$, and the infimun of two element $b, c \in [a)$ is given by $b \wedge_a c = (b \rightarrow (c \rightarrow a)) \rightarrow a$. Conversely, if $\langle A, \vee, 1\rangle$ is a join-semilattice such that $[a)$ is a Boolean algebra for each element $a \in A$, then one can define an implication $\rightarrow$ as $x \rightarrow y = \neg_y (x \vee y)$, where $\neg_y$ is the Boolean negation in $[y)$. So, $\langle A, \rightarrow, 1\rangle$ is a Tarski algebra.

A non-empty subset $F$ in a Tarski algebra $A = \langle A, \rightarrow, 1\rangle$ is a called *an implicative filter, or filter,* if $1 \in F$, and for any $a, b \in A$, if $a, a \rightarrow b \in F$, then $b \in F$. The set of all filters of $A$ is denoted by $\mathrm{Fi}(A)$. It is known that $\mathrm{Fi}(A)$ is a distributive lattice (see [7] or [8]). A proper filter $F \subseteq A$ is *maximal* if and only if for any $H \in \mathrm{Fi}(A)$ such that $F \subseteq H$, we have $F = H$ or $H = A$. The set of all maximal filters is denoted by $X(A)$. We recall that in a Tarski algebra $A$ a filter $P$ is maximal iff it is *prime*, i.e., for all $a, b \in A$ if $a \vee b \in P$, then $a \in P$ or $b \in P$.

**Definition 2** Let $A$ be a Tarski algebra. A filter $F$ of $A$ is *complete*, if for each $D \subseteq F$, if there exists the infimum $\bigwedge D$, then $\bigwedge D \in F$. A filter $P$ of $A$ is called a *completely prime* if $P$ is proper, and for each $D \subseteq A$, such that there exists $\bigvee D$ and $\bigvee D \in P$, then $D \cap P \neq \emptyset$.

We denote by $\mathrm{pt}(A)$ the set of all completely prime filters of $A$. We note that every completely prime filter is maximal, because it is prime, i.e., $\mathrm{pt}(A) \subseteq X(A)$.

Let $A$ be a Tarski algebra. A subset $I$ of $A$ is called an *ideal* of $A$ if $b \in I$ and $a \leq b$, then $a \in I$, and $a \vee b \in I$, for all $a, b \in I$. The following result can be found in [5] for Hilbert algebras.

**Theorem 3** *Let $A$ be a Tarski algebra. Let $D \in \mathrm{Fi}(A)$ and let $I$ be an ideal of $A$ such that $D \cap I = \emptyset$. Then there exists $P \in X(A)$ such that $D \subseteq P$ and $P \cap I = \emptyset$.*

From this Theorem we can give the following result.

**Theorem 4** *Let $A$ be a Tarski algebra. Then*

(1) *For all $a, b \in A$, if $a \nleq b$ there exists $P \in X(A)$ such that $a \in P$ and $b \notin P$.*
(2) *If $P \in X(A)$, then $a \rightarrow b \notin P$ if and only if $a \in P$ and $b \notin P$.*

## 3 Atomic and complete Tarski algebras

It is a known fact that every Boolean algebra is isomorphic to a field of sets (of some set). But, if a Boolean algebra $B$ is atomic and complete, then it is isomorphic to powerset of some set with the usual set-theoretic operations of union, intersection, and complement. Now we will introduce a class of Tarski algebras which is a generalization of the atomic and complete Boolean algebras.

**Definition 5** Let $A$ be a Tarski algebra. We shall say that $a \in A - \{1\}$ is a *dual atom*, if for all $x \in A$ such that $a \leq x \leq 1$, then $a = x$ or $x = 1$.

The set of all dual atoms of a Tarski algebra $A$ will be denoted by $\mathrm{At}_d(A)$.

**Lemma 6** *Let $A$ be a Tarski algebra. Then $a$ is a dual atom iff for any $b \in A$, if $b \nleq a$, then $b \to a = a$.*

***Proof*** Assume that $a$ is a dual atom of $A$. As $a \leq b \to a$, and $a$ is a dual atom, and $b \nleq a$, we have that $a = b \to a$. Conversely. Let $a \leq b < 1$. If $b \nleq a$, then $b \to a = a$. So, $b = a \vee b = (b \to a) \to a = a \to a = 1$, which is a contradiction. Thus, $a = b$, and consequently $a$ is a dual atom. □

***Remark 7*** We note that for all $a, b, c \in A$ if $b \to a = a$, and $(b \to c) \to a = a$, then $c \to a = a$. Otherwise, by Theorem 3, there exists $P \in X(A)$ such that $c \to a \in P$ and $a \notin P$. Since $P$ is maximal, then $b \in P$ and $b \to c \in P$, and as $P$ is a filter, $a \in P$, which is impossible. Thus, $c \to a = a$.

**Definition 8** A Tarski algebra $A$ is *complete* if for each non-empty set $D \subseteq A$ there exists the supremum $\bigvee D$.

We shall say that a complete Tarski algebra $A$ is *atomic* if for each $a \neq 1$, there exists a set $G \subseteq \mathrm{At}_{\mathrm{d}}(A)$ such that $x = \bigwedge G$. In other words, $A$ is *atomic* iff each element $a \neq 1$ is infimum of the dual atoms.

Now let us give the most important examples of complete and atomic Tarski algebras. Let $X$ be a non-empty set. It is known that $\langle \mathcal{P}(X), \Rightarrow, X \rangle$ is a Tarski algebra, where the implication $\Rightarrow$ is defined by

$$U \Rightarrow V = (X - U) \cup V,$$

for $U, V \in \mathcal{P}(X)$.

It is known that for every finite Boolean algebra $A$ there exists a finite set $X$ such that $A \cong \mathcal{P}(X)$ (for example, the set $X = X(A)$). The converse is also valid: if $X$ is a finite non-empty set, then $X \cong X(\mathcal{P}(X))$. But in general, these good correspondences are not held for Tarski algebras. Figure 1 shows a finite Tarski algebra $A = B \cup \{d\} = \{a, b, c, d, 1\}$, where $B = \{a, b, c, 1\}$ is the Boolean algebra with two atoms (with implication denoted by $\to_B$), and the implication $\to$ is defined by:

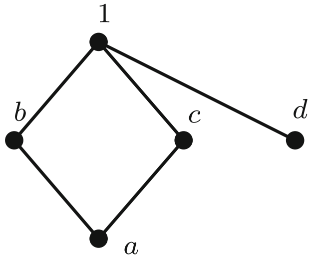
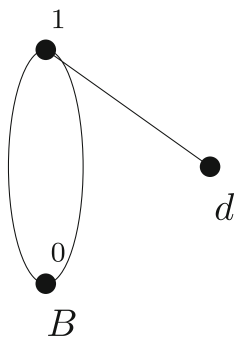


**Fig. 1** Finite Tarski

**Fig. 2** Atomic Tarski




$$x \to y = \begin{cases} x \to_B y & if \quad x, y \in B \\ d & if \quad x \in B \text{ and } y = d \\ y & if \quad y \in B \text{ and } x = d \end{cases}, \tag{1}$$

for $x, y \in A$. It is easy to see that $X(A) = \{\{c, d, 1\}, \{b, d, 1\}, \{a, b, c, 1\}\}$, and $A$ is neither isomorphic to $\mathcal{P}(X(A))$ nor to $\mathcal{P}(X(A)) - \{\emptyset\}$. We note $\mathrm{At}_d(A) = \{b, c, d\}$. This algebra is complete and atomic.

This example can be generalized. We can define a structure of Tarski algebra taking as reduct a set $B \cup \{d\}$, where $B$ is the reduct of a complete and atomic Boolean algebra, $d \notin B$, $d < 1$, $d$ is incomparable with the other elements of $B - \{1\}$, and the implication $\to$ is defined as in (1) (see Fig. 2)

**Theorem 9** *Let $A$ be a complete and atomic Tarski algebra. Then a filter $P$ is completely prime if and only if there exists $a \in \mathrm{At}_d(A)$ such that $P = A - (a] = (a]^c$. Thus, the sets $\mathrm{At}_d(A)$ and $\mathrm{pt}(A)$ are isomorphic by means of the map*

$$f : \mathrm{At}_d(A) \to \mathrm{pt}(A)$$

*given by*

$$f(a) = (a]^c,$$

*for each $a \in \mathrm{At}_d(A)$.*

***Proof*** Let $P$ be a completely prime filter. Consider the set $D = \{b \in A : b \notin P\}$. As $A$ is complete, there exists $a = \bigvee \{d \in A : d \notin P\}$. We prove that $(a]^c = P$. If $x \in P$ but $x \leq a$, then $\bigvee \{d \in A : d \notin P\} \in P$, and as $P$ is completely prime, there exists $d \in D$ such that $d \in P$, which is impossible. Thus, $P \subseteq (a]^c$. If $x \nleq a$, but $x \notin P$, then $x \leq \bigvee \{d \in A : d \notin P\} = a$, which is impossible. Thus, $P = (a]^c$.

We prove that $a$ is a dual atom. Let $b \in A$ such that $b \nleq a$. So, $b \in (a]^c$. If $b \to a \nleq a$, we have $b \to a \in (a]^c$, as since $(a]^c$ is an implicative filter, $a \in (a]^c$, which is impossible. Thus, by Lemma 6 $a$ is a dual atom.

Assume that $a \in \mathrm{At}_d(A)$. We prove that $(a]^c$ is filter. Let $b, b \to c \in (a]^c$ .Then $b \nleq a$ and $b \to c \nleq a$. By Lemma 6, $b \to a = a$ and $(b \to c) \to a = a$. By

Remark 7, $c \to a = a$. As $a$ is a dual atom, $a \neq 1$, i.e., $c \to a \neq 1$, and consequently $c \in (a]^c$ . Then $(a]^c$ is a filter.

We prove that $(a]^c$ is completely prime. Let $D \subseteq A$ such that $\bigvee \{d \in A : d \in D\} \in (a]^c$ . If $d \notin (a]^c$ ,for all $d \in D$, then $d \leq a$, for all $d \in D$. So, $\bigvee d \leq a$, which is a contradiction. Thus, there exist $d \in D$ such that $d \in (a]^c$.

Finally, it is clear that the map $f : \mathrm{At}_d(A) \to \mathrm{pt}(A)$ given by $f(a) = (a]^c$ is bijective. □

**Lemma 10** *Le $A$ be a complete and atomic Tarski algebra. Then for all $a, b \in A$ such that $a \not\leq b$, there exists $P \in \mathrm{pt}(A)$ such that $a \in P$ and $b \notin P$.*

***Proof*** The proof is immediate taking into account Theorem 9 and the fact that $A$ is atomic. □

**Lemma 11** *Let $A$ be a complete and atomic Tarski algebra. Then*

$$\langle [a), \vee, \wedge_a, \neg_a, a, 1\rangle$$

*is a complete and atomic Boolean algebra for each $a \in A$.*

***Proof*** Let $D \subseteq [a)$. As $A$ is complete there exists $\bigvee D$. It is clear that $\bigvee D \in [a)$. We prove that there exists the infimum of $D$ in $[a)$ and it is $(\bigvee \neg_a d) \to a = \bigwedge_a D$. We note that $(\bigvee \neg_a d) \to a \leq d$, for all $d \in D$,otherwise there exists $d_0 \in D$, and there exists $P \in \mathrm{pt}(A)$ such that $(\bigvee \neg_a d) \to a \in P$ and $d_0 \notin P$. As $a \leq d_0$, $a \notin P$. Since $P$ is an implicative filter, $\bigvee \neg_a d \notin P$, and as $P$ is completely prime, $\neg_a d = d \to a \notin P$, for all $d \in D$. In particular, $d_0 \to a \notin P$, but $P$ is maximal, we have $d_0 \in P$, which is impossible. Thus $\bigwedge_a D \leq d$, for all $d \in D$. Finally, if $a \leq x \leq d$, for all $d \in D$, then $\bigvee \neg_a d \leq \neg_a x$. Thus, $\neg_a \neg_a x = x \leq (\bigvee \neg_a d) \to a = \bigwedge_a D$. □

**Definition 12** An implicative filter $P$ of a complete and atomic Tarski algebra $A$ is *complete* iff for each $D \subseteq P$ if there exists $\bigwedge D$, then $\bigwedge D \in P$.

**Lemma 13** *Let $A$ be a complete and atomic Tarski algebra.*

(1) *Let $\{a_i : i \in I\} \subseteq A$. For each $x \in A$, there exists $\bigwedge \{a_i \to x : i \in I\}$ and $\bigwedge \{a_i \to x : i \in I\} = \bigvee \{a_i : i \in I\} \to x$.*
(2) *Let $P$ be a filter. Then, $P$ is a complete maximal filter iff it is completely prime.*
(3) *Let $D \subseteq A$ such that there exists $\bigwedge D$. Then, $\bigvee \{d \to a : d \in D\} = \bigwedge \{d : d \in D\} \to a$, for all $a \in A$.*

***Proof*** (1) As $a_i \leq \bigvee \{a_i : i \in I\}$, we get $c = \bigvee \{a_i : i \in I\} \to x \leq a_i \to x$, for each $i \in I$. So, $\{(a_i \to x : i \in I\} \in [c)$, and as $[c)$ is a complete Boolean algebra there exists the infimum $\bigwedge \{(a_i \to x : i \in I\}$. Then, $\bigvee \{a_i : i \in I\} \to x \leq \bigwedge \{(a \to x : i \in I\}$.

Since $\bigwedge \{(a \to x : i \in I\} \leq a_i \to x$, we have $a_i \leq \bigwedge \{(a \to x : i \in I\} \to x$, for each $i \in I$. Then, $\bigvee \{a_i : i \in I\} \leq \bigwedge \{(a \to x : i \in I\} \to x$, and so $\bigwedge \{(a \to x : i \in I\} \leq \bigvee \{a_i : i \in I\} \to x$. Thus, $\bigwedge \{(a \to x : i \in I\} = \bigvee \{a_i : i \in I\} \to x$.

(2) Assume that $P$ is a complete maximal filter. Let $\bigvee \{a_i : i \in I\} \in P$. Suppose that $a_i \notin P$, for all $i \in I$. As $P$ is proper, there exists $x \in A$ such that $x \notin P$. As $P$ is

maximal, $a_i \to x \in P$, for each $i \in I$, by item (1) and taking into account that $P$ is complete, $\bigwedge \{(a_i \to x : i \in I\} = \bigvee \{a_i : i \in I\} \to x \in P$. Then by modus ponens, $x \in P$, which is a contradiction.Then there exists $i \in I$ such that $a_i \in P$.

Assume that $P \in \mathrm{pt}(A)$. Let $D \subseteq P$ and we suppose that there exists $\bigwedge D$. Let $a = \bigwedge D$. By Lemma 11 $\langle [a), \vee, \wedge_a, \neg_a, a, 1\rangle$ is a complete and atomic Boolean algebra. It is clear that $\bigwedge D = \bigwedge_a D$. So, $1 = a \to a = \bigwedge_a D \to a = \neg_a \bigwedge D = \bigvee_a \{\neg_a d : d \in D\} = \bigvee \{\neg_a d : d \in D\} \in P$. Then there exists $d \in D$, such that $\neg_a d = d \to a \in P$. As $d \in P$, we have $a = \bigwedge D \in P$. Thus, $P$ is complete.

(3) Let $a \in A$, $D \subseteq A$ and suppose that there exists $\bigwedge D$. We note that $a \leq \bigwedge D \to a$ and $a \leq \bigvee \{d \to a : d \in D\}$. By Lemma 11 $\langle [a), \vee, \wedge_a, \neg_a, a, 1\rangle$ is a complete and atomic Boolean algebra. Thus, $\bigvee \{d \to a : d \in D\} = \bigwedge D \to a$ Suppose that $\bigwedge D \to a \nleq \bigvee \{d \to a : d \in D\}$. By Lemma 10 there exists $P \in \mathrm{pt}(A)$ such that $\bigwedge D \to a \in P$ and $\bigvee \{d \to a : d \in D\} \notin P$. So, $d \to a \notin P$, for all $d \in D$. As $P$ is maximal, $a \notin P$, and $d \in P$, for all $d \in D$. Then $\bigwedge D \in P$, because $P$ is complete by (2). As $\bigwedge D \to a \in P$, we get that $a \in P$, which is a contradiction. □

## 4 Representation and duality

In this section we shall prove the mentioned representation for complete and atomic Tarski algebras. First, we define the objects that are duals of complete and atomic Tarski algebras.

**Definition 14** A *Tarski set* is a pair $\langle X, \mathcal{K}\rangle$ where $X$ is a non-empty set and $\mathcal{K}$ is a non-empty subset of $\mathcal{P}(X)$. We shall say that $\langle X, \mathcal{K}\rangle$ is *a covering* if for every $x \in X$ there exists $W \in \mathcal{K}$ such that $x \in W$, i.e., $\bigcup \mathcal{K} = X$.

The dual of a Tarski set $\langle X, \mathcal{K}\rangle$ is the subset $T_{\mathcal{K}}(X)$ of $\mathcal{P}(X)$ defined by:

$$T_{\mathcal{K}}(X) = \left\{U \in \mathcal{P}(X) : \exists W \in \mathcal{K} \,\&\, \exists S \subseteq W \,(U = W^c \cup S) \right\}.$$

Any subalgebra of Tarski algebra of the form $T_{\mathcal{K}}(X)$ is called a Tarski algebra of sets. We note that if $X \in \mathcal{K}$,then $T_{\mathcal{K}}(X) = \mathcal{P}(X)$, because for any $U \subseteq X$, $U = X^c \cup U$. Thus, all results given for Tarski algebra of sets are valid for Boolean algebras of sets.

**Theorem 15** *Let $\langle X, \mathcal{K}\rangle$ be a covering Tarski set. Then $\langle T_{\mathcal{K}}(X), \Rightarrow, X\rangle$ is a complete and atomic Tarski algebra and the map*

$$\varepsilon_X : X \to \mathrm{pt}(T_{\mathcal{K}}(X))$$

*given by*

$$\varepsilon_X(x) = \{U \in T_{\mathcal{K}}(X) : x \in U\},$$

*for each $x \in X$, is a bijection.*

***Proof*** In [6] it was proved that $T_{\mathcal{K}}(X)$ is a Tarski algebra. We prove that it is complete. Let $D \subseteq T_{\mathcal{K}}(X)$. Take any $U \in D$. Then there exists $W \in \mathcal{K}$ and $S \subseteq W$ such that $U = W^c \cup S$. So,

$$W^c \cup \left(W \cap \left(\bigcup D\right)\right) = (W^c \cup W) \cap \left(W^c \cup \bigcup D\right) = \bigcup D.$$

As $W \cap \left(\bigcup D\right) \subseteq W$, we have that $\bigcup D \in T_{\mathcal{K}}(X)$, i.e., $T_{\mathcal{K}}(X)$ is complete. To prove that $T_{\mathcal{K}}(X)$ is atomic we prove first that

$$U \in \mathrm{At}_d(T_{\mathcal{K}}(X)) \text{ iff } \exists x \in X\, (U = \{x\}^c).$$

Let $U \in \mathrm{At}_d(T_{\mathcal{K}}(X))$. As $U \neq X$, then there exists $x \notin U$. So, $U \subseteq \{x\}^c$. Moreover, as $U \in T_{\mathcal{K}}(X)$, there exists $W \in \mathcal{K}$ and $S \subseteq W$ such that $U = W^c \cup S$. Then, $x \in W$ and $x \notin S$. Then $S \cup \{x\} \subseteq W$. So, $U_x = W^c \cup S \cup \{x\} = U \cup \{x\} \in T_{\mathcal{K}}(X)$. Since $U \subset U_x$ and $U$ is a dual atom, $U_x = X$. Thus, for every $y \in X$ such that $y \neq x$, we get $y \in U$, i.e., $U = \{x\}^c$.

Let $x \in X$. As $\langle X, \mathcal{K}\rangle$ is a covering, there exists $W \in \mathcal{K}$ such that $x \in W$. Let $S_x = W - \{x\}$. So, $S_x \subset W$ and $W^c \cup S_x = \{x\}^c \in T_{\mathcal{K}}(X)$. It is clear that $\{x\}^c \in \mathrm{At}_d(T_{\mathcal{K}}(X))$. Moreover, if $U \in T_{\mathcal{K}}(X)$, for each $x \notin U$, we get $U \subseteq \{x\}^c$, and thus

$$U = \bigcap \left\{\{x\}^c : x \notin U\right\},$$

i.e., $U$ is infimum of dual atoms. So, $T_{\mathcal{K}}(X)$ is a complete and atomic Tarski algebra.

It is easy to see that $\varepsilon_X(x) \in \mathrm{pt}(T_{\mathcal{K}}(X))$ for each $x \in X$ (see Theorem 9). Then the map $\varepsilon$ is well-defined. We prove that $\varepsilon_X$ is injective. Let $x, y \in X$. If $x \neq y$, then clearly $\{x\}^c \in \varepsilon_X(y)$, and $\{x\}^c \notin \varepsilon_X(x)$. Thus, $x \neq y$.

We prove that $\varepsilon_X$ is onto. Let $P \in \mathrm{pt}(T_{\mathcal{K}}(X))$. By Theorem 9, there exists $U \in \mathrm{At}(T_{\mathcal{K}}(X))$ such that $P = (U]^c$. Since $U$ is a dual atom of $T_{\mathcal{K}}(X)$, there exists $x \in X$ such that $U = \{x\}^c$. Then it is easy to see that $\varepsilon_X(x) = P$, and thus $\varepsilon_X$ is onto. □

***Remark 16*** Let $\langle X, \mathcal{K}\rangle$ be a covering Tarski set. We note that for each $x \in X$ and for each $U \in T_{\mathcal{K}}(X))$ we have the following equivalences:

$$\begin{aligned} U \in \varepsilon_X(x) &\Leftrightarrow x \in U && \Leftrightarrow U \cap \{x\} \neq \emptyset \\ &\Leftrightarrow U \subseteq \{x\}^c && \Leftrightarrow\ U \notin \left(\{x\}^c\right] \\ &\Leftrightarrow U \in \left(\{x\}^c\right]^c \end{aligned}$$

Thus, instead of the map $\varepsilon_X : X \to \mathrm{pt}(T_{\mathcal{K}}(X))$ we can use the map

$$\lambda_X : X \to \mathrm{At}_d(T_{\mathcal{K}}(X))$$

defined by

$$\lambda_X(x) = \{x\}^c.$$

Let $A$ be a complete and atomic Tarski algebra. Let us consider the mapping

$$\varphi_A : A \to \mathcal{P}(\mathrm{pt}(A)),$$

defined by

$$\varphi_A(b) = \{P \in \mathrm{pt}(A) : b \in P\},$$

for each $b \in A$. Let us consider the family of subsets $\mathcal{K}_A = \{\varphi_A(b)^c : b \in A\}$. The pair

$$\langle \mathrm{pt}(A), \mathcal{K}_A \rangle$$

is a Tarski set called the *dual* Tarski set of $A$ or the *associated* Tarski set of $A$. As $\mathrm{At}_d(A) \cong \mathrm{pt}(A)$, we have $\mathcal{P}(\mathrm{At}_d(A)) \cong \mathcal{P}(\mathrm{pt}(A))$. Alternatively, we may consider the map

$$\phi_A : A \to \mathcal{P}(\mathrm{At}_d(A))$$

define as

$$\phi_A(b) = \left\{a \in \mathrm{At}_d(A) : b \nleq a\right\} = (b]^c \cap \mathrm{At}_d(A).$$

**Definition 17** Let $A$ and $B$ be two complete and atomic Tarski algebras. A *complete homomorphism* between $A$ and $B$ is a Tarski homomorphism $h : A \to B$ such that $h(\bigvee \{d : d \in D\} = \bigvee \{h(d) : d \in D\}$, for any non-empty set $D \subseteq A$.

**Theorem 18** *Let $A$ be a complete and atomic Tarski algebra. Then, $\langle \mathrm{pt}(A), \mathcal{K}_A \rangle$ is a covering Tarski set, and the map $\varphi_A$ is a complete isomorphism of Tarski algebras.*

***Proof*** We first note that for every $a \in A$, $\varphi_A(a) \in T_{\mathcal{K}_A}(\mathrm{pt}(A))$, because $\varphi_A(a) = \varphi_A(a) \cup \emptyset$. Let $a, b \in A$ and $P \in \mathrm{pt}(A)$. Taking into account that $P$ is a maximal implicative filter, we have $a \to b \notin P$ iff $a \in P$ and $b \notin P$ iff $P \in \varphi_A(a)$ and $P \notin \varphi_A(b)$ iff $P \notin \varphi_A(a)^c \cup \varphi_A(b) = \varphi_A(a) \Rightarrow \varphi_A(b)$. So, $\varphi_A$ is a Tarski homomorphism. Suppose that $a \nleq b$. As $A$ is atomic, there exists $p \in \mathrm{At}_d(A)$ such that $a \nleq p$ and $b \leq p$. So, $(p]^c \in \varphi_A(a)$ and $(p]^c \notin \varphi_A(b)$, i.e., $\varphi_A(a) \nsubseteq \varphi_A(b)$. Thus, $\varphi_A$ is injective. As each $P \in \mathrm{pt}(A)$ is proper, then there exists $b \in A$ such that $b \notin P$, i.e., $P \in \varphi_A(b)^c$. So, $\langle \mathrm{pt}(A), \mathcal{K}_A \rangle$ is covering.

We prove that $\varphi_A$ is onto. Let $U \in T_{\mathcal{K}_A}(\mathrm{pt}(A))$. Then there exists $\varphi(a)^c \in \mathcal{K}_A$ and $S \subseteq \varphi(a)^c$ such that $U = \varphi_A(a) \cup S$. For each $P \in S$ there exists a unique $b \in \mathrm{At}_d(A)$ such that $P = (b]^c$. Then

$$S = \bigcup \{\{P\} : P \in S\} = \bigcup \left\{\left\{(b]^c\right\} : (b]^c \in S\right\} = \bigcup \left\{\varphi_A(b)^c : (b]^c \in S\right\}.$$

Consider the set $B = \left\{b : (b]^c \in S\right\}$. As $S \subseteq \varphi_A(a)^c$, we get $a \leq b$ for all $b \in B$. Since $A$ is complete, $[a)$ is a complete Boolean algebra. So there exists $c = \bigwedge \{b : b \in B\}$. Thus, $a \leq c$, and

$$S = \bigcup \left\{\varphi_A(b)^c : (b]^c \in S\right\} = (\bigcap \{\varphi_A(b) : b \in B\})^c = \varphi_A(c)^c.$$

Then $U = \varphi_A(a) \cup \varphi_A(c)^c = \varphi_A(c \to a)$. So, $\varphi_A$ is onto, and thus $\varphi_A[A] = T_{\mathcal{K}_A}(\mathrm{pt}(A))$. □

By Theorems 18 and 15 we can identify a complete atomic Tarski algebra $A$ with the Tarski algebra $T_{\mathcal{K}_A}(\mathrm{pt}(A))$. This means we need no longer to consider abstract complete atomic Tarski algebras, but only those of the form $T_{\mathcal{K}}(X)$ for a covering Tarski set $\langle X, \mathcal{K}\rangle$. This correspondence can be extended to a categorical duality.

The following result characterizes the complete homomorphisms in term of completely prime filters.

**Lemma 19** *Let $A$ and $B$ be two complete and atomic Tarski algebras. A Tarski homomorphism $h : A \to B$ is complete iff $h^{-1}(P) \in \mathrm{pt}(A)$ for each $P \in \mathrm{pt}(B)$.*

***Proof*** $\Rightarrow$) Assume that $h$ is a complete homomorphism. Let $P \in \mathrm{pt}(B)$. If $\bigvee \{d : d \in D\} \in h^{-1}(P)$, then $h(\bigvee \{d : d \in D\}) = \bigvee(\{h(d) : d \in D\} \in P$, and so $h(d) \in P$, for some $d \in D$. So, $d \in h^{-1}(P)$, for some $d \in P$. Thus, $h^{-1}(P)$ is completely prime.

$\Leftarrow$) We note that $h$ is monotonic, because it is a Tarski homomorphism. So, $\bigvee(\{h(d) : d \in D\} \leq h(\bigvee \{d : d \in D\})$.

Suppose that $h(\bigvee \{d : d \in D\}) \nleq \bigvee(\{h(d) : d \in D\}$. As $B$ is atomic, there exist $b \in \mathrm{At}_d(B)$ such that $h(\bigvee \{d : d \in D\} \nleq b$ and $\bigvee(\{h(d) : d \in D\} \leq b$. Then $h(\bigvee \{d : d \in D\} \in (b]^c$ and $\bigvee(\{h(d) : d \in D\} \notin (b]^c$. As $(b]^c$ is completely prime, $h^{-1}((b]^c)$ is completely prime. Then there exists $d \in D$ such that $d \in h^{-1}((b]^c)$ i.e., $h(d) \in (b]^c$. So, $h(d) \nleq b$, which is impossible. Thus, $h$ is complete. □

**Definition 20** Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two covering Tarski sets. A map $f : X_1 \to X_2$ is a *Tarski* map iff $f^{-1}(U) \in \mathcal{K}_1$, for each $U \in \mathcal{K}_2$.

**Lemma 21** *Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two covering Tarski sets. If $f : X_1 \to X_2$ is a Tarski map then the map*

$$\mathrm{S}(f) : T_{\mathcal{K}_2}(X_2) \to T_{\mathcal{K}_1}(X_1)$$

*defined by*

$$\mathrm{S}(f)(U) = f^{-1}(U),$$

*for each $U \in T_{\mathcal{K}_2}(X_2)$, is a complete homomorphism.*

***Proof*** Assume that $f : X_1 \to X_2$ is a Tarski map. If $U \in T_{\mathcal{K}_2}(X_2)$. So there exists $W \in \mathcal{K}_2$ and $S \subseteq W$ such that $U = W^c \cup S$. So $\mathrm{S}(f)(U) = f^{-1}(U) = f^{-1}(W^c \cup S) = f^{-1}(W^c) \cup f^{-1}(S) \in T_{\mathcal{K}_1}(X_1)$, because $f^{-1}(W) \in \mathcal{K}_1$ and $f^{-1}(S) \subseteq f^{-1}(W)$. So, $\mathrm{S}(f)$ is well-defined. Moreover, it is clear that $\mathrm{S}(f)(X_2) = X_1$ and that

$$\mathrm{S}(f)\left(\bigcup \left\{U_i : U_i \in D \subseteq T_{\mathcal{K}_2}(X_2)\right\} = \bigcup \left\{\mathrm{S}(f)(U_i) : U_i \in T_{\mathcal{K}_2}(X_2)\right\},$$

for any $D \subseteq T_{\mathcal{K}_2}(X_2)$. If $U, V \in T_{\mathcal{K}_2}(X_2)$, then $\mathrm{S}(f)(U \Rightarrow V) = \mathrm{S}(f)(U) \Rightarrow \mathrm{S}(f)(V)$, for all $U, V \in T_{\mathcal{K}_2}(X_2)$. Thus, $\mathrm{S}(f)$ is a complete homomorphism. □

**Lemma 22** *Let $A$ and $B$ be two complete and atomic Tarski algebras. If $h : A \to B$ is a complete homomorphism, then the function*

$$\mathrm{T}(h) : \mathrm{pt}(B) \to \mathrm{pt}(A),$$

*defined by*

$$\mathrm{T}(h)(P) = h^{-1}(P),$$

*for each $P \in \mathrm{pt}(B)$, is a Tarski map.*

***Proof*** By Lemma 19 $\mathrm{T}(h)(P) = h^{-1}(P)$ is a completely prime filter of $A$, for each $P \in \mathrm{pt}(B)$. So, $\mathrm{T}(h)$ is well-defined. Let $a \in A$. Then $P \in \mathrm{T}(h)^{-1}(\varphi_A(a)^c)$ iff $\mathrm{T}(h)(P) = h^{-1}(P) \notin \varphi_A(a)$ iff $a \notin h^{-1}(P)$ iff $h(a) \notin P$ iff $P \in \varphi_A(h(a))^c$. So, $\mathrm{T}(h)^{-1}(\varphi_A(a)^c) = \varphi_B(h(a))$. Thus, $\mathrm{T}(h)$ is a Tarski map. □

Let **CTA** be the category whose objects are the complete and atomic Tarski algebras and the arrows of it are all complete homomorphisms between complete and atomic Tarski algebras. Let **CTS** be the category whose objects are the covering Tarski sets and the arrows are the Tarski maps between covering Tarski sets.

We define a contravariant funtor $\mathrm{S} : \mathbf{CTA} \to \mathbf{CTS}$ as follows. For each Tarski algebra $A$ we consider the covering Tarski set

$$\mathrm{S}(A) = \langle \mathrm{pt}(A), \mathcal{K}_A \rangle,$$

and for each morphism of **CTA** $h : A \to B$ we consider the Tarski map

$$\mathrm{S}(h) : \mathrm{S}(B) \to \mathrm{S}(A)$$

defined by $\mathrm{S}(h)(P) = h^{-1}(P)$, for each $P \in \mathrm{pt}(B)$.

We define a contravariant funtor $\mathrm{T} : \mathbf{CTS} \to \mathbf{CTA}$ as follows. For each covering Tarski set $\langle X, \mathcal{K} \rangle$ we consider the Tarski algebra of set

$$\mathrm{T}(X) = \langle \mathrm{T}_{\mathcal{K}}(X), \Rightarrow, X \rangle,$$

and for each morphism of **CTS** $f : X_1 \to X_2$, where $\langle X_1, \mathcal{K}_1 \rangle$ and $\langle X_2, \mathcal{K}_2 \rangle$ are covering Tarski sets, we consider the complete homomorphism

$$\mathrm{T}(f) : \mathrm{T}(X_2) \to \mathrm{T}(X_1)$$

defined by

$$T(f)(U) = f^{-1}(U),$$

for each $U \in \mathrm{T}_{\mathcal{K}}(X_2)$.

If $A$ is a Tarski algebra, then by Theorem 18, $\varphi_A : A \to \mathrm{T}(\mathrm{S}(A))$ is a complete isomorphism of Tarski algebras. Thus, $\varphi$ is a natural transformation from $\mathrm{Id}_{\mathbf{CTA}}$ to $\mathrm{T} \circ \mathrm{S}$.

**Lemma 23** *Let $\langle X, \mathcal{K}\rangle$ be a covering Tarski set. Then the map $\varepsilon_X : X \to \mathrm{pt}(T_{\mathcal{K}}(X))$ defined by $\varepsilon_X(x) = \{U \in T_{\mathcal{K}}(X) : x \in U\}$ is a Tarski map.*

***Proof*** We prove that $\varepsilon^{-1}(H) \in \mathcal{K}$, for each $H \in \mathcal{K}_{T_{\mathcal{K}}(X)}$. Indeed. If $H \in \mathcal{K}_{T_{\mathcal{K}}(X)}$, then there exists $U \in T_{\mathcal{K}}(X)$ such that $H = \varphi_{T_{\mathcal{K}}(X)}(U)^c$. As $U \in T_{\mathcal{K}}(X)$ there exists $W \in \mathcal{K}$ and $S \subseteq W$ such that $U = W^c \cup S$. So, we have the folllowing equivalence:

$$\begin{array}{lll} x \in \varepsilon^{-1}(H) & \text{iff} & \varepsilon(x) \in H \\ & \text{iff} & \varepsilon(x) \notin \varphi_{T_{\mathcal{K}}(X)}(U) \\ & \text{iff} & U \notin \varepsilon(x) \\ & \text{iff} & x \notin U = W^c \cup S \\ & \text{iff} & x \in W. \end{array}$$

□

By Theorem 15 and Lemma 23, the map $\varepsilon : X \to \mathrm{S}(\mathrm{T}(X))$ is a natural transformation from $\mathrm{Id}_{\mathbf{CTS}}$ to $\mathrm{S} \circ \mathrm{T}$. Therefore we can formulate the following result.

**Theorem 24** *The categories* **CTA** *and* **CTS** *are dually equivalent.*

## 5 Complete Modal Tarski algebras

A modal operator $\Diamond$ defined in a Boolean algebra $A$ is completely additive if it distributes over the join of every subset of the algebra. Dually, the modal operator $\Box$ defined by $\Box a = \neg\Diamond\neg a$, distributes over the meet of every subset of the algebra. It was proved by Thomason that the category of all completely additive complete atomic modal algebras (CAMA) is dually equivalent to the category of all Kripke frames [14] (see also [9,11,12]). In this section we are going to extend this representation to the class of algebras that correspond to the $\{\Box, \to\}$-reduct of complete atomic modal algebras.

**Definition 25** A *modal* Tarski algebra is an algebra $\langle A, \Box\rangle$ where $A$ is a Tarski algebra and $\Box$ is a unary operator defined in $A$ such that it verifies the following conditions:

*MT1* $\Box 1 = 1$,
*MT2* $\Box(a \to b) \leq \Box a \to \Box b$, for all $a, b \in A$.

Modal Tarski algebras were studied in [6] as a generalization of the modal algebras. The particular case of monadic Tarski algebras was studied in [1]. The class of modal Tarski algebras is a variety denoted by $\mathcal{MT}$.

A *relational frame or frame* is a pair $\langle X, R\rangle$ where $X$ is a set and $R$ is a binary relation defined in $X$. Given a frame $\langle X, R\rangle$, define a modal operator

$$\Box_R : \mathcal{P}(X) \to \mathcal{P}(X)$$

by

$$\Box_R(U) = \{x \in X : R(x) \subseteq U\}.$$

Then $\langle \mathcal{P}(X), \Box_R, \Rightarrow, X\rangle$ is a modal Tarski algebra. Now we define proper modal Tarski algebras, i.e, not Boolean algebras, by means of Tarski sets endowed with a relation.

**Definition 26** A *relational Tarski set* is a triple $\langle X, \mathcal{K}, R\rangle$ such that $\langle X, \mathcal{K}\rangle$ is a Tarski set and $R$ is a binary relation of $X$ such that $\Box_R(U) \in T_{\mathcal{K}}(X)$, for each $U \in T_{\mathcal{K}}(X)$.

**Lemma 27** *If $\langle X, \mathcal{K}, R\rangle$ is relational Tarski set, then $\langle T_{\mathcal{K}}(X), \Box_R, \Rightarrow, X\rangle$ is a modal Tarski algebra.*

***Proof*** It is obvious. □

Let $\langle X, \mathcal{K}, R\rangle$ be a relational Tarski set. Any modal subalgebra of a modal Tarski algebra of the form $\langle T_{\mathcal{K}}(X), \Box_R\rangle = \langle T_{\mathcal{K}}(X), \Box_R, \Rightarrow, X\rangle$ will be called a *modal Tarski algebra of sets*.

Suppose that $D \subseteq T_{\mathcal{K}}(X)$ such that $\bigcap D \in T_{\mathcal{K}}(X)$. So, it is easy to see that

$$\Box_R\left(\bigcap D\right) = \bigcap\{\Box_R(U) : U \in D\} \in T_{\mathcal{K}}(X).$$

This fact motivates the following definition.

**Definition 28** A complete and atomic modal Tarski algebra is a modal Tarski algebra $\langle A, \Box\rangle$ such that $A$ is complete, and atomic, and $\Box$ is complete modal operator, i.e., $\Box(\bigwedge D) = \bigwedge\{\Box d : d \in D\}$, for any set $D \subseteq A$ such that there exists $\bigwedge D$.

The category of complete and atomic modal Tarski algebras we will denoted by CMTA.

**Definition 29** A relational Tarski set $\langle X, \mathcal{K}, R\rangle$ is *complete* if $\langle T_{\mathcal{K}}(X), \Box_R\rangle \in$ CMTA.

Now we shall study the representation of modal Tarski algebras.

Let $\langle A, \Box\rangle \in$ CMTA. Let us define a binary relation $R_A \subseteq \mathrm{pt}(A) \times \mathrm{pt}(A)$ by:

$$(P, Q) \in R_A \Leftrightarrow \Box^{-1}(P) \subseteq Q,$$

where $\Box^{-1}(P) = \{a \in A : \Box a \in P\}$. Since there exists a bijective correspondence between $\mathrm{pt}(A)$ and $\mathrm{At}_d(A)$, we have that for $P, Q \in \mathrm{pt}(A)$ there exist $p, q \in \mathrm{At}_d(A)$ such that $P = (p]^c$ and $Q = (q]^c$. So,

$$\begin{aligned}
\Box^{-1}(P) \subseteq Q &\Leftrightarrow \Box^{-1}\left((p]^c\right) \subseteq (q]^c \\
&\Leftrightarrow \forall x \in A\,(\Box x \in (p]^c \Rightarrow x \in (q]^c) \\
&\Leftrightarrow \forall x \in A\,(\Box x \nleq p \Rightarrow x \nleq q) \\
&\Leftrightarrow \forall x \in A\,(x \leq q \Rightarrow \Box x \leq p) \\
&\Leftrightarrow \Box q \leq p.
\end{aligned}$$

Therefore, instead of the relation $R_A$ given in $\mathrm{pt}(A)$ we can define a relation

$$S_A \subseteq \mathrm{At}_d(A) \times \mathrm{At}_d(A)$$

as

$$(p, q) \in R_A \Leftrightarrow \Box q \leq p.$$

The structure

$$\langle \mathrm{pt}(A), R_A, \mathcal{K}_A \rangle ,$$

is called the *relational Tarski frame* associated to $A$.

***Remark 30*** Let $\langle A, \Box \rangle \in$ CMTA. For each $P \in \mathrm{pt}(A)$, $\Box^{-1}(P)$ is a complete filter. Indeed. Let $D \subseteq \Box^{-1}(P)$ and we suppose that there exists $\bigwedge D$. So, $\Box d \in P$, for all $d \in D$. By Lemma 13, $P$ is complete. Then, $\bigwedge \{\Box d : d \in D\} = \Box(\bigwedge D) \in P$, i.e., $\bigwedge D \in \Box^{-1}(P)$. Thus, $\Box^{-1}(P)$ is a complete filter.

**Lemma 31** *Let* $\langle A, \Box \rangle \in$ CMTA. *Let* $P \in \mathrm{pt}(A)$ *and* $x \in A$. *Then,* $\Box x \notin P$ *if and only if there exists* $Q \in \mathrm{pt}(A)$ *such that* $(P, Q) \in R_A$ *and* $x \notin Q$.

***Proof*** Let us suppose that $\Box x \notin P$. By Theorem 9, there exists $a \in \mathrm{At}_d(A)$ such that $P = (a]^c$ . Then, $\Box x \leq a$. As $x \neq 1$, $x = \bigwedge [x) \cap \mathrm{At}_d(A)$. We prove that there exists $b \in [x) \cap \mathrm{At}_d(A)$ such that $\Box b \leq a$. Suppose the contrary. Then by Lemma 6, $\Box b \to a = a$, for each $b \in [x) \cap \mathrm{At}_d(A)$. So $\bigvee \{\Box b \to a : b \in [x) \cap \mathrm{At}_d(A)\} = a$. By Lemma 13, $\bigwedge \Box b \to a = \Box x \to a = a$, i.e., $\Box x \nleq a$, which is impossible. Thus, there exists $b \in [x) \cap \mathrm{At}_d(A)$ such that $\Box b \leq a$. So, $x \notin (b]^c = Q$ and $\Box^{-1}(P) \subseteq Q$. The other direction is immediate. □

Let $\langle X_1, \mathcal{K}_1, R_1 \rangle$ and $\langle X_2, \mathcal{K}_2, R_2 \rangle$ be two relational Tarski sets. A relational isomorphism is a bijective Tarski map $f : X_1 \to X_2$ such that

$$(x, y) \in R_1 \text{ iff } (f(x), f(y)) \in R_2,$$

for all $x, y \in X_1$.

**Theorem 32** *Let* $\langle X, \mathcal{K}, R \rangle$ *be a complete and covering relational Tarski set. Then* $\langle T_{\mathcal{K}}(X), \Box_R \rangle$ *is a complete Tarski modal algebra and the map* $\varepsilon_X : X \to \mathrm{pt}(T_{\mathcal{K}}(X))$*, given by* $\varepsilon_X(x) = \{U \in T_{\mathcal{K}}(X) : x \in U\}$*, is a relational isomorphism.*

***Proof*** By Theorem 15, $T_{\mathcal{K}}(X)$ is a complete and atomic Tarski algebra and the map $\varepsilon_X$ is a bijection. By Lemma 23 $\varepsilon_X$ is a Tarski map. Let $R_{\mathcal{K}} = R_{T_{\mathcal{K}}(X)}$. We recall that $U \in \varepsilon_X(x)$ iff $U \in \left(\{x\}^c\right]^c$. We note that for any $x, y \in X$ the following equivalences are valid:

$$\begin{aligned}(\varepsilon_X(x), \varepsilon_X(y)) \in R_{\mathcal{K}} &\Leftrightarrow \Box_R^{-1}(\varepsilon_X(x)) \subseteq \varepsilon_X(y) &&\Leftrightarrow \Box_R^{-1}(\left(\{x\}^c\right]^c) \subseteq \left(\{y\}^c\right]^c \\ &\Leftrightarrow \Box_R(\{y\}^c) \subseteq \{x\}^c &&\Leftrightarrow \Box_R(\{y\}^c) \cap \{x\} = \emptyset \\ &\Leftrightarrow x \notin \Box_R(\{y\}^c) &&\Leftrightarrow R(x) \nsubseteq \{y\}^c \\ &\Leftrightarrow R(x) \cap \{y\} \neq \emptyset &&\Leftrightarrow (x, y) \in R.\end{aligned}$$

Thus, $\varepsilon_X$ is a relational isomorphism. □

**Theorem 33** *Let* $\langle A, \Box\rangle \in$ CMTA. *Then,* $\langle \mathrm{pt}(A), \mathcal{K}_A, R_A\rangle$ *is a covering relational Tarski set and the map* $\varphi_A : A \to T_{\mathcal{K}}(\mathrm{pt}(A))$ *is an isomorphism.*

***Proof*** By Theorem 18, $\langle \mathrm{pt}(A), \mathcal{K}_A\rangle$ is a covering Tarski set and $\varphi_A$ is an isomorphism of Tarski algebras. Let $a \in A$, and let $P \in \mathrm{pt}(A)$. Then $\Box a \notin P$ iff there exists $Q \in X(A)$ such that $(P, Q) \in R_A$ and $a \notin Q$. Thus, $\varphi_A(\Box a) = \Box_{R_A}(\varphi(a))$. So, $\varphi_A$ is a modal homomorphism. □

A map $f : X_1 \to X_2$ between two relational Tarski sets $\langle X_1, \mathcal{K}_1, R_1\rangle$ and $\langle X_2, \mathcal{K}_2, R_2\rangle$ is a *bounded morphism* if $f$ satisfies the following conditions:

(TMO) $f$ is a Tarski map,
(MF1) If $(x, y) \in R_1$, then $(f(x), f(y)) \in R_2$,
(MF2) If $(f(x), z) \in R_2$, then there exists $y \in X_1$ such that $(x, y) \in R_1$ and $f(y) = z$.

Let $\langle A, \Box\rangle$ and $\langle B, \Box\rangle$ be two complete modal Tarski algebras. A *complete modal homomorphism* between $\langle A, \Box\rangle$ and $\langle B, \Box\rangle$ is a complete homomorphism $h : A \to B$ such that $h(\Box a) = \Box h(a)$, for any $a \in A$.

**Theorem 34** *Let* $\langle X_1, \mathcal{K}_1, R_1\rangle$ *and* $\langle X_2, \mathcal{K}_2, R_2\rangle$ *be relational covering Tarski sets. If* $f : X_1 \to X_2$ *is a bounded morphism, then the map* $\mathrm{T}(f) : T_{\mathcal{K}_2}(X_2) \to T_{\mathcal{K}_1}(X_1)$ *defined by* $\mathrm{T}(U) = f^{-1}(U)$*, for each* $U \in T_{\mathcal{K}_2}(X_2)$*, is a complete modal homomorphism.*

***Proof*** It is clear that T is a complete Tarski homomorphism. By condition (MF1), it is easy to see that $f^{-1}(\Box_{R_2}(U)) \subseteq \Box_{R_1}(f^{-1}(U))$, and by condition (MF2) we can prove that $\Box_{R_1}(f^{-1}(U)) \subseteq f^{-1}(\Box_{R_2}(U))$, for each $U \in T_{\mathcal{K}_2}(X_2)$. □

**Theorem 35** *Let* $\langle A, \Box\rangle$ *and* $\langle B, \Box\rangle$ *be two complete modal Tarski algebras. If* $h : A \to B$ *is a complete modal homomorphism, then the function* $S : \mathrm{pt}(B) \to \mathrm{pt}(A)$ *defined by* $S(P) = h^{-1}(P)$*, for each* $P \in \mathrm{pt}(B)$*, is a bounded morphism.*

***Proof*** Condition (MF1) can be simply checked. We prove the condition (MF2). Let $(h^{-1}(P), Q) \in R_A$, i.e., $\Box^{-1}(h^{-1}(P)) \subseteq Q$. We recall that the filter $\Box^{-1}(P)$ is complete. Moreover, $(h(Q^c)] = \{b \in B : \exists x \notin Q\, (b \leq h(x)\}$ is an ideal of $B$. It is easy to see that $\Box^{-1}(P) \cap (h(Q^c)] = \emptyset$. So, by Lemma 13, there exists a completely prime filter $D$ of $A$ such that $\Box^{-1}(P) \subseteq D$ and $(h(Q^c)] \cap D = \emptyset$. As $D$ is maximal, $h^{-1}(D) = Q$. Thus, $S$ is a bounded morphism. □

Let **CTMA** be the category whose objects are the complete and atomic modal Tarski algebras and the arrows of it are all complete modal homomorphisms between complete and atomic modal Tarski algebras. Let **CTMS** be the category whose objects are the relational covering Tarski sets and the arrows are bounded morphism between relational covering Tarski sets. Taking into account Theorem 24 we can prove the following result.

**Theorem 36** *The categories* **CTMA** *and* **CTMS** *are dually equivalent.*

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