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On uniformly continuous functions between pseudometric spaces and the Axiom of Countable Choice

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Abstract

In this note we show that the Axiom of Countable Choice is equivalent to two statements from the theory of pseudometric spaces: the first of them is a well-known characterization of uniform continuity for functions between (pseudo)metric spaces, and the second declares that sequentially compact pseudometric spaces are **UC**—meaning that all real valued, continuous functions defined on these spaces are necessarily uniformly continuous.

Keywords Axiom of Countable Choice · Pseudometric spaces · Uniform continuity

Mathematics Subject Classification Primary 03E25; Secondary 54E35

1 Introduction

In what follows, **ZF** is Zermelo-Fraenkel Set Theory, **AC** is the Axiom of Choice and **ZFC** = **ZF** + **AC**. In this paper we deal with a particular weak choice principle, namely the *Axiom of Countable Choice*, which we denote by $\mathbf{AC}_\omega$. As usual in the research on weak choice principles, all theorems in this paper are **ZF** theorems.

$\mathbf{AC}_\omega$ declares that every countable family of non-empty sets has a choice function. In the standard reference [4], this choice principle is referred to as Form 8.

$\mathbf{AC}_\omega$ is a key ingredient of many classical, **ZFC** results on metric and pseudometric spaces (and we assume the reader is familiar with the usual terminology concerning such topological spaces). For instance, as shown by Bentley and Herrlich in [1], $\mathbf{AC}_\omega$ is enough to prove, in the realm of pseudometric spaces, the equivalence between the

This paper is dedicated to the memory of Prof. Horst Herrlich (1937–2015).

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notions of: (i) separability; (ii) having a countable base; and (iii) being a Lindelöf space (Proposition 1.2 of Bentley, Herrlich, op.cit.), and, moreover, $\mathbf{AC}_\omega$ is, in fact, *equivalent* to each of the statements "All pseudometric spaces with a countable base are separable" and "All Lindelöf pseudometric spaces are separable" (Theorem 1.7 of Bentley, Herrlich, op.cit). A similar phenomenon happens if one considers the statements "Sequentially compact pseudometric spaces are totally bounded", "Totally bounded, complete pseudometric spaces are compact" and "Sequentially compact, pseudometric spaces are compact" (see Theorem 4.3 of Bentley, Herrlich, op.cit). More information on pseudometric spaces within **ZF** may be found in the papers [3,5].

In the papers [6,7], Keremedis has investigated the class of metric spaces which are *Atsuji*, or **UC** – i.e., the class of metric spaces satisfying the following property: all continuous real valued functions defined on them are, necessarily, uniformly continuous. Among many other results, the following theorem was established by Keremedis:

Theorem 1 ([6]) *Let* $\mathbf{M} = (M, d)$ *be a metric space. Under* $\mathbf{AC}_\omega$*, the following statements are equivalent:*

- (i) $\mathbf{M}$ *is Lebesgue—i.e., every open cover of* $\mathbf{M}$ *has a Lebesgue covering number.*
- (ii) $\mathbf{M}$ *is* **UC***.*
- (iii) $\mathbf{M}$ *is uniformly normal—i.e., the distance between any two disjoint, non-empty closed subsets of* $\mathbf{M}$ *is strictly positive.*

□

It is worthwhile remarking that, in the previous theorem, implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ hold in **ZF**. It is also shown in [6] that, in **ZF**, countably compact metric spaces are Lebesgue—whereas the negation of the statement "All sequentially compact metric spaces are uniformly normal" is consistent with **ZF**.[1]

In this work we show that, if we consider the property of being **UC** in the class of *pseudometric* spaces, we get that the statement "All sequentially compact, pseudometric spaces are **UC**" is, itself, an *equivalent* of the Axiom of Countable Choice. We also show that $\mathbf{AC}_\omega$ is equivalent to a well-known characterization (via pairs of sequences) of uniform continuity of real valued continuous functions defined on pseudometric spaces.

As the reader will be able to check in the proof of the main theorem, the importance of working with pseudometric spaces rather than metric spaces comes from the following observation: a pseudometric may allow you to "enlarge" the points of a metric space to "blobs" of diameter zero—in such a way that, within **ZF**, certain countable choices are no longer possible. This "impossibility of countable choices" we have just mentioned is crystallized by the following result, which is widely used in the context of the research on weak choice principles:

Theorem 2 ("Folklore") *Under the failure of the Axiom of Countable Choice, there is a countable family* $\mathcal{F} = \{X_n : n \geqslant 1\}$ *of non-empty sets such that no infinite subfamily of* $\mathcal{F}$ *has a choice function.* □

[1] Notice that, as the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ hold in **ZF**, the negation of "All sequentially compact metric spaces are **UC**" is consistent with **ZF** as well.

Indeed, if the statement

(*) *"All countable families of non-empty sets have infinite subfamilies which have choice functions."*

holds, then, given any countable family $\{A_n : n \geqslant 1\}$ of non-empty sets, an element $g \in \prod_{n \geqslant 1} A_n$ is easily produced, after applying (*) to the countable family of non-empty sets $\{X_n : n \geqslant 1\}$ given by $X_1 = A_1$ and $X_n = \prod_{i \leqslant n} A_i$ for all $n \geqslant 2$; indeed, if f is an infinite partial function for the X_n's, then let $g(n)$ be the n^{th} component of $f(k)$, where k is the first natural number $\geqslant n$ in the domain of f.

We finish this introduction with two easy **ZF** remarks on pseudometric spaces.

Fact 3 *If M, N are pseudometric spaces and $f : M \to N$ is continuous at a point $a \in M$, then $d(f(a), f(b)) = 0$ whenever $d(a, b) = 0$.*

Indeed, if $d(a, b) = 0$ and f is continuous at a then $d(f(a), f(b)) < \varepsilon$ for every $\varepsilon > 0$.

Finally, notice that the following fact also holds in **ZF**:

Fact 4 *If M, N are pseudometric spaces and $f : M \to N$ is continuous at a point $x_0 \in M$, then $f(x_n) \to f(x_0)$ (in N) whenever (x_n) is a sequence of points of M with $x_n \to x_0$ (in M).*

In fact, what needs Countable Choice to be established is the converse statement.

2 The main theorem

The following theorem is the main result of this paper. The two statements which are referred to in the abstract are (ii) and (v).

Theorem 5 *The following statements are equivalent:*

- (i) $\mathbf{AC}_\omega$.
- (ii) *If M, N are pseudometric spaces and $f : M \to N$ is a continuous function, f is uniformly continuous if, and only if, for every pair of (not necessarily convergent) sequences $(x_n), (y_n) \in M$ such that $d_M(x_n, y_n) \to 0$ one has $d_N(f(x_n), f(y_n)) \to 0$.*
- (iii) *If M is a pseudometric space and $f : M \to \mathbb{R}$ is a continuous function, f is uniformly continuous if, and only if, for every pair of (not necessarily convergent) sequences $(x_n), (y_n) \in M$ such that $d_M(x_n, y_n) \to 0$ one has $|f(x_n) - f(y_n)| \to 0$.*
- (iv) *Let M, N be pseudometric spaces and assume M to be sequentially compact. Then, for every function $f : M \to N$, f is continuous if, and only if, it is uniformly continuous.*
- (v) *Sequentially compact, pseudometric spaces are* **UC**.

Let us proceed with the proof of the above equivalences. $(ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v)$ are obvious. Proofs of $(i) \Rightarrow (ii) \Rightarrow (iv)$ are apparent; let us present them, by the sake of completeness.

Proof of $(i) \Rightarrow (ii)$***:*** The "only if" part is obvious. For the "if" part, we argue contrapositively: assume f is not uniformly continuous. So, there is some $\varepsilon > 0$ such that for all $\delta > 0$ there are points $x, y \in M$ with $d_M(x, y) < \delta$ and $d_N(f(x), f(y)) \geqslant \varepsilon$. This gives us a family $\{A_n : n \geqslant 1\}$ of non-empty subsets of $M \times M$ such that, for any $n \geqslant 1$,

$$A_n = \left\{(x, y) \in M \times M : d_M(x, y) < \frac{1}{n} \text{ and } d_N(f(x), f(y)) \geqslant \varepsilon\right\}.$$

Let $z_n = (x_n, y_n) \in A_n$ be given by Countable Choice; we have that $d_M(x_n, y_n) \to 0$ whereas $d_N(f(x_n), f(y_n)) \not\to 0$, as desired. □

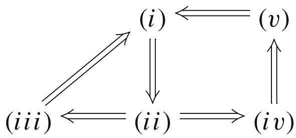
Proof of $(ii) \Rightarrow (iv)$ The "if" part is obvious. For the "only if" part, let M, N be pseudometric spaces, with M sequentially compact, and let $f : M \to N$ be a continuous function. Assuming (ii), what we have to check is that

$$d_N(f(x_n), f(y_n)) \to 0$$

whenever $(x_n), (y_n)$ are sequences in M satisfying $d_M(x_n, y_n) \to 0$.

So, let $(x_n), (y_n) \in M$ satisfy $d_M(x_n, y_n) \to 0$. Assume towards a contradiction that $d_N(f(x_n), f(y_n)) \not\to 0$; without loss of generality, we may assume there is some $\varepsilon > 0$ such that $d_N(f(x_n), f(y_n)) \geqslant \varepsilon$ for all $n \geqslant 1$. By sequential compactness, we may also assume, without loss of generality, the sequences $(x_n), (y_n)$ to be convergent—and therefore there are points $x, y \in M$ such that $x_n \to x$ and $y_n \to y$; as $d_M(x_n, y_n) \to 0$ we have, necessarily, $d_M(x, y) = 0$. The continuity of f ensures that $d(f(x), f(y)) = 0$, by Fact 3. However, the continuity of f also ensures $f(x_n) \to f(x)$ and $f(y_n) \to f(y)$ in N, by Fact 4. Then, necessarily one has $d_N(f(x_n), f(y_n)) \to 0$, but this is an absurdity since $d_N(f(x_n), f(y_n)) \geqslant \varepsilon$ for all $n \geqslant 1$. It follows that the desired implication holds. □

To be done with the equivalences, what is left to do is to prove $(iii) \Rightarrow (i)$ and $(v) \Rightarrow (i)$.[2]



[2] One is able to check in the display that (ii) is the crucial statement in our context—at least in the etymological sense of the word "crucial". Indeed, the kind of cross-shaped figure centered at statement (ii) (in the above display) is a resemblance of the original impetus for the research reported in this paper, which was to prove that (ii) is an equivalent of the Axiom of Countable Choice; our proof of $(ii) \Rightarrow (i)$ was made using (iii), (iv) and (v) as intermediate results, and (fortunately to the author) those intermediate results turned out to be much more interesting (and relevant) than statement (ii) itself.

Both remaining proofs will be done using the following construction (which is essentially due to Bentley and Herrlich [1]):

Example 6 [1] Under the failure of the Axiom of Countable Choice, there is a non-compact, non-separable, sequentially compact pseudometric space.

Construction: Suppose $\mathbf{AC}_\omega$ fails. Let $\mathcal{F} = \{X_n : n \geqslant 1\}$ be as in Theorem 2; we may assume w.l.g. that $\mathcal{F}$ is a disjoint family. Let $M = \bigcup_{n \geqslant 1} X_n$, and define $d : M \times M \to \mathbb{R}$ as follows: for any $x, y \in X$,

$$d(x, y) = \begin{cases} 0 & \text{if there is } m \geqslant 1 \text{ such that } x, y \in X_m; \text{ and} \\ |\frac{1}{n} - \frac{1}{m}| & \text{if } n \neq m, x \in X_n \text{ and } y \in X_m. \end{cases}$$

The above function d is easily seen to be a pseudometric—which "enlarges" the metric space $\{\frac{1}{n} : n \geqslant 1\}$ (endowed with the usual induced metric) to "blobs" of diameter zero. M is sequentially compact since, for any sequence $(y_n)_{n \geqslant 1}$ of points of X, necessarily the set

$$A = \left\{m \geqslant 1 : X_m \text{ contains points of the sequence } (y_n)_{n \geqslant 1}\right\}$$

is finite (otherwise the infinite subfamily of $\mathcal{F}$ given by $\{X_m : m \in A\}$ would have—clearly—a choice function). It follows that $(y_n)_{n \geqslant 1}$ has at least one subsequence $(y_{n_k})_{k \geqslant 1}$ satisfying $d(y_{n_k}, y_{n_j}) = 0$ for all $k, j \geqslant 1$ and any such subsequence converges. A similar reasoning shows that M is not separable. To see that M is not compact, notice that the open cover given by $\{X_n : n \geqslant 1\}$ has no finite subcover. □

In fact, a minor modification of the above construction was used by Bentley and Herrlich in [1] (see Theorems 2.4 and 4.3, op.cit.) to prove that the Axiom of Countable Choice is equivalent to both statements *"Sequentially compact pseudometric spaces are compact"* and *"Totally bounded pseudometric spaces are separable"*—see also Theorems 3.26 and 3.27 of [2].

Proofs of $(iii) \Rightarrow (i)$ ***and*** $(v) \Rightarrow (i)$ Arguing contrapositively, it is enough to show that, under the failure of $\mathbf{AC}_\omega$, there is sequentially compact pseudometric space M with a continuous, non-uniformly continuous real valued function $f : M \to \mathbb{R}$ satisfying, for all pairs of sequences (x_n), (y_n),

$$d_M(x_n, y_n) \to 0 \implies |f(x_n) - f(y_n)| \to 0.$$

As expected, let M be the sequentially compact pseudometric space constructed in Example 6. We define a function $f : M \to \mathbb{R}$ in the following way: for every $x \in X$,

$$f(x) = n \iff x \in X_n.$$

f is continuous (since it is locally constant) and a straigthforward calculation shows that it is not uniformly continuous.

Let (x_n), (y_n) be sequences in M such that $d_M(x_n, y_n) \to 0$. Let A_x and A_y denote the finite subsets of ω given by

$$A_x = \left\{m \geqslant 1 : X_m \cap \mathrm{im}(x) \neq \emptyset\right\} \text{ , and } A_y = \left\{m \geqslant 1 : X_m \cap \mathrm{im}(y) \neq \emptyset\right\} .$$

If $d_M(x_n, y_n) \to 0$, then necessarily there is some $m \in A_x \cap A_y$ and some $n_0 \geqslant 1$ such that, for every $n \geqslant n_0$, both points x_n and y_n are elements of X_m. It follows that $f(x_n) = f(y_n) = m$ for all $n \geqslant n_0$, and so we are done. □

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