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On positive local combinatorial dividing-lines in model theory

Vincent Guingona[2] · **Cameron Donnay Hill**[1]

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Abstract
We introduce the notion of positive local combinatorial dividing-lines in model theory. We show these are equivalently characterized by indecomposable algebraically trivial Fraïssé classes and by complete prime filter classes. We exhibit the relationship between this and collapse-of-indiscernibles dividing-lines. We examine several test cases, including those arising from various classes of hypergraphs.

Keywords Indiscernibles · Dividing-lines · Combinatorics · Fraisse class · Complexity · Hierarchies

Mathematics Subject Classification 03C45 · 03C64 · 03C20 · 03C68

1 Introduction

A model-theoretic dividing-line is a partitioning of the class of all complete theories into two parts—a tame part and a wild part. In some way, theories that lie in the wild part are much harder to analyze than those in the tame part, and by identifying the dividing-line explicitly, we hope to understand exactly what makes the qualitative divergence occur. Ofttimes, we also seem to prefer that a dividing-line is associated with some partial ordering $\leq$ of the class of theories, so that $T_1 \leq T_2$ implies that T_2 is more complex than T_1.[1] Thus, the wild part of a dividing-line should be a filter $\mathfrak{C}$—closed upwards relative to the associated ordering $\leq$—while the tame part $N\mathfrak{C}$ should be just the complementary ideal to $\mathfrak{C}$. In this introduction, we will first try

[1] Actually, we want a pre-ordering, since in practice, we must allow distinct theories to be equivalent modulo the ordering.

✉ Vincent Guingona
vguingona@towson.edu

1 Department of Mathematics and Computer Science, Wesleyan University, 655 Exley Science Tower, 265 Church Street, Middletown, CT 06459, USA

2 Department of Mathematics, Towson University, 8000 York Road, Towson, MD 21252, USA

to provide some context for our work, hopefully demonstrating how the viewpoint developed here differs meaningfully from some others and captures quite different tameness/wildness separations.

The demand for a partial ordering $\leq$ on theories is relaxed in what we will here call "geometric" dividing-lines. The collection of geometric dividing-lines would include, say, the super-simple theories within the class of simple theories and the super-rosy theories within the class of rosy theories. These seem to be determined by the behavior of a geometry formulated in terms of an abstract independence relation (as in Adler [1,2] and Cassanovas [5]). In particular, they don't seem to be well explained in terms of formula-by-formula or local combinatorial analyses. In this paper, we will have nothing more to say about geometric dividing-lines.

The saturation-of-ultrapowers program of Malliaris and Shelah, carried out in [17–19] and several other papers, is local in nature in that (as shown in [16]) the underlying Keisler ordering $\leq_K$ compares theories on a formula-by-formula basis. The saturation-of-ultrapowers program has at least that much in common with the framework that we propose and study here. Like that program, we also base our work on an ordering $\trianglelefteq$ that compares two theories on a formula-by-formula basis. Speaking roughly, we take $T_1 \trianglelefteq T_2$ to mean that, up to matching formulas of T_1 to formulas of T_2, "every" finite configuration that appears in a model of T_1 also appears in a model of T_2. (We write "every" because we will conscientiously avoid involving algebraic configurations.) Our formulation of $\trianglelefteq$ is almost immediately linked to the idea of a family of Fraïssé classes involved in a theory.

The work in this paper originated in an attempt to build a general framework for the collapse-of-indiscernibles dividing-lines addressed in [7,23]. In connection with collapse-of-indiscernibles, an immediate question is, "Which $\trianglelefteq$-dividing-lines $\mathfrak{C}$ have a *single* characteristic Fraïssé class (if not a Ramsey class)?" Since there may be dividing-lines that require several characteristic Fraïssé classes, a second question might be, "What are the 'irreducible' $\trianglelefteq$-dividing-lines, and what does it mean for $\mathfrak{C}$ to be irreducible?" Addressing the second part of the second question, we try (and hopefully succeed) to capture irreducibility in the notion of a complete prime filter class. We prove (Theorem 2.17) that the complete prime filter classes are *precisely* the classes of theories that have a single characteristic indecomposable Fraïssé class. It follows that if $\mathscr{F}$ is the family of all complete prime filter classes—i.e., the class of all irreducible $\trianglelefteq$-dividing-lines—then $|\mathscr{F}| \leq 2^{\aleph_0}$.

We also carry out a detailed analysis of the Fraïssé classes $\mathbf{H}_r$ of r-hypergraphs (for each $2 \leq r < \omega$) in the context of $\trianglelefteq$. These are all indecomposable Fraïssé classes, and one result of this analysis is the finding that $\mathscr{F}$ is infinite; in fact, it contains an infinite strictly-nested chain (cf. [19]). We also find that excluding/forbidding a k-clique ($k > r$, obtaining another Fraïssé class $\mathbf{H}_{r,k}$) has no effect on the associated dividing-line. Somewhat surprisingly, adding an order to $\mathbf{H}_r$ in an unconstrained way, obtaining the class $\mathbf{H}_r^<$ of ordered r-hypergraphs, also has no effect on the associated dividing-line. This is a special case of a general phenomenon: we prove that for a Fraïssé class $\mathbf{K}$, if the generic theory $T_{\mathbf{K}}$ is unstable, then $\mathbf{K}$ and $\mathbf{K}^<$ yield the same dividing-line. We also find that adding additional symmetric irreflexive relations does not change the associated dividing-line; that is to say, for a Fraïssé class $\mathbf{S}$ of societies (in the terminology of [22]), **S** characterizes the same dividing-line as does $\mathbf{H}_r$, where r is the maximum arity of a relation symbol in the language of **S**.

The attempt to build a general framework for collapse-of-indiscernibles dividing-lines specifically seems to have failed for the moment because, as far as we know at present, an indecomposable Fraïssé class need not be Ramsey or even Ramsey-expandable.[2] However, we do manage to prove that any dividing-line arising from an unstable Ramsey-expandable Fraïssé class *is* a collapse-of-indiscernibles dividing-line. Since all of the $\mathbf{H}_r$'s are unstable Ramsey-expandable Fraïssé classes, it follows that there is an infinite strict chain of collapse-of-indiscernibles dividing-lines, all of which are irreducible.

1.1 Outline of the paper

Section 1: In the remainder of this section, we introduce some notation, basic definitions, and conventions that are used throughout the rest of the paper.
Section 2: We introduce the infrastructure both for discussing the (positive) local combinatorics of a first-order theory and for comparing two theories based on their local combinatorics, arriving at the ordering $\trianglelefteq$. Around this ordering, we define precisely what is meant by a "complete prime filter class" after discussing at some length why, we believe, this formulation captures the idea of an irreducible dividing-line. We also state and prove Theorem 2.17, showing that the irreducible dividing-lines are precisely the ones characterized by a single indecomposable Fraïssé class. In all of our definitions to this point, we will have used only one-sorted languages for Fraïssé classes, and to conclude this section, we show that there would be, in fact, nothing gained by allowing multiple sorts.
Section 3: In this section, we reconnect with the collapse-of-indiscernibles dividing-lines that impelled this project to begin with. It is a fact that the generic model of an algebraically trivial Fraïssé class admits a generic linear ordering. It is also a fact that the generic model of a Ramsey class carries a 0-definable linear ordering, and many of the best-known Ramsey classes are obtained just by adding a linear order "freely" to a Fraïssé class whose generic model is NSOP (as in the passage from the class **G** of finite graphs to the class $\mathbf{G}^<$ of finite ordered graphs). We examine in detail whether or not the generic order-expansion of a Fraïssé class induces a different dividing-line from its precursor's, showing that if the precursor's generic theory is unstable, then the additional ordering adds nothing. This allows us to show that if **K** is a Ramsey-*expandable* Fraïssé classes with unstable generic theory, then $\mathfrak{C}_{\mathbf{K}}$ is a collapse-of-indiscernibles dividing-line.
Section 4: By way of some case studies, we examine the structure of the $\trianglelefteq$-ordering and the set of irreducible dividing-lines. To start, we identify some theories that are maximum under $\trianglelefteq$. We then examine the classes $\mathbf{H}_r$ $(0 < r < \omega)$ of hypergraphs, verifying that they are all indecomposable and that they induce a strict chain of irreducible dividing-lines. Along the way, we also show that certain natu-

[2] Here, a class **K** is "Ramsey-expandable" if the class $\mathbf{K}^<$ obtained by adding an order in an unconstrained way is a Ramsey class. This definition is not standard.

ral perturbations of hypergraphs—excluding cliques, adding additional irreflexive symmetric relations—do not have any effect on the associated dividing-lines.
Section 5: To conclude the paper, we discuss some open questions about the structure of the $\trianglelefteq$-ordering and the irreducible classes generated by this ordering. We also explore questions around the relationship between these classes and collapse-of-indiscernibles dividing-lines.

1.2 Notation and conventions

Definition 1.1 We write $\mathbb{T}$ for the class of all complete theories with infinite models that eliminate imaginaries. If left unspecified, the language of a theory $T \in \mathbb{T}$ is $\mathscr{L}_T$, which may be, and usually is, many-sorted.

Convention When a theory T arises that may not eliminate imaginaries, we freely identify T with T^{eq}, which of course eliminates imaginaries.

Definition 1.2 Let $\mathscr{L}$ be a finite relational language (meaning that its signature sig($\mathscr{L}$) consists of finitely many relation symbols and no function symbols or constant symbols). $\mathbf{Fin}(\mathscr{L})$ denotes the class of all finite $\mathscr{L}$-structures. A Fraïssé class in $\mathscr{L}$ is a sub-class $\mathbf{K} \subseteq \mathbf{Fin}(\mathscr{L})$ satisfying the following:

Heredity (HP): For all $A, B \in \mathbf{Fin}(\mathscr{L})$, if $A \leq B$ and $B \in \mathbf{K}$, then $A \in \mathbf{K}$.
Joint-embedding (JEP): For any two $B_1, B_2 \in \mathbf{K}$, there are $C \in \mathbf{K}$ and embeddings $B_1 \to C$ and $B_2 \to C$.
Amalgamation (AP): For any $A, B_1, B_2 \in \mathbf{K}$ and any embeddings $f_i : A \to B_i$ $(i = 1, 2)$, there are $C \in \mathbf{K}$ and embeddings $f_i' : B_i \to C$ $(i = 1, 2)$ such that $f_1' \circ f_1 = f_2' \circ f_2$.

According to [9], a Fraïssé class $\mathbf{K}$ has a generic model (or Fraïssé limit) $\mathcal{A}$ satisfying the following properties:

- $\mathbf{K}$-universality: For every $B \in \mathbf{K}$, there is an embedding $B \to \mathcal{A}$.
- $\mathbf{K}$-closedness: For every finite $X \subset A$, the induced substructure $\mathcal{A} \restriction X$ is in $\mathbf{K}$.
- Ultrahomogeneity (or $\mathbf{K}$-homogeneity): For every $X \subset_{\mathrm{fin}} A$ and every embedding $f : \mathcal{A} \restriction X \to \mathcal{A}$, there is an automorphism $g \in Aut(\mathcal{A})$ extending f.

Since we are working with only finite relational languages (for Fraïssé classes), the generic theory $T_{\mathbf{K}} = \mathrm{Th}(\mathcal{A})$ of $\mathbf{K}$ is always $\aleph_0$-categorical and eliminates quantifiers.

In general, a theory T is *algebraically trivial* if $\mathrm{acl}^{\mathcal{M}}(A) = A$ whenever $\mathcal{M} \models T$ and $A \subseteq M$. A Fraïssé class $\mathbf{K}$ is called *algebraically trivial* just in case $T_{\mathbf{K}}$ is algebraically trivial, and this condition can be characterized by strengthened amalgamation conditions:

Disjoint-JEP: For any two $B_1, B_2 \in \mathbf{K}$, there are $C \in \mathbf{K}$ and embeddings $f_1 : B_1 \to C$ and $f_2 : B_2 \to C$ such that $f_1 B_1 \cap f_2 B_2 = \emptyset$.
Disjoint-AP: For any $A, B_1, B_2 \in \mathbf{K}$ and any embeddings $f_i : A \to B_i$ $(i = 1, 2)$, there are $C \in \mathbf{K}$ and embeddings $f_i' : B_i \to C$ $(i = 1, 2)$ such that $f_1' \circ f_1 = f_2' \circ f_2$ and $f_1' B_1 \cap f_2' B_1 = f_1' f_1 A = f_2' f_2 A$.

That is, a Fraïssé class $\mathbf{K}$ is algebraically trivial if and only if it has disjoint-JEP and disjoint-AP (see [3,4]).

Convention For Fraïssé classes, we allow only finite relational signatures.

Definition 1.3 Let $\mathbf{K}_0, \dots, \mathbf{K}_{n-1}$ be algebraically trivial Fraïssé classes in languages $\mathscr{L}_0, \dots, \mathscr{L}_{n-1}$, respectively, such that $\mathrm{sig}(\mathscr{L}_i) \cap \mathrm{sig}(\mathscr{L}_j) = \emptyset$ whenever $i < j < n$. Let $\mathcal{A}_0, \dots, \mathcal{A}_{n-1}$ be the generic models of $\mathbf{K}_0, \dots, \mathbf{K}_{n-1}$, respectively. Let $\Pi_i \mathscr{L}_i$ be the language with signature $\bigcup_i \mathrm{sig}(\mathscr{L}_i)$. We define $\Pi_i \mathbf{K}_i$ to be the class of finite $\Pi_i \mathscr{L}_i$-structures of the form $B = B_0 \times \cdots \times B_{n-1}$, where $B_0 \in \mathbf{K}_0, \dots, B_{n-1} \in \mathbf{K}_{n-1}$, with interpretations

$$R^B = \left\{ (\overline{a}_0, \dots, \overline{a}_{r-1}) \in B^r : (a_{0,i}, \dots, a_{r-1,i}) \in R^{B_i} \right\}$$

for each $i < n$ and $R^{(r)} \in \mathrm{sig}(\mathscr{L}_i)$.

In general, $\Pi_i \mathbf{K}_i$ is not a Fraïssé class because it need not have HP, but it is not hard to verify that it does have AP and JEP. (To see that $\Pi_i \mathbf{K}_i$ need not have HP, consider the case where $\mathbf{K}_0, \mathbf{K}_1$ are two copies of the class of all finite linear orders with signatures $\{<_0\}, \{<_1\}$, respectively. Given, say, $A_0 = \{0 <_0 1\} \in \mathbf{K}_0$ and $A_1 = \{0 <_1 1\} \in \mathbf{K}_1$, the induced substructure of $A_0 \times A_1$ on the subset $\{(0,0), (0,1), (1,0)\}$ is not in $\mathbf{K}_0 \times \mathbf{K}_1$.) Since $\Pi_i \mathbf{K}_i$ has AP and JEP, it does have a well-defined generic model, and one can show that this generic model is $\aleph_0$-categorical (by a result of [15]) and algebraically trivial.

We have yet to define the ordering $\trianglelefteq$, but with the reader's indulgence, we use it now to define indecomposability for algebraically trivial Fraïssé classes.

Definition 1.4 Given an arbitrary algebraically trivial Fraïssé class $\mathbf{K}$ in $\mathscr{L}$ with generic model $\mathcal{A}$, a *factorization of* $\mathbf{K}$ is a list $(\mathbf{K}_0, \dots, \mathbf{K}_{n-1})$ of algebraically trivial Fraïssé classes $\mathbf{K}_0, \dots, \mathbf{K}_{n-1}$ $(n > 1)$ for which there is an injection $u : A \to B$, where $\mathcal{B}$ is the generic model of $\Pi_i \mathbf{K}_i$, such that for all k and $\overline{a}, \overline{a}' \in A^k$,

$$\mathrm{qftp}^{\mathcal{A}}(\overline{a}) = \mathrm{qftp}^{\mathcal{A}}(\overline{a}') \Leftrightarrow \mathrm{tp}^{\mathcal{B}}(u\overline{a}) = \mathrm{tp}^{\mathcal{B}}(u\overline{a}').$$

We then say that $\mathbf{K}$ *is indecomposable* if for any factorization $(\mathbf{K}_0, \dots, \mathbf{K}_{n-1})$ of $\mathbf{K}$, there is an $i < n$ such that $T_{\mathbf{K}} \trianglelefteq T_{\mathbf{K}_i}$ (see Definition 2.7).

2 Comparing the local combinatorics of theories

In this section, we develop and formalize the idea of (positive) local combinatorics of first-order theories, showing that the local combinatorics of a theory on the finite level is captured by Fraïssé classes "embedded" in models of that theory.

Convention For all of this section, we fix a countably infinite set A. It will serve as the universe of generic models of Fraïssé classes involved in various theories $T \in \mathbb{T}$.

2.1 Infrastructure

We will understand the local combinatorics of a theory T as a collection of functions from finite subsets of A into models of T—see Definition 2.1. (In order to avoid the "geometric" behavior of T, we will require that these functions have strong algebraic-triviality properties.) Such a function and a list of formulas of the language of T will induce a structure on the domain of the function, and really, it is this structure that encodes some of the combinatorics of T—see Definition 2.2.

Definition 2.1 Let $T \in \mathbb{T}$.

- We define $\mathbf{F}(T)$ to be the set of injections $f : B \to \|\mathcal{M}\|$ where:
  - $B \subset_{\text{fin}} A$ and $\mathcal{M} \models T$.
  - For all $a, b \in B$, $f(a)$ and $f(b)$ are in the same sort of $\mathcal{M}$.
  - For every $B_0 \subsetneq B$, $\text{tp}(f(B)/f(B_0))$ is non-algebraic.
- $\mathbf{F}(T)$ is the set of functions $F : A \to \|\mathcal{M}\|$, for some $\mathcal{M} \models T$, such that $F{\restriction}B \in \mathbf{F}(T)$ for all $B \subset_{\text{fin}} A$.

Convention Let $T \in \mathbb{T}$. Consider a sequence

$$\boldsymbol{\varphi} = (\varphi_0(x_0, \ldots, x_{n_0-1}), \ldots, \varphi_{N-1}(x_0, \ldots, x_{n_{N-1}-1}))$$

of $\mathscr{L}_T$-formulas. Whenever we speak of sequences of formulas, we understand that the sequence is finite and all of the free variables range over a single common sort which we call *the sort of* $\boldsymbol{\varphi}$. Associated with $\boldsymbol{\varphi}$, we have a language $\mathscr{L}_{\boldsymbol{\varphi}}$ with signature $\left\{R_0^{(n_0)}, \ldots, R_{N-1}^{(n_{N-1})}\right\}$. If $\boldsymbol{\varphi}^1$ and $\boldsymbol{\varphi}^2$ have the same length and coordinate-wise have the same arities, then the signatures of $\mathscr{L}_{\boldsymbol{\varphi}^1}$ and $\mathscr{L}_{\boldsymbol{\varphi}^2}$ are identical, so we identify $\mathscr{L}_{\boldsymbol{\varphi}^1}$ and $\mathscr{L}_{\boldsymbol{\varphi}^2}$ to compare structures in these languages.

Definition 2.2 Let $T \in \mathbb{T}$, and $\boldsymbol{\varphi}$ be a sequence of formulas of $\mathscr{L}_T$.

- $\mathbf{F}_{\boldsymbol{\varphi}}(T)$ is the set of all $f \in \mathbf{F}(T)$ such that for each $a \in \text{dom}(f)$, $f(a)$ is in the sort of $\boldsymbol{\varphi}$. We define $\mathbf{F}_{\boldsymbol{\varphi}}(T)$ similarly. In both cases, $f \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$ or $F \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$ is said to be *compatible with* $\boldsymbol{\varphi}$.
- Let $f : B \to \|\mathcal{M}\|$ be in $\mathbf{F}_{\boldsymbol{\varphi}}(T)$ (for some $B \subset_{\text{fin}} A$ and $\mathcal{M} \models T$). Then, we write $B_{\boldsymbol{\varphi}}(f)$ for the $\mathscr{L}_{\boldsymbol{\varphi}}$-structure with universe B and interpretations

$$R_i^{B_{\boldsymbol{\varphi}}(f)} = \left\{\overline{a} \in B^{n_i} : \mathcal{M} \models \varphi_i(f\overline{a})\right\}.$$

  We then define $\mathsf{Age}_{\boldsymbol{\varphi}}(T) = \left\{B_{\boldsymbol{\varphi}}(f) : f \in \mathbf{F}_{\boldsymbol{\varphi}}(T)\right\}$ up to isomorphism.
- For $F \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$, we define an $\mathscr{L}_{\boldsymbol{\varphi}}$-structure $\mathcal{A}_{\boldsymbol{\varphi}}(F)$ with universe A and interpretations

$$R_i^{\mathcal{A}_{\boldsymbol{\varphi}}(F)} = \left\{\overline{a} \in A^{n_i} : \mathcal{M} \models \varphi_i(f\overline{a})\right\}.$$

  We then define $\mathsf{Age}_{\boldsymbol{\varphi}}(F) = \mathsf{Age}(\mathcal{A}_{\boldsymbol{\varphi}}(F))$.

- Let $F \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$. We say that F is *$\boldsymbol{\varphi}$-resolved* if for any $B \subsetneq B' \subset_{\text{fin}} A$, there are $B'_0, \ldots, B'_i, \ldots, \subset_{\text{fin}} A$, such that $B'_i \cap B'_j = B$ whenever $i < j < \omega$, and $\text{tp}_{\boldsymbol{\varphi}}(F(B'_i)/F(B)) = \text{tp}_{\boldsymbol{\varphi}}(F(B')/F(B))$ for all $i < \omega$.
- Finally, we define $\mathbf{S}_{\boldsymbol{\varphi}}(T) = \left\{\mathcal{A}_{\boldsymbol{\varphi}}(F) : F \in \mathbf{F}_{\boldsymbol{\varphi}}(T), \boldsymbol{\varphi}\text{-}\textit{resolved}\right\}$.

We have now defined the various objects that encode the positive local combinatorics of a theory T. In the proposition below, we show that all of this is, essentially, explicable on the level of algebraically trivial Fraïssé classes "embedded" in models of T as $\mathcal{A}_{\boldsymbol{\varphi}}(F)$'s.

Proposition 2.3 *Let $T \in \mathbb{T}$, and let $\boldsymbol{\varphi} = (\varphi_0(x_0, \ldots, x_{n_0-1}), \ldots, \varphi_{N-1}(x_0, \ldots, x_{n_{N-1}-1}))$ be a sequence of $\mathscr{L}_T$-formulas. Let $F \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$.*

1. *There is a $\boldsymbol{\varphi}$-resolved $F' \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$ such that $\mathsf{Age}_{\boldsymbol{\varphi}}(F) \subseteq \mathsf{Age}_{\boldsymbol{\varphi}}(F')$.*
2. *If F is $\boldsymbol{\varphi}$-resolved, then $\mathsf{Age}_{\boldsymbol{\varphi}}(F)$ is an algebraically trivial Fraïssé class and $\mathcal{A}_{\boldsymbol{\varphi}}(F)$ is the Fraïssé limit of $\mathsf{Age}_{\boldsymbol{\varphi}}(F)$.*

Proof For Item 1, let F be given. Let $\sim$ be the equivalence relation on $\mathbf{F}_{\boldsymbol{\varphi}}(T)$ given by

$$F_1 \sim F_2 \Leftrightarrow (\forall B \subset_{\text{fin}} A)\left[B_{\boldsymbol{\varphi}}(F_1 \restriction B) = B_{\boldsymbol{\varphi}}(F_2 \restriction B)\right].$$

Let $\mathbb{X} = \mathbf{F}_{\boldsymbol{\varphi}}(T)/{\sim}$. For $f \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$, we define $[f]$ to be the set of classes $F'/{\sim}$ such that $B_{\boldsymbol{\varphi}}(f) = B_{\boldsymbol{\varphi}}(F' \restriction \text{dom}(f))$; then $\tau_0 = \left\{[f] : f \in \mathbf{F}_{\boldsymbol{\varphi}}(T)\right\}$ may be viewed as a base of clopen sets for a Stone topology on $\mathbb{X}$ (τ_0 generates a boolean algebra and $\mathbb{X}$ is homeomorphic to its Stone space). We define two families of subsets of $\mathbf{F}_{\boldsymbol{\varphi}}(T)$ as follows:

- Let $B_0, B \in \mathsf{Age}_{\boldsymbol{\varphi}}(F)$, f_0, u_0 be such that $B_0 \leq B$, and let $f_0 \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$ such that $u_0 : B_0 \cong B_{\boldsymbol{\varphi}}(f_0)$.
  Then, $f \in R^n(B, u_0, f_0)$ iff the following holds:
  If $f_0 \subseteq f$ and f is compatible with $\boldsymbol{\varphi}$, **then** there are $f_0 \subseteq f_1, \ldots, f_n \subseteq f$ and $u_0 \subseteq u_1, \ldots, u_n$ such that $u_i : B \cong B_{\boldsymbol{\varphi}}(f_i)$ for each $1 \leq i \leq n$ and $\text{img}(f_i) \cap \text{img}(f_j) = \text{img}(f_0)$ for all $1 \leq i < j \leq n$.
- Let $B \in \mathsf{Age}_{\boldsymbol{\varphi}}(F)$. Then $f \in R_F(B)$ iff f is compatible with $\boldsymbol{\varphi}$ and there is some $f_0 \subseteq f$ such that $B_{\boldsymbol{\varphi}}(f_0) \cong B$

By the definition of $\mathbf{F}_{\boldsymbol{\varphi}}(T)$ and non-algebraicity, it is not difficult to check that

$$\begin{aligned}\Gamma_F = \{[f] : f \in R_F(B) \text{ for some } B\} \cup \\ \times \left\{[f] : f \in R^n(B, u_0, f_0) \text{ for some } n, B, u_0, f_0\right\}\end{aligned}$$

is a countable family of dense-open sets, so as $\mathbb{X}$ has the Baire property (because it is compact Hausdorff), the intersection $\bigcap \Gamma_F$ is non-empty. For any $F'/{\sim}$ in $\bigcap \Gamma_F$, F' satisfies the requirements of Item 1.

For Item 2: First, we observe that for any $F' \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$, $\mathsf{Age}_{\boldsymbol{\varphi}}(F')$ has HP. Moreover, if F is $\boldsymbol{\varphi}$-resolved, then by definition, $\mathsf{Age}_{\boldsymbol{\varphi}}(F)$ has disjoint-JEP and disjoint-AP—so $\mathsf{Age}_{\boldsymbol{\varphi}}(F)$ is an algebraically trivial Fraïssé class. Since F is $\boldsymbol{\varphi}$-resolved, $\mathcal{A}_{\boldsymbol{\varphi}}(F)$ is the generic model of $\mathsf{Age}_{\boldsymbol{\varphi}}(F)$. □

Corollary 2.4 *Let $T \in \mathbb{T}$, and let $\boldsymbol{\varphi}$ be a sequence of $\mathscr{L}_T$-formulas. Then, for any $f \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$, there is a $\boldsymbol{\varphi}$-resolved $F \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$ such that $B_{\boldsymbol{\varphi}}(f) \in \mathsf{Age}_{\boldsymbol{\varphi}}(F)$.*

Proof Given $f \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$, we choose any $F_0 \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$ such that $f \subset F_0$. Applying Proposition 2.3, we then obtain a $\boldsymbol{\varphi}$-resolved $F \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$ such that $B_{\boldsymbol{\varphi}}(f) \in \mathsf{Age}_{\boldsymbol{\varphi}}(F_0) \subseteq \mathsf{Age}_{\boldsymbol{\varphi}}(F)$. □

Observation 2.5 Let $T \in \mathbb{T}$, and let $\boldsymbol{\varphi}$ be a sequence of $\mathscr{L}_T$-formulas. Let $F_0, \ldots, F_n, \ldots \in \mathbf{F}_{\varphi}(T)$. Then, there are a partition $\{A_n\}_{n<\omega}$ of A into infinite classes, injections $u_n : A \to A_n$, and an $F \in \mathbf{F}_{\varphi}(T)$ such that for each $n < \omega$, $\mathcal{A}_{\boldsymbol{\varphi}}(F_n) = \mathcal{A}_{\boldsymbol{\varphi}}(F \circ u_n)$.

Proof Let $F_0, F_1, \ldots, F_n, \ldots \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$. By compactness and the definition of $\mathbf{F}_{\boldsymbol{\varphi}}(T)$, we may assume that, for $i < \omega$, pairwise distinct $j_0, \ldots, j_{n-1} < \omega$, $B, C_0, \ldots, C_{n-1} \subset_{\text{fin}} A$:

- If $i \notin \{j_0, \ldots, j_{n-1}\}$, then $\text{tp}\left(F_i(B) \big/ F_{i_0}(C_0) \cup \ldots \cup F_{i_{n-1}}(C_{n-1})\right)$ is non-algebraic
- If $i = j_0$, then $\text{tp}\left(F_i(B \backslash C_0) \big/ F_{i_0}(C_0) \cup \ldots \cup F_{i_{n-1}}(C_{n-1})\right)$ is non-algebraic.

Now, we arbitrarily choose a partition $\{A_n\}_n$ of A in which each A_n is infinite, and for each n, we choose a bijection $v_n : A_n \to A$. Then, we define $F \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$ by setting $F(a) = F_n(v_n(a))$ whenever $a \in A_n$. For each n, $F \circ v_n^{-1} = F_n$, and it is clear that $\mathcal{A}_{\boldsymbol{\varphi}}(F_n) = \mathcal{A}_{\boldsymbol{\varphi}}(F \circ v_n^{-1})$. □

Observation 2.6 The space $\mathbb{X}_{\boldsymbol{\varphi}}(T)$ (denoted $\mathbb{X}$) in the proof of Proposition 2.3 has a countable dense subset $W_{\boldsymbol{\varphi}}(T)$ such that F is $\boldsymbol{\varphi}$-resolved for each $F/\sim$ in $W_{\boldsymbol{\varphi}}(T)$.

2.2 The ordering $\trianglelefteq$ (and some more infrastructure)

We have discussed and formalized our notion of local combinatorics of first-order theories, and now, we use these ideas to formulate an ordering $\trianglelefteq$ of $\mathbb{T}$ that will allow us to compare theories based on their local combinatorics. Initially, we present a definition in which T_1 and T_2 are compared by way of finite structures, $B_{\boldsymbol{\varphi}}(f)$'s, but in Proposition 2.9, we demonstrate (unsurprisingly, given Proposition 2.3) that this is equivalent to comparing T_1 and T_2 based on (generic models of) algebraically trivial Fraïssé classes.

Definition 2.7 Let $T_1, T_2 \in \mathbb{T}$. Then we assert $T_1 \trianglelefteq T_2$ if for every finite sequence $\boldsymbol{\varphi}^1 = (\varphi^1_0, \ldots, \varphi^1_{N-1})$ of $\mathscr{L}_{T_1}$-formulas, there is $\boldsymbol{\varphi}^2 = (\varphi^2_0, \ldots, \varphi^2_{N-1})$ in $\mathscr{L}_{T_2}$, coordinate-wise of the same arities, such that $\mathsf{Age}_{\boldsymbol{\varphi}^1}(T_1) \subseteq \mathsf{Age}_{\boldsymbol{\varphi}^2}(T_2)$.

Observation 2.8 Let $T_1, T_2 \in \mathbb{T}$. If T_1 is interpretable in T_2, then $T_1 \trianglelefteq T_2$.

Proposition 2.9 *Let $T_1, T_2 \in \mathbb{T}$. The following are equivalent:*

1. $T_1 \trianglelefteq T_2$.
2. *For every finite sequence $\boldsymbol{\varphi}^1 = (\varphi^1_0, \ldots, \varphi^1_{N-1})$ of $\mathscr{L}_{T_1}$-formulas, there is $\boldsymbol{\varphi}^2 = (\varphi^2_0, \ldots, \varphi^2_{N-1})$ in $\mathscr{L}_{T_2}$, coordinate-wise of the same arities, such that for every $\boldsymbol{\varphi}^1$-resolved $F_1 \in \mathbf{F}_{\boldsymbol{\varphi}^1}(T_1)$, there is a $\boldsymbol{\varphi}^2$-resolved $F_2 \in \mathbf{F}_{\boldsymbol{\varphi}^2}(T_2)$ such that $\mathcal{A}_{\boldsymbol{\varphi}^1}(F_1) = \mathcal{A}_{\boldsymbol{\varphi}^2}(F_2)$.*

3. *For every finite sequence* $\boldsymbol{\varphi}^1 = (\varphi_0^1, \ldots, \varphi_{N-1}^1)$ *of* $\mathscr{L}_{T_1}$*-formulas, there is* $\boldsymbol{\varphi}^2 = (\varphi_0^2, \ldots, \varphi_{N-1}^2)$ *in* $\mathscr{L}_{T_2}$*, coordinate-wise of the same arities, such that* $\mathbf{S}_{\boldsymbol{\varphi}^1}(T_1) \subseteq \mathbf{S}_{\boldsymbol{\varphi}^2}(T_2)$.

Proof 2⇔3 is by definition of $\mathbf{S}_{\boldsymbol{\varphi}}(T)$. Let us write $T_1 \trianglelefteq' T_2$ to mean that the condition expressed in item 2 holds.

On the one hand, suppose $T_1 \trianglelefteq' T_2$, and let $\boldsymbol{\varphi}^1 = (\varphi_0^1, \ldots, \varphi_{N-1}^1)$, a sequence of $\mathscr{L}_{T_1}$-formulas, be given. Since $T_1 \trianglelefteq' T_2$, let $\boldsymbol{\varphi}^2$ be the promised sequence of $\mathscr{L}_{T_2}$-formulas. To show that $\mathsf{Age}_{\boldsymbol{\varphi}^1}(T_1) \subseteq \mathsf{Age}_{\boldsymbol{\varphi}^2}(T_2)$, let $f_1 \in \mathbf{F}_{\boldsymbol{\varphi}^1}(T_1)$ be given. By Corollary 2.4, we obtain a $\boldsymbol{\varphi}^1$-resolved $F_1 \in \mathbf{F}_{\boldsymbol{\varphi}^1}(T_1)$ such that $B_{\boldsymbol{\varphi}^1}(f_1) \in \mathsf{Age}_{\boldsymbol{\varphi}^1}(F_1)$. By our choice of $\boldsymbol{\varphi}^2$, then, there is a $\boldsymbol{\varphi}^2$-resolved $F_2 \in \mathbf{F}_{\boldsymbol{\varphi}^2}(T_2)$ such that $\mathcal{A}_{\boldsymbol{\varphi}^1}(F_1) = \mathcal{A}_{\boldsymbol{\varphi}_2}(F_2)$. We have $B_{\boldsymbol{\varphi}^1}(f_1) \in \mathsf{Age}_{\boldsymbol{\varphi}^2}(F_2) \subseteq \mathsf{Age}_{\boldsymbol{\varphi}^2}(T_2)$. We have shown that $T_1 \trianglelefteq T_2$, which proves 2, 3 ⇒ 1.

For 1 ⇒ 2, 3, suppose $T_1 \trianglelefteq T_2$. Again, let $\boldsymbol{\varphi}^1 = (\varphi_0^1, \ldots, \varphi_{N-1}^1)$, a sequence of $\mathscr{L}_{T_1}$-formulas, be given. Let $\boldsymbol{\varphi}^2$ be the sequence of $\mathscr{L}_{T_2}$-formulas promised by $T_1 \trianglelefteq T_2$. Suppose $F_1 \in \mathbf{F}_{\boldsymbol{\varphi}^1}(T_1)$ is $\boldsymbol{\varphi}^1$-resolved.

Let $a_0, a_1, \ldots, a_n, \ldots$ be an enumeration of A, and for each $n < \omega$, let $f_n = F_1 \restriction \{a_0, \ldots, a_{n-1}\}$. Since $T_1 \trianglelefteq T_2$, for each n, we obtain $f_n' \in \mathbf{F}_{\boldsymbol{\varphi}^2}(T_2)$ with domain $\{a_0, \ldots, a_{n-1}\}$ such that $B_{\boldsymbol{\varphi}^1}(f_n) = B_{\boldsymbol{\varphi}^2}(f_n')$. By definition of $\mathbf{F}_{\boldsymbol{\varphi}^2}(T_2)$ and non-algebraicity, we can also ensure that $f_n' \subset f_{n+1}'$ for all $n < \omega$, so $F_2 = \bigcup_n f_n'$ is in $\mathbf{F}_{\boldsymbol{\varphi}^2}(T_2)$. It is not difficult to verify that F_2 is $\boldsymbol{\varphi}^2$-resolved and that $\mathcal{A}_{\boldsymbol{\varphi}^1}(F_1) = \mathcal{A}_{\boldsymbol{\varphi}^2}(F_2)$. Thus, $T_1 \trianglelefteq' T_2$—as desired. □

Of course, generic models and generic theories of algebraically trivial Fraïssé classes will play a key role in our work later in this paper. We make one more convenient definition and two observations about $\trianglelefteq$ as it pertains to generic theories of Fraïssé classes. These two observations—2.11 and 2.12—will be used repeatedly in the sequel, often without comment.

Definition 2.10 For each theory $T \in \mathbb{T}$, we define

$$Q_T = \left\{ \mathrm{Th}(\mathcal{A}_{\boldsymbol{\varphi}}(F))^{\mathrm{eq}} : \begin{array}{l} \boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_{N-1}) \text{ in } \mathscr{L}_T, \\ F \in \mathbf{F}_{\boldsymbol{\varphi}}(T) \ \boldsymbol{\varphi}\text{-resolved}, \end{array} \right\}$$

Later on, it will also be convenient to work with the following sub-class of theories:

$$\tilde{\mathbb{T}} := \left\{ T_{\mathbf{K}}^{\mathrm{eq}} \ : \ \mathbf{K} \text{ is an algebraically trivial Fraïssé class} \right\}.$$

Observation 2.11 In the definition of Q_T, each F is $\boldsymbol{\varphi}$-resolved, so by Proposition 2.3, $\mathcal{A}_{\boldsymbol{\varphi}}(F)$ is the Fraïssé limit of $\mathsf{Age}_{\boldsymbol{\varphi}}(F)$, an algebraically trivial Fraïssé class. Hence, $Q_T \subseteq \tilde{\mathbb{T}}$. Notice further that $T_0 \trianglelefteq T$ for all $T_0 \in Q_T$.

Observation 2.12 Let $\mathbf{K}$ be an algebraically trivial Fraïssé class in a finite relational language $\mathscr{L}$. Then, for any complete 1-sorted theory T, the following are equivalent:

1. $T_{\mathbf{K}} \trianglelefteq T$.
2. There are $0 < m < \omega$, formulas $\varphi_R(\overline{x}_0, \ldots, \overline{x}_{r-1}) \in \mathscr{L}_T$ for each $R^{(r)} \in \mathrm{sig}(\mathscr{L})$ (where each $\overline{x}_j$ is a non-repeating m-tuple of variables), and $F \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$ such that $\mathbf{K} = \mathsf{Age}_{\boldsymbol{\varphi}}(F)$.

(The number m is then said to *witness* $T_{\mathbf{K}} \trianglelefteq T$.) In particular, for algebraically trivial Fraïssé classes $\mathbf{K}_1, \mathbf{K}_2$ in languages $\mathscr{L}_1, \mathscr{L}_2$, with generic models $\mathcal{A}_1, \mathcal{A}_2$, respectively, the following are equivalent:

1. $T_{\mathbf{K}_1} \trianglelefteq T_{\mathbf{K}_2}$.
2. There are $0 < m < \omega$, an injection $u : A_1 \to A_2^m$, and quantifier-free formulas $\theta_R(\overline{x}_0, \ldots, \overline{x}_{r-1})$ of $\mathscr{L}_2$ ($|\overline{x}_i| = m$, $R^{(r)} \in \mathrm{sig}(\mathscr{L}_1)$) such that for all $R^{(r)} \in \mathrm{sig}(\mathscr{L}_1)$ and $a_0, \ldots, a_{r-1} \in A_1$,

$$\mathcal{A}_1 \models R(a_0, \ldots, a_{r-1}) \Leftrightarrow \mathcal{A}_2 \models \theta_R(u(a_0), \ldots, u(a_{r-1})).$$

2.3 Irreducible model-theoretic dividing-lines

2.3.1 Discussion: What is an "irreducible" dividing-line?

As we have already discussed at some length, a model-theoretic dividing-line amounts to a partition of $\mathbb{T}$ into two sub-classes—a sub-class $\mathfrak{C}$ of "wild" theories (unstable, IP, unsimple,...) and a complementary class $N\mathfrak{C} := \mathbb{T} \backslash \mathfrak{C}$ of "tame" theories. Although we prefer to work with theories from $N\mathfrak{C}$ in practice, we can characterize a dividing-line purely in terms of the "wild" class $\mathfrak{C}$. In principle, $\mathfrak{C}$ could be any sub-class of $\mathbb{T}$, but we have to demand more from $\mathfrak{C}$ if $N\mathfrak{C}$ is to have any practical value. This already suggests the most primitive requirement we make on irreducible dividing-lines (relative to any ordering $\leq$ of $\mathbb{T}$):

1. *Existence*: $\mathfrak{C}$ is not empty.
2. *Upward-closure*: If $T_1 \leq T_2$ and $T_1 \in \mathfrak{C}$, then $T_2 \in \mathfrak{C}$.

If we are to call $\mathfrak{C}$ an irreducible dividing-line, there are several "indivisibility" or "non-transience" requirements that seem unavoidable:

3. Consider a theory T obtained as a "disjoint union" of a family of theories $\{T_i\}_{i \in I}$; say, T has a family of sorts for each T_i, and these sorts are orthogonal in T. For the sake of irreducibility, if $T \in \mathfrak{C}$, then we should expect that at least one of the T_i's is in $\mathfrak{C}$. Otherwise membership in $\mathfrak{C}$ would appear to depend on two or more phenomena that occur (or not) independently of each other.
   That is to say, $\mathfrak{C}$ should be *prime*.
4. Consider a finite family of theories $T_0, \ldots, T_{n-1} \in \mathfrak{C}$. It would be very strange to require that *every* theory T that lies $\leq$-below each of $T_0, \ldots, T_{n-1}$ to be in $\mathfrak{C}$; if $\leq$ is in any way natural, this would presumably place the theory of an infinite set in $\mathfrak{C}$. However, for the sake of irreducibility, we should expect the fact that $T_0, \ldots, T_{n-1}$ are all in $\mathfrak{C}$ to have a single common explanation. That is, there should be *some* T in $\mathfrak{C}$ and that lies $\leq$-below each of $T_0, \ldots, T_{n-1}$.
   Thus, $\mathfrak{C}$ should have some sort of "finite intersection property".

5. Consider a descending chain $T_0 \geq T_1 \geq T_2 \geq \cdots$ of members of $\mathfrak{C}$. Again, we should expect the fact that $T_0, T_1, \ldots$ are all in $\mathfrak{C}$ to have a single common explanation, not infinitely many different explanations that, for no particular reason, happened upon a $\leq$-chain. Thus, we expect that this common explanation is witnessed through $\leq$ by a theory—that is, there should be some $T \in \mathfrak{C}$ that lives $\leq$-below all T_i's.
   Therefore, $\mathfrak{C}$ should have some sort of "completeness" property.

If we accept these strictures for a definition of "irreducible" dividing-line relative to an ordering $\leq$ of $\mathbb{T}$, then our definition in the next subsection is forced on us. If we settle on this (or any) definition of irreducible dividing-line, then it is reasonably natural to ask if irreducible dividing-lines admit "characterizing objects", and we address this question in Theorem 2.17.

2.3.2 Irreducibles: complete prime filter classes

Based on our discussion in Sect. 2.3.1, we now formalize the notion of an irreducible dividing-line relative to our ordering $\trianglelefteq$ of $\mathbb{T}$; this formalization is given in Definition 2.14 below. (Definition 2.13, which precedes it, just establishes some helpful notation.) Given the definition of $\trianglelefteq$, it is probably not surprising that classes $\mathfrak{C}_{\mathbf{K}} \subset \mathbb{T}$, defined from Fraïssé classes $\mathbf{K}$, will play an important role in the development, so we make these $\mathfrak{C}_{\mathbf{K}}$'s formal in Definition 2.16. Finally, in Theorem 2.17, we state the main result of this section, identifying irreducible dividing-lines with classes of theories defined from indecomposable Fraïssé classes.

Definition 2.13 For a set $S \subset \mathbb{T}$, we define

$$\downarrow S = \{T \in \mathbb{T} : (\forall T_1 \in S)\, T \trianglelefteq T_1\}$$
$$\uparrow S = \{T \in \mathbb{T} : (\forall T_0 \in S)\, T_0 \trianglelefteq T\}$$

which are the lower- and upper-cones of S.

Definition 2.14 Let $\mathfrak{C} \subseteq \mathbb{T}$. We say that $\mathfrak{C}$ is *irreducible* (or less succinctly, is a *complete prime filter class*) if the following hold:

- (Existence) $\mathfrak{C}$ is non-empty.
- (Filter properties)
  - If $T_0 \in \mathfrak{C}$, $T \in \mathbb{T}$, and $T_0 \trianglelefteq T$, then $T \in \mathfrak{C}$.
  - If $T_0, T_1, \ldots, T_{n-1} \in \mathfrak{C}$, then $\mathfrak{C} \cap \downarrow\{T_0, \ldots, T_{n-1}\} \neq \emptyset$.
- (Completeness) If $(T_i)_{i \in I}$ is a non-empty $\trianglelefteq$-chain of theories in $\mathfrak{C}$ (i.e., $I = (I, <)$ is a non-empty linear order, and for all $i, j \in I$, $i < j \Rightarrow T_i \trianglelefteq T_j$), then $\mathfrak{C} \cap \downarrow \{T_i\}_{i \in I} \neq \emptyset$.
- (Primality) For any set $S \subset \mathbb{T}$, if $\uparrow S \subseteq \mathfrak{C}$, then $\mathfrak{C} \cap S \neq \emptyset$.

Remark 2.15 Our completeness axiom is definitely stronger than required: In fact, the following $(2^{\aleph_0})^+$-*completeness condition* would suffice to obtain Theorem 2.17 below:

For any positive ordinal $\alpha < (2^{\aleph_0})^+$, for any descending $\trianglelefteq$-chain $(T_i)_{i<\alpha}$ (i.e., $i < j < \alpha \Rightarrow T_j \trianglelefteq T_i$) of members of $\mathfrak{C}$ of length α, $\mathfrak{C} \cap \downarrow \{T_i\}_{i<\alpha} \neq \emptyset$.

We conjecture that even this $(2^{\aleph_0})^+$-completeness condition is stronger than necessary to prove Theorem 2.17.

Definition 2.16 Let $\mathbf{K}$ be an algebraically trivial Fraïssé class. Then $\mathfrak{C}_{\mathbf{K}}$ is the class of theories $T \in \mathbb{T}$ for which there are a sequence of formulas $\boldsymbol{\varphi} = (\varphi_R(x_0, \ldots, x_{r-1}))_{R^{(r)} \in \mathrm{sig}(\mathscr{L}_{\mathbf{K}})}$ in $\mathscr{L}_T$ and a $\boldsymbol{\varphi}$-resolved $F \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$ such that $\mathrm{Age}_{\boldsymbol{\varphi}}(F) = \mathbf{K}$. In other words,

$$\mathfrak{C}_{\mathbf{K}} = \{T \in \mathbb{T} : T_{\mathbf{K}} \trianglelefteq T\}.$$

We note that if $\mathbf{K}_1$ and $\mathbf{K}_2$ are two algebraically trivial Fraïssé classes, then

$$T_{\mathbf{K}_1} \trianglelefteq T_{\mathbf{K}_2} \Leftrightarrow \mathfrak{C}_{\mathbf{K}_2} \subseteq \mathfrak{C}_{\mathbf{K}_1}.$$

As promised, we now state the main result of this section, which says that irreducible dividing-lines (relative to $\trianglelefteq$) have fairly concrete "characterizing objects", namely indecomposable Fraïssé classes, and that any class defined from one of these is, indeed, the "wild" class of an irreducible dividing-line. Of course, this reduces the project to identifying the indecomposable Fraïssé classes, and we take small steps in this project in the ensuing sections of the paper. The proof of Theorem 2.17 is given in the next subsection.

Theorem 2.17 *Let $\mathfrak{C}$ be a non-empty class of theories. The following are equivalent:*

1. *$\mathfrak{C}$ is irreducible.*
2. *$\mathfrak{C} = \mathfrak{C}_{\mathbf{K}}$ for some indecomposable algebraically trivial Fraïssé class $\mathbf{K}$.*

To conclude this subsection, we observe that the identification of irreducible dividing-lines with certain kinds of Fraïssé classes yields an easy upper bound on the number of such dividing-lines.

Corollary 2.18 *There are no more than $2^{\aleph_0}$ indecomposable algebraically trivial Fraïssé classes (in finite relational languages). So by Theorem 2.17, if $\mathscr{F}$ denotes the family of all irreducible dividing-lines (complete prime filter classes), then $|\mathscr{F}| \leq 2^{\aleph_0}$.*

Proof Let S be the set of functions $s : \omega \to \omega$ of finite support. For each $s \in S$, let $\mathrm{sig}(\mathscr{L}_s)$ be the signature with relation symbols $R_i^{(s(i))}$ for each $i \in \mathrm{supp}(s)$, and let $\mathscr{K}_s$ be the set of all algebraically trivial Fraïssé classes of finite $\mathscr{L}_s$-structures. We define $\mathscr{K}_s^*$ to be the set of all indecomposable algebraically trivial Fraïssé classes of finite $\mathscr{L}_s$-structures. Then

$$|\mathscr{F}| \leq \left|\bigcup_{s \in S} \mathscr{K}_s^*\right| \leq \aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0}.$$

∎

2.4 Proof of Theorem 2.17

We now turn to the proof of Theorem 2.17, which of course has two directions. The proof of 2 ⇒ 1 in Theorem 2.17 is quite short, so we give it immediately in the form of Proposition 2.19. The proof of 1 ⇒ 2 (Proposition 2.20, below) is somewhat more involved.

Proposition 2.19 *Let* $\mathbf{K}$ *be an indecomposable algebraically trivial Fraïssé class of finite hen* $\mathfrak{C}_{\mathbf{K}}$ *is irreducible.*

Proof Let $\mathscr{L}$ be the language of $\mathbf{K}$, say $\mathrm{sig}(\mathscr{L}) = \left\{R_0^{(r_0)}, \ldots, R_{k-1}^{(r_{k-1})}\right\}$. Only the primality of $\mathfrak{C}_{\mathbf{K}}$ is not altogether obvious. To prove primality, let $S \subset \mathbb{T}$ be a set of theories. Without loss of generality, we assume that for each $T \in S$, $\mathscr{L}_T$ is purely relational. We define a language $\mathscr{L}_S$ as follows:

- For each $T \in S$, for each sort X of $\mathscr{L}_T$, $\mathscr{L}_S$ has a sort $\mathsf{Y}_{T:\mathsf{X}}$
- For each $T \in S$, for each relation symbol $R \subseteq \mathsf{X}_0 \times \cdots \times \mathsf{X}_{n-1}$ of $\mathscr{L}_T$, $\mathscr{L}_S$ has a relation symbol $R_T \subseteq \mathsf{Y}_{T:\mathsf{X}_0} \times \cdots \times \mathsf{Y}_{T:\mathsf{X}_{n-1}}$

For an $\mathscr{L}_S$-structure $\mathcal{M}$, we take $\mathcal{M}_T$ to denote the restriction/reduct of $\mathcal{M}$ to the sorts $\mathsf{Y}_{T:\mathsf{X}}$ and symbols R_T associated with T. We define T_S to be the theory of $\mathscr{L}_S$-structures $\mathcal{M}$ such that $\mathcal{M}_T \vDash T$ for every $T \in S$. For $S_0 \subseteq S$, we define $\mathcal{M}_{S_0}$ similarly.

It is not hard to see that $T_S \in {\uparrow}S$. It is routine to verify that T_S is complete, that up to the obvious translations of formulas, $\bigcup_{T \in S} \mathscr{L}_T$ is an elimination set of T_S, and that T_S eliminates imaginaries (so $T_S \in \mathbb{T}$). Finally, we observe that for pairwise distinct $T_0, \ldots, T_{n-1} \in S$ and $\mathcal{M} \vDash T_S$, $\mathcal{M}_{T_0}, \ldots, \mathcal{M}_{T_{n-1}}$ are orthogonal in the sense that any 0-definable set D of $\mathcal{M}_{\{T_i\}_i}$ is equal to a union of sets of the form $D_0 \times \cdots \times D_{n-1}$, where D_i is a 0-definable set of $\mathcal{M}_{T_i}$ for each $i < n$.

Now, suppose that ${\uparrow}S \subseteq \mathfrak{C}_{\mathbf{K}}$—so of course, $T_S \in \mathfrak{C}_{\mathbf{K}}$. We must show that there is some $T \in S$ such that $T \in \mathfrak{C}_{\mathbf{K}}$. Since $T_S \in \mathfrak{C}_{\mathbf{K}}$, there are $0 < m < \omega$, formulas $\varphi_i(\overline{x}_0, \ldots, \overline{x}_{r_i-1}) \in \mathscr{L}_S$ for each $i < k$ (where each $\overline{x}_j$ is a non-repeating m-tuple of variables, say in the sort $\mathsf{Y}_{T_0:\mathsf{X}_0} \ldots \mathsf{Y}_{T_{m-1}:\mathsf{X}_{m-1}}$), and $F \in \mathbf{F}_{\boldsymbol{\varphi}}(T_S)$ such that $\mathbf{K} = \mathsf{Age}_{\boldsymbol{\varphi}}(F)$. For each $j < m$ and $i < k$, let φ_i^j be the reduct of φ_i to $\mathsf{Y}_{T_j:\mathsf{X}_j}$, let F_j be the restriction of F to $\mathsf{Y}_{T_j:\mathsf{X}_j}$, and let $\mathbf{K}_j = \mathsf{Age}_{\boldsymbol{\varphi}^j}(F_j)$. Then, $(\mathbf{K}_0, \ldots, \mathbf{K}_{m-1})$ is a factorization of $\mathbf{K}$. As $\mathbf{K}$ is indecomposable, it follows that $T_{\mathbf{K}} \trianglelefteq T_{\mathbf{K}_j} \trianglelefteq T_j$ for some $j < m$, and then $T_j \in \mathfrak{C}_{\mathbf{K}}$—as required. □

Proposition 2.20 *If* $\mathfrak{C}$ *is irreducible, then there is an indecomposable algebraically trivial Fraïssé class* $\mathbf{K}$ *such that* $\mathfrak{C} = \mathfrak{C}_{\mathbf{K}}$.

For the rest of this subsection (the proof of Proposition 2.20), we fix a complete prime filter class $\mathfrak{C}$.

The first important step in the proof of Proposition 2.20 is to identify the role of the Fraïssé class $\mathbf{K}$ in $\mathfrak{C}$ in terms of $\trianglelefteq$. Unsurprisingly, we find that $\mathbf{K}$ is chosen so that $T_{\mathbf{K}}$ is the $\trianglelefteq$-minimum element of $\tilde{\mathfrak{C}} = \tilde{\mathbb{T}} \cap \mathfrak{C}$, and we then demonstrate (Lemma 2.22) being minimum for an irreducible class $\mathfrak{C}$ is sufficient for indecomposability.

Observation 2.21 Let $\mathbf{K}_0, \ldots, \mathbf{K}_{n-1}$ be algebraically trivial Fraïssé classes, and let $\mathcal{B}$ be the generic model of $\prod_i \mathbf{K}_i$. For $T \in \mathbb{T}$, if $T_{\mathbf{K}_i} \trianglelefteq T$ for each $i < n$, then $\mathrm{Th}(\mathcal{B}) \trianglelefteq T$.

Lemma 2.22 *Let* $\mathbf{K}$ *be an algebraically trivial Fraïssé class such that* $T_{\mathbf{K}}$ *is* $\trianglelefteq$*-minimum in* $\tilde{\mathfrak{C}} = \tilde{\mathbb{T}} \cap \mathfrak{C}$. *Then* $\mathbf{K}$ *is indecomposable.*

Proof Let $\mathcal{A}$ be the generic model of $\mathbf{K}$. Suppose $(\mathbf{K}_0, \dots, \mathbf{K}_{n-1})$ is a factorization of $\mathbf{K}$ via an injection $u : A \to B$, where $\mathcal{B}$ is the generic model of $\Pi_i \mathbf{K}_i$. Obviously, $T_{\mathbf{K}} \trianglelefteq \mathrm{Th}(\mathcal{B})$.

Let $S = \{T_{\mathbf{K}_i}\}_{i<n}$. We have observed that if $T_{\mathbf{K}_i} \trianglelefteq T$ for each $i < n$, then $\mathrm{Th}(\mathcal{B}) \trianglelefteq T$. Thus, for any $T \in \uparrow S$, we have $T_{\mathbf{K}} \trianglelefteq \mathrm{Th}(\mathcal{B}) \trianglelefteq T$, so $\uparrow S \subseteq \mathfrak{C}$. Since $\mathfrak{C}$ is prime, it follows that $T_{\mathbf{K}_i} \in \mathfrak{C}$ for some $i < n$. Since $T_{\mathbf{K}}$ is $\trianglelefteq$-minimum in $\tilde{\mathfrak{C}}$, we find that $T_{\mathbf{K}} \trianglelefteq T_{\mathbf{K}_i}$—as required. □

By Lemma 2.22, we now know that in order to prove Proposition 2.20, it is sufficient just to prove that $\tilde{\mathfrak{C}}$ has a $\trianglelefteq$-minimum element, and that is what we do in the rest of the proof. This amounts to demonstrating, first, that a $\trianglelefteq$-minim*al* element of $\tilde{\mathfrak{C}}$ is already $\trianglelefteq$-minim*um*, and second, that $\tilde{\mathfrak{C}}$ must indeed have $\trianglelefteq$-minim*al* element. The first project accounts for Lemmas 2.23, 2.24, and 2.25. The second part accounts for Lemma 2.27 and Corollary 2.28.

Lemma 2.23 *For any* $T \in \mathbb{T}$, $T \in \mathfrak{C}$ *if and only if* $Q_T \cap \mathfrak{C} \neq \emptyset$.

Proof Clearly, if $Q_T \cap \mathfrak{C}$ is non-empty, then $T \in \mathfrak{C}$, so we just need to deal with the converse. Suppose $T \in \mathfrak{C}$.

Let $\boldsymbol{\varphi}$ be a sequence of $\mathscr{L}_T$-formulas. By Observation 2.6, $\mathbb{X}_{\boldsymbol{\varphi}}(T)$, the space representing $\mathbf{F}_{\varphi}(T)$ up to "isomorphism" from the proof of Proposition 2.3, has a countable dense subset $W_{\boldsymbol{\varphi}}(T)$ such that F is $\boldsymbol{\varphi}$-resolved whenever $F/\!\sim\, \in W_{\boldsymbol{\varphi}}(T)$. By Observation 2.5 and Proposition 2.3, there is a $\boldsymbol{\varphi}$-resolved $F_{\boldsymbol{\varphi}} \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$ such that for every $F/\!\sim\, \in W_{\boldsymbol{\varphi}}(T)$, $\mathcal{A}_{\boldsymbol{\varphi}}(F)$ embeds into $\mathcal{A}_{\boldsymbol{\varphi}}(F_{\boldsymbol{\varphi}})$. One easily verifies, then, that for every $\boldsymbol{\varphi}$-resolved $F \in \mathbf{F}_{\boldsymbol{\varphi}}(T)$, $\mathcal{A}_{\boldsymbol{\varphi}}(F)$ embeds into $\mathcal{A}_{\boldsymbol{\varphi}}(F_{\boldsymbol{\varphi}})$. Now, we observe that for an arbitrary theory $T' \in \mathbb{T}$,

$$T' \in \uparrow Q_T \implies (\forall \boldsymbol{\varphi} \text{ of } \mathscr{L}_T)\, Th(\mathcal{A}_{\boldsymbol{\varphi}}(F_{\boldsymbol{\varphi}})) \trianglelefteq T' \implies T \trianglelefteq T' \implies T' \in \mathfrak{C}.$$

More succinctly, we have shown that $\uparrow Q_T \subseteq \mathfrak{C}$. Since $\mathfrak{C}$ is prime, $Q_T \cap \mathfrak{C} \neq \emptyset$, as desired. □

Lemma 2.24 *The sub-class* $\tilde{\mathfrak{C}} = \mathfrak{C} \cap \tilde{\mathbb{T}}$ *is a complete filter class (but not necessarily prime) relative to* $\tilde{\mathbb{T}}$.

Proof Since $\mathfrak{C}$ is non-empty, we may choose $T \in \mathfrak{C}$. Clearly, $Q_T \cap \mathfrak{C} \subseteq \tilde{\mathfrak{C}}$, so as $Q_T \cap \mathfrak{C}$ is non-empty, $\tilde{\mathfrak{C}} \neq \emptyset$ as well.

For the first filter requirement, let $T_1, T_2 \in \tilde{\mathbb{T}}$, and suppose that $T_1 \in \tilde{\mathfrak{C}}$ and $T_1 \trianglelefteq T_2$. Since $\mathfrak{C}$ is a filter class, $T_2 \in \mathfrak{C}$, so $T_2 \in \mathfrak{C} \cap \tilde{\mathbb{T}} = \tilde{\mathfrak{C}}$. For the second filter requirement, let $T_0, \dots, T_{n-1} \in \tilde{\mathfrak{C}}$. We claim that $\tilde{\mathfrak{C}} \cap \downarrow \{T_0, \dots, T_{n-1}\}$ is non-empty. Since $\mathfrak{C}$ is a filter class, let $T' \in \mathfrak{C} \cap \downarrow \{T_0, \dots, T_{n-1}\}$. By Lemma 2.23, $Q_{T'} \cap \mathfrak{C}$ is non-empty, so let $T'_0 \in Q_{T'} \cap \mathfrak{C}$. Since $Q_{T'} \subseteq \tilde{\mathbb{T}}$, we have $T'_0 \in \tilde{\mathfrak{C}} \cap \downarrow \{T_0, \dots, T_{n-1}\}$, as required.

For the completeness of $\tilde{\mathfrak{C}}$, let κ be a positive ordinal, and let $(T_i)_{i<\kappa}$ be a descending $\trianglelefteq$-chain of members of $\tilde{\mathfrak{C}}$. By the completeness of $\mathfrak{C}$, there is a theory $T \in \mathfrak{C}$ that is $\trianglelefteq$-below all of the T_i's ($i < \kappa$). By Lemma 2.23 again, $Q_T \cap \mathfrak{C}$ is non-empty, and any $T^* \in Q_T \cap \mathfrak{C}$ is in $\tilde{\mathfrak{C}}$ and also $\trianglelefteq$-below all of the T_i's. □

Lemma 2.25 *If $\tilde{\mathfrak{C}}$ has at least one $\trianglelefteq$-minimal element, then it has a $\trianglelefteq$-minim*um *element.*

Proof Let $\mathbf{K}_0$ be an algebraically trivial Fraïssé class such that $T_{\mathbf{K}_0}$ is a $\trianglelefteq$-minimal element of $\tilde{\mathfrak{C}}$. If $\mathbf{K}_0$ is not $\trianglelefteq$-minim*um*, then there is some $\mathbf{K}_1$ such that $T_{\mathbf{K}_1} \in \tilde{\mathfrak{C}}$ such that $T_{\mathbf{K}_0} \not\trianglelefteq T_{\mathbf{K}_1}$. Since $\tilde{\mathfrak{C}}$ is a filter class (specifically, the second requirement), there is an algebraically trivial Fraïssé class $\mathbf{K}^*$ such that $T_{\mathbf{K}^*}$ is in $\tilde{\mathfrak{C}}$, $T_{\mathbf{K}^*} \trianglelefteq T_{\mathbf{K}_0}$, and $T_{\mathbf{K}^*} \trianglelefteq T_{\mathbf{K}_1}$. Since $T_{\mathbf{K}_0}$ is $\trianglelefteq$-minimal, we have $T_{\mathbf{K}_0} \trianglelefteq T_{\mathbf{K}^*} \trianglelefteq T_{\mathbf{K}_1}$ – a contradiction. Thus, $T_{\mathbf{K}_0}$ is in fact a $\trianglelefteq$-minimum element of $\tilde{\mathfrak{C}}$. □

We have verified that a $\trianglelefteq$-minimal element of $\tilde{\mathfrak{C}}$ is already $\trianglelefteq$-minimum, and now we need to show that $\tilde{\mathfrak{C}}$ does indeed have $\trianglelefteq$-minimal element. The proof of this fact goes through showing that the lack of a $\trianglelefteq$-minimal element implies the existence of long descending chains in $\tilde{\mathfrak{C}}$, which violates the following easy observation.

Observation 2.26 Since there are, at most, $2^{\aleph_0}$-many algebraically trivial Fraïssé classes (see the proof of Corollary 2.18), $|\tilde{\mathbb{T}}| \le 2^{\aleph_0}$. Therefore, $\tilde{\mathbb{T}}$ contains no strictly descending $\trianglelefteq$-chains of length greater than $2^{\aleph_0}$.

Lemma 2.27 *If $\tilde{\mathfrak{C}}$ does not have a $\trianglelefteq$-minimal element, then it contains a strictly descending $\trianglelefteq$-chain of length* $(2^{\aleph_0})^+$.

Proof At each stage $s < (2^{\aleph_0})^+$ of the following process, we will have a strictly descending $\trianglelefteq$-chain, so that

$$k < \ell \le s \Rightarrow T_{\mathbf{K}_k} \rhd T_{\mathbf{K}_\ell}.$$

- Choose $\mathbf{K}_0$ arbitrarily subject to $T_{\mathbf{K}_0} \in \tilde{\mathfrak{C}}$.
- At a successor stage $i+1$, since $\tilde{\mathfrak{C}}$ does not have any $\trianglelefteq$-minimal elements, $T_{\mathbf{K}_i}$ is not $\trianglelefteq$-minimal in $\tilde{\mathfrak{C}}$, and we may choose $\mathbf{K}_{i+1}$ such that $T_{\mathbf{K}_{i+1}} \lhd T_{\mathbf{K}_i}$ and $T_{\mathbf{K}_{i+1}} \in \tilde{\mathfrak{C}}$.
- At a limit stage ℓ, we are faced with a chain

  $$T_{\mathbf{K}_0} \rhd \cdots \rhd T_{\mathbf{K}_i} \rhd T_{\mathbf{K}_{i+1}} \rhd \cdots$$

  in $\tilde{\mathfrak{C}}$. Since $\tilde{\mathfrak{C}}$ is complete, we may choose $\mathbf{K}_\ell$ such that $T_{\mathbf{K}_\ell} \in \tilde{\mathfrak{C}}$ and $T_{\mathbf{K}_\ell} \trianglelefteq T_{\mathbf{K}_i}$ for all $i < \ell$. We observe that if $T_{\mathbf{K}_i} \trianglelefteq T_{\mathbf{K}_\ell}$ for some $i < \ell$, then we would find

  $$T_{\mathbf{K}_i} \trianglelefteq T_{\mathbf{K}_\ell} \trianglelefteq T_{\mathbf{K}_{i+1}} \lhd T_{\mathbf{K}_i}$$

  which is impossible; hence $T_{\mathbf{K}_\ell} \lhd T_{\mathbf{K}_i}$ for all $i < \ell$.

□

Corollary 2.28 *$\tilde{\mathfrak{C}}$ has a $\trianglelefteq$-minimum element.*

Proof By Observation 2.26 and Lemma 2.27, $\tilde{\mathfrak{C}}$ has a $\trianglelefteq$-minimal element, say $T_{\mathbf{K}}$, and by Lemma 2.25, $T_{\mathbf{K}}$ is $\trianglelefteq$-minimum in $\tilde{\mathfrak{C}}$. □

Proof of Proposition 2.20 Let $\mathbf{K}$ be the algebraically trivial Fraïssé class such that $T_{\mathbf{K}}$ is $\trianglelefteq$-minimum in $\tilde{\mathfrak{C}}$. Fix any $T \in \mathfrak{C}$. Fix $T_0 \in Q_T \cap \mathfrak{C} \subseteq \tilde{\mathfrak{C}}$. Then, $T_{\mathbf{K}} \trianglelefteq T_0 \trianglelefteq T$, so $T \in \mathfrak{C}_{\mathbf{K}}$. Conversely, fix $T \in \mathfrak{C}_{\mathbf{K}}$. Then, $T_{\mathbf{K}} \trianglelefteq T$ and $\mathfrak{C}$ is upward closed, so $T \in \mathfrak{C}$. Therefore, $\mathfrak{C} = \mathfrak{C}_{\mathbf{K}}$. □

This completes the proof of Theorem 2.17.

2.5 Reduction to one sort

So far, the classes $\mathbf{K}$ considered are always Fraïssé classes in languages with one sort. One might ask, "what if we allowed $\mathbf{K}$ to have several (but finitely many) sorts?" In this subsection, we show that allowing $\mathbf{K}$ to have multiple sorts actually produces no additional generality with regards to dividing lines. We remark that the number of sorts of target theories is not at issue—we assume those theories have elimination of imaginaries and carry however many sorts are required for that (perhaps uncountably many).

Theorem 2.29 *Let $\mathbf{K}$ be an algebraically trivial Fraïssé class in a p-sorted language $\mathscr{L}$ (sorts $\mathsf{S}_0, \ldots, \mathsf{S}_{p-1}$) with generic model $\mathcal{B}$, and let $T \in \mathbb{T}$. Then there is an algebraically trivial Fraïssé class $\tilde{\mathbf{K}}$ in a 1-sorted language such that the following are equivalent:*

1. *There are saturated $\mathcal{M} \vDash T$, $X_0, \ldots, X_{p-1}$ definable sets of $\mathcal{M}$, $f_i : \mathsf{S}_i(\mathcal{B}) \to X_i$ injections ($i < p$), and for each $R \subseteq \mathsf{S}_{i_0} \times \cdots \times \mathsf{S}_{i_{r-1}}$ in $\mathrm{sig}(\mathscr{L})$, a formula $\varphi_R(x_0, \ldots, x_{r-1})$ of $\mathscr{L}_T$ such that*

$$\mathcal{B} \vDash R(b_0, \ldots, b_{r-1}) \Leftrightarrow \mathcal{M} \vDash \varphi_R(f_{i_0}(b_0), \ldots, f_{i_{r-1}}(b_{r-1})).$$

2. $T \in \mathfrak{C}_{\tilde{\mathbf{K}}}$.

The proof of Theorem 2.29, of course, requires that we define a Fraïssé class $\tilde{\mathbf{K}}$ and determine how it is related to $\mathbf{K}$ itself. Lemma 2.31 is a transfer result matching members of $\mathbf{K}$ directly with members of $\tilde{\mathbf{K}}$

Definition 2.30 Let $\mathscr{L}$ be a p-sorted finite relational language with sorts $\mathsf{S}_0, \ldots, \mathsf{S}_{p-1}$, and let $\mathbf{K}$ be a Fraïssé class of finite $\mathscr{L}$-structures. Then, let $\mathscr{L}_{\mathbf{K}}$ be the one-sorted language with relation symbols $R_q^{(r)}$ for each irreflexive quantifier-free-complete type $q \in S_r^{\mathsf{qf}}(T_{\mathbf{K}})$, $r \leq \mathrm{ari}(\mathscr{L})$.

- For each $B \in \mathbf{K}$, we define an $\mathscr{L}_{\mathbf{K}}$-structure A_B and a family of maps $u_i^B : \mathsf{S}_i(B) \to A_B$ as follows:
  - $A_B = \bigcup_{i<p} \left(\{i\} \times \mathsf{S}_i(B)\right)$ as a set.
  - For each $i < p$, $u_i^B : \mathsf{S}_i(B) \to A_B$ is given by $u_i^B(b) = (i, b)$.
  - $R_q^{A_B} = \left\{\left((i_0, b_0), \ldots, (i_{r-1}, b_{r-1})\right) : \mathrm{qftp}^B(b_0, \ldots, b_{r-1}) = q\right\}$ for each irreflexive $q \in S_r^{\mathsf{qf}}(T_{\mathbf{K}})$, $r \leq \mathrm{ari}(\mathscr{L})$.
- We define $\tilde{\mathbf{K}}$ to be the isomorphism-closure of $\{C : C \leq A_B, B \in \mathbf{K}\}$.

- For each $C \in \tilde{\mathbf{K}}$, we define an $\mathscr{L}$-structure B^C and a family of partial maps $v_i^C : C \rightharpoonup \mathsf{S}_i(B^C)$ as follows:
  - For $q \in S_1^{\mathrm{qf}}(T_{\mathbf{K}})$, let $i_q < p$ be the index such that $q \models \mathsf{S}_{i_q}$.
  - For each $i < p$, let $\mathsf{S}_i(B^C) = \bigcup \left\{ R_q^C : q \in S_1^{\mathrm{qf}}(T_{\mathbf{K}}),\ i_q = i \right\}$ and let v_i^C be the identity mapping.
  - For $R \subseteq \mathsf{S}_{i_0} \times \cdots \times \mathsf{S}_{i_{r-1}}$ in $\mathrm{sig}(\mathscr{L})$, let $Q_R = \left\{ q \in S_r^{\mathrm{qf}}(T_{\mathbf{K}}) : q \models R(x_0, \ldots, x_{r-1}) \right\}$ and let $R^{B^C} = \bigcup \left\{ R_q^C : q \in Q_R \right\}$.
- For an $\mathscr{L}_{\mathbf{K}}$-structure $\mathcal{M}$ such that $\mathsf{Age}(\mathcal{M}) \subseteq \tilde{\mathbf{K}}$, we define $B^{\mathcal{M}}$ similarly.

Lemma 2.31 *Let* $\mathbf{K}$ *be an algebraically trivial Fraïssé class in a* p*-sorted language* $\mathscr{L}$*. Then:*

1. *If* $B \in \mathbf{K}$*, then* $B \cong B^{A_B}$ *via* $b \mapsto (i_b, b)$ *where* $i_\bullet : b \mapsto i_b$ *is such that* $b \in \mathsf{S}_{i_b}(B)$ *for each* b.
2. *If* $C \in \tilde{\mathbf{K}}$*, then* $C \cong A_{B^C}$ *via* $c \mapsto (i_{\mathrm{qftp}(c)}, c)$.

Proof The proofs of the two items of the lemma are very similar, so we will just prove item 1. Let $B \in \mathbf{K}$ be given. The map $f : b \mapsto (i_b, b)$ is actually the union $f = \bigcup_{i<p}(v_i^{A_B} \circ u_i^B)$, and it is clear that f is a bijection between B and B^{A_B}. To see that f is an isomorphism, let $R \subseteq \mathsf{S}_{i_0} \times \cdots \times \mathsf{S}_{i_{r-1}}$ be in $\mathrm{sig}(\mathscr{L})$, and let $b_j \in \mathsf{S}_{i_j}(B)$ for each $j < r$. Let $q = \mathrm{qftp}^B(b_0, \ldots, b_{r-1})$, so that $\overline{b} \in R_q^{A_B}$ by definition. Then

$$\begin{aligned} B \models R(b_0, \ldots, b_{r-1}) &\Leftrightarrow R(x_0, \ldots, x_{r-1}) \in q \\ &\Leftrightarrow q \in Q_R \\ &\Leftrightarrow R_q^{A_B} \subseteq R^{B^{A_B}} \end{aligned}$$

and it follows that $B \models R(\overline{b}) \Leftrightarrow B^{A_B} \models R(f\overline{b})$. This completes the proof. ◻

Corollary 2.32 *Let* $\mathbf{K}$ *be an algebraically trivial Fraïssé class in a* p*-sorted language* $\mathscr{L}$.

1. *If* $C \in \tilde{\mathbf{K}}$*, then* $B^C \in \mathbf{K}$.
2. $\tilde{\mathbf{K}}$ *is an algebraically trivial Fraïssé class.*
3. $\mathbf{K}$ *is the isomorphism-closure of* $\left\{ B^C : C \in \tilde{\mathbf{K}} \right\}$

Proof For Item 1: Given $C \in \tilde{\mathbf{K}}$, by definition, there is some $B_0 \in \mathbf{K}$ such that $C \leq A_{B_0}$. One easily verifies that $B^C \leq B^{A_{B_0}} \cong B_0$, so as $\mathbf{K}$ is a Fraïssé class, we find that $B^C \in \mathbf{K}$. For Item 2: HP for $\tilde{\mathbf{K}}$ is built in to its definition, and for JEP and AP, one simply transfers the discussion from $\tilde{\mathbf{K}}$ to $\mathbf{K}$ via $C \mapsto B^C$, applies JEP or AP there, and transfers it back to $\tilde{\mathbf{K}}$ via $B \mapsto A_B$. Item 3 is immediate from Lemma 2.31. ◻

The remainder of the proof of Theorem 2.29 is encoded in the following proposition, which extends to the transfer between the two Fraïssé classes $\mathbf{K}$ and $\tilde{\mathbf{K}}$ to the level of their generic models.

Proposition 2.33 *Let $\mathbf{K}$ be an algebraically trivial Fraïssé class in a p-sorted language $\mathscr{L}$. Let $\mathcal{B}$ be the generic model of $\mathbf{K}$, and let $\mathcal{M}$ be generic model of $\tilde{\mathbf{K}}$. Then $B^{\mathcal{M}} \cong \mathcal{M}$.*

It follows that there are injections $u_i : S_i(\mathcal{B}) \to M$ $(i < p)$ and a surjective mapping $i_\bullet : S_1^{qf}(T_{\tilde{\mathbf{K}}}) \to p : q \mapsto i_q$ such that:

- $M = \dot{\bigcup}_{i<p} \mathrm{img}(u_i)$, *and* $\mathrm{img}(u_i) = \bigcup \{q(\mathcal{M}) : i_q = i\}$ *for each* $i < p$.
- *For each irreflexive* $q \in S_r^{qf}(T_{\mathbf{K}})$, $r \leq \mathrm{ari}(\mathscr{L})$, *if* $q \vDash S_{i_0} \times \cdots \times S_{i_{r-1}}$ *and* $(b_0, \ldots, b_{r-1}) \in \prod_{j<r} S_{i_j}(\mathcal{B})$

$$\mathcal{B} \vDash q(b_0, \ldots, b_{r-1}) \Leftrightarrow \mathcal{M} \vDash R_q(u_{i_0}(b_0), \ldots, u_{i_{r-1}}(b_{r-1})).$$

Proof One verifies that $B^{\mathcal{M}}$ is $\mathbf{K}$-universal and $\mathbf{K}$-homogeneous, and that $A \in \mathbf{K}$ whenever A is a finite subset of $B^{\mathcal{M}}$. It follows that $B^{\mathcal{M}} \cong \mathcal{B}$ by the uniqueness of the generic model of an amalgamation class. □

3 Linear orderings and the connection to collapse-of-indiscernibles dividing-lines

The starting point for the research in this paper was an attempt to generalize the collapse-of-indiscernible dividing-lines results from [7,23]. These results are themselves a generalization of Shelah's classification of stable theories in terms of indiscernible sequences: A theory is stable if and only if every indiscernible sequence is an indiscernible set [24]. In this section, we connect our discussion with this concept and explore the relationship between positive-local-combinatorial and collapse-of-indiscernible dividing-lines.

3.1 Definitions and previously known facts

We begin our discussion by recalling definitions around generalized indiscernibles. This starts with the definition of a Ramsey class; as it turns out these are exactly the classes which produce well-behaved indiscernibles (i.e., ones that have the Patterning property). From there, we discuss (un-collapsed) indiscernible pictures. We then define the Patterning property, and recall a theorem from [7], stating that the Patterning property is equivalent to the Ramsey property.

Definition 3.1 (*Ramsey property, Ramsey class*) Let $\mathbf{K}$ be a Fraïssé class.

- For $A \in \mathbf{K}$, we say that $\mathbf{K}$ has the *A-Ramsey property* if, for any $0 < k < \omega$ and any $B \in \mathbf{K}$, there is some $C = C(A, B, k) \in \mathbf{K}$ such that, for any coloring $\xi : \mathrm{Emb}(A, C) \to k$, there is an embedding $u : B \to C$ such that ξ is constant on $\mathrm{Emb}(A, uB)$.

- **K** is said to have the *Ramsey property* if it has the A-Ramsey property for every $A \in \mathbf{K}$. When **K** has the Ramsey property, then we also say that **K** is a Ramsey class.

Definition 3.2 Let **K** be a Fraïssé class with generic model $\mathcal{A}$. Let $\mathcal{M}$ be an infinite $\mathscr{L}$-structure for some language $\mathscr{L}$.

- A *picture of* $\mathcal{A}$ *in* $\mathcal{M}$, $\gamma : \mathcal{A} \to \mathcal{M}$, is a just an injective mapping of A into (a single sort of) $\mathcal{M}$.
- A picture $\gamma : \mathcal{A} \to \mathcal{M}$ of $\mathcal{A}$ in $\mathcal{M}$ is *indiscernible* if for all $n \in \mathbb{N}$, $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1}$ in A,

$$\mathrm{qftp}^{\mathcal{A}}(\overline{a}) = \mathrm{qftp}^{\mathcal{A}}(\overline{b}) \;\Rightarrow\; \mathrm{tp}^{\mathcal{M}}(\gamma\overline{a}) = \mathrm{tp}^{\mathcal{M}}(\gamma\overline{b}).$$

  (For $\Delta \subseteq \mathscr{L}$, Δ-indiscernible pictures are defined similarly.)
- An indiscernible picture $\gamma : \mathcal{A} \to \mathcal{M}$ is called *un-collapsed* if for all $n \in \mathbb{N}$, $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1}$ in A,

$$\mathrm{tp}^{\mathcal{M}}(\gamma\overline{a}) = \mathrm{tp}^{\mathcal{M}}(\gamma\overline{b}) \;\Rightarrow\; \mathrm{qftp}^{\mathcal{A}}(\overline{a}) = \mathrm{qftp}^{\mathcal{A}}(\overline{b}).$$

  Of course, we say that γ *collapses* if it is not un-collapsed.

Usually, we will denote indiscernible pictures with the letters I or J instead of γ.

Definition 3.3 (*Patterning property*) Let **K** be a Fraïssé class with generic model $\mathcal{A}$. Let $\mathcal{M}$ be an infinite $\mathscr{L}$-structure for some language $\mathscr{L}$.

Let $\gamma : \mathcal{A} \to \mathcal{M}$ be a picture, and let $I : \mathcal{A} \to \mathcal{M}$ be an indiscernible picture. We say that I *is patterned on* γ if for every $\Delta \subset_{\text{fin}} \mathscr{L}$, every $n \in \mathbb{N}$, and all $a_0, \ldots, a_{n-1} \in A$, there is an embedding $f = f_{\Delta,\overline{a}} : \mathcal{A}\restriction\overline{a} \to \mathcal{A}$ such that

$$\mathrm{tp}_{\Delta}^{\mathcal{M}}(I\overline{a}) = \mathrm{tp}_{\Delta}^{\mathcal{M}}(\gamma f\overline{a}).$$

Now, we say that **K** *has the Patterning property* if for every picture $\gamma : \mathcal{A} \to \mathcal{M}$, there is an indiscernible picture $I : \mathcal{A} \to \mathcal{M}$ of $\mathcal{A}$ patterned on γ.

The existence of indiscernible sequences is usually stated (as in [20]) with less precision than is actually required in practice. The existence statement in full precision, but generalized to objects richer than pure linear orders, is the following theorem due to [23].

Theorem 3.4 **K** *has the Ramsey property if and only if it has the Patterning property.*

Thus, if we wish to consider Fraïssé classes that produce a coherent theory of indiscernibles, we are compelled to look at Ramsey classes. Furthermore, as the next theorem will show, looking at Ramsey classes forces us to consider algebraically trivial classes which carry a 0-definable linear order. Therefore, in this section, we will be primarily interested in studying algebraically trivial Fraïssé classes that, when one adds a generic linear order, become Ramsey classes.

Theorem 3.5 *Let* $\mathbf{K}$ *be a Fraïssé class with disjoint-JEP, and let* $\mathcal{A}$ *be the generic model of* $\mathbf{K}$*. If* $\mathbf{K}$ *has the Ramsey property, then:*

- ([21]) $\mathbf{K}$ *is algebraically trivial.*
- ([13]) $\mathcal{A}$ *carries a 0-definable linear ordering.*

Definition 3.6 Let $\mathbf{K}$ be an algebraically trivial Fraïssé class in the language $\mathscr{L}_{\mathbf{K}}$, and assume that $|S_1(T_{\mathbf{K}})| = 1$. Let $\mathscr{L}_{\mathbf{K}}^{<}$ be the language obtained by adding one new binary relation $<$ to the signature of $\mathscr{L}_{\mathbf{K}}$, and let $\mathbf{K}^{<}$ be the class of all finite $\mathscr{L}_{\mathbf{K}}^{<}$-structures B such that $B \restriction \mathscr{L}_{\mathbf{K}} \in \mathbf{K}$ and $<^B$ is a linear ordering of B. Then $\mathbf{K}^{<}$ is also an algebraically trivial Fraïssé class with $|S_1(T_{\mathbf{K}^{<}})| = 1$, and if $\mathcal{B}$ is its generic model, then $<^{\mathcal{B}}$ is a dense linear ordering of B without endpoints (see, for example, [3,4]).

The class $\mathbf{K}^{<}$ is the *generic order-expansion* of $\mathbf{K}$. We will say that the original class $\mathbf{K}$ is *simply Ramsey-expandable* if $\mathbf{K}^{<}$ has the Ramsey property. (A more general notion of Ramsey-expandable would allow arbitrary expansions by finitely many relation symbols; this topic is addressed in, for example, [10,11], under the name of "having a Ramsey-lift".)

3.2 Order-expansions

A priori, adding a generic linear order to an algebraically trivial Fraïssé class could undermine the project of classifying dividing-lines by changing the corresponding class of theories. Luckily, that seems not to be the case. We will show, in Corollary 3.10, that, given any algebraically trivial Fraïssé class $\mathbf{K}$, if $T_{\mathbf{K}}$ has the order property, then $\mathbf{K}$ and $\mathbf{K}^{<}$ correspond to the same dividing-line. Therefore, as long as the generic model of $\mathbf{K}$ is unstable, we need not worry about adding a generic order to $\mathbf{K}$.

We begin with the definition of "coding orders", which is precisely what is needed to show that adding a generic order will have no effect on the corresponding dividing-line. Then, we show that a class codes orders if and only if its generic model has the order property.

Definition 3.7 Let $\mathbf{K}$ be an algebraically trivial Fraïssé class of $\mathscr{L}$-structures. We say that $\mathbf{K}$ *codes orders* if there are $0 < n < \omega$, quantifier-free formulas $\theta_R(\overline{x}_0, \ldots, \overline{x}_{r-1})(R^{(r)} \in \mathrm{sig}(\mathscr{L}), |\overline{x}_i| = n)$, and a quantifier-free formula $\theta_<(\overline{x}, \overline{y})$ such that for every $B \in \mathbf{K}^{<}$, there are $C \in \mathbf{K}$ and an injection $f : B \to C^n$ such that

$$B \vDash R(b_0, \ldots, b_{r-1}) \Leftrightarrow C \vDash \theta_R(f(b_0), \ldots, f(b_{r-1}))$$

for each $R^{(r)} \in \mathrm{sig}(\mathscr{L})$ and all $b_0, \ldots, b_{r-1} \in B$, and

$$b <^B b' \Leftrightarrow C \vDash \theta_<(f(b), f(b'))$$

for all $b, b' \in B$.

Observation 3.8 Let $\mathbf{K}$ be an algebraically trivial Fraïssé class that codes orders, and let $\mathcal{A}^{<} = (\mathcal{A}, <)$ be the generic model of $\mathbf{K}^{<}$. Then there are $0 < n < \omega$, quantifier-free formulas $\theta_R(\overline{x}_0, \ldots, \overline{x}_{r-1})$ $(R^{(r)} \in \mathrm{sig}(\mathscr{L}), |\overline{x}_i| = n)$, a quantifier-free formula $\theta_<(\overline{x}, \overline{y})$, and an injection $u : A \to A^n$ such that

$$\mathcal{A} \models R(a_0, \ldots, a_{r-1}) \Leftrightarrow \mathcal{A} \models \theta_R(u(a_0), \ldots, u(a_{r-1}))$$

for each $R^{(r)} \in \text{sig}(\mathscr{L})$ and all $a_0, \ldots, a_{r-1} \in A$, and

$$a < a' \Leftrightarrow \mathcal{A} \models \theta_<(u(b), u(b'))$$

for all $a, a' \in A$.

Proposition 3.9 *Let* $\mathbf{K}$ *be an algebraically trivial Fraïssé class. Then,* $\mathbf{K}$ *codes orders if and only if* $T_{\mathbf{K}}$ *has the order property.*

Proof Obviously, if $\mathbf{K}$ codes orders, $\theta_<(\overline{x}, \overline{y})$ as in the definition of coding orders has the order property in $T_{\mathbf{K}}$.

Conversely, suppose $T_{\mathbf{K}}$ has the order property, and let $\mathcal{A} \models T_{\mathbf{K}}$ be the generic model. Since $\mathcal{A}$ has the order property and $T_{\mathbf{K}}$ eliminates quantifiers, there is a quantifier-free $\mathscr{L}$-formula $\psi(\overline{x}; \overline{y})$, $|\overline{x}| = |\overline{y}| = m$, such that for every n, there is a sequence $(\overline{a}_i)_{i<n}$ of m-tuples from $\mathcal{A}$ such that, for all $i, j < n$,

$$\mathcal{A} \models \psi(\overline{a}_i; \overline{a}_j) \Leftrightarrow i < j.$$

Let $\theta_<(x'\overline{x}, y'\overline{y}) = \psi(\overline{x}, \overline{y})$, and for each $R^{(r)} \in \text{sig}(\mathscr{L})$, let

$$\theta_R(x_0'\overline{x}_0, x_1'\overline{x}_1, \ldots, x_{r-1}'\overline{x}_{r-1}) = R(x_0', \ldots, x_{r-1}')$$

Now, let $B \in \mathbf{K}^<$ be given—and say, $n = |B|$ and $B = \{b_0 <^B \cdots <^B b_{n-1}\}$. Since $\mathcal{A}$ is the generic model of $\mathbf{K}$, we can choose a sequence $(\overline{a}_i)_{i<n}$ of m-tuples from $\mathcal{A}$ such that $\mathcal{A} \models \psi(\overline{a}_i; \overline{a}_j) \Leftrightarrow i < j$ for all $i, j < n$, and an embedding $u_0 : B{\restriction}\mathscr{L} \to \mathcal{A}$. We define $f : B \to A^{m+1}$ by setting $f(b_i) = (u_0(b_i), \overline{a}_i)$ for each $i < n$, and we take C to be the induced substructure of $\mathcal{A}$ on $u_0 B \cup \bigcup_i \overline{a}_i$. It is routine to verify that C and f meet the requirements of coding orders for $B \in \mathbf{K}^<$. As B was arbitrary, $\mathbf{K}$ indeed codes orders. □

These are the ingredients needed to prove the main result of this subsection: For an algebraically trivial Fraïssé class whose generic model is unstable, adding a generic linear order does not change the corresponding dividing-line.

Corollary 3.10 *Let* $\mathbf{K}$ *be an algebraically trivial Fraïssé class with generic model* $\mathcal{A}$ *and generic order-expansion* $\mathbf{K}^<$ *(whose generic model is* $\mathcal{A}^< = (\mathcal{A}, <)$*). If* $T_{\mathbf{K}}$ *is unstable, then* $\mathfrak{C}_{\mathbf{K}} = \mathfrak{C}_{\mathbf{K}^<}$.

Proof $\mathfrak{C}_{\mathbf{K}^<} \subseteq \mathfrak{C}_{\mathbf{K}}$ is trivial because $\mathcal{A}^<$ is an expansion of $\mathcal{A}$. To show $\mathfrak{C}_{\mathbf{K}} \subseteq \mathfrak{C}_{\mathbf{K}^<}$, let $T \in \mathfrak{C}_{\mathbf{K}}$, $\mathcal{M} \models T$ saturated, and let $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_{m-1})$ and $F : \mathcal{A} \to \mathcal{M}$ be a $\boldsymbol{\varphi}$-resolved member of $\mathbf{F}_{\boldsymbol{\varphi}}(T)$ such that $\mathcal{A}_{\boldsymbol{\varphi}}(F) = \mathcal{A}$ up to an identification of relation symbols. Also, let $0 < n < \omega$, θ_R, $\theta_<$, and $u : A \to A^n$ be as described in Observation 3.8. It is not hard to see that $T_{\mathbf{K}^<} \in Q_T$ via

$$F' = (\underbrace{F, \ldots, F}_{n \text{ times}}) \circ u$$

so $T \in \mathfrak{C}_{\mathbf{K}^<}$. □

We can apply this result to the Fraïssé class of finite sets with k independent linear orders. These classes were studied, for example, in Section 3 of [7] in the classification of op-dimension. It turns out that the dividing-line corresponding to k independent linear orders is the same as the dividing line corresponding to one linear order.

Definition 3.11 For $0 < k < \omega$, let $\mathscr{L}_k$ be the language whose signature consists of binary relation symbols $<_0, \ldots, <_{k-1}$. Let $\mathbf{MO}_k$ be the class of all finite $\mathscr{L}_k$-structures B such that $<_i^B$ is a linear ordering of B for each $i < k$. Members of $\mathbf{MO}_k$ are called k-multi-orders, and it is not difficult to verify that $\mathbf{MO}_k$ is an algebraically trivial Fraïssé class. Clearly, $\mathbf{MO}_1$ is just the class $\mathbf{LO}$ of all finite linear orders.

Proposition 3.12 *For every* $0 < k < \omega$, $\mathbf{MO}_{k+1}$ *is the generic order-expansion of* $\mathbf{MO}_k$*:* $\mathbf{MO}_{k+1} = \mathbf{MO}_k^<$. *Since each* $\mathbf{MO}_k$ *is unstable, we find that* $\mathfrak{C}_{\mathbf{MO}_{k+1}} = \mathfrak{C}_{\mathbf{MO}_k^<} = \mathfrak{C}_{\mathbf{MO}_k}$. *Thus,* $\mathfrak{C}_{\mathbf{MO}_k} = \mathfrak{C}_{\mathbf{LO}}$ *for every* $k > 0$.

This is, perhaps, not surprising. As noted in Section 3 of [7], $\mathbf{MO}_k$ relates to op-dimension $\geq k$, and a theory T has sorts of arbitrarily high op-dimension if and only if T is unstable (i.e., $T \in \mathfrak{C}_{\mathbf{LO}}$).

Observation 3.13 $\mathbf{LO}$ is indecomposable.

The proof of this observation is similar to the proof of Proposition 4.9 below (in the setup of that proof, for $a_0 < a_1$ from $\mathcal{A}$, $\mathrm{tp}^{\mathcal{B}}(ua_0a_1) \neq \mathrm{tp}^{\mathcal{B}}(ua_1a_0)$, so it is witnessed in a reduct of $\mathcal{B}$ to $\mathscr{L}_{i_0}$ for some $i_0 < n$).

3.3 Collapse of indiscernibles

In this subsection, we make explicit the connection between positive-local-combinatorial and collapse-of-indiscernible dividing-lines. In Theorem 3.14, we show that the dividing-line corresponding to an algebraically trivial Fraïssé class with the Ramsey property is characterized by the existence of an un-collapsed indiscernible picture. In Theorem 3.15, we generalize this to simply Ramsey-expandable classes. This partially captures a generalization of the standard collapse-of-indiscernible results found in the literature [6,7,23].

Theorem 3.14 *Let* $\mathbf{K}$ *be an algebraically trivial Fraïssé class with generic model* $\mathcal{A}$ *in a language* $\mathscr{L}$*. If* $\mathbf{K}$ *is a Ramsey class, then for every* $T \in \mathbb{T}$*, the following are equivalent:*

1. $T \in \mathfrak{C}_{\mathbf{K}}$
2. *There is an un-collapsed indiscernible picture of* $\mathcal{A}$ *in a model of* T*.*

Proof 2 $\Rightarrow$ 1 is trivial. For 1 $\Rightarrow$ 2, let $T \in \mathbb{T}$, $\mathcal{M} \models T$ saturated, and let $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_{m-1})$ and $F : \mathcal{A} \to \mathcal{M}$ be a $\boldsymbol{\varphi}$-resolved member of $\mathbf{F}_{\boldsymbol{\varphi}}(T)$ such that $\mathcal{A}_{\boldsymbol{\varphi}}(F) = \mathcal{A}$ up to an identification of relation symbols.

Let $I : \mathcal{A} \to \mathcal{M}$ be an indiscernible picture of $\mathcal{A}$ patterned on F. We claim that I is un-collapsed. Towards a contradiction, suppose there are $0 < k < \omega$ and $\overline{a}, \overline{a}' \in A^k$ such that $\mathrm{qftp}^{\mathcal{A}}(\overline{a}) \neq \mathrm{qftp}^{\mathcal{A}}(\overline{a}')$ but $\mathrm{tp}^{\mathcal{M}}(I\overline{a}) = \mathrm{tp}^{\mathcal{M}}(I\overline{a}')$. Since $\boldsymbol{\varphi}$ is finite and I is patterned on F, there is an $\mathscr{L}$-embedding $u : \overline{aa}' \to \mathcal{A}$ such that $\mathrm{tp}_{\boldsymbol{\varphi}}^{\mathcal{M}}(I\overline{aa}') = \mathrm{tp}_{\boldsymbol{\varphi}}^{\mathcal{M}}(Fu\overline{aa}')$, and it follows that

$$\mathrm{tp}_{\boldsymbol{\varphi}}^{\mathcal{M}}(Fu\overline{a}) = \mathrm{tp}_{\boldsymbol{\varphi}}^{\mathcal{M}}(I\overline{a}) = \mathrm{tp}_{\boldsymbol{\varphi}}^{\mathcal{M}}(I\overline{a}') = \mathrm{tp}_{\boldsymbol{\varphi}}^{\mathcal{M}}(Fu\overline{a}').$$

Since $\mathcal{A} = \mathcal{A}_{\boldsymbol{\varphi}}(F)$, we find that $\mathrm{qftp}^{\mathcal{A}}(u\overline{a}) = \mathrm{qftp}^{\mathcal{A}}(u\overline{a}')$, and since u is an embedding and $\mathcal{A} = \mathcal{A}_{\boldsymbol{\varphi}}(F)$, we find that $\mathrm{qftp}^{\mathcal{A}}(\overline{a}) = \mathrm{qftp}^{\mathcal{A}}(\overline{a}')$—a contradiction. Thus, I must be un-collapsed. □

Though Theorem 3.14 is certainly an interesting result in its own right, it does not match the flavor of collapse-of-indiscernible results discovered thus far (in, say, [6,7,23]). "Natural" dividing-lines typically do not involve linear orders. For example, the independence property naturally corresponds to the Fraïssé class of finite graphs (and not finite ordered graphs). However, in light of Corollary 3.10, adding generic linear orders has no effect on the dividing-line. Therefore, we get a stronger result, more in the spirit of typical collapse-of-indiscernibles results.

Theorem 3.15 *Let* $\mathbf{K}$ *be an algebraically trivial Fraïssé class. If* $T_{\mathbf{K}}$ *is unstable and* $\mathbf{K}$ *is simply Ramsey-expandable, then the following are equivalent:*

1. $T \in \mathfrak{C}_{\mathbf{K}}$
2. *There is an un-collapsed indiscernible picture of* $\mathcal{A}^<$ *in a model of* T*, where* $\mathcal{A}^<$ *is the generic model of* $\mathbf{K}^<$.

Proof By Theorem 3.14 and Corollary 3.10. □

Theorem 3.15 is, in spirit, a generalization of the collapse-of-indiscernible results from [6,7,23]. The following example explores this connection.

Example 3.16 By choosing the appropriate $\mathbf{K}$, we obtain the following corollaries of Theorem 3.15:

1. If $\mathbf{K} = \mathbf{H}_{r+1}$ is the class of finite $(r+1)$-hypergraphs, then $\mathfrak{C}_{\mathbf{K}}$ is the class of theories with r-IP (see Proposition 4.31). Therefore, T has r-IP if and only if there is an un-collapsed indiscernible picture of a model of the generic ordered $(r+1)$-hypergraph in a model of T. This is analogous to Theorem 5.4 in [6].
2. In particular, consider $r = 1$. Then, T has IP if and only if there is an un-collapsed indiscernible picture of a model of the generic ordered graph in a model of T. This is analogous to the main result of [23].

One thing missing from the discussion here is precisely how these indiscernibles resist collapse. The results from [6,7], and [23] all provide a more precise reason for the lack of collapse. For example, in [23], we see that a theory has NIP if and only if all ordered graph indiscernibles collapse to ordered indiscernibles. It is precisely the collapse of the edge relation that determines whether or not a theory has NIP. In future work, it would be interesting to explore exactly what reducts of $\mathcal{A}^{<}$ characterize whether or not a theory belongs to $\mathfrak{C}_{\mathbf{K}}$ (in Theorem 3.15).

4 Case studies

In this section, to get a better feel for the quasi-ordering $\trianglelefteq$, we study some examples of classes in the $\trianglelefteq$-ordering.

4.1 The top of the $\trianglelefteq$-order

In some sense, the top of the $\trianglelefteq$-order is not surprising. If this is a sensible ordering on the local complexity of theories, the maximal class must include the theory of true arithmetic. Moreover, the top class must also include the theory of hereditarily finite sets since these two theories are bi-interpretable. We show that every algebraically trivial Fraïssé class is subordinate to this theory, and we conclude from this that all theories lie $\trianglelefteq$-below it. Finally, we give an example of a theory in the top class which interprets neither hereditarily finite sets nor true arithmetic.

Definition 4.1 Let $\mathscr{L}_0$ be the language of set theory, with signature $\{\in\}$. We define $\mathcal{H} = (H, \in^{\mathcal{H}})$, the hereditarily finite sets, as follows:

- $H_0 = \emptyset$, and for each $n < \omega$, $H_{n+1} = H_n \cup \mathcal{P}(H_n)$
- Then $H = \bigcup_n H_n$, and $\in^{\mathcal{H}}$ is the usual membership relation.

We write T_{FS} for the complete theory of $\mathcal{H}$—the theory of finite sets.

Proposition 4.2 *$T_{\mathbf{K}} \trianglelefteq T_{\text{FS}}$ for every algebraically trivial Fraïssé class $\mathbf{K}$, and the sequence of formulas involved depends only on the signature of $\mathbf{K}$, not on $\mathbf{K}$ itself. Thus, $T \trianglelefteq T_{\text{FS}}$ for every $T \in \mathbb{T}$.*

Proof Let $\mathbf{K}$ be an algebraically trivial Fraïssé class in a language $\mathscr{L}$ with signature $\text{sig}(\mathscr{L}) = \{R_0, \ldots, R_{n-1}\}$, and let $\mathcal{A}$ be its generic model. Let $m = n + 1$, and for each $i < n$, if $r = \text{ari}(R_i)$, then let $\theta_i(\overline{x}_0, \ldots, \overline{x}_{r-1})$, where $|\overline{x}_i| = m$, be the formula asserting

$$(x_{0,0}, \ldots, x_{r-1,0}) \in x_{0,i+1}.$$

Now, if $B \in \mathbf{K}$, we define a certain injection $u : B \to H$. First, let $u_0 : B \to H$ be any injection at all, and then define $u : B \to H$ by

$$u(b) = \left(u_0(b), u_0\left(R_0^B\right), \ldots, u_0\left(R_{n-1}^B\right)\right).$$

We observe that for each $i < n$, for all $b_0, \dots, b_{r-1} \in A$ where $r = \mathrm{ari}(R_i)$,

$$\begin{aligned} B \vDash R_i(b_0, \dots, b_{r-1}) &\Leftrightarrow (u_0(b_0), \dots, u_0(b_{r-1})) \in u_0\left(R_i^B\right) \\ &\Leftrightarrow (u(b_0)_0, \dots, u(b_{r-1})_0) \in u(b_0)_{i+1} \\ &\Leftrightarrow \mathcal{H} \vDash \theta_i(u(b_0), \dots, u(b_{r-1})). \end{aligned}$$

Since B was an arbitrary member of $\mathbf{K}$, by compactness, there are an elementary extension $\mathcal{H}'$ of $\mathcal{H}$ and an injection $F : A \to H'$ such that for all $i < n$, $r = \mathrm{ari}(R_i)$, for all $a_0, \dots, a_{r-1} \in A$,

$$\mathcal{A} \vDash R_i(a_0, \dots, a_{r-1}) \Leftrightarrow \mathcal{H}' \vDash \theta_i(F(a_0), \dots, F(a_{r-1})).$$

It follows that $T_{\mathbf{K}} \trianglelefteq T_{\mathrm{FS}}$. □

Corollary 4.3 *Let* $TA = \mathrm{Th}(\mathbb{N}, +, \cdot, 0, 1)$*, often called "true arithmetic". Since* T_{FS} *is interpretable in TA, it follows that* $T \trianglelefteq TA$ *for every* $T \in \mathbb{T}$.

Proof In general, we know that for any $T_1, T_2 \in \mathbb{T}$, if T_1 is interpretable in T_2, then $T_1 \trianglelefteq T_2$. The corollary then follows immediately from Theorem 4.2 and the fact that T_{FS} and TA are bi-interpretable [12]. □

Examining the proof of Corollary 2.18 and the proof of Proposition 2.19 gives another example of a theory T^* that belongs to the $\trianglelefteq$-maximal class. However, T^* does not interpret T_{FS} or TA, showing that there are theories in the top class that are not bi-interpretable.

Definition 4.4 Following the proof of Corollary 2.18, let S be the set of functions $s : \omega \to \omega$ of finite support. For each $s \in S$, let $\mathrm{sig}(\mathscr{L}_s)$ be the signature with relation symbols $R_i^{(s(i))}$ for each $i \in \mathrm{supp}(s)$, and let $\mathscr{K}_s$ be the set of all algebraically trivial Fraïssé classes of finite $\mathscr{L}_s$-structures. Then, let T^* be the theory T_S constructed from $S = \{T_{\mathbf{K}} : \mathbf{K} \in \mathscr{K}_s, s \in S\}$ as in the proof of Proposition 2.19.

Fact 4.5 $T \trianglelefteq T^*$ for every $T \in \mathbb{T}$, so in particular, $T_{\mathrm{FS}}, TA \trianglelefteq T^*$. However T^* does not interpret T_{FS} or TA.

Proof For the first part of the claim, we notice that for every algebraically trivial Fraïssé class $\mathbf{K}$, $T_{\mathbf{K}} \trianglelefteq T^*$ by construction. For non-interpretability, just notice that any reduct of T^* to finitely many of its sorts is $\aleph_0$-categorical, but obviously T_{FS} is not $\aleph_0$-categorical (use compactness to add an infinite "set"). □

4.2 Hypergraphs

In this subsection, we explore hypergraphs, showing that the Fraïssé class of r-hypergraphs is indecomposable (i.e., corresponds to an irreducible dividing-line). We show that adding a generic order or prohibiting cliques of a fixed size does not alter the corresponding dividing-line. We also show that these hypergraph classes form a strict chain. We begin the discussion with the definition of the relevant algebraically trivial Fraïssé classes, $\mathbf{H}_r$, $\mathbf{H}_{r,k}$, and $\mathbf{H}_r^*$.

Definition 4.6 Let $2 \leq r < \omega$, and let $\mathscr{L}_r$ be the language whose signature is a single r-ary relation symbol R.

- $\mathbf{H}_r$ is the class of all finite r-hypergraphs—i.e., finite models of the two sentences

$$\forall \overline{x} \left(R(\overline{x}) \to \bigwedge_{i<j} x_i \neq x_j \right), \ \forall \overline{x} \left(R(\overline{x}) \to \bigwedge_{\sigma \in \mathrm{Sym}(r)} R(x_{\sigma(0)}, \ldots, x_{\sigma(r-1)}) \right).$$

- For $k > r$, $\mathbf{H}_{r,k}$ is the sub-class consisting of those $A \in \mathbf{H}_r$ that exclude $K_k(r)$, the complete r-hypergraph on k vertices. So, $\mathbf{H}_{r,k}$ is the class of finite models of the previous two sentences and the sentence

$$\forall x_0, \ldots, x_{k-1} \left(\bigwedge_{i<j} x_i \neq x_j \to \bigvee_{i_0 < \cdots < i_{r-1} < k} \neg R(x_{i_0}, \ldots, x_{i_{r-1}}) \right).$$

One easily verifies that all $\mathbf{H}_r$'s and $\mathbf{H}_{r,k}$'s are algebraically trivial Fraïssé classes.

Now, let $\mathscr{L}_r^*$ be the expansion of $\mathscr{L}_r$ with new unary predicate symbols $U_0, \ldots, U_{r-1}$, and let $\mathbf{H}_r^*$ be the set of models of the first two sentences and the sentences

$$\forall x \bigvee_{i<r} \left(U_i(x) \wedge \bigwedge_{j \neq i} \neg U_j(x) \right),$$

$$\forall x_0 \ldots x_{r-1} \left(R(\overline{x}) \to \bigvee_{\sigma \in \mathrm{Sym}(r)} \bigwedge_{i<r} U_{\sigma(i)}(x_i) \right).$$

Again, it is not hard to see that $\mathbf{H}_r^*$ is, again, an algebraically trivial Fraïssé class.

We make a few observations about the relationship between these Fraïssé classes.

Observation 4.7 Let $2 \leq r < k < \omega$, and let $\mathcal{A}^<, \mathcal{B}^<$ be the generic models of $\mathbf{H}_{r,k}^<, \mathbf{H}_r^<$, respectively, and $\mathcal{A} = \mathcal{A}^< \restriction \mathscr{L}_r$, $\mathcal{B} = \mathcal{B}^< \restriction \mathscr{L}_r$. Then, there are embeddings $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A}^< \to \mathcal{B}^<$. It follows that $\mathfrak{C}_{\mathbf{H}_r} \subseteq \mathfrak{C}_{\mathbf{H}_{r,k}}$ and $\mathfrak{C}_{\mathbf{H}_r^<} \subseteq \mathfrak{C}_{\mathbf{H}_{r,k}^<}$ whenever $2 \leq r < k < \omega$.

It is not difficult to check the following observation (see, for example, [6]).

Observation 4.8 For every $2 \leq r < \omega$, $\mathfrak{C}_{\mathbf{H}_r^*} = \mathfrak{C}_{\mathbf{H}_r}$.

As promised, we show that the Fraïssé class of hypergraphs of a fixed arity is indecomposable. Since this is an algebraically trivial Fraïssé class, it corresponds to an irreducible dividing-line (by Theorem 2.17).

Proposition 4.9 *For every* $2 \leq r < \omega$, $\mathbf{H}_r$ *is indecomposable.*

Proof Let $(\mathbf{K}_0, \ldots, \mathbf{K}_{n-1})$ be a factorization of $\mathbf{H}_r$. Let $\mathcal{A}$ be the generic model of $\mathbf{H}_r$, and for each $i < n$, let $\mathcal{A}_i$ be the generic model of $\mathbf{K}_i$. Let $\mathcal{B}$ be the generic model of $\Pi_i \mathbf{K}_i$, and let $u : A \to B$ be an injection such that for all k and all $\overline{a}, \overline{a}' \in A^k$, $\mathrm{qftp}^{\mathcal{A}}(\overline{a}) = \mathrm{qftp}^{\mathcal{A}}(\overline{a}') \Leftrightarrow \mathrm{tp}^{\mathcal{B}}(u\overline{a}) = \mathrm{tp}^{\mathcal{B}}(u\overline{a}')$. We make three observations:

- If $k < r$ and $\overline{a}, \overline{a}' \in A^k$ are both non-repeating, then $\mathrm{qftp}^{\mathcal{A}}(\overline{a}) = \mathrm{qftp}^{\mathcal{A}}(\overline{a}')$.
- If $\overline{a} \in A^r$ is non-repeating, then $\mathrm{qftp}^{\mathcal{A}}(\overline{a}) = \mathrm{qftp}^{\mathcal{A}}(a_{\sigma(0)}, \ldots, a_{\sigma(r-1)})$ for any $\sigma \in \mathrm{Sym}(r)$.
- For any k and any $\overline{a} \in A^k$, $\mathrm{qftp}^{\mathcal{A}}(\overline{a}) = \bigcup_{I \in \binom{k}{\leq r}} \mathrm{qftp}^{\mathcal{A}}(\overline{a}_{\restriction I})$.

(We can collapse repeating instances and only deal with non-repeating tuples from $\mathcal{A}$.) Let $\overline{a} \in R^{\mathcal{A}}$ and $\overline{a}' \in A^r \backslash R^{\mathcal{A}}$ non-repeating, and for each $i < n$, let $\theta_i(x_0, \ldots, x_{r-1}) = \mathrm{tp}^{\mathcal{B}}(u\overline{a}) \restriction \mathscr{L}_i$. Then for some $i_0 < n$, $\neg\theta_{i_0} \in \mathrm{tp}^{\mathcal{B}}(u\overline{a}')$. From this, we see that θ_{i_0} witnesses $T_{\mathbf{H}_r} \trianglelefteq T_{\mathbf{K}_{i_0}}$. □

We now establish the fact that hypergraphs form a strict chain (Corollary 4.13). In Lemma 4.10, we show that classes defined from hypergraphs form a decreasing chain and, in Lemma 4.11, we establish that this chain is strict. Together, these yield Theorem 4.12 and Corollary 4.13. In particular, this shows that there are infinitely many irreducible dividing-lines, as noted in Corollary 4.14.

Lemma 4.10 *For all* $2 \leq r_1 < r_2 < \omega$, $\mathfrak{C}_{\mathbf{H}_{r_2}} \subseteq \mathfrak{C}_{\mathbf{H}_{r_1}}$.

Proof For $i = 1, 2$, let $\mathcal{A}_i$ be the generic model of $\mathbf{H}_{r_i}$. Consider the $\mathscr{L}_{r_2}$-formula

$$\theta(x_0\overline{y}_0, \ldots, x_{r_1-1}\overline{y}_{r_1-1}) = R(x_0, \ldots, x_{r_1-1}, \overline{y}_0)$$

where $|\overline{y}_t| = r_2 - r_1$ for all $t < r_1 - 1$. For any $B_1 \in \mathbf{H}_{r_1}$, construct an $\mathscr{L}_{r_2}$-structure B_2 as follows:

- As a set, $B_2 = B_1 \cup \{\overline{c}\}$ for some $\overline{c}$ where $|\overline{c}| = r_2 - r_1$ and $\overline{c} \cap B_1 = \emptyset$.
- R^{B_2} is $\{(b_0, \ldots, b_{r_1-1}, \overline{c}) : (b_0, \ldots, b_{r_1-1}) \in R^{B_1}\}$, closed under symmetry.
- Consider the injection $u : B_1 \to B_2^{r_2-r_1+1}$ given by $u(b) = (b, \overline{c})$.

Then, for all $b_0, \ldots, b_{r_1-1} \in B_1$,

$$B_1 \vDash R(b_0, \ldots, b_{r_1-1}) \Leftrightarrow B_2 \vDash \theta(u(b_0), \ldots, u(b_{r_1-1})).$$

By compactness, we get an injection $u : A_1 \to A_2^{r_2-r_1+1}$ such that, for all $a_0, \ldots, a_{r-1} \in A_1$

$$A_1 \vDash R(a_0, \ldots, a_{r_1-1}) \Leftrightarrow A_2 \vDash \theta(u(a_0), \ldots, u(a_{r_1-1})).$$

Therefore, $T_{\mathbf{H}_{r_1}} \trianglelefteq T_{\mathbf{H}_{r_2}}$, so $\mathfrak{C}_{\mathbf{H}_{r_2}} \subseteq \mathfrak{C}_{\mathbf{H}_{r_1}}$. □

Lemma 4.11 *If* $2 \leq r_1 < r_2 < \omega$, *then* $\mathfrak{C}_{\mathbf{H}_{r_1}} \nsubseteq \mathfrak{C}_{\mathbf{H}_{r_2}}$.

Proof For $i = 1, 2$, let $\mathcal{A}_i$ be the generic model of $\mathbf{H}_{r_i}$. Towards a contradiction, suppose $\mathfrak{C}_{\mathbf{H}_{r_1}} \subseteq \mathfrak{C}_{\mathbf{H}_{r_2}}$, (i.e., $T_{\mathbf{H}_{r_2}} \trianglelefteq T_{\mathbf{H}_{r_1}}$)—that is, there are $0 < m < \omega$, an injection $u : A_2 \to A_1^m$, and a quantifier-free formula $\theta(\overline{x}_0, \ldots, \overline{x}_{r_2-1})$ such that $|\overline{x}_i| = m$ for each $i < r_2$, and for all $a_0, \ldots, a_{r_2-1}$ in A_2,

$$\mathcal{A}_2 \vDash R(a_0, \ldots, a_{r_2-1}) \Leftrightarrow \mathcal{A}_1 \vDash \theta(u(a_0), \ldots, u(a_{r_2-1}))$$

where R is the single r_2-ary relation symbol of the language $\mathbf{H}_{r_2}$. Since $r_1 < r_2$, we may choose a number $N < \omega$ such that $\binom{Nm}{r_1} < \binom{N}{r_2}$—so that $2^{\binom{Nm}{r_1}} < 2^{\binom{N}{r_2}}$. Now, let $B_0, \ldots, B_{k-1}$ (where $k = 2^{\binom{N}{r_2}}$) be an enumeration of N-element substructures of $\mathcal{A}_2$ up to isomorphism—that is, an enumeration of all r_2-hypergraphs on N vertices. For each $i < k$, let B_i^u be the r_2-hypergraph with universe $u(B_i) \subset A_1^m$ and interpretation $R^{B_i^u} = \theta(\mathcal{A}_1) \cap u(B_i)^m$. Then we find that

$$2^{\binom{N}{r_2}} \leq \left| \left\{ B_i^u : i < k \right\} / \cong \right| \leq 2^{\binom{Nm}{r_1}} < 2^{\binom{N}{r_2}}$$

which is impossible. Thus $\mathfrak{C}_{\mathbf{H}_{r_1}} \nsubseteq \mathfrak{C}_{\mathbf{H}_{r_2}}$, as claimed. □

Theorem 4.12 *For all* $2 \leq r_1 < r_2 < \omega$, $\mathfrak{C}_{\mathbf{H}_{r_2}} \subsetneq \mathfrak{C}_{\mathbf{H}_{r_1}}$.

Proof By Lemmas 4.10 and 4.11. □

Corollary 4.13 $\mathfrak{C}_{\mathbf{H}_2} \supsetneq \mathfrak{C}_{\mathbf{H}_3} \supsetneq \cdots \supsetneq \mathfrak{C}_{\mathbf{H}_r} \supsetneq \cdots$. *Thus, there is a strict nested chain of irreducible dividing-lines.*

Corollary 4.14 *If* $\mathscr{F}$ *denotes the family of all irreducible dividing-lines, then* $\aleph_0 \leq |\mathscr{F}| \leq 2^{\aleph_0}$.

Proof Combine Corollaries 2.18 and 4.13 □

Now that we have established that the $\mathfrak{C}_{\mathbf{H}_r}$'s form a strictly decreasing chain, one might wonder what the intersection of these classes looks like. It turns out that there is a theory that characterizes the intersection of all classes corresponding to hypergraphs, $\bigcap_{2 \leq r < \omega} \mathfrak{C}_{\mathbf{H}_r}$. We build this theory in the obvious manner, by disjointly "gluing" together the generic hypergraphs of each arity.

Definition 4.15 Let $\mathscr{L}$ be the language with, for each $2 \leq r < \omega$, a sort H_r and an r-ary relation R_r on H_r. Let $\mathcal{M}$ be the $\mathscr{L}$-structure such that $\mathsf{H}_r(\mathcal{M})$ is the generic model of $\mathbf{H}_r$ for each $2 \leq r < \omega$, and let $T_{\mathrm{HYP}} = \mathrm{Th}(\mathcal{M})$.

Observation 4.16 Let $T \in \mathbb{T}$. The following are equivalent:

1. $T_{\mathrm{HYP}} \trianglelefteq T$;
2. $T \in \bigcap_{2 \leq r < \omega} \mathfrak{C}_{\mathbf{H}_r}$.

We return our attention to exploring the relationship between $\mathbf{H}_r$, $\mathbf{H}_{r,k}$, $\mathbf{H}_r^<$, and $\mathbf{H}_{r,k}^<$, culminating in Theorem 4.20, which states that they all correspond to the same irreducible dividing line.

Observation 4.17 For every $2 \leq r < k < \omega$, $\mathbf{H}_r$ and $\mathbf{H}_{r,k}$ code orders because $T_{\mathbf{H}_r}$ and $T_{\mathbf{H}_{r,k}}$ both have the order property. Hence, $\mathfrak{C}_{\mathbf{H}_r} = \mathfrak{C}_{\mathbf{H}_r^<}$ and $\mathfrak{C}_{\mathbf{H}_{r,k}} = \mathfrak{C}_{\mathbf{H}_{r,k}^<}$.

Next, we show that prohibiting a k-clique has no effect on the corresponding dividing line.

Lemma 4.18 *Let $2 \leq r < k < \omega$. Then there is a quantifier-free formula $\theta(\overline{x}_0, \ldots, \overline{x}_{r-1})$, $|\overline{x}_i| = k-1$ such that for every $B \in \mathbf{H}_r$, there are $C \in \mathbf{H}_{r,k}$ and an injection $u : B \to C^{k-1}$ such that for all $b_0, \ldots, b_{r-1} \in B$,*

$$B \vDash R(b_0, \ldots, b_{r-1}) \Leftrightarrow C \vDash \theta(u(b_0), \ldots, u(b_{r-1})).$$

Proof Let $\theta(\overline{x}_0, \ldots, \overline{x}_{r-1}) =$

$$\bigwedge_{\text{inj. } s : r \to k-1} R(x_{s(0),0}, \ldots, x_{s(r-1),r-1}).$$

Given $B \in \mathbf{H}_r$, we define $C \in \mathbf{H}_r$ as follows:

- $C = B \times \{0, 1, \ldots, k-2\}$ as a set.
- $R^C =$

$$\left\{ \big((b_0, i_0), \ldots, (b_{r-1}, i_{r-1})\big) : \begin{array}{l} (b_0, \ldots, b_{r-1}) \in R^B, \\ i_0, \ldots, i_{r-1} < k-1 \text{ pairwise distinct} \end{array} \right\}$$

We claim that C is $K_k(r)$-free, where $K_k(r)$ is the complete r-hypergraph on $\{0, 1, \ldots, k-1\}$. Towards a contradiction, suppose $X \in \binom{C}{k}$ is such that $C \restriction X \cong K_k(r)$. Let $(b_0, i_0), \ldots, (b_{k-1}, i_{k-1})$ be pairwise distinct elements of X. By the pigeonhole principle, there are $s < t < k$ such that $i_s = i_t$. Selecting pairwise distinct $j_0, \ldots, j_{r-3} \in k \backslash \{s, t\}$ arbitrarily, we have

$$\big((b_{j_0}, i_{j_0}), \ldots, (b_{j_{r-3}}, i_{j_{r-3}}), (b_s, i_s), (b_t, i_t)\big) \in R^C$$

because $C \restriction X$ is complete. But this contradicts the definition of R^C. Thus, C is $K_k(r)$-free as claimed.

Finally, we define $u : B \to C^{k-1}$ by setting $u(b) = \big((b, 0), \ldots, (b, k-2)\big)$. For $b_0, \ldots, b_{r-1} \in B$, we see that for every injection $s : r \to k-1$

$$(b_0, \ldots, b_{r-1}) \in R^B \Leftrightarrow \big((b_0, s(0)), (b_1, s(1)), \ldots, (b_{r-1}, s(r-1))\big) \in R^C$$

so

$$(b_0, \ldots, b_{r-1}) \in R^B \Leftrightarrow C \vDash \theta(u(b_0), \ldots, u(b_{r-1}))$$

as desired. □

Corollary 4.19 *Let $2 \le r < k < \omega$. Then there are a quantifier-free formula $\theta(\overline{x}_0, \ldots, \overline{x}_{r-1})$ ($|\overline{x}_i| = k-1$) and an injection $u : A_r \to A_{r,k}^{k-1}$ such that for all $b_0, \ldots, b_{r-1} \in B$,*

$$\mathcal{A}_r \models R(b_0, \ldots, b_{r-1}) \Leftrightarrow \mathcal{A}_{r,k} \models \theta(u(b_0), \ldots, u(b_{r-1})).$$

It follows that $\mathfrak{C}_{\mathbf{H}_{r,k}} \subseteq \mathfrak{C}_{\mathbf{H}_r}$.

By combining these ideas, we get the desired theorem, showing that the irreducible dividing-line corresponding to the class of all finite r-hypergraphs is also characterized by any one of the other classes of (ordered) r-hypergraphs mentioned above.

Theorem 4.20 $\mathfrak{C}_{\mathbf{H}_r} = \mathfrak{C}_{\mathbf{H}_r^<} = \mathfrak{C}_{\mathbf{H}_{r,k}} = \mathfrak{C}_{\mathbf{H}_{r,k}^<}$ *whenever* $2 \le r < k < \omega$.

Proof Clearly $\mathbf{H}_{r,k} \subseteq \mathbf{H}_r$, so $\mathfrak{C}_{\mathbf{H}_r} \subseteq \mathfrak{C}_{\mathbf{H}_{r,k}}$. By the previous corollary, we conclude $\mathfrak{C}_{\mathbf{H}_r} = \mathfrak{C}_{\mathbf{H}_{r,k}}$. The remainder of the theorem follows by Observation 4.17. □

Remark 4.21 From Theorem 4.20, we learn that the class/dividing-line $\mathfrak{C} = \{$unsimple theories$\}$ is not irreducible in the sense of this paper. To see this, suppose $\mathfrak{C}$ *were* irreducible—say $\mathfrak{C} = \mathfrak{C}_{\mathbf{K}}$ for some indecomposable algebraically trivial Fraïssé class $\mathbf{K}$. Since $T_{\mathbf{H}_{2,3}}$, the theory of the Henson graph, is in $\mathfrak{C}$ (see, for example, [8,14]), we find that $T_{\mathbf{K}} \trianglelefteq T_{\mathbf{H}_{2,3}} \trianglelefteq T_{\mathbf{H}_2}$, so $T_{\mathbf{H}_2} \in \mathfrak{C}$—i.e., $T_{\mathbf{H}_2}$ is unsimple. But $T_{\mathbf{H}_2}$ is the theory of the random graph, which certainly *is* simple—a contradiction.

4.3 Societies

Theorem 4.20 tells us that, given any $k > r \ge 2$, the class of theories corresponding to the algebraically trivial Fraïssé class of finite (ordered) r-hypergraphs (omitting k-cliques) coincide. What happens if we have more than one hyperedge relation, each acting independently? Do we get a new dividing-line? As it turns out, we get nothing new; adding new hyperedge relations of smaller or equal arity does not change the corresponding dividing-line. We begin by formally defining the notion of a society, which captures the idea of having multiple independent hyperedge relations.

Definition 4.22 Let $\mathscr{L}$ be a finite relational language in which all relation symbols have arity ≥ 2. For each $R^{(n)} \in \mathrm{sig}(\mathscr{L})$, let φ_R be the sentence

$$\forall \overline{x}\left(R(\overline{x}) \to \bigwedge_{i<j<n} x_i \neq x_j\right) \wedge \forall \overline{x}\left(R(\overline{x}) \to \bigwedge_{\sigma \in \mathrm{Sym}(n)} R(x_{\sigma(0)}, \ldots, x_{\sigma(n-1)})\right)$$

and let $\Sigma_{\mathscr{L}}$ be the set of sentences $\{\varphi_R : R \in \mathrm{sig}(\mathscr{L})\}$. Following [22], we write $\mathbf{S}_{\mathscr{L}}$ for the class of $\mathscr{L}$-societies—that class of all finite models of $\Sigma_{\mathscr{L}}$. (One easily verifies that $\mathbf{S}_{\mathscr{L}}$ is an algebraically trivial Fraïssé class.)

Observation 4.23 Let $\mathscr{L}$ be a finite relational language in which all relation symbols have arity ≥ 2, and let r be the maximum arity among relation symbols in $\mathscr{L}$; then

$T_{\mathbf{H}_r} \trianglelefteq T_{\mathbf{S}_{\mathscr{L}}}$. To see this, just fix a relation $R \in \operatorname{sig}(\mathscr{L})$ of arity r—then the reduct of the generic model $\mathcal{A}$ of $\mathbf{S}_{\mathscr{L}}$ to the signature $\{R\}$ actually is the generic model of the class $\mathbf{H}_r$ of r-hypergraphs.

In the following theorem, we show that the converse is also true; i.e., the dividing-line corresponding to a class of societies is the same as the class of r-hypergraphs, where r is the largest arity in the language. This conclusion is formally stated in Corollary 4.25.

Theorem 4.24 *Let $\mathscr{L}$ be a finite relational language in which all relation symbols have arity ≥ 2, and let $r = \operatorname{ari}(\mathscr{L})$. Then $T_{\mathbf{S}_{\mathscr{L}}} \trianglelefteq T_{\mathbf{H}_r}$.*

Proof We assume that the signature of $\mathbf{H}_r$ is $\{R^{(r)}\}$. Let $m = \sum_{Q \in \operatorname{sig}(\mathscr{L})} \operatorname{ari}(Q)$, and let $(I_Q)_{Q \in \operatorname{sig}(\mathscr{L})}$ be a partition of m such that for each $Q^{(n)} \in \operatorname{sig}(\mathscr{L})$, we have $I_Q = \{i_0(Q) < \cdots < i_{n-1}(Q)\}$ and an enumeration $j_0(Q) < \cdots < j_{m-n-1}(Q)$ of $m \backslash I_Q$. For each $Q \in \operatorname{sig}(\mathscr{L})$ of arity n, let

$$\theta_Q(\overline{x}_0, \ldots, \overline{x}_{n-1}) = R(x_{0,i_0(Q)}, \ldots, x_{n-1,i_{n-1}(Q)}, x_{0,j_0(Q)}, \ldots, x_{0,j_{r-n-1}(Q)}).$$

Given $C \in \mathbf{S}_{\mathscr{L}}$, we define $B_C \in \mathbf{H}_r$ as follows:

- $B_C = C \times m$
- For $Q^{(n)} \in \operatorname{sig}(\mathscr{L})$, we define the intermediate relation R_Q^C to be the symmetric closure of

$$\Big\{ ((c_0, i_0(Q)), \ldots, (c_{n-1}, i_{n-1}(Q)), (c_0, j_0(Q)), \ldots, (c_0, j_{r-n-1}(Q))) : (c_0, \ldots, c_{n-1}) \in Q^C \Big\}$$

  Then we define $R^{B_C} = \bigcup_{Q \in \operatorname{sig}(\mathscr{L})} R_Q^C$.

We define $u : C \to B_C^m$ by $u(c) = \big((c, 0), \ldots, (c, m-1)\big)$. Let $Q^{(n)} \in \operatorname{sig}(\mathscr{L})$ and $c_0, \ldots, c_{n-1} \in C$ be given. Firstly, we have

$$\begin{aligned} &C \vDash Q(c_0, \ldots, c_{n-1}) \\ &\quad \Rightarrow B_C \vDash R((c_0, i_0(Q)), \ldots, (c_{n-1}, i_{n-1}(Q)), (c_0, j_0(Q)), \ldots, (c_0, j_{r-n-1}(Q))) \\ &\quad \Leftrightarrow B_C \vDash \theta_Q(u(c_0), \ldots, u(c_{n-1})) \end{aligned}$$

Now, we claim that if

$$B_C \vDash R((c_0, i_0(Q)), \ldots, (c_{n-1}, i_{n-1}(Q)), (c_0, j_0(Q)), \ldots, (c_0, j_{r-n-1}(Q))),$$

then $(c_0, \ldots, c_{n-1})$ is in Q^C. If not, then for some $Q_1^{(n_1)} \in \operatorname{sig}(\mathscr{L})$ different from Q, we have

$$\big((c_0, i_0(Q)), \ldots, (c_{n_1-1}, i_{n_1-1}(Q)), (c_0, j_0(Q)), \ldots, (c_0, j_{r-n_1-1}(Q))\big) \in R_{Q_1}^C.$$

Since Q_1^C contains only non-repeating tuples, it must then be that $I_{Q_1} \subseteq I_Q$—contradicting the fact that $(I_Q)_{Q \in \mathrm{sig}(\mathscr{L})}$ is partition of m. Thus, $(c_0, \ldots, c_{n-1})$ is in Q^C, and we have proven that

$$C \vDash Q(c_0, \ldots, c_{n-1}) \Leftrightarrow B_C \vDash \theta_Q(u(c_0), \ldots, u(c_{n-1})).$$

As $C \in \mathbf{S}_{\mathscr{L}}$ was arbitrary, we have shown that $T_{\mathbf{S}_{\mathscr{L}}} \trianglelefteq T_{\mathbf{H}_r}$. □

Corollary 4.25 *Let $\mathscr{L}$ be a finite relational language in which all relation symbols have arity ≥ 2. Then, $\mathfrak{C}_{\mathbf{S}_{\mathscr{L}}} = \mathfrak{C}_{\mathbf{H}_{\mathrm{ari}(\mathscr{L})}}$.*

Proof By Observation 4.23 and Theorem 4.24. □

As we shall see in the following subsection, $\mathfrak{C}_{\mathbf{H}_{r+1}}$ is exactly equal to the theories that have r-IP. One might hope that, by studying societies in general, one might find a whole zoo of generalizations of the independence property, akin to r-IP. However, Corollary 4.25 says that no such thing seems to exist. Adding more relation symbols (i.e., considering more formulas) does nothing to alter the positive local complexity of the theory. In some sense, the r-IP's are the only generalizations of the independence property of this type.

4.4 Multi-partite multi-concept classes

In this subsection, we show that the irreducible dividing-line corresponding to $(r+1)$-hypergraphs is precisely the same as r-IP, the r-independence property. We begin by discussing p-partite concept classes, which is a generalization of (standard) concept classes used to study the independence property.

Definition 4.26 If W is a set, then $S \subseteq_m W$ means that S is a multi-set all of whose elements are members of W, each with finite multiplicity.

Definition 4.27 For $0 < p < \omega$, $\mathscr{L}_p$ is the $(p+1)$-sorted language (with sorts $\mathsf{S}_0, \mathsf{S}_1, \ldots, \mathsf{S}_p$) and just one relation symbol $R \subseteq \mathsf{S}_0 \times \mathsf{S}_1 \times \cdots \times \mathsf{S}_p$. The Fraïssé class $\mathbf{Fin}(\mathscr{L}_p)$ of all finite $\mathscr{L}_p$-structures is also known as the class of $(p+1)$-partite $(p+1)$-hypergraphs.

Definition 4.28 For $0 < p < \omega$, a finite p-partite concept class is a multi-set $\mathscr{C} \subseteq_m \mathcal{P}(X_1 \times \cdots \times X_p)$.

Given a finite p-partite concept class $\mathscr{C} \subseteq_m \mathcal{P}(X_1 \times \cdots \times X_p)$ (multi-set of subsets of $X_1 \times \cdots \times X_p$), we define an $\mathscr{L}_p$-structure $B_{\mathscr{C}}$ with $\mathsf{S}_0(B_{\mathscr{C}}) = \mathscr{C}$, $\mathsf{S}_i(B_{\mathscr{C}}) = X_i$ for each $i = 1, \ldots, p$, and

$$R^{B_{\mathscr{C}}} = \left\{(S, x_1, \ldots, x_p) : (x_1, \ldots, x_p) \in S \in \mathscr{C}\right\}.$$

Let $\mathbf{J}_p$ be the isomorphism-closure of

$$\{B_{\mathscr{C}} : \mathscr{C} \text{ is a finite } p\text{-partite concept class}\}.$$

Observation 4.29 For every $0 < p < \omega$, $\mathbf{J}_p = \mathbf{Fin}(\mathscr{L}_p)$ and $\mathbf{H}^*_{p+1} = \widetilde{\mathbf{Fin}(\mathscr{L}_p)}$ (in the notation of Sect. 2.5).

We define the r-independence property (r-IP), which generalizes the usual independence property. Then, in Proposition 4.31, we show that this exactly equals the dividing-line corresponding to $(r+1)$-hypergraphs.

Definition 4.30 Let $T \in \mathbb{T}$, and let $\mathcal{M} \vDash T$ be $\aleph_1$-saturated. We say that T has the *r-independence property* (r-IP) if there is a formula $\varphi(x_0, \ldots, x_{r-1}; y) \in \mathscr{L}_T$ such that there are $a_{0,i}, \ldots, a_{r-1,i}$ in $\mathcal{M}$ ($i < \omega$) such that for any finite $X \subset \omega^r$, there is some b_X in $\mathcal{M}$ such that for all $(i_0, \ldots, i_{n-1}) \in \omega^r$,

$$\mathcal{M} \vDash \varphi(a_{0,i_0}, \ldots, a_{r-1,i_{r-1}}; b_X) \Leftrightarrow (i_0, \ldots, i_{r-1}) \in X.$$

Let $\mathbf{IP}_r = \{T \in \mathbb{T} : T \text{ has } r\text{-IP}\}$. Note that the usual independence property is 1-IP.

Proposition 4.31 ([6]) $\mathbf{IP}_r = \mathfrak{C}_{\mathbf{H}_{r+1}}$.

Proof Let $T \in \mathbb{T}$. If $T \in \mathfrak{C}_{\mathbf{H}_{r+1}}$ is witnessed by $F : A \to M$ (where $\mathcal{M} \vDash T$) and $\varphi(x_0, \ldots, x_{r-1}, x_r) \in \mathscr{L}_T$, then it is easy to see that φ witnesses the fact that T has the r-independence property. Conversely, if T has r-IP, then there are a formula $\varphi(x_0, \ldots, x_{r-1}; y) \in \mathscr{L}_T$, $(a_{0,i}, \ldots, a_{r-1,i})_{i<\omega}$ and $(b_X)_{X \subset_{\text{fin}} \omega^r}$ in some $\mathcal{M} \vDash T$ witnessing this. Then, if $\mathcal{B}$ is the generic model of $\mathbf{J}_r$, we have injections $u_i : \mathsf{S}_i(B) \to M$ ($i \leq r$) such that for all $b_0 \in \mathsf{S}_0(\mathcal{B}), \ldots, b_r \in \mathsf{S}_r(\mathcal{B})$, $\mathcal{B} \vDash R(b_0, \ldots, b_r) \Leftrightarrow \mathcal{M} \vDash \varphi(u_0(b_0), \ldots, u_r(b_r))$. By Theorem 2.29, it follows that $T \in \mathfrak{C}_{\tilde{\mathbf{J}}_r} = \mathfrak{C}_{\mathbf{H}^*_{r+1}} = \mathfrak{C}_{\mathbf{H}_{r+1}}$. □

Combining this with Corollary 4.13, we see that the r-independence property forms a strictly decreasing chain of irreducible dividing-lines.

Corollary 4.32 *Every $\mathbf{IP}_r$ ($2 \leq r < \omega$) is an irreducible dividing-line, and*

$$\mathbf{IP} = \mathbf{IP}_1 \supsetneq \mathbf{IP}_2 \supsetneq \cdots \supsetneq \mathbf{IP}_r \supsetneq \cdots .$$

The fact that these are strict was already well-known (e.g., [6]). However, that each is an irreducible dividing-line is an interesting fact, providing evidence that our definition of irreducible is the "right" one. Indeed, irreducibility really should encompass all known positive local dividing-lines.

5 Open questions

When looking at Sect. 3, one notices that, if $\mathbf{K}$ is an algebraically trivial indecomposable Fraïssé class and $T_{\mathbf{K}}$ is unstable, then $\mathfrak{C}_{\mathbf{K}}$ is characterized by a collapse of indiscernibles when $\mathbf{K}^<$ is a Ramsey class. So a natural question arises:

Open Question 5.1 *Let $\mathbf{K}$ be an algebraically trivial indecomposable Fraïssé class such that $T_{\mathbf{K}}$ is unstable and $|S_1(T_{\mathbf{K}})| = 1$. When is $\mathbf{K}^<$ a Ramsey class? Is there a model-theoretic characterization of this?*

Also in that section, one notices a difference between our general result for collapse-of-indiscernibles (Theorem 3.15) and the specific results found in the literature (e.g., [6,7,23]).

Open Question 5.2 *Suppose* $\mathbf{K}$ *is an algebraically trivial indecomposable Fraïssé class such that* $\mathbf{K}^<$ *is a Ramsey class. Can one find a specific reduct* T_0 *of* $T_\mathbf{K}$ *such that a theory* T *lies outside of* $\mathfrak{C}_\mathbf{K}$ *if and only if every* $\mathbf{K}^<$*-indiscernible in* T *collapses to* T_0*?*

We would like to better understand the quasi-ordering $\trianglelefteq$ and the irreducible dividing-lines it generates. For example, deciding which classes $\mathbf{K}$ are equivalent vis-à-vis the class $\mathfrak{C}_\mathbf{K}$ seems to be an interesting project. Which are equivalent to the trivial dividing-line?

Open Question 5.3 *Suppose* $\mathbf{K}$ *is an indecomposable algebraically trivial Fraïssé class such that* $|S_1(T_\mathbf{K})| = 1$ *and* $T_\mathbf{K}$ *is stable. Then, do we have* $\mathfrak{C}_\mathbf{K} = \mathbb{T}$*? If not, can we characterize which* $\mathbf{K}$ *yield the trivial dividing line?*

(Obviously, if $\mathbf{K}_=$ *is the Fraïssé class of finite pure sets (in the empty signature); then* $\mathfrak{C}_{\mathbf{K}_=} = \mathbb{T}$*.)*

Another question revolves around the number of irreducible dividing lines. By Corollary 4.14, we know there are between $\aleph_0$ and $2^{\aleph_0}$ such, but can we get a better estimate?

Open Question 5.4 *Is the set of irreducible dividing-lines countable?*

During the first attempt at categorizing irreducibility for classes of theories, we replaced "completeness" with "countable completeness" in Definition 2.14. Although the proof of Lemma 2.27 seems to require at least "$(2^{\aleph_0})^+$-completeness", is this actually necessary?

Open Question 5.5 *Let* $\mathfrak{C} \subset \mathbb{T}$ *be a prime filter class. Is* $\mathfrak{C}$ *complete if and only if* $\mathfrak{C}$ *is countably-complete (i.e., every descending* $\trianglelefteq$*-chain* $(T_n)_{n<\omega}$ *of members of* $\mathfrak{C}$*, there is some* $T \in \mathfrak{C}$ *such that* $T \trianglelefteq T_n$ *for all* $n < \omega$*)?*

Notice that $\trianglelefteq$ relates any sort of one theory to any sort of another (which is why, in Proposition 3.12, we find that $\mathfrak{C}_{\mathbf{MO}_k} = \mathfrak{C}_{\mathbf{LO}}$ for all $k > 0$). What would happen if one restricted the sorts under consideration? For example, could one recover a generalized collapse-of-indiscernible result on sorts (or partial types) akin to the one for op-dimension in [7]? This may be related to examining the witness number from Observation 2.12. We hope to explore this (and the other questions in this section) in future papers.

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