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Splitting idempotents in a fibered setting

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Abstract By splitting idempotent morphisms in the total and base categories of fibrations we provide an explicit elementary description of the Cauchy completion of objects in the categories **Fib**($\mathbb{B}$) of fibrations with a fixed base category $\mathbb{B}$ and **Fib** of fibrations with any base category. Two universal constructions are at issue, corresponding to two fibered reflections involving the fibration of fibrations **Fib** $\to$ **Cat**.

Keywords Category · Idempotent · Cauchy completion · Fibration

Mathematics Subject Classification 1802 · 18A15 · 18A05 · 18B99

1 Introduction

In any category, an idempotent is an endomorphism a such that $aa = a$. An idempotent is said to split if there exist morphisms r, s such that $a = sr$ and rs is an identity. A pair r, s as above is a splitting of a. A category in which the idempotents belonging to a certain class split, is said to be Cauchy complete with respect to that class of idempotents. There is a well-known construction, see [6,7], producing a Cauchy complete category with respect to a class of idempotents, from any category, see Sect. 2.1. In such Cauchy complete category the idempotents in the class under consideration acquire a canonical, that is choice independent, splitting; moreover, that category is universal among the categories in which the idempotents in question split, in the sense of Proposition 1. Although the splitting of idempotent morphisms in a fibered setting is maybe a folklore matter, we felt the need to provide a systematic

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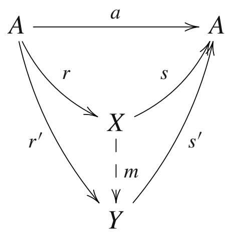
description of constructions that we have not been able to find elsewhere. Beyond these special need, general motivations for pursueing the investigation which is the matter of this paper come from the will to adopt the foundational attitude that characterizes the employment of fibrational tools in approaching fundamental subjects in category theory, see [1], as Cauchy completion certainly is, see [2]. More to the point, we provide an explicit elementary description of the Cauchy completion of fibrations with a fixed base category $\mathbb{B}$ as objects of the category **Fib**($\mathbb{B}$) and of fibrations with any base category as objects of the category **Fib**. The first construction arises by splitting reindexing stable vertical idempotent morphisms in the total category of fibrations and is described in Sect. 3.1, Proposition 2. About this construction we hasten to say that we encountered the description of a particular case of it in [9], but also to say that we described it independently and in more general hypothesis. The case described in *loc. cit.* is the one obtained by splitting every vertical idempotent in the total category of a fibration, thus making the mentioned stability condition automatically satisfied, as observed in Remark 5. Furthermore, we use Proposition 2 in Sect. 3.2 to provide a description of the fiberwise exact completion of a fiberwise regular fibration. As far as we know, this construction has not been described yet. Splitting all the vertical idempotents in the total categories of fibrations is a functorial process that gives rise to a fibered reflection from the fibration of fibrations **Fib** $\to$ **Cat**, see Proposition 3. The ordinary reflection at the terminal category $\mathbb{1}$ of the fibered reflection at issue provides the functorialization of the process of Cauchy-completing a category with respect to the class of all of its idempotents, see Remark 6. The second construction that we referred to above arises by simultaneously splitting idempotents in the total and base categories of fibrations, and is described in Sect. 3.3, Proposition 4, which, as far as we know, is new. For a given fibration, it provides the description of the Cauchy complete one out of it in **Fib**, with respect to a class of idempotents in the base category and a class of idempotents in the total category of the original fibration. This process is functorial too and gives rise to a fibered reflection from the fibration of fibrations, see Proposition 5. The first construction, dealing specially with the splitting of vertical idempotents, is a particular case of this second, see Remark 8.

2 Categorical preliminaries

We assume that the reader is already familiar with the basics of ordinary and fibered category theory, but see [1,10,13–15]. In any case, in this section we recall some of the relevant notions that will be useful in the sequel of this paper, also to establish terminology and notations.

2.1 The Cauchy completion of a category

Let $\mathbb{C}$ be a category. In $\mathbb{C}$ an *idempotent* is an endomorphism, say $a : A \to A$, such that $aa = a$; think of identity morphisms, in particular. In $\mathbb{C}$, *idempotents split* if for every idempotent $a : A \to A$ there are an object X and a pair of morphisms $r : A \to X, s : X \to A$ such that $a = sr$ and $id_X = rs$. For an idempotent $a : A \to A$, a triple $(X, r : A \to X, s : X \to S)$ with r, s as above, is a *splitting* of a. The splittings of an idempotent are determined up to a unique isomorphism, in the sense that if $(Y, r' : A \to Y, s' : Y \to A)$ is another splitting of a, then there exists a unique mediating isomorphism $m : X \to Y$ making the diagram



commute.

Remark 1 In any category, for every object A, the identity morphism id_A has a canonical, that is choice independent, splitting, which is (A, id_A, id_A). Henceforth, by referring to a splitting of an identity morphism we will assume to be the canonical one, if not differently and explicitly intended.

In the sequel of the paper, whenever confusion is not likely to arise, rather than providing the whole data for the components of a splitting of an idempotent a, we will more briefly write that $a = sr$ is a splitting of a, or even that a splits as $a = sr$. If $\mathcal{A}$ is a class of idempotent morphisms of $\mathbb{C}$, henceforth always assumed to contain all the identity morphisms, then $\mathbb{C}$ is said to be *Cauchy complete with respect to* $\mathcal{A}$ if every idempotent in $\mathcal{A}$ splits in $\mathbb{C}$ and *Cauchy complete* without any further specification if it is Cauchy complete with respect to the class of all of its idempotent morphisms. The Definition 1 below introduces the Cauchy completion of a category with respect to a class of its idempotent morphisms, but see also [2,6,7,11].

Definition 1 Let $\mathbb{C}$ be a category and $\mathcal{A}$ be a class of idempotent morphisms of $\mathbb{C}$. The *Cauchy completion* of $\mathbb{C}$ with respect to $\mathcal{A}$ is the category $\mathbb{C}[\check{\mathcal{A}}]$ identified by the following data:

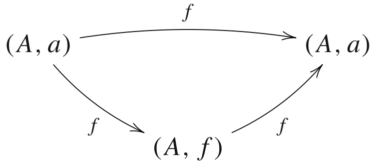
Objects:	pairs (A, a) with $a : A \to A$ an idempotent in $\mathcal{A}$.
Morphisms:	$f : (A, a) \to (B, b)$ is $f : A \to B$ in $\mathbb{C}$ such that $bfa = f$ or, equivalently, such that $bf = f$ and $fa = f$.
Composition:	inherited from $\mathbb{C}$.
Identities:	for every object (A, a), $id_{(A,a)} \doteq a : (A, a) \to (A, a)$.

The Cauchy completion of a category $\mathbb{C}$ with respect to the class of all of its idempotent morphisms will be henceforth denoted $\check{\mathbb{C}}$.

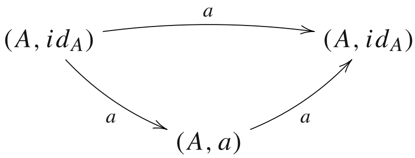
Proposition 1 below provides a list of well-known facts which are worth to be recalled.

Proposition 1 *Let* $\mathbb{C}$ *be a category and* $\mathcal{A}$ *be a class of idempotent morphisms of* $\mathbb{C}$. *The following facts hold.*

1. *In* $\mathbb{C}[\check{\mathcal{A}}]$*, every idempotent morphism, say* $f : (A, a) \to (A, a)$*, with* f *in* $\mathcal{A}$*, splits canonically as*

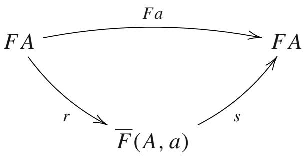


2. *For every object* A *and morphism* $f : A \to B$ *of* $\mathbb{C}$*, the assignments* $A \mapsto (A, id_A)$ *and* $f \mapsto f : (A, id_A) \to (B, id_B)$ *identify a full and faithful functor* $I_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}[\check{\mathcal{A}}]$*.*
3. *The functor* $I_{\mathbb{C}}$ *maps every idempotent in* $\mathcal{A}$ *to an idempotent that splits canonically in* $\mathbb{C}[\check{\mathcal{A}}]$*: for* $a : A \to A$ *in* $\mathcal{A}$*, a canonical splitting of* $I_{\mathbb{C}}(a) = a : (A, id_A) \to (A, id_A)$ *is*

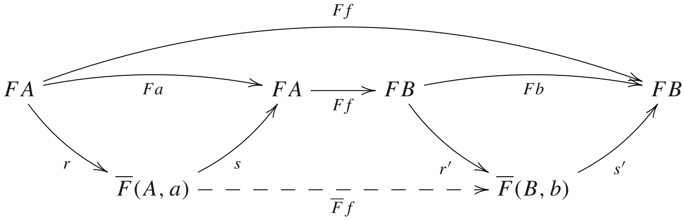


4. *The functor* $I_{\mathbb{C}}$ *is universal with the property described in 3: for every category* $\mathbb{D}$ *and functor* $F : \mathbb{C} \to \mathbb{D}$*, if* F *maps every idempotent in* $\mathcal{A}$ *to an idempotent that splits in* $\mathbb{D}$*, then* F *extends, up to a unique natural isomorphism, to a functor* $\overline{F} : \mathbb{C}[\check{\mathcal{A}}] \to \mathbb{D}$*.*

Proof Points 1, 2, 3 are straightforward. For point 4: for every object (A, a), let



be a chosen splitting of Fa in $\mathbb{D}$. The assigment $(A, a) \mapsto \overline{F}(A, a)$ uniquely extends to morphisms: for every $f : (A, a) \to (B, b)$ in $\mathbb{C}[\check{\mathcal{A}}]$, let $\overline{F} f$ be the uniquely induced dashed morphism in the commutative diagram



that is $\overline{F} f = r'(Ff)s$. □

Corollary 1 *Let $\mathbb{C}$ be a category and $\mathcal{A}$ a class of idempotent morphisms of $\mathbb{C}$. The full and faithful functor $I_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}[\check{\mathcal{A}}]$ is an equivalence of categories if and only if every idempotent in $\mathcal{A}$ already splits in $\mathbb{C}$.*

Proof We suppose that every idempotent in $\mathcal{A}$ already splits in $\mathbb{C}$ and show that $I_{\mathbb{C}}$ is essentially surjective on objects. Let (X, x) be an object in $\mathbb{C}[\check{\mathcal{A}}]$ and let $(A, r : X \to A, s : A \to X)$ be a splitting of $x : X \to X$ in $\mathbb{C}$. The objects (X, x), (A, id_A) are isomorphic in $\mathbb{C}[\check{\mathcal{A}}]$, as shown by the commutative diagram

$$\overset{x}{\circlearrowright}(X, x) \underset{r}{\overset{s}{\leftrightarrows}} (A, id_A)\overset{id_A}{\circlearrowright}$$

The converse is obvious by virtue of Proposition 1. □

Remark 2 As a further consequence of Proposition 1 and of Corollary 1, one has that the Cauchy completion of a category with respect to a class of its idempotent morphisms is an idempotent construction up to equivalence of categories, which fact provides the reason why it is referred to as a completion.

Remark 3 Let **Cat** be the category of categories and functors between them. Let $\mathbf{Cat_{Cc}}$ be the full subcategory of **Cat** identified by the Cauchy complete categories. The universal property described in Proposition 1 amounts to say that the assignment $\mathbb{C} \mapsto \check{\mathbb{C}}$ extends to a functor which is left adjoint to the inclusion of $\mathbf{Cat_{Cc}}$ in **Cat**. That is, one has a reflection

$$\mathbf{Cat_{Cc}} \underset{\hookrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Cat} \tag{1}$$

whose unit, for every category $\mathbb{C}$, is the functor $I_{\mathbb{C}} : \mathbb{C} \to \check{\mathbb{C}}$, whose counit, for every Cauchy complete category $\mathbb{C}$ is the equivalence between $\mathbb{C}$ and $\check{\mathbb{C}}$ from Corollary 1.

2.2 Fibered category theory

In this subsection we briefly recall some basics of fibered category theory.

Definition 2 Let $P : \mathbb{X} \to \mathbb{B}$ be a functor. A morphism $\varphi : X \to Y$ of $\mathbb{X}$ is *P-cartesian* if for every morphism $v : K \to PX$ of $\mathbb{B}$, for every morphism $g : Z \to Y$ with $Pg = (P\varphi)v$, there exists a unique morphism $h : Z \to X$ such that $\varphi h = g$ and $Ph = v$. The functor P is a *fibration* if for every object Y of $\mathbb{X}$, for every morphism $u : I \to PY$, there exists a cartesian morphism $\varphi : X \to Y$ such that $P\varphi = u$. The domain of a fibration is its *total category*. The codomain of a fibration is its *base category*.

The notion of fibration axiomatically captures how to deal with the reindexing of internally indexed families of objects and morphisms of a category. For $P : \mathbb{X} \to \mathbb{B}$

a fibration, the cartesian morphism $\varphi : X \to Y$ which is required to exist and fit in a situation like

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & & \\ I & \xrightarrow[u]{} & PY \end{array}$$

with $P\varphi = u$, for an arbitrary object Y of $\mathbb{X}$ and morphism u of $\mathbb{B}$, is a *P-reindexing* of Y along u. We will also say that a fibration has *enough reindexings*. The morphisms of $\mathbb{X}$ whose image via P is an identity of $\mathbb{B}$ are said to be *P-vertical*. For every object I of $\mathbb{B}$, the subcategory of $\mathbb{X}$ which is completely identified by the P-vertical morphisms whose image via P is id_I will be referred to as the *fiber category of P at I* and henceforth denoted P_I. In the total category of a fibration every morphism f factors as a vertical morphism $\overline{f}$ followed by a cartesian morphism φ, that is $f = \varphi \circ \overline{f}$ with φ a cartesian morphism such that $P\varphi = Pf$ and $\overline{f}$ uniquely induced by virtue of the P-cartesianness of φ. By means of the axiom of choice for classes, P-reindexings can be chosen. For every morphism $u : I \to J$ in $\mathbb{B}$, for every object Y in P_J a chosen P-reindexing of Y along u will be henceforth denoted as $y_u : u^*Y \to Y$. A fibration with chosen reindexings is said to be *cloven*. If P is a cloven fibration, then the assignment $Y \mapsto u^*Y$ extends to a *reindexing functor* $u^* : P_J \to P_I$ which, in case P is fiberwise structured, is required to be coherently structure-preserving. For instance, if P is a cloven and fiberwise finitely complete fibration then u^* is required to preserve finite limits. For $P : \mathbb{X} \to \mathbb{B}$ a cloven fibration and I an object of $\mathbb{B}$, the reindexing functors $(id_I)^*, id_{P_I} : P_I \to P_I$ are naturally isomorphic. Without any employment of the axiom of choice for classes it can be assumed that they actually coincide. Henceforth, we will communicate this assumption by saying that P is *normally cloven*. Moreover, for morphisms $u : I \to J$, $v : J \to K$ of $\mathbb{B}$, the reindexing functors $u^*v^*, (vu)^* : P_K \to P_I$ are naturally isomorphic by means of a natural isomorphism that we will henceforth denote $\mu_{u,v} : u^*v^* \Rightarrow (vu)^*$. For fibrations $P : \mathbb{X} \to \mathbb{B}$ and $Q : \mathbb{Y} \to \mathbb{B}$, a *fibered functor* $F : P \to Q$ is a functor $F : \mathbb{X} \to \mathbb{Y}$ such that $QF = P$ and mapping P-cartesian morphisms to Q-cartesian morphisms. For $F, G : P \to Q$ a pair of parallel fibered functors, a *fibered natural transformation* from F to G is a natural transformation $\alpha : F \Rightarrow G$ whose components are Q-vertical morphisms. Fibrations with a fixed base category $\mathbb{B}$, fibered functors and fibered natural transformations form a 2-category **Fib**($\mathbb{B}$). For fibrations $P : \mathbb{X} \to \mathbb{B}$ and $Q : \mathbb{Y} \to \mathbb{C}$, a *morphism of fibrations* $(F, G) : P \to Q$ is a pair of functors $F : \mathbb{X} \to \mathbb{Y}, G : \mathbb{B} \to \mathbb{C}$ such that $QF = GP$ with F mapping P-cartesian morphisms to Q-cartesian morphisms. For $(F, G), (F', G') : P \to Q$ a pair of parallel morphisms of fibrations, a *fibered natural transformation* from (F, G) to (F', G') is a pair (α, β) with $\alpha : F \Rightarrow F', \beta : G \Rightarrow G'$ with, for every object X of $\mathbb{X}$, $Q\alpha_X = \beta_{PX}$. Fibrations, morphisms of fibrations and fibered natural transformations between them form a 2-category **Fib**. It is well-known that for every fibration $P : \mathbb{X} \to \mathbb{B}$, the assignment $P \mapsto \mathbb{B}$ extends to a functor **Fib** $\to$ **Cat** which is a fibration thanks to the stability of fibrations under change of base, which is the operation of pulling back a fibration along a functor, see point (iii) of examples 2.3 below. For every category $\mathbb{B}$, **Fib**($\mathbb{B}$) occurs as the fiber at $\mathbb{B}$ of **Fib** $\to$ **Cat**. For further details on **Fib** as a 2-category fibered over **Cat** see [8].

2.3 Examples of fibrations

(i) Let **Sets** be the category of sets and functions between them. For every category $\mathbb{C}$, let $Fam(\mathbb{C})$ be the category of set-indexed families of objects of $\mathbb{C}$ and set-indexed families of morphisms of $\mathbb{C}$ between them. Explicitly, the objects of $Fam(\mathbb{C})$ are families $(A_i)_{i\in I}$ with I a set and, for every $i \in I$, A_i an object of $\mathbb{C}$, whereas its morphisms are pairs $(u, f) : (A_i)_{i\in I} \to (B_j)_{j\in J}$, with $u : I \to J$ a function and $f = (f_i : A_i \to B_{u(i)})_{i\in I}$ an I-indexed familiy of morphisms of $\mathbb{C}$. For every object $(A_i)_{i\in I}$, the assignment $(A_i)_{i\in I} \mapsto I$ extends to a fibration $P_{\mathbb{C}} : Fam(\mathbb{C}) \to$ **Sets**. For every object $(B_j)_{j\in J}$ of $Fam(\mathbb{C})$ and for every function $u : I \to J$, it can be verified that

$$\begin{array}{ccc} (B_{u(i)})_{i\in I} & \xrightarrow{(u,id)} & (B_j)_{j\in J} \\ \\ I & \xrightarrow[u]{} & J \end{array}$$

with $id = (id_{B_{u(i)}})_{i\in I}$, is a $P_{\mathbb{C}}$-reindexing of $(B_j)_{j\in J}$ along u. $P_{\mathbb{C}}$ is often referred to as the "naive" indexing of $\mathbb{C}$; $P_{\mathbb{C}}$ is the fibration which is very often implicitly involved in the categorial arguments that more or less hiddenly rely on set-theory. For I a set, the fiber category of $P_{\mathbb{C}}$ at I is $\mathbb{C}^I$, essentially.

(ii) For every category $\mathbb{B}$, let $\mathbb{B}^{\to}$ be the category identified by the following data:

- Objects: the morphisms of $\mathbb{B}$.
- Morphisms: a morphisms from $a : A \to I$ to $b : B \to J$ is $(u, f) : a \to b$, with $u : I \to J$ and $f : A \to B$ in $\mathbb{B}$, such that $bf = ua$.
- Composition: inherited from $\mathbb{B}$, componentwise.
- Identities: inherited from $\mathbb{B}$, componentwise.

For every object $a : A \to I$ of $\mathbb{B}^{\to}$, the assignment $a \mapsto I$ extends to a functor $cod_{\mathbb{B}} : \mathbb{B}^{\to} \to \mathbb{B}$ which is a fibration if and only $\mathbb{B}$ is a category with pullbacks. In this case, $cod_{\mathbb{B}}$ is often referred to as "the fundamental fibration over $\mathbb{B}$" because, by thinking of $\mathbb{B}$ as a sort of category-theoretic base universe, $cod_{\mathbb{B}}$ provides the categorial universe of internally indexed families of objects and morphisms of $\mathbb{B}$ itself; that is, as it is usually said, $cod_{\mathbb{B}}$ allows to consider $\mathbb{B}$ as indexed over itself. For I an object of $\mathbb{B}$, the fiber category of $cod_{\mathbb{B}}$ at I is the slice category $\mathbb{B}/I$.

(iii) We previously observed that **Fib** is fibered over **Cat** because fibrations are stable under change of base. Explicitly, this means that for every fibration $P : \mathbb{X} \to \mathbb{B}$ and functor $F : \mathbb{A} \to \mathbb{B}$, if

$$\begin{array}{ccc} \mathbb{A} \times_{\mathbb{B}} \mathbb{X} & \xrightarrow{G} & \mathbb{X} \\ {\scriptstyle P_F}\downarrow & & \downarrow{\scriptstyle P} \\ \mathbb{A} & \xrightarrow[F]{} & \mathbb{B} \end{array} \tag{2}$$

is a pullback diagram in **Cat**, then P_F is a fibration. Here $\mathbb{A} \times_{\mathbb{B}} \mathbb{X}$ is the category identified by the following data:

Objects: are pairs (A, X) with A an object from $\mathbb{A}$ and X an object from $\mathbb{X}$, such that $FA = PX$.

Morphisms: are pairs (u, f) with u a morphism from $\mathbb{A}$ and f a morphism from $\mathbb{X}$, such that $Pf = Fu$.

Composition: inherited from $\mathbb{A}$ in the first component and from $\mathbb{X}$ in the second component.

Identities: inherited from $\mathbb{A}$ in the first component and from $\mathbb{X}$ in the second component.

In diagram (2) P_F is the first projection functor and G the second projection functor. For every object A of $\mathbb{A}$, the fiber category of P_F at A is essentially the fiber category of P at FA, more precisely $(P_F)_A$ is isomorphic to P_{FA}.

2.4 Fibered adjunctions

For fibrations $P : \mathbb{X} \to \mathbb{B}$ and $Q : \mathbb{Y} \to \mathbb{B}$, a *fibered adjunction* between P and Q in **Fib**($\mathbb{B}$) is an adjunction

$$\mathbb{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{Y} \quad (F \dashv G)$$

with P-vertical unit, Q-vertical counit and $F : P \to Q, G : Q \to P$ fibered functors. For fibrations $P : \mathbb{X} \to \mathbb{B}$ and $Q : \mathbb{Y} \to \mathbb{C}$, a *fibered adjunction* between P and Q in **Fib** is a pair of adjunctions

$$\mathbb{X} \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{Y} \quad (F \dashv G)$$

$$\mathbb{B} \underset{K}{\overset{H}{\rightleftarrows}} \mathbb{C} \quad (H \dashv K)$$

with $P\eta = \eta P$, $Q\varepsilon = \varepsilon Q$ and $(F, H) : P \to Q$, $(G, K) : Q \to P$ morphisms of fibrations.

3 Splitting idempotents in a fibered setting

In Sect. 3.1 we prove Proposition 2 about the splitting of vertical idempotents in the total category of a fibration. As a result, a description of the Cauchy completion of an object of **Fib**($\mathbb{B}$), for $\mathbb{B}$ a fixed base category, is provided together with an application of such construction in Sect. 3.2: the obtainment of the fiberwise exact completion of a fibration out of a fiberwise regular fibration. Section 3.3 generalizes Sect. 3.1. In it we prove Proposition 4 about the splitting of whatever idempotents in the total category of a fibration. As a result a description of the Cauchy completion of an object of **Fib** is provided.

Remark 4 Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration and let $x : X \to X$ be a P-vertical idempotent. We assume that $Px = id_{PX}$ splits canonically, in accordance with Remark 1. As a consequence, if x splits as $x = sr$, then $Ps = Pr = id_{PX}$. That is, vertical idempotents split in the fiber category of which they are morphisms.

3.1 Splitting vertical idempotents

Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration and $\mathcal{E}$ be a class of P-vertical idempotents of $\mathbb{X}$. For every object I of $\mathbb{B}$ let $\mathcal{E}_I$ be the subclass of $\mathcal{E}$ consisting of the members of $\mathcal{E}$ which are in the fiber P_I. Suppose that idempotents in $\mathcal{E}$ are stable under P-reindexing, as follows: in every diagram like

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[\varphi]{} & Y \end{array} \qquad (3)$$

in which y is in $\mathcal{E}_{PY}$ and φ is P-cartesian, the uniquely induced P-vertical idempotent x which makes the whole diagram commute is in $\mathcal{E}_{PX}$.

Remark 5 If $\mathcal{E}$ is the class of all the P-vertical idempotents, then the previous stability condition is automatically satisfied.

Proposition 2 *Under the hypothesis on P and $\mathcal{E}$ described above, the following facts hold.*

1. *By virtue of the universal property described in Proposition 1 there exists a unique functor $\overline{P} : \mathbb{X}[\check{\mathcal{E}}] \to \mathbb{B}$.*
2. *The functor $\overline{P}$ is a fibration and $I_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}[\check{\mathcal{E}}]$ identifies a full and faithful fibered functor $I_{\mathbb{X}} : P \to \overline{P}$.*
3. *Up to equivalence, for every object I of $\mathbb{B}$, $\overline{P}_I$ is the Cauchy completion of P_I with respect to $\mathcal{E}_I$.*
4. *The fibered functor $I_{\mathbb{X}} : P \to \overline{P}$ maps every idempotent in $\mathcal{E}$ to an idempotent that splits in $\mathbb{X}[\check{\mathcal{E}}]$ and is universal with this property: for every fibration $Q : \mathbb{Y} \to \mathbb{B}$ and fibered functor $F : P \to Q$, if F maps every idempotent in $\mathcal{E}$ to an idempotent*

that splits in $\mathbb{Y}$, then F extends, up to a unique fibered natural isomorphism, to a fibered functor $\overline{F} : \overline{P} \to Q$.

Proof 1. Since the idempotents in $\mathcal{E}$ are P-vertical morphisms, P maps them to idempotent morphisms that split in $\mathbb{B}$. Thus, by virtue of the universal property described in Proposition 1 there exists a unique up to natural isomorphism functor that makes the diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{I_{\mathbb{X}}} & \mathbb{X}[\check{\mathcal{E}}] \\ & \searrow^{P} & \downarrow{\scriptstyle \overline{P}} \\ & & \mathbb{B} \end{array}$$

commute but, by virtue of the fact that identity morphisms split canonically, the functor $\overline{P}$ can be uniquely identified by the assignments

$$(X, x) \mapsto PX$$

on objects, and

$$f : (X, x) \to (Y, y) \quad \mapsto \quad Pf : PX \to PY$$

on morphisms.

2. We prove that the functor $\overline{P}$ has enough reindexings, in the sense explained in Sect. 2.2. Let (Y, y) be an object of $\mathbb{X}[\check{\mathcal{E}}]$ and $u : I \to PY$ be a morphism of $\mathbb{B}$. Let $\varphi : X \to Y$ be a P-cartesian lifting of Y along u and x be uniquely induced as in diagram (3), so that $y\varphi = \varphi x$. We claim that

$$\begin{array}{c} (X, x) \xrightarrow{\varphi x} (Y, y) \\ \\ I \xrightarrow[u]{} PY \end{array} \tag{4}$$

is a $\overline{P}$-cartesian lifting of (Y, y) along u. For every $v : K \to I$ and for every $g : (Z, z) \to (Y, y)$ in $\mathbb{X}[\check{\mathcal{E}}]$ with $Pg = uv$, let $\overline{g} : Z \to X$ be the unique morphism such that $\varphi\overline{g} = g$ and $P\overline{g} = v$, by virtue of the P-cartesianness of φ. One has

$$P(x(\overline{g}z)z) = v = P(\overline{g}z)$$

and

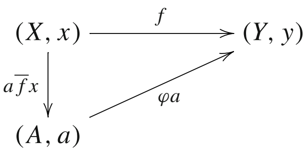
$$\varphi(x(\overline{g}z)z) = y\varphi\overline{g}z = ygz = g = \varphi(\overline{g}z)$$

so that $\overline{g}z : (Z, z) \to (X, x)$. Also one has

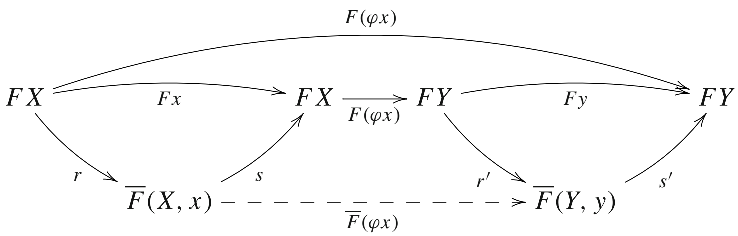
$$(\varphi x)(\overline{g}z) = (y\varphi)(\overline{g}z) = ygz = g$$

and uniqueness of $\overline{g}z$ can be proved as follows: if $f : (Z, z) \to (X, x)$ is such that $Pf = v$ and $(\varphi x)f = g$, then $\varphi(xf) = g$, consequently $f = xf = \overline{g}$ but $\overline{g} = \overline{g}z$ because $P(\overline{g}z) = v$ and $\varphi(\overline{g}z) = gz = g$. Finally, to verify that $I_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}[\check{\mathcal{E}}]$ identifies a full and faithful fibered functor from P to $\overline{P}$ is straightforward.

3. Let I be an object of $\mathbb{B}$. It is immediate to see that $\overline{P}_I$ is isomorphic to $P_I[\check{\mathcal{E}}_I]$. Now, if $f : (X, x) \to (X, x)$ is an idempotent in $\mathbb{X}[\check{\mathcal{E}}]$, with $f : X \to X$ in $\mathcal{E}$, then it splits as an idempotent in $P_{PX}[\check{\mathcal{E}}_{PX}]$.
4. In the way just explained, every idempotent in $\mathcal{E}$, say $x : X \to X$, acquires, via $I_{\mathbb{X}}$, a splitting in $\mathbb{X}[\check{\mathcal{E}}]$. Under the stated hypothesis the functor F extends to a functor $\overline{F} : \mathbb{X}[\check{\mathcal{E}}] \to \mathbb{Y}$ as shown in Proposition 1 and it is immediate to verify that $Q\overline{F} = \overline{P}$. It remains to show that $\overline{F}$ maps $\overline{P}$-cartesian morphism to Q-cartesian morphism. For this, we observe that if $f : (X, x) \to (Y, y)$ is a morphism in $\mathbb{X}[\check{\mathcal{E}}]$, then it factors as in the commutative diagram



where $\varphi : A \to Y$ is a P-cartesian morphism over Pf, $\varphi\overline{f} = f$ with $\overline{f}$ a P-vertical morphism and $a : A \to A$ a P-vertical idempotent in $\mathcal{E}$ obtained by P-reindexing of y, in a diagram such as (3). Now, if f is a $\overline{P}$-cartesian morphism, then there exists a $\overline{P}$-vertical morphism $g : (A, a) \to (X, x)$ such that $fg = \varphi a$, and it can be easily verified that $(a\overline{f}x)g = a$ and that $g(a\overline{f}x) = x$, so that g and $a\overline{f}x$ are each other inverses in $\mathbb{X}[\check{\mathcal{E}}]$, vertically over PX. As a consequence of all this, a $\overline{P}$-cartesian morphism can be assumed to be as in diagram (4). For one such $\overline{P}$-cartesian morphism, $\overline{F}(\varphi x)$ is the unique dashed morphism that makes the diagram



that is $\overline{F}(\varphi x) = r'F(\varphi x)s$. Now, commute for every morphism $u : I \to PX$ and $g : Z \to \overline{F}(Y, y)$ with

$$Qg = Q(\overline{F}(\varphi x))u = (P\varphi)u$$

let $\overline{g} : Z \to FX$ be the uniquely induced morphism such that $Q\overline{g} = u$ and $(F\varphi)\overline{g} = s'g$, by virtue of the Q-cartesianness of $F\varphi$. One has, $Q(r\overline{g}) = u$ and

$$(\overline{F}(\varphi x))r\overline{g} = r'(F(\varphi x))\overline{g} = r'F(y\varphi)\overline{g} = r's'r'F\varphi\overline{g} = r's'g = g$$

Finally, if $h : Z \to \overline{F}(X, x)$ is such that $Qh = u$ and $\overline{F}(\varphi x)h = g$, then

$$s'g = s'\overline{F}(\varphi x)h = s'r'F(\varphi x)sh = F(y\varphi)sh = F(\varphi x)sh = F(\varphi)srsh = F(\varphi)sh$$

Thus, by virtue of the Q-cartesianness of $F(\varphi)$, $sh = \overline{g}$, so $h = r\overline{g}$. □

Corollary 2 *Under the hypothesis of Proposition 2, the fibrations P and $\overline{P}$ are equivalent if and only if every idempotent in $\mathcal{E}$ already splits in P.*

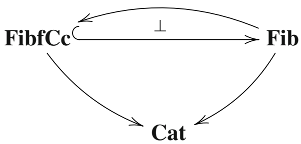
Proof The result follows from Proposition 2 and Corollary 1. □

On the base of Proposition 2, Corollary 2 and Remark 2 we give the following

Definition 3 Under the hypothesis assumed at the beginning of Sect. 3.1, the fibration $\overline{P}$ described in Proposition 2 is the *fiberwise Cauchy completion* of P as an object of **Fib**($\mathbb{B}$), with respect to $\mathcal{E}$.

Proposition 3 *Let* **FibfCc** *be the full subcategory of* **Fib** *identified by the fiberwise Cauchy complete fibrations. The following facts hold:*

1. *There is a fibration* **FibfCc** $\to$ **Cat**.
2. *There is a fibered reflection*



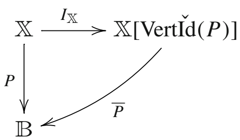
(5)

Proof 1. The fibrations in **FibfCc** are stable under change of base, so they identify a full subfibration **FibfCc** $\to$ **Cat** of **Fib** $\to$ **Cat**.

2. For every fibration $P : \mathbb{X} \to \mathbb{B}$, let VertId($P$) be the class of all P-vertical idempotents in $\mathbb{X}$. By virtue of the universal property described in Proposition 2, for every fibration $P : \mathbb{X} \to \mathbb{B}$, the assignment

$$P \quad \mapsto \quad \overline{P} : \mathbb{X}[\mathrm{Vert}\check{\mathrm{I}}\mathrm{d}(P)] \to \mathbb{B}$$

extends to a fibered functor from **Fib** $\to$ **Cat** to **FibfCc** $\to$ **Cat** and, moreover, the fibered functor



behaves like a unit for the seeked fibered reflection. □

Remark 6 The fibered reflection (5) amounts to the "continuos glueing" of a family of reflections

$$\mathbf{FibfCc}(\mathbb{B}) \underset{\bot}{\hookrightarrow} \mathbf{Fib}(\mathbb{B})$$

with $\mathbb{B}$ a category in **Cat** and $\mathbf{FibfCc}(\mathbb{B})$ the fiber of $\mathbf{FibfCc} \to \mathbf{Cat}$ at $\mathbb{B}$. The reflection 1 is obtained when $\mathbb{B}$ is the terminal category $\mathbb{1}$.

3.2 An application: the fiberwise ex/reg completion of a fibration

In this section we briefly describe a straightforward application of Proposition 2: a generalization to fibrations of the construction of an exact category out of a regular category. The construction at issue is also known as *ex/reg completion*. About regular categories, exact categories and the ex/reg completion we will recall just the main facts and refer the reader to suitable bibliographic references for more details. A *regular category* is a category with finite limits, in which every morphism factors as a regular epi followed by a monomorphism and with pulback-stable regular epis. In a regular category every kernel pair has a coequalizer and an *exact category* is a regular category in which every equivalence relation (see below) has a coequalizer. A *regular functor* between regular categories is one which preserves finite limits and regular epis. For $\mathbb{C}$ a category, for A, B any objects of $\mathbb{C}$, a *relation* r from A to B, often indicated as $r : A \longrightarrow\!\!\!\!\!\!\!\!\!+\;\; B$, is represented by a *jointly monic* span $A \xleftarrow{r_0} R \xrightarrow{r_1} B$; that is, one such that if $r_0 u = r_0 v$ and $r_1 u = r_1 v$, then $u = v$, whenever the composites make sense. Two jointly monic spans represent the same relation from an object to another if and only if they factor through each other. If $\mathbb{C}$ is a category with binary products, then a span such as the previous is jointly monic if and only if the morphism $< r_0, r_1 >: R \to A \times B$ is monic. Coherently, relations should be thought of as subobjects. For A an object of $\mathbb{C}$, the *diagonal relation* on A is the relation δ_A represented by the span (id_A, id_A). In any category with pullbacks, an *equivalence relation* on A is a relation $e : A \longrightarrow\!\!\!\!\!\!\!\!\!+\;\; A$, say represented by $A \xleftarrow{e_0} E \xrightarrow{e_1} A$, satisfying the following conditions:

- Reflexivity: there exists a morphism $\rho : A \to E$ such that $e_0 \rho = id_A$ and $e_1 \rho = id_A$.
- Symmetry: there exists a morphism $\sigma : E \to E$ such that $e_0 \sigma = e_1$ and $e_1 \sigma = e_0$.
- Transitivity: construct a pullback of e_0 against e_1, such as

$$\begin{array}{ccc} E \star E & \xrightarrow{\pi_1} & E \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle e_0} \\ E & \xrightarrow[e_1]{} & A \end{array}$$

> Transitivity of e means that there exists a morphism $\tau : E \star E \to E$ such that $e_0\tau = e_0\pi_0$ and $e_1\tau = e_1\pi_1$.

Diagonal relations are equivalence relations and, in any category with pullbacks, kernel pairs are equivalence relations. In a suitable sense, the structure of a regular category is precisely what is needed in order to implement a *calculus of relations*, see [12]. For $\mathbb{C}$ a regular category it is possibile to consider the category $Rel(\mathbb{C})$ with the same objects as $\mathbb{C}$ and relations between them as morphisms; given that relations $r : A \mathrel{-\!\!\!\mapstochar\!\!\rightarrow} B$, $s : B \mathrel{-\!\!\!\mapstochar\!\!\rightarrow} C$ compose like this: first construct a pullback diagram of s_0 against r_1 such as

$$\begin{array}{ccc} S \star R & \xrightarrow{\pi_1} & S \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle s_0} \\ R & \xrightarrow[r_1]{} & B \end{array}$$

and then let the composite relation $s \bullet r : A \mathrel{-\!\!\!\mapstochar\!\!\rightarrow} C$ be the one represented by the jointly monic span provided by the monic morphism $< t_0, t_1 >$ in the regular epi-mono factorization

$$\begin{array}{ccc} S \star R & \xrightarrow{<r_0\pi_0, s_1\pi_1>} & A \times C \\ & {\scriptstyle q}\searrow \qquad \nearrow{\scriptstyle <t_0,t_1>} & \\ & T & \end{array}$$

Now, as already observed at the beginning of this subsection, in a regular category every kernel pair has a coequalizer or, in other words, a quotient. An exact category is a regular category in which every equivalence relation is a kernel pair. Thus, in an exact category every equivalence relation has a quotient. The ex/reg completion of a regular category $\mathbb{C}$ is the process of freely adding quotients to the equivalence relations in it, and is achieved in two steps: first, consider the Cauchy completion of $Rel(\mathbb{C})$ with respect to the class $\mathcal{Q}$ of equivalence relations (these are idempotent with respect to the law of composition described above), that is the category $Rel(\mathbb{C})[\check{\mathcal{Q}}]$, then take the subcategory of maps of this category (see below for a reference to an explicit description of that subcategory). The category $Map(Rel(\mathbb{C})[\check{\mathcal{Q}}])$ is exact and provides the ex/reg completion of $\mathbb{C}$. It is usually more briefly denoted $\mathbb{C}_{ex/reg}$. An explicit description of $\mathbb{C}_{ex/reg}$ and a thourough discussion of the universal property it enjoys as a free category can be found in [3,5]. For the sake of completeness, we observe that more in general any category with (weak) finite limits can be freely completed to an exact category, see [4,5]. At last, thanks to Proposition 2 the ex/reg completion of a regular category generalizes to *fiberwise regular fibrations*. By these we mean fibrations with fibers which are regular categories whose structure is stable by reindexing. For $P : \mathbb{X} \to \mathbb{B}$ a fiberwise regular fibration, it means in particular that in every diagram like

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[\psi]{} & Y \end{array}$$

in which y is a P-vertical regular epi in the regular category P_{PY} and φ, ψ are P-cartesian and such that $P\varphi = P\psi$, the uniquely induced P-vertical morphism x which makes the whole diagram commute is a regular epi in the regular category P_{PX}. For instance, if $\mathbb{B}$ is a regular category, then $cod_{\mathbb{B}} : \mathbb{B}^{\to} \to \mathbb{B}$ is fiberwise regular. Now, let $P : \mathbb{X} \to \mathbb{B}$ be a fiberwise regular fibration which we suppose to be normally cloven. The fiberwise ex/reg completion of P can be achieved in three steps:

1. Construct the *fibration of relations* out of P. This is $Rel(P) : \mathbb{Y} \to \mathbb{B}$, where $\mathbb{Y}$ is the category identified by the following data:
   Objects: are pairs (I, A) with I an object of $\mathbb{B}$ and A an object of P_I.
   Morphisms: a morphism from (I, A) o (J, B) is a pair (u, r) with $u : I \to J$ a morphism of $\mathbb{B}$ and $r : A \longrightarrow\!\!\!\!\!\!\!\mid\;\; u^*B$ a morphism of $Rel(P_I)$.
   Composition: for $(u, r) : (I, A) \to (J, B)$, $(v, s) : (J, B) \to (K, C)$ as above, their composite is $(vu, s \bullet r) : (I, A) \to (K, C)$, where $s \bullet r$ is now the relation in $Rel(P_I)$ obtained as follows: if $B \xleftarrow{s_0} S \xrightarrow{s_1} v^*C$ is a jointly monic span representing the relation s, then consider a reindexing of it along u, say

$$u^*B \xleftarrow{u^*s_0} u^*S \xrightarrow{u^*s_1} u^*(v^*C)$$

   then compose u^*s_1 with the isomorphism $\mu_{u,v}(C) : u^*(v^*C) \to (vu)^*C$ to obtain the jointly monic span

$$u^*B \xleftarrow{u^*s_0} u^*S \xrightarrow{\mu_{u,v}(C)u^*s_1} (vu)^*C$$

   and finally compose the relation represented by this span with r.
   Identities: for every object (I, A) the morphism (id_I, δ_A) acts as identical with respect to the law of composition just described.

   For every object (I, A) of $\mathbb{Y}$, the assignment $(I, A) \mapsto I$ extends to a functor $Rel(P) : \mathbb{Y} \to \mathbb{B}$ which is a fibration, because, for every morphism $v : J \to K$ and object (K, C), it can be verified that a $Rel(P)$-reindexing of (K, C) along v is given by the morphism $(v, \delta_{v^*C}) : (J, v^*C) \to (K, C)$. Moreover, there is a fibered functor

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\Gamma} & \mathbb{Y} \\ & {\scriptstyle P}\searrow & \downarrow{\scriptstyle Rel(P)} \\ & & \mathbb{B} \end{array}$$

that to every morphism $f : X \to Y$ of $\mathbb{X}$ assigns the morphism

$$(Pf, \gamma_{\overline{f}}) : (PX, X) \to (PY, Y)$$

of $\mathbb{Y}$, with $\overline{f} : X \to (Pf)^*Y$ the vertical component of a vertical-cartesian factorization of f in P, and $\gamma_{\overline{f}} : X \longrightarrow\!\!\!\!\!\!\mapsto (Pf)^*Y$ the relation represented by the graph of $\overline{f}$, that is the jointly monic span $(id_X, \overline{f})$.

2. For $\mathcal{Q}$ the class of $Rel(P)$-vertical equivalence relations construct the fiberwise Cauchy completion of $Rel(P)$ with respect to $\mathcal{Q}$ in accordance with Proposition 2. This is the fibration $\overline{Rel(P)} : \mathbb{Y}[\check{\mathcal{Q}}] \to \mathbb{B}$ whose fiber at an object I of $\mathbb{B}$ is $Rel(P_I)[\check{\mathcal{Q}}_I]$, essentially.
3. Consider the subfibration of $\overline{Rel(P)}$ identified by the subcategories

$$(P_I)_{ex/reg} \doteq Map(Rel(P_I)[\check{\mathcal{Q}}_I]) \hookrightarrow Rel(P_I)[\check{\mathcal{Q}}_I]$$

with I varying among the objects of $\mathbb{B}$. Let it be $P_{ex/reg} : \mathbb{Z} \to \mathbb{B}$.

As a whole the situation appears like this: there are fibered functors

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\Gamma} & \mathbb{Y} \\ & {\scriptstyle P}\searrow & \downarrow {\scriptstyle Rel(P)} \\ & & \mathbb{B} \end{array} \qquad \begin{array}{ccc} \mathbb{Z} & \hookrightarrow & \mathbb{Y}[\check{\mathcal{Q}}] \\ & {\scriptstyle P_{ex/reg}}\searrow & \downarrow {\scriptstyle \overline{Rel(P)}} \\ & & \mathbb{B} \end{array} \qquad \begin{array}{ccc} \mathbb{Y} & \xrightarrow{I_{\mathbb{Y}}} & \mathbb{Y}[\check{\mathcal{Q}}] \\ & {\scriptstyle Rel(P)}\searrow & \downarrow {\scriptstyle \overline{Rel(P)}} \\ & & \mathbb{B} \end{array}$$

and it can be verified that the fibered functor $I_{\mathbb{Y}} \circ \Gamma$ takes values in $P_{ex/reg}$, is fiberwise regular and, for every fiberwise exact fibration Q with base category $\mathbb{B}$, it induces an equivalence between the category of fibered and fiberwise regular functors from P to Q and the category of fibered and fiberwise regular functors from $P_{ex/reg}$ to Q. Finally, again for the sake of completeness, we point out that the construction of the free fiberwise exact fibration out of a fiberwise finitely complete one is considered in [16], thus generalizing the constructions described in the already cited papers [4] and [5].

3.3 Splitting whatever idempotents

In this section we generalize the results in Sect. 3.1. Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration, $\mathcal{E}$ a class of idempotent morphisms of $\mathbb{X}$, $\mathcal{F}$ a class of idempotent morphisms of $\mathbb{B}$. Suppose that every idempotent in $\mathcal{E}$ is mapped by P to an idempotent in $\mathcal{F}$. Furthermore, suppose that the idempotents in $\mathcal{E}$ are stable under P-reindexing as follows: for every idempotent in $\mathcal{E}$, say $y : Y \to Y$, for every commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & PY \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle Py} \\ A & \xrightarrow[u]{} & PY \end{array}$$

in $\mathbb{B}$, with a an idempotent in $\mathcal{F}$, the uniquely induced idempotent x with $Px = a$ that makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[\varphi]{} & Y \end{array} \tag{6}$$

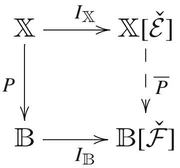
commute, with φ a P-cartesian lifting of Y along u, is an idempotent in $\mathcal{E}$.

Remark 7 If $\mathcal{E}$ is the class of all idempotent morphisms of $\mathbb{X}$ and if every idempotent in $\mathcal{E}$ is mapped by P to an idempotent in $\mathcal{F}$, then the previous stability condition is automatically satisfied.

Proposition 4 *Under the hypothesis on P, $\mathcal{E}$ and $\mathcal{F}$ described above, the following facts hold.*

1. *By virtue of the universal property described in Proposition 1 there exists a unique functor $\overline{P} : \mathbb{X}[\check{\mathcal{E}}] \to \mathbb{B}[\check{\mathcal{F}}]$.*
2. *The functor $\overline{P}$ is a fibration and $I_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}[\check{\mathcal{E}}]$, $I_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}[\check{\mathcal{F}}]$ identify a full and faithful morphism of fibrations $(I_{\mathbb{X}}, I_{\mathbb{B}}) : P \to \overline{P}$.*
3. *Every idempotent $f : (X, x) \to (X, x)$ of $\mathbb{X}[\check{\mathcal{E}}]$ with f in $\mathcal{E}$ splits canonically and every idempotent $u : (A, a) \to (A, a)$ of $\mathbb{B}[\check{\mathcal{F}}]$ with u in $\mathcal{F}$ splits canonically. Furthermore, $\overline{P}$ maps canonical splittings to canonical splittings.*
4. *The morphism of fibrations $(I_{\mathbb{X}}, I_{\mathbb{B}}) : P \to \overline{P}$ maps every idempotent in $\mathcal{E}$ to an idempotent that splits in $\mathbb{X}[\check{\mathcal{E}}]$ and every idempotent in $\mathcal{F}$ to an idempotent that splits in $\mathbb{B}[\check{\mathcal{F}}]$, and is universal with this property: for every fibration $Q : \mathbb{Y} \to \mathbb{C}$ and morphism of fibrations $(F, G) : P \to Q$ if F maps every idempotent in $\mathcal{E}$ to an idempotent that splits in $\mathbb{Y}$, G maps every idempotent in $\mathcal{F}$ to an idempotent that splits in $\mathbb{C}$, and Q maps chosen splittings to chosen splittings, then (F, G) extends, up to a unique fibered natural isomorphism, to a morphism of fibrations $(\overline{F}, \overline{G}) : \overline{P} \to Q$.*

Proof 1. P maps idempotent morphisms in $\mathcal{E}$ to idempotent morphisms in $\mathcal{F}$, which in turn are mapped by $I_{\mathbb{B}}$ to idempotent morphisms that split in $\mathbb{B}[\check{\mathcal{F}}]$. So, by virtue of the universal property described in Proposition 1, there exists a unique up to natural isomorphism functor that makes the diagram



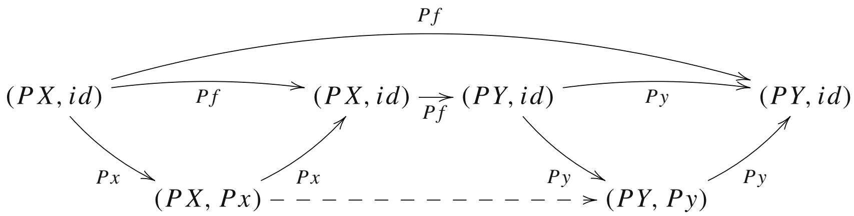
commute but, by virtue of the fact that idempotent moprhisms in $\mathcal{F}$ have canonical splittings in $\mathbb{B}[\check{\mathcal{F}}]$, the functor $\overline{P}$ can be identified by the assignments

$$(X, x) \mapsto (PX, Px)$$

on objects, and

$$f : (X, x) \to (Y, y) \quad \mapsto \quad Pf : (PX, Px) \to (PY, Py)$$

on morphisms. This is like that because, on the base of Proposition 1, to every object (X, x) of $\mathbb{X}[\check{\mathcal{E}}]$, the functor $\overline{P}$ must assign the object arising in a chosen splitting of $I_{\mathbb{B}} Px$, or better in the canonical splitting of $Px : (PX, id_{PX}) \to (PX, id_{PX})$ in $\mathbb{B}[\check{\mathcal{F}}]$; that object is precisely (PX, Px). On the other hand, to every morphism $f : (X, x) \to (Y, y)$, the functor $\overline{P}$ must assign the unique dashed morphism that makes the diagram



commute, that is $\overline{P} f = Py Pf Px$.

2. We prove that the functor $\overline{P}$ has enough reindexings. Let be (Y, y) be an object of $\mathbb{X}[\check{\mathcal{E}}]$ and $u : (A, a) \to (PY, Py)$ be a morphism of $\mathbb{B}[\check{\mathcal{F}}]$. Let $\varphi : X \to Y$ be a P-cartesian lifting of Y along $u : A \to PY$ and x uniquely induced as in diagram (6), so that $y\varphi = \varphi x$ and $Px = a$. We claim that

$$\begin{array}{c}(X, x) \xrightarrow{\ \varphi x\ } (Y, y) \\ \\ (A, a) \xrightarrow[\ u\ ]{} (PY, Py)\end{array} \qquad (7)$$

is a $\overline{P}$-cartesian lifting of (Y, y) along u. For every $v : (B, b) \to (A, a)$ in $\mathbb{B}[\check{\mathcal{F}}]$ and for every $g : (Z, z) \to (Y, y)$ in $\mathbb{X}[\check{\mathcal{E}}]$ with $Pg = uv$, let $\overline{g} : Z \to X$ be the unique morphism such that $P\overline{g} = v$ and $\varphi\overline{g} = g$, by virtue of the P-cartesianness of φ. Now, one has

$$P(x(\overline{g}z)z) = avbb = v = P(\overline{g}z)$$

and

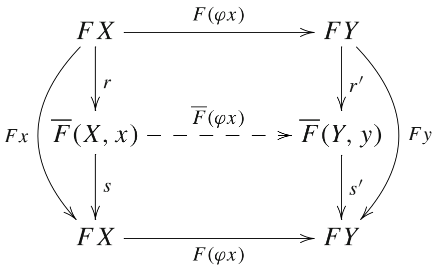
$$\varphi(x(\overline{g}z)z) = y\varphi\overline{g}zz = ygz = g = gz = \varphi(\overline{g}z)$$

so that $\overline{g}z : (Z, z) \to (X, x)$ and also one has

$$(\varphi x)(\overline{g}z) = y\varphi\overline{g}z = ygz = g$$

and uniqueness of $\overline{g}z$ can be proved as follows: if $f : (Z, z) \to (X, x)$ is such that $Pf = v$ and $(\varphi x)f = g$, then $\varphi(xf) = g$, consequently $f = xf = \overline{g}$ but $\overline{g} = \overline{g}z$ because $P(\overline{g}z) = vb = v$ and $\varphi(\overline{g}z) = gz = g$.
To verify that $I_{\mathbb{X}}$ and $I_{\mathbb{B}}$ identify a full and faithful morphism of fibrations $(I_{\mathbb{X}}, I_{\mathbb{B}}) : P \to \overline{P}$ is straightforward.

3. The first statement is just point 1 of Proposition 1 applied to the total and base categories of $\overline{P}$. The verification of the second statement is straightforward.
4. The first statement is straightforward. It is just point 2 of Proposition 1 applied to the total and base categories of $\overline{P}$. For what concern the stated universal propery we observe that under the assumed hypothesis the functors F, G extend to functors $\overline{F} : \mathbb{X}[\check{\mathcal{E}}] \to \mathbb{Y}$ and $\overline{G} : \mathbb{B}[\check{\mathcal{F}}] \to \mathbb{C}$ as described in Proposition 1, respectively, and it is immediate to verify that $Q\overline{F} = \overline{G}\,\overline{P}$. It remains to show that $\overline{F}$ maps $\overline{P}$-cartesian morphisms to Q-cartesian morphisms. So, for $\varphi x : (X, x) \to (Y, y)$ a $\overline{P}$-cartesian morphism, $\overline{F}(\varphi x)$ is the uniquely induced morphism in the commutative diagram



that is $\overline{F}(\varphi x) = r'F(\varphi x)s$. Now, for every morphism $v : K \to Q\overline{F}(X, x)$ and $g : Z \to \overline{F}(Y, y)$ such that $Qg = Q(\overline{F}(\varphi x))v$, let $\overline{g} : Z \to FX$ be the uniquely induced morphism such that $Q\overline{g} = (Qs)v$ and $(F\varphi)\overline{g} = s'g$, by virtue of the Q-cartesianness of $F\varphi$. Now, $Q(r\overline{g}) = Qr\,Q\overline{g} = (Qr)(Qs)v = v$ and

$$\begin{aligned}(\overline{F}(\varphi x))r\overline{g} &= (r'F(\varphi x)s)(r\overline{g}) = r'F(\varphi x)Fx\overline{g} = r'F(\varphi x)\overline{g} = r'F(y\varphi)\overline{g}\\ &= r's'r'(F\varphi)\overline{g} = r's'g = g\end{aligned}$$

Finally, if $h : Z \to \overline{F}(X, x)$ is such that $Qh = v$ and $(\overline{F}(\varphi x))h = g$ one has

$$Q(sh) = QsQh(Qs)v = Q\overline{g}$$

and

$$s'g = s'(\overline{F}(\varphi x))h = s'(r'F(\varphi x)s)h = FyF(y\varphi)(sh) = F(y\varphi)(sh)$$
$$= F(\varphi x)sh = F(\varphi)srsh = F(\varphi)(sh)$$

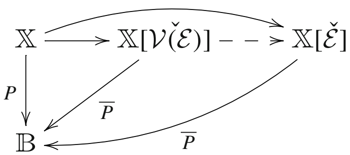
so that, by the Q-cartesianness of $F(\varphi)$, $sh = \overline{g}$, and $h = r\overline{g}$ in turn. □

Corollary 3 *Let P, $\mathcal{E}$, $\mathcal{F}$ and $\overline{P}$ be as in Proposition 4. The fibrations P and $\overline{P}$ are equivalent if and only if idempotent morphisms in $\mathcal{E}$ already split in $\mathbb{X}$ and idempotent morphisms in $\mathcal{F}$ already split in $\mathbb{B}$.*

Proof The result follows from Proposition 4 and Corollary 1. □

Remark 8 Proposition 2 is Proposition 4 in the special case in which every idempotent morphism in $\mathcal{F}$ splits in $\mathbb{B}$ and every idempotent morphism in $\mathcal{E}$ is P-vertical.

Remark 9 Assume the hypothesis of Proposition 4, and moreover assume that every idempotent morphism in $\mathcal{F}$ splits in $\mathbb{B}$. Let $\mathcal{V}(\mathcal{E})$ be the class of P-vertical idempotents in $\mathcal{E}$. Under these assumptions, because of the universal property described in Proposition 2, there is a uniquely induced comparison fibered functor $\mathbb{X}[\mathcal{V}\check{(}\mathcal{E})] \to \mathbb{X}[\check{\mathcal{E}}]$ as in the commutative diagram



which is nothing but the inclusion functor of the category $\mathbb{X}[\mathcal{V}\check{(}\mathcal{E})]$ in the category $\mathbb{X}[\check{\mathcal{E}}]$. That functor is full and faithful; it is an equivalence if and only if every idempotent morphism in $\mathcal{E}$ is P-vertical. The previous considerations hold in particular when $\mathbb{B}$ is Cauchy complete and $\mathcal{E}$ is the class of all the idempotent morphisms in $\mathbb{X}$.

On the base of Proposition 4, Corollary 3 and Remark 2 we give the following

Definition 4 Under the hypothesis assumed at the beginning of Sect. 3.3, the fibration $\overline{P}$ described in Proposition 4 is the *Cauchy completion* of P as an object of **Fib** with respect to a class $\mathcal{E}$ of idempotent morphisms in its total category and a class $\mathcal{F}$ of idempotent morphisms in its base category.

Proposition 5 *Let* $\mathbf{Fib}^{\mathbf{Cc}}_{\mathbf{Cc}}$ *be the full subcategory of* **Fib** *identified by the fibrations which have a Cauchy complete total category and a Cauchy complete base category. The following facts hold:*

1. *There is a fibration* $\mathbf{Fib}^{\mathbf{Cc}}_{\mathbf{Cc}} \to \mathbf{Cat}_{\mathbf{Cc}}$.
2. *There is a fibered reflection*

$$\begin{array}{ccc} \mathbf{Fib}^{\mathbf{Cc}}_{\mathbf{Cc}} & \underset{\perp}{\hookrightarrow} & \mathbf{Fib} \\ \downarrow & & \downarrow \\ \mathbf{Cat}_{\mathbf{Cc}} & \underset{\perp}{\hookrightarrow} & \mathbf{Cat} \end{array} \tag{8}$$

Proof 1. The fibrations in $\mathbf{Fib}^{\mathbf{Cc}}_{\mathbf{Cc}}$ are stable under change of base along functors between Cauchy complete categories.

2. For every fibration $P : \mathbb{X} \to \mathbb{B}$ in **Fib** let $\overline{P} : \check{\mathbb{X}} \to \check{\mathbb{B}}$ be its Cauchy completion in the sense of Proposition 4, with $\mathcal{E}$, resp. $\mathcal{F}$, the class of all the idempotent morphisms in $\mathbb{X}$, resp. $\mathbb{B}$. By virtue of the universal property described in Proposition 4, for every fibration P, the assignment $P \mapsto \overline{P}$ extends to a morphism of fibrations from $\mathbf{Fib} \to \mathbf{Cat}$ to $\mathbf{Fib}^{\mathbf{Cc}}_{\mathbf{Cc}} \to \mathbf{Cat}_{\mathbf{Cc}}$ and the morphism of fibrations

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{I_{\mathbb{X}}} & \check{\mathbb{X}} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle \overline{P}} \\ \mathbb{B} & \xrightarrow[I_{\mathbb{B}}]{} & \check{\mathbb{B}} \end{array} \tag{9}$$

behaves like a unit for the seeked reflection. □

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