Arch. Math. Logic (2018) 57:795–807
https://doi.org/10.1007/s00153-017-0606-z

Mathematical Logic

Homomorphism reductions on Polish groups

Konstantinos A. Beros[1]

Received: 13 February 2017 / Accepted: 11 December 2017 / Published online: 15 December 2017

Abstract In an earlier paper, we introduced the following pre-order on the subgroups of a given Polish group: if G is a Polish group and $H, L \subseteq G$ are subgroups, we say H is *homomorphism reducible* to L iff there is a continuous group homomorphism $\varphi : G \to G$ such that $H = \varphi^{-1}(L)$. We previously showed that there is a K_σ subgroup L of the countable power of any locally compact Polish group G such that every K_σ subgroup of G^ω is homomorphism reducible to L. In the present work, we show that this fails in the countable power of the group of increasing homeomorphisms of the unit interval.

Keywords Polish groups · Homomorphism reductions · K_σ subgroups

Mathematics Subject Classification 03E15 · 54H11 · 22A05 · 57S05

1 Introduction

In the study of definable equivalence relations, the equivalence relations arising as coset equivalence relations on Polish groups have played a key role. For instance, consider the equivalence relation E_0 on 2^ω where

$$x E_0 y \iff x(n) = y(n) \text{ for all but finitely many } n$$

This equivalence relation may be regarded as the coset equivalence relation of the subgroup

✉ Konstantinos A. Beros
berosk@miamioh.edu

1 Miami University, Oxford, OH, USA

$$\mathsf{Fin} = \{x \in 2^\omega : x(n) = 0 \text{ for all but finitely many } n\}$$

of the Polish group 2^ω with bitwise addition modulo 2. Many other important equivalence relations arise in this way, e.g., E_1, E_2, E_3, and ℓ^∞.

The most important means of comparing equivalence relations is the quasiorder of *Borel reducibility*. That is, if E and F are equivalence relations on Polish spaces X and Y, respectively, one says that E is *Borel reducible* to F iff there is a Borel map $\varphi : X \to Y$ such that, for all $x, y \in X$,

$$x E y \iff \varphi(x) F \varphi(y).$$

The map φ is called a *Borel reduction* of E to F. The present work and our earlier paper [1] are motivated by the observation that in many cases, Borel reductions between coset equivalence relations can be witnessed by continuous group homomorphisms between appropriate groups. For instance, let

$$H_1 = \{y \in 2^{\omega\times\omega} : (\forall^\infty m)(\forall n)(y(m,n) = 0)\},$$

and let E_1 be the coset relation of H_1. It is well-known that E_0 is Borel reducible to E_1. A Borel map witnessing this reduction is $\varphi : 2^\omega \to 2^{\omega\times\omega}$ given by

$$\varphi(x)(m,n) = x(m)$$

for $x \in \mathbb{Z}_2^\omega$ and $m, n \in \omega$. In fact, $\mathsf{Fin} = \varphi^{-1}(H_1)$, i.e., φ is a continuous reduction of the subgroup Fin to H_1. Furthermore, the map φ induces an injective homomorphism

$$\tilde{\varphi} : 2^\omega/\mathsf{Fin} \to 2^{\omega\times\omega}/H_1$$

such that the diagram

$$\begin{array}{ccc} 2^\omega & \xrightarrow{\varphi} & 2^{\omega\times\omega} \\ \downarrow p & & \downarrow q \\ 2^\omega/\mathsf{Fin} & \xrightarrow{\tilde{\varphi}} & \mathbb{Z}_2^{\omega\times\omega}/H_1 \end{array}$$

commutes. Here p and q denote the appropriate factor maps.

Abstracting from such specific examples leads to the following definition introduced in our paper [1].

Definition 1.1 Let G be a Polish group. For subgroups H and L of G, we say that H is *group homomorphism reducible* to L iff there is a continuous group homomorphism $\varphi : G \to G$ such that $H = \varphi^{-1}(L)$. As a shorthand, we write $H \le_{\mathrm{g}} L$.

As an alternative formulation, observe that $H \le_{\mathrm{g}} L$ iff there is an embedding of G/H into G/L with a lifting which is a continuous endomorphism of G. Viewed in this light, our definition is not without precedent. For instance, Farah [2] considers maps between factors of Boolean algebras with liftings which are algebra homomorphisms.

The K_σ equivalence relations play an important role in the general theory of equivalence relations, e.g., E_0, E_1 and ℓ^∞ are all K_σ. (Recall that a relation is K_σ if it is the countable union of compact sets). In fact, much study has been done of such relations. See for instance Rosendal [4]. In light of this, we directed much of our attention to the K_σ subgroups of Polish groups in [1]. For a large class of groups G, we showed that there are $\leq_g$-maximal K_σ subgroups of G^ω and referred to such subgroups as *universal K_σ subgroups* of G^ω. The equivalence relation analog is that of a Borel-complete K_σ equivalence relation, though the requirement of universality is much stronger. For instance, ℓ^∞ is a Borel-complete K_σ equivalence relation, but it is not universal in our sense.

As a counterpoint, we showed that there are no universal K_σ subgroups of S_∞. That is, for any K_σ subgroup H of S_∞, there is a K_σ subgroup $L \subseteq S_\infty$ such that $L \neq \varphi^{-1}(H)$ for every continuous homomorphism $\varphi : S_\infty \to S_\infty$. Since S_∞ continuously embeds its own countable power, it followed immediately that S_∞^ω has no universal K_σ subgroup.

The cases in which universal K_σ subgroups do not exist are interesting in that they indicate a degree of rigidity of the group in question. That is, if G does not have a universal K_σ subgroup, there must be (in some sense) a shortage of continuous endomorphisms of G. It is worth remarking upon the fact that S_∞ does have $\leq_g$-maximal analytic subgroups. Thus, the rigidity which precludes universal K_σ subgroups of S_∞ is not sufficient to rule out universal analytic subgroups.

In the vein of our earlier results, we prove here an analogous theorem for the transformation group $H_+([0, 1])$ of increasing homeomorphisms of the unit interval.

Theorem 1.2 *There is no universal K_σ subgroup of $H_+([0, 1])$.*

The proof of this result is in Sect. 3. As an aside, it follows from our earlier paper [1, Theorem 1.7] that there is an F_σ subgroup of $H_+([0, 1])$ of which every K_σ subgroup is a continuous homomorphic pre-image. By Theorem 1.2, such a "universal F_σ subgroup for K_σ" cannot be K_σ itself.

A direct corollary of Theorem 1.2 is

Corollary 1.3 *There is no universal K_σ subgroup of $H_+([0, 1])^{\mathbb{N}}$.*

Proof Assume Theorem 1.2 holds. Observe that $H_+([0, 1])^{\mathbb{N}}$ is isomorphic to the closed subgroup

$$C = \{f \in H_+([0, 1]) : (\forall n)(f(1/n) = 1/n)\}$$

of $H_+([0, 1])$. Let $\varphi : H_+([0, 1])^{\mathbb{N}} \to C$ be an isomorphism and let ψ be an isomorphism between $H_+([0, 1])$ and a closed subgroup of $H_+([0, 1])^{\mathbb{N}}$. Towards a contradiction, suppose that $H_+([0, 1])^{\mathbb{N}}$ has a universal K_σ subgroup H. Let $\tilde{H}$ be the K_σ sugbroup $\varphi(H)$ of $H_+([0, 1])$. Fix an arbitrary K_σ subgroup L of $H_+([0, 1])$. Note that $\psi(L)$ is a K_σ subgroup of $H_+([0, 1])^{\mathbb{N}}$ and hence, by the universality of H, there is a continuous endomorphism ρ of $H_+([0, 1])^{\mathbb{N}}$ such that $\psi(L) = \rho^{-1}(H)$. This implies that $L = (\varphi \circ \rho \circ \psi)^{-1}(\tilde{H})$. As L was arbitrary, this shows that $\tilde{H} = \varphi(H)$ is a universal K_σ subgroup of $H_+([0, 1])$, contradicting Theorem 1.2. □

In our study of K_σ subgroups from [1], we also considered situations in which every K_σ subgroup of a given Polish group is compactly generated. This is true in $\mathbb{Z}^\omega$ for instance. In the final section of this paper, we expand upon these ideas and describe a large class of groups in which every K_σ subgroup is group homomorphism reducible to a compactly generated subgroup, i.e., the compactly generated subgroups are $\leq_g$-cofinal in the K_σ subgroups.

2 Preliminaries and notation

The group $H_+([0, 1])$ of increasing homeomorphisms of $[0, 1]$ is a Polish group when equipped with the topology of uniform convergence, i.e., the relative topology inherited from $C([0, 1])$. It is a peculiarity of $H_+([0, 1])$ that this topology coincides with the topology of pointwise convergence. In the present setting, the latter is much easier to manipulate. The basic open sets in $H_+([0, 1])$ are thus of the form

$$U = \{f \in H_+([0, 1]) : (\forall i \leq k)(f(r_i) \in I_i)\}$$

where $r_0, \dots, r_k \in [0, 1]$ and $I_0, \dots, I_k \subseteq [0, 1]$ are open intervals.

For $f \in H_+([0, 1])$, let $\operatorname{supp}(f) = \{x \in [0, 1] : f(x) \neq x\}$ be the *support* of f.

Let $\mathcal{N}$ denote the Baire space, i.e., $\mathbb{N}^\mathbb{N}$ with the product topology. For $\alpha \in \mathcal{N}$ and a finite sequence s of natural numbers, $\alpha \supset s$ indicates that s is an initial segment of α.

Definition 2.1 Given $f \in H_+([0, 1])$ and $\alpha \in \mathcal{N}$, define f to be *α-continuous* iff, for each $k \in \mathbb{N}$ with $k \geq 1$, if $x, y \in [0, 1]$ are such that $|x - y| < 1/\alpha(k)$, then $|f(x) - f(y)| < 1/k$.

Recall that a family $F \subseteq H_+([0, 1])$ is *equicontinuous* iff there is an $\alpha \in \mathcal{N}$ such that each $f \in F$ is α-continuous. For each $\alpha \in \mathcal{N}$, let

$$K_\alpha = \{f \in H_+([0, 1]) : f, f^{-1} \text{ are } \alpha\text{-continuous}\}.$$

By definition, each K_α is equicontinuous.

Lemma 2.2 *Suppose $f_0, f_1, \dots \in H_+([0, 1])$ and each f_n is α-continuous. If $f_n \to f$ pointwise, then f is also α-continuous.*

Proof Fix $k \in \mathbb{N}$ and $x, y \in [0, 1]$ such that $|x - y| < 1/\alpha(k)$. Assume that $k, \alpha(k) > 1$, as otherwise the desired conclusion is automatic. Assume that $x < y$ and, since $1/\alpha(k) \leq 1/2$, we may assume that $x \neq 0$ or $y \neq 1$. These two cases are analogous, so assume $y \neq 1$. Let $y' > y$ be such that $|x - y'| < 1/\alpha(k)$. For each n, $|f_n(x) - f_n(y')| < 1/k$ and hence

$$|f(x) - f(y')| = \lim_n |f_n(x) - f_n(y')| \leq 1/k.$$

Since f is a strictly increasing function,

$$|f(x) - f(y)| < |f(x) - f(y')| \leq 1/k.$$

This completes the proof. □

Each K_α is thus closed and, since the K_α are also equicontinuous, it follows from the Arzelà-Ascoli Theorem (see [5, §7.10]) that each K_α is compact. Conversely,

Lemma 2.3 *If $K \subseteq H_+([0,1])$ is compact, then there exists $\alpha \in \mathcal{N}$ such that each $f \in K$ is α-continuous.*

Proof Fix $k \in \mathbb{N}$, it suffices to determine a value for $\alpha(k)$ which is sufficiently large that $|f(x) - f(y)| < 1/k$ whenever $f \in K$ and $|x - y| < 1/\alpha(k)$. Indeed, fix open intervals $I_0, \ldots, I_n \subseteq (0,1)$ which cover $(0,1)$ and such that, if each $I_i = (a_i, b_i)$, then

$$a_i < a_{i+1} < b_i$$

for $i \leq n-1$. Moreover, choose the I_i such that $|b_{i+2} - a_i| < 1/k$ for each $i \leq n-2$. For each increasing tuple $r = \langle r(0), r(1), \ldots, r(n)\rangle \in (0,1)^{n+1}$, define the open set

$$U_r = \{f \in H_+([0,1]) : (\forall i \leq n)(f(r(i)) \in I_i)\}$$

and observe that the U_r cover $H_+([0,1])$. Since K is compact, there exist $r_0 < r_1 < \cdots < r_p$ such that $U_{r_0}, \ldots, U_{r_p}$ cover K. Let

$$\varepsilon_k = \min(\{|r_j(i) - r_j(i+1)| : i < n \;\&\; j \leq p\})$$

and note that, for each $f \in K$ and $x, y \in [0,1]$, if $|x - y| < \varepsilon_k$, then $|f(x) - f(y)| < 1/k$. Choose $\alpha(k) \geq 1/\varepsilon_k$. This completes the proof. □

If $K \subseteq H_+([0,1])$ is compact, then so is $\{f^{-1} : f \in K\}$. It thus follows from the last lemma that $K \subseteq K_\alpha$ for some $\alpha \in \mathcal{N}$.

In the interest of clarity, if f, g are functions, fg throughout denotes the composite function $f \circ g$ and f^n denotes the n-fold iterate of f (for $n \in \mathbb{N}$). Regarding elements of $\mathcal{N}$ as functions on $\mathbb{N}$, this notation applies equally to members of the Baire space. Similarly, if $f \in H_+([0,1])$ and $\varphi : H_+([0,1]) \to H_+([0,1])$ is an endomorphism, φf denotes the image of f under φ, i.e., $\varphi(f)$.

The following facts are direct consequences of the definitions of α-continuity and the K_α.

(1) If f is α-continuous and g is β-continuous, fg is $\beta\alpha$-continuous.
(2) If $f \in K_\alpha$ and $g \in K_\beta$, then $fg \in K_{\max(\alpha\beta, \beta\alpha)}$, where $\max(\alpha\beta, \beta\alpha) \in \mathcal{N}$ is such that $\max(\alpha\beta, \beta\alpha)(k) = \max(\alpha\beta(k), \beta\alpha(k))$ for each $k \in \mathbb{N}$.
(3) If $f \in K_\alpha$ and $|x - y| \geq 1/k$, for some k, then $|f(x) - f(y)| \geq 1/\alpha(k)$.

Finally, for $A \subseteq H_+([0,1])$, let $\langle A\rangle$ designate the subgroup generated by A.

3 Proof of Theorem 1.2

Lemma 3.1 *There is an increasing homeomorphism $\sigma : [0,1] \to [0,1]$ such that σ is supported on $[1/2, 3/4]$ and the conjugacy class of σ is dense.*

Proof Glasner-Weiss [3] have shown that $H_+([0, 1])$ contains a dense conjugacy class. Therefore, fix $\sigma \in H_+([0, 1])$ such that the conjugacy class of σ is dense. The objective of the proof is to modify σ and produce τ which is supported on $[1/2, 3/4]$ and still has dense conjugacy class.

Let $h : [0, 1] \to [1/2, 3/4]$ be a linear bijection. Define $\tau \in H_+([0, 1])$ by

$$\tau(x) = \begin{cases} h\sigma h^{-1}(x) & \text{if } x \in [1/2, 3/4], \\ x & \text{otherwise.} \end{cases}$$

It remains to show that the conjugacy class of τ is dense in $H_+([0, 1])$. To this end, fix a nonempty basic open set $U \subseteq H_+([0, 1])$. Shrinking U if necessary, choose $0 < r_0 < \cdots < r_k < 1$ and nonempty open intervals $I_0, \ldots, I_k \subseteq (0, 1)$ such that

$$\sup(I_0) < \sup(I_1) < \cdots < \sup(I_k)$$

and

$$U = \{f \in H_+([0, 1]) : (\forall i \leq k)(f(r_i) \in I_i)\}.$$

With no loss of generality, $\sup(I_i) < 1$ and $\inf(I_i) > 0$ for each $i \leq k$ as this again only shrinks the open set U. Fix $a, b \in (0, 1)$ such that $0 < a < r_0$, $r_k < b < 1$, $a < \min_{i \leq k}(\inf(I_i))$ and $b > \max_{i \leq k}(\sup(I_i))$. Let g be the piecewise linear map defined such that the graph of g has vertices $(0, 0)$, $(1/2, a)$, $(3/4, b)$, $(1, 1)$. Note that $g \in H_+([0, 1])$.

For each $i \leq k$, let $s_i = h^{-1}g^{-1}(r_i)$ and $J_i = h^{-1}g^{-1}(I_i)$. Since the conjugacy class of σ is dense, there exists $f \in H_+([0, 1])$ such that $f\sigma f^{-1}(s_i) \in J_i$ for each $i \leq k$. Define $f_1 \in H_+([0, 1])$ by

$$f_1(x) = \begin{cases} hfh^{-1}(x) & \text{if } x \in [1/2, 3/4], \\ x & \text{otherwise.} \end{cases}$$

To complete the proof, it suffices to show that $gf_1\tau f_1^{-1}g^{-1} \in U$. Fix $i \leq k$ and observe that

$$\begin{aligned} gf_1\tau f_1^{-1}g^{-1}(r_i) &= gf_1\tau f_1^{-1}h(s_i) \\ &= ghfh^{-1}h\sigma h^{-1}hf^{-1}h^{-1}h(s_i) \\ &= ghf\sigma f^{-1}(s_i) \\ &\in gh(J_i) \\ &= I_i. \end{aligned}$$

This completes the proof. □

Let σ_1 be as in the previous lemma, with dense conjugacy class and $\mathrm{supp}(\sigma_1) \subseteq [1/2, 3/4]$. Define $h \in H_+([0, 1])$ such that the graph of h is polygonal and the vertices of h are the elements of the set

$$\{(1 - 2^{-n}, 1 - 2^{-n-1}) : n \in \mathbb{N}, n \geq 1\}$$

of points in $[0, 1]^2$, i.e., h has vertices $(1/2, 3/4)$, $(3/4, 7/8)$, ... To simplify notation, let I_n denote the interval $[1 - 2^{-n}, 1 - 2^{-n-1}]$. In other words, $I_0 = [0, 1/2]$, $I_1 = [1/2, 3/4]$ and so forth. For $n \geq 1$, define $\sigma_n = h^{n-1} \sigma_1 h^{1-n}$ and observe that σ_n is supported on the interval I_n.

Given $\alpha \in \mathcal{N}$, define $L_\alpha \subseteq H_+([0, 1])$ to be the set of those $f \in H_+([0, 1])$ such that $f \upharpoonright I_0 = \mathrm{id} \upharpoonright I_0$ and, for each $n \geq 1$, there is an $i \leq \alpha(n)$ with $f \upharpoonright I_n = \sigma_n^i \upharpoonright I_n$. If $f \in L_\alpha$ is the (unique) member of L_α such that $f \upharpoonright I_n = \sigma_n^{\delta(n)} \upharpoonright I_n$ for a given $\delta \leq \alpha$, denote f by f_δ. Observe that L_α is the continuous image of the compact space

$$X_\alpha = \prod_n \{0, \ldots, \alpha(n)\}$$

under the map $\delta \mapsto f_\delta$. Each L_α is therefore compact.

In fact, the map $\delta \mapsto f_\delta$ is a homeomorphism between X_α and L_α with its subspace topology. It follows that the sets

$$U_s = \{f_\delta : \delta \leq \alpha \text{ and } \delta \supset s\},$$

with s a finite sequence bounded by α, form a basis for the relative topology on L_α.

Given a K_σ subgroup $H \subseteq H_+([0, 1])$, the objective of the proof is to show that H is not a universal subgroup. For this it is sufficient to show that there exists a compact set $L \subseteq H_+([0, 1])$ such that, for every non-trivial continuous endomorphism φ of $H_+([0, 1])$, the image of $\langle L \rangle$ under φ is not contained in H and hence φ is not a homomorphism reduction of $\langle L \rangle$ to H. Note that restricting to non-trivial endomorphisms of $H_+([0, 1])$ is appropriate since if $\langle L \rangle = \varphi^{-1}(H)$, then $\ker(\varphi) \subseteq \langle L \rangle$ and hence $\ker(\varphi) \neq H_+([0, 1])$ because $H_+([0, 1])$ is not itself K_σ.

To further simplify the argument, assume that H is of the form $\bigcup_m K_{\beta_m}$ where K_{β_m} is defined as in Sect. 2. It is possible to make this assumption as H is contained in a set of the form $\bigcup_m K_{\beta_m}$. In addition, assume that $\{\beta_m : m \in \mathbb{N}\}$ is closed under composition and taking pointwise maxima, i.e., for each m, n there exist p, q such that $\beta_m \beta_n = \beta_p$ and $\max(\beta_m, \beta_n) = \beta_q$. The only purpose of these assumptions is to guarantee that, for any m, n, there exists p such that, if $f \in K_{\beta_m}$ and $g \in K_{\beta_n}$, then $fg, gf \in K_{\beta_p}$.

Observe that these simplifying assumptions introduce no loss of generality since they only enlarge the set $\bigcup_m K_{\beta_m}$.

Henceforth, fix $\beta_0, \beta_1, \ldots \in \mathcal{N}$ as above. Choose $\alpha \in \mathcal{N}$ such that, for each $n \in \mathbb{N}$,

$$\alpha(n) = \max(\{\beta_i^j(k) : i, j, k \leq n\}) + 1.$$

To complete the proof, it suffices to show that there is no continuous endomorphism of $H_+([0, 1])$ which maps the compactly generated subgroup $\langle \{h\} \cup L_\alpha \rangle$ into $\bigcup_m K_{\beta_m}$. (Here h is the map described above such that $\sigma_{n+1} = h\sigma_n h^{-1}$.) Indeed, suppose that, on the contrary, there exists an endomorphism φ of $H_+([0, 1])$ such that φ maps $\langle \{h\} \cup L_\alpha \rangle$ into $\bigcup_m K_{\beta_m}$.

Lemma 3.2 *There exists m such that* $\varphi(\{h\} \cup L_\alpha) \subseteq K_{\beta_m}$.

Proof As a compact subset of a Polish space, $\varphi(L_\alpha)$ is Polish in its relative topology and thus, by the Baire Category Theorem, there exists an open set $V \subseteq H_+([0, 1])$ and $m_0 \in \mathbb{N}$ such that $\varphi(L_\alpha) \cap V$ is nonempty and $K_{\beta_{m_0}}$ is comeager on $\varphi(L_\alpha) \cap V$. Since $K_{\beta_{m_0}}$ is a closed set, $K_{\beta_{m_0}}$ must in fact contain $\varphi(L_\alpha) \cap V$. Let $U = \varphi^{-1}(V)$. It follows that $U \cap L_\alpha$ is nonempty and hence $\varphi(U_s) \subseteq K_{\beta_{m_0}}$ for some finite sequence s which is bitwise bounded by α. Observe that any $f \in L_\alpha$ is of the form

$$\sigma_1^{i_1} \sigma_2^{i_2} \dots \sigma_{|s|}^{i_{|s|}} g \tag{3.1}$$

for some $g \in U_s$ and $i_1, \dots, i_{|s|} \in \mathbb{Z}$ with each $|i_p| \leq 2\alpha(p)$. Note that

$$\sigma_1^{i_1}, \sigma_2^{i_2}, \dots \sigma_{|s|}^{i_{|s|}} \in \langle L_\alpha \rangle$$

for all such $i_1, \dots, i_{|s|}$. Since $\varphi(\langle L_\alpha \rangle) \subseteq \bigcup_m K_{\beta_m}$, it now follows from the properties of $\beta_0, \beta_1, \dots$ that there exists $m_1 \in \mathbb{N}$ such that $K_{\beta_{m_1}}$ contains all φf where f is of the form (3.1). In other words, $K_{\beta_{m_1}}$ contains $\varphi(L_\alpha)$. Recall that φf denotes the image $\varphi(f)$ of f under φ.

Finally, since $\varphi h \in \bigcup_m K_{\beta_m}$, there exists m_2 with $\varphi h \in K_{\beta_{m_2}}$. It follows from the closure properties of the β_i that $\varphi(\{h\} \cup L_\alpha) \subseteq K_{\beta_{m_3}}$ for some $m_3 \in \mathbb{N}$. This proves the lemma. □

Let β_m be as in the previous lemma, with $\varphi(\{h\} \cup L_\alpha) \subseteq K_{\beta_m}$. For simplicity, write $\beta = \beta_m$.

Since the conjugacy class of σ_1 is dense, if $\varphi\sigma_1 = \text{id}$, then $\varphi f = \text{id}$ for each $f \in H_+([0, 1])$. Indeed, if $\sigma_1 \in \ker(\varphi)$, then $\ker(\varphi)$ contains the entire conjugacy class of σ_1, as it is a normal subgroup of $H_+([0, 1])$. This implies that $\ker(\varphi) = H_+([0, 1])$, since $\ker(\varphi)$ is also closed. On the other hand, this violates the assumption that φ is non-trivial. Therefore, choose $x_1 \in [0, 1]$ and $k \in \mathbb{N}$ such that

$$|x_1 - \varphi\sigma_1(x_1)| \geq 1/k.$$

Define $x_{n+1} = \varphi h(x_n)$ for each $n \geq 1$ where h is again as above. Observe that

$$\begin{aligned} |x_2 - \varphi\sigma_2(x_2)| &= |\varphi h(x_1) - (\varphi\sigma_2)\,(\varphi h)(x_1)| \\ &= |\varphi h(x_1) - (\varphi h)\,(\varphi\sigma_1)\,(\varphi h^{-1})\,(\varphi h)(x_1)| \\ &= |\varphi h(x_1) - (\varphi h)\,(\varphi\sigma_1)(x_1)| \\ &\geq 1/\beta(k), \end{aligned}$$

since $\varphi h \in K_\beta$ and hence φh^{-1} is β-continuous. Iterating this argument, it follows that

$$|x_n - \varphi\sigma_n(x_n)| \geq 1/\beta^{n-1}(k) \tag{3.2}$$

for each $n \geq 1$.

Let $n \geq k, m$ and let $p = \alpha(n)$. Recall that, by the choice of α,

$$p > \beta^n(k).$$

Consider the points $x_n, \varphi\sigma_n(x_n), \varphi\sigma_n^2(x_n), \dots, \varphi\sigma_n^p(x_n)$ where $p \geq 1$. Because $\varphi\sigma_n$ is an increasing homeomorphism of [0, 1] and $\varphi\sigma_n(x_n) \neq x_n$, either

$$x_n < \varphi\sigma_n(x_n) < \varphi\sigma_n^2(x_n) < \cdots < \varphi\sigma_n^p(x_n)$$

or

$$x_n > \varphi\sigma_n(x_n) > \varphi\sigma_n^2(x_n) > \cdots > \varphi\sigma_n^p(x_n).$$

In either case, there must exist $i < p$ such that

$$\left|\varphi\sigma_n^i(x_n) - \varphi\sigma_n^{i+1}(x_n)\right| \leq 1/p$$

since the length of [0, 1] is 1. Since $1/p < 1/\beta^n(k)$, it follows that $\varphi\sigma_n^{-i}$ is not β-continuous since

$$\left|\varphi\sigma_n^i(x_n) - \varphi\sigma_n^{i+1}(x_n)\right| \leq 1/p < 1/\beta^n(k),$$

but

$$\left|(\varphi\sigma_n^{-i})\,(\varphi\sigma_n^i)(x_n) - (\varphi\sigma_n^{-i})\,(\varphi\sigma_n^{i+1})(x_n)\right| = |x_n - \varphi\sigma_n(x_n)| \not< 1/\beta^{n-1}(k)$$

by (3.2). Thus, $\varphi\sigma_n^i \notin K_\beta$. This is a contradiction since the choice of α implies that $\sigma_n^i \in L_\alpha$ and φ maps L_α into K_β. □

4 Compactly generated subgroups

We consider a criterion on a group G which guarantees that the compactly generated subgroups of G^ω are $\leq_g$-cofinal in the K_σ subgroups. We begin with a couple of motivating examples.

Example 4.1 We showed in [1] that every K_σ subgroup of $\mathbb{Z}^\omega$ is compactly generated. The argument was similar in character to the proof that $\mathbb{Z}$ is a principal ideal domain. For instance, the countable dense subgroup $\{x \in \mathbb{Z}^\omega : (\forall^\infty n)(x(n) = 0)\}$ is generated by the compact set

$$\{0^n {}^\frown 1 {}^\frown \bar{0} : n \in \omega\} \cup \{\bar{0}\}.$$

For the sake of the next example, we say that x is *divisible in H* (where H is a group) iff, for each n, there exists $y \in H$ such that $y^n = x$.

Example 4.2 Consider the Polish group $\mathbb{Q}^\omega$, where $\mathbb{Q}$ is equipped with the discrete topology. There are K_σ subgroups, e.g.,

$$H = \{x \in \mathbb{Q}^\omega : (\forall n)(n \geq 1 \implies x(n) = 0)\},$$

which are K_σ, but not compactly generated. In fact, the subgroup H is not $\leq_{\mathrm{g}}$-reducible to any compactly generated subgroup of $\mathbb{Q}^\omega$. To see this, suppose that $\varphi : \mathbb{Q}^\omega \to \mathbb{Q}^\omega$ is a group homomorphism with $H = \varphi^{-1}(\langle K \rangle)$ for some compact set $K \subseteq \mathbb{Q}^\omega$. Note that, since H is a divisible subgroup of $\mathbb{Q}^\omega$, the subgroup $\langle K \rangle$ must contain elements which are divisible in $\langle K \rangle$. This, however, is impossible since it would imply that there is some $n \in \omega$ such that $\{x(n) : x \in K\}$ is infinite.

The results in this section were motivated by the observation that the last example hinges on the fact that there are no compactly generated subgroups H of $\mathbb{Q}^\omega$ such that some element of H is divisible in H. There are groups where this is not the case. For instance, in the permutation group of the natural numbers, there are compactly generated subgroups which have divisible elements. Consider the following example.

Example 4.3 Let $a_0, a_1, \ldots, b_0, b_1, \ldots \in \omega$ be distinct. Let $\pi_i \in S_\infty$ be the product

$$\pi_1 = \prod_{i \in \omega} [a_i, b_i]$$

of transpositions. For each $n > 1$, let

$$\pi_n = \prod_{i \in \omega} [a_{ni}, a_{ni+1}, \ldots, a_{ni+(n-1)}, b_{ni}, b_{ni+1}, \ldots, b_{ni+(n-1)}].$$

Notice that π_n is a product of countably many disjoint $2n$-cycles and $(\pi_n)^n = \pi_1$ for each $n > 1$. It follows that π_1 is divisible in the subgroup generated by the π_n. If we let H be the subgroup generated by the π_n together with the finite support permutations, we obtain a compactly generated subgroup with a divisible element. Indeed, H is compactly generated since it is generated by the compact set

$$\{[n, n+1] : n \in \omega\} \cup \{\pi'_n : n \in \omega\} \cup \{\mathrm{id}\}$$

where

$$\pi'_n = \prod_{i \geq n} [a_{ni}, a_{ni+1}, \ldots, a_{ni+(n-1)}, b_{ni}, b_{ni+1}, \ldots, b_{ni+(n-1)}].$$

In turns out that while there are non-compactly generated K_σ subgroups of S_∞, every K_σ subgroup of S_∞ is in fact $\leq_{\mathrm{g}}$-reducible to a compactly generated subgroup. This is a consequence of Theorem 4.5 below.

Definition 4.4 A subgroup H of a topological group G is *almost compactly generated* iff there is a a compact set $K \subseteq G$ and a continuous, injective group homomorphism $\varphi : G \to G$ such that $H = \varphi^{-1}(\langle K \rangle)$.

The following theorem and remark together characterize those Polish groups of the form G^ω in which every K_σ subgroup is almost compactly generated.

Theorem 4.5 *Let G be a Polish group. If G has a dense subgroup which is a one-to-one continuous homomorphic pre-image of some compactly generated subgroup of G^ω, then every K_σ subgroup of G^ω is almost compactly generated.*

Remark Suppose that the hypothesis of Theorem 4.5 fails, i.e., there is no dense subgroup of G which is an injective homomorphic pre-image of some compactly generated subgroup of G^ω. One can define a K_σ subgroup of G^ω which is not almost compactly generated by taking a dense K_σ subgroup $D \subseteq G$ and letting

$$\tilde{D} = \{\xi \in G^\omega : (\forall i)(\xi(i) = \xi(0) \in D)\}.$$

It follows that $\tilde{D}$ is not almost compactly generated. Otherwise, there would exist an injective continuous endomorphism $\varphi : G^\omega \to G^\omega$ and a compact set $K \subseteq G^\omega$ such that $\tilde{D} = \varphi^{-1}(\langle K\rangle)$. Let $\psi : G \to G^\omega$ be given by $\psi(x)(i) = x$, for all i. Then, contrary to assumption, $D = (\varphi \circ \psi)^{-1}(\langle K\rangle)$.

Theorem 4.5 yields the following immediate corollary.

Corollary 4.6 *If G is a Polish group with a dense compactly generated subgroup, then every K_σ subgroup of G^ω is almost compactly generated.*

In general, we can consider other groups besides those of the form G^ω.

Corollary 4.7 *Suppose that G is a Polish group such that G^ω is isomorphic to a subgroup of G. If G has a dense almost compactly generated subgroup, then every K_σ subgroup of G is almost compactly generated.*

Proof Let $\varphi_1 : G \to G^\omega$ and $\varphi_2 : G^\omega \to G$ be isomorphic embeddings. Fix a K_σ subgroup $H \subseteq G$. By Theorem 4.5, there exists a compact set $K \subseteq G^\omega$ and a continuous injective homomorphism $\varphi : G^\omega \to G^\omega$ such that $\varphi_1(H) = \varphi^{-1}(\langle K\rangle)$. It now follows from the injectivity of φ_2 that

$$H = (\varphi_2 \circ \varphi \circ \varphi_1)^{-1}(\langle \varphi_2(K)\rangle).$$

In other words, H is almost compactly generated. □

Proof of Theorem 4.5 We start with a special case. Assume there is a dense subgroup $D \subseteq G$ and a continuous, injective homomorphism $\psi : G \to G$ such that $D = \psi^{-1}(\langle K\rangle)$, for some compact set $K \subseteq G$.

Let e denote the identity element of G. By a theorem of Birkhoff and Kakutani, assume that G is equipped with a left-invariant metric (which need not be complete). For $\varepsilon > 0$ and $x \in G$, let $B_\varepsilon(x)$ denote the ε-ball centered at x with respect to this metric.

We will show that every K_σ subgroup of G is a one-to-one continuous homomorphic preimage of some compactly generated subgroup of G^ω. To this end, fix a K_σ subgroup

$H = \bigcup_n H_n \subseteq G$, with each H_n compact. For each n, let $F_n \subseteq D$ be a finite $\frac{1}{n}$-net for H_n. Let

$$\tilde{H}_n = \bigcup_{y \in F_n} y^{-1} \cdot \mathrm{cl}(B_{\frac{1}{n}}(y) \cap H_n)$$

and define $C_n \subseteq G^\omega$ to be the set of $\xi \in G^\omega$ such that

(1) $(\forall i < n)(\xi(i) \in \psi(\tilde{H}_n))$ and
(2) $(\forall i \geq n)(\xi(i) \in \psi(H_n))$.

As a product of compact sets, each C_n is a compact subset of G^ω. In fact, $\bigcup_n C_n$ is itself compact. To see this, suppose that $\xi_0, \xi_1, \ldots \in \bigcup_n C_n$. In the first place, suppose that there is an m such that infinitely many $\xi_n \in C_m$. In this case, there is a convergent subsequence of (ξ_n), since each C_m is compact. On the other hand, suppose that, for each m, there are only finitely many n such that $\xi_n \in C_m$. Extract subsequences $n_0 < n_1 < \cdots$ and $s_0 < s_1 < \cdots$ such that $\xi_{n_p} \in C_{s_p}$, for each $p \in \omega$. For each p, let $\eta_p \in G^\omega$ be such that

$$\xi_{n_p}(i) = \psi(\eta_p(i)),$$

for each $i \in \omega$. By the injectivity of ψ, we must have

$$\eta_p(i) \in \tilde{H}_{s_p} \subseteq B_{\frac{1}{s_p}}(e)$$

for each p and $i < p$. It follows that $\eta_p \to \overline{e}$ as $p \to \infty$. Hence, $\xi_{n_p} \to \overline{e}$ since ψ is a continuous homomorphism.

Now define a compact set $K^* \subseteq G^\omega$ by

$$K^* = \{\xi \in K^\omega : (\exists^{\leq 1} i)(\xi(i) \neq e)\}$$

and let $\varphi : G \to G^\omega$ be given by $\varphi(x) = \overline{\psi(x)}$. Note that φ is injective since ψ is. We check that

$$H = \varphi^{-1}\left(\left\langle K^* \cup \bigcup_n C_n \right\rangle\right).$$

In the first place, suppose that $x \in H$, say $x \in H_n$. Let $y \in F_n$ and $z \in \tilde{H}_n$ be such that $y^{-1}x = z$. It follows that

$$\begin{aligned}\varphi(x) &= \overline{\psi(x)} \\ &= \underbrace{\left((\psi(y))^{n\frown}\overline{e}\right)}_{\in \langle K^* \rangle} \cdot \underbrace{\left((\psi(z))^{n\frown}\overline{\psi(x)}\right)}_{\in C_n}.\end{aligned}$$

Hence, $\varphi(x) \in \langle K^* \cup \bigcup_n C_n \rangle$.

On the other hand, suppose that $\varphi(x) \in \langle K^* \cup \bigcup_n C_n \rangle$. Let $n_0 \in \omega$ and w be a group word, with $\xi_0, \dots \xi_k \in \bigcup_{n \leq n_0} C_n$ and $\eta_0 \dots, \eta_p \in K^*$ such that

$$\varphi(x) = w(\xi_0, \dots, \xi_k, \eta_0, \dots, \eta_p).$$

Let $i \geq n_0$ be large enough that

$$\eta_0(i) = \cdots = \eta_p(i) = e.$$

It follows that

$$\begin{aligned} \psi(x) &= \varphi(x)(i) \\ &= w(\xi_0(i), \dots, \xi_k(i), \eta_0(i), \dots, \eta_p(i)) \\ &= w(\xi_0(i), \dots, \xi_k(i), e, \dots, e) \in \left\langle \bigcup_{n \leq n_0} \psi(H_n) \right\rangle. \end{aligned}$$

Since ψ is injective, it follows that $x \in \left\langle \bigcup_{n \leq n_0} H_n \right\rangle$.

We now turn to the general case in which there is a dense subgroup $D \subseteq G$, a compact set $K \subseteq G^\omega$, and an injective homomorphism $\psi : G \to G^\omega$ such that $D = \psi^{-1}(\langle K \rangle)$.

Consider the subgroup

$$D^* = \{\xi \in D^\omega : (\forall^\infty i)(\xi(i) = e)\}$$

and observe that D^* is dense in G^ω. Define $\psi^* : G^\omega \to (G^\omega)^\omega$ by

$$\psi^*(\xi)(n)(i) = \psi(\xi(i)).$$

Now let

$$K^* = \{\eta \in K^\omega : (\exists^{\leq 1} n)(\eta(n) \neq \overline{e})\}.$$

Note that K^* is compact and $D^* = (\psi^*)^{-1}(\langle K^* \rangle)$. We may therefore apply the special case above to the group G^ω and use the fact that $G^\omega \cong (G^\omega)^\omega$ to complete the proof. □

References

1. Beros, K.A.: Universal subgroups of Polish groups. J. Symb. Log. **79**(4), 1148–1183 (2014)
2. Farah, I.: Completely additive liftings. Bull. Symb. Log. **4**(1), 37–54 (1998)
3. Glasner, E., Weiss, B.: The topological Rohlin property and topological entropy. Am. J. Math. **123**(6), 1055–1070 (2001)
4. Rosendal, C.: Cofinal families of Borel equivalence relations and quasiorders. J. Symb. Log. **70**(4), 1325–1340 (2005)
5. Royden, H.L.: Real Analysis, 3rd edn. Prentice-Hall, Englewood Cliffs (1988)