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A weird relation between two cardinals

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Abstract For a set M, let $\mathrm{seq}(M)$ denote the set of all finite sequences which can be formed with elements of M, and let $[M]^2$ denote the set of all 2-element subsets of M. Furthermore, for a set A, let $\overline{\overline{A}}$ denote the cardinality of A. It will be shown that the following statement is consistent with Zermelo–Fraenkel Set Theory ZF: There exists a set M such that $\overline{\overline{\mathrm{seq}(M)}} < \overline{\overline{[M]^2}}$ and no function $f : [M]^2 \to \mathrm{seq}(M)$ is finite-to-one.

Keywords Permutation models · Cardinals in ZF · Consistency results

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1 Introduction

Let M be a set. Then $\mathrm{fin}(M)$ denotes the set of all finite subsets of M, $[M]^2$ denotes the set of all 2-element subsets of M, and $\mathrm{seq}(M)$ denotes the set of all finite sequences which can be formed with elements of M.

For a set A, let $\overline{\overline{A}}$ denote the cardinality of A. We write $\overline{\overline{A}} = \overline{\overline{B}}$, if there exists a bijection between A and B, and we write $\overline{\overline{A}} \leq \overline{\overline{B}}$, if there exists a bijection between A and a subset $B' \subseteq B$ (i.e., $\overline{\overline{A}} \leq \overline{\overline{B}}$ if and only if there exists an injection from A into B). Finally, we write $\overline{\overline{A}} < \overline{\overline{B}}$ if $\overline{\overline{A}} \leq \overline{\overline{B}}$ and $\overline{\overline{A}} \neq \overline{\overline{B}}$. By the Cantor- Bernstein Theorem, which is provable in ZF only (i.e., without using the Axiom of Choice), we get that $\overline{\overline{A}} \leq \overline{\overline{B}}$ and $\overline{\overline{A}} \geq \overline{\overline{B}}$ implies $\overline{\overline{A}} = \overline{\overline{B}}$.

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In [1], Shelah constructed a permutation model in which there exists an infinite set M, such that $\overline{\overline{\mathrm{seq}(M)}} < \overline{\overline{\mathrm{fin}(M)}}$ (see [1, Theorem 2] or [3, Proposition 7.17]). Later in [2] it was shown that a modification of that permutation model gives a model in which there exists an infinite set M, such that $\overline{\overline{M \times M}} < \overline{\overline{[M]^2}}$ (see [2, Proposition 7.3.1] or [3, Proposition 7.18]). In this note we shall see that a further modification of that model gives a model in which there is an infinite set M, such that $\overline{\overline{\mathrm{seq}(M)}} < \overline{\overline{[M]^2}}$. Notice that $\overline{\overline{\mathrm{seq}(M)}} < \overline{\overline{[M]^2}}$ implies $\overline{\overline{\mathrm{seq}(M)}} < \overline{\overline{\mathrm{fin}(M)}}$ as well as $\overline{\overline{M \times M}} < \overline{\overline{[M]^2}}$.

2 Permutation models

In order to make the paper self-contained, we give a brief introduction to permutation models (see also [3] or [4]). First we introduce models of ZFA, which is set theory with atoms. Set theory with atoms is characterized by the fact that it admits objects other than sets, namely **atoms**. Atoms are objects which do not have any elements but which are distinct from the empty-set. The development of the theory ZFA is essentially the same as that of ZF (except for the definition of ordinals, where we have to require that an ordinal does not have atoms among its elements). Let A be a set. Then by transfinite recursion on $\alpha \in \Omega$ we can define $\mathscr{P}^{\alpha}(S)$ as follows: $\mathscr{P}^{\emptyset}(S) := S$, $\mathscr{P}^{\alpha+1}(S) := \mathscr{P}^{\alpha}(S) \cup \mathscr{P}(\mathscr{P}^{\alpha}(S))$ and $\mathscr{P}^{\alpha}(S) := \bigcup_{\beta\in\alpha} \mathscr{P}^{\alpha}(S)$ when α is a limit ordinal. Further let $\mathscr{P}^{\infty}(S) := \bigcup_{\alpha\in\Omega} \mathscr{P}^{\alpha}(S)$. If $\mathcal{M}$ is a model of ZFA and A is the set of atoms of $\mathcal{M}$, then we have $\mathcal{M} := \mathscr{P}^{\infty}(A)$. The class $M_0 := \mathscr{P}^{\infty}(\emptyset)$ is a model of ZF and is called the **kernel**. Notice that all ordinals belong to the kernel.

The underlying idea of permutation models, which are models of ZFA, is the fact that the axioms of ZFA do not distinguish between the atoms, and so a permutation of the set of atoms induces an automorphism of the universe. The method of permutation models was introduced by Adolf Fraenkel and, in a precise version (with supports), by Andrzej Mostowski. The version with filters is due to Ernst Specker.

In order to construct a permutation model, we usually start with a set of atoms A and then define a group G of permutations or automorphisms of A (where a permutation of A is a one-to-one mapping from A onto A). However, we can also build the set of atoms A and the permutation group G simultaneously step by step, as in Shelah's construction.

We say that a set $\mathscr{F}$ of subgroups of G is a **normal filter** on G if for all subgroups H, K of G we have:

(A) $G \in \mathscr{F}$
(B) if $H \in \mathscr{F}$ and $H \subseteq K$, then $K \in \mathscr{F}$
(C) if $H \in \mathscr{F}$ and $K \in \mathscr{F}$, then $H \cap K \in \mathscr{F}$
(D) if $\pi \in G$ and $H \in \mathscr{F}$, then $\pi H \pi^{-1} \in \mathscr{F}$
(E) for each $a \in A$, $\{\pi \in G : \pi a = a\} \in \mathscr{F}$

Let $\mathscr{F}$ be a normal filter on G. We say that x is **symmetric** (with respect to G) if the group

$$\mathrm{sym}_G(x) := \{\pi \in G : \pi x = x\}$$

belongs to $\mathscr{F}$. By (E) we have that every $a \in A$ is symmetric.

Let $\mathcal{V}$ be the class of all hereditarily symmetric objects; then $\mathcal{V}$ is a transitive model of ZFA. We call $\mathcal{V}$ a permutation model. Because every $a \in A$ is symmetric, we get that the set of atoms A belongs to $\mathcal{V}$.

Now, every $\pi \in G$ induces an $\in$-automorphism of the universe $\mathcal{V}$. Because $\emptyset$ is hereditarily symmetric and for all ordinals α the set $\mathscr{P}^{\alpha}(\emptyset)$ is hereditarily symmetric too, the class $V := \mathscr{P}^{\infty}(\emptyset)$ is a class in $\mathcal{V}$ which is equal to the kernel M_0. In particular, for every $\pi \in G$ and every ordinal $\alpha \in \Omega$ we have $\pi\alpha = \alpha$.

Since the atoms $x \in A$ do not contain any elements, but are distinct from the emptyset, the permutation models are not models of ZF. However, with the JECH- SOCHOR EMBEDDING THEOREM (see for example [4] or [3]) one can embed arbitrarily large fragments of a permutation model in a well-founded model of ZF. In particular, if we can prove that in a permutation model a certain relation between two cardinalities holds, then this relation is consistent with ZF.

Most of the well-known permutation models are of the following simple type: Let G be a group of permutations of A. For each finite set $E \in \text{fin}(A)$, let

$$\text{Fix}_G(E) := \left\{\pi \in G : \forall a \in E\, (\pi a = a)\right\},$$

and let $\mathscr{F}$ be the filter on G generated by the subgroups $\{\text{Fix}_G(E) : E \in \text{fin}(A)\}$. Then $\mathscr{F}$ is a normal filter and x is symmetric if and only if there exists a set of atoms $E_x \in \text{fin}(A)$ such that

$$\text{Fix}_G(E_x) \subseteq \text{sym}_G(x).$$

We say that E_x is a **support** of x. So, a set x belongs to the permutation model $\mathcal{V}$ (with respect to G and $\mathscr{F}$), if and only if x has a finite support $E_x \in \text{fin}(A)$.

3 A Shelah-type permutation model

As mentioned above, Shelah construced in [1] a permutation model in which there is a set M with $\overline{\overline{\text{seq}(M)}} < \overline{\overline{\text{fin}(M)}}$. We give now a modified version of this model and show that for its set of atoms A we have $\overline{\overline{\text{seq}(A)}} < \overline{\overline{[A]^2}}$ and no function $f : [A]^2 \to \text{seq}(A)$ is finite-to-one.

The set of atoms of this Shelah-type permutation model is built by induction, where every atom encodes a finite sequences of atoms on lower levels.

The atoms of the model are constructed as follows:

(α) Let A_0 be an arbitrary infinite set.

(β) G_0 is the group of *all* permutations of A_0.

(γ) $A_{n+1} := A_n \cup \left\{(n+1, p, \varepsilon) : p \in \bigcup_{k=0}^{n+1} A_n^k \wedge \varepsilon \in \{0, 1\}\right\}$.

(δ) G_{n+1} is the subgroup of the permutation group of A_{n+1} containing all permutations σ for which there are $\pi_\sigma \in G_n$ and $\varepsilon_\sigma \in \{0, 1\}$ such that

$$\sigma(x) = \begin{cases} \pi_\sigma(x) & \text{if } x \in A_n, \\ (n+1, \pi_\sigma(p), \varepsilon_\sigma +_2 \varepsilon) & \text{if } x = (n+1, p, \varepsilon), \end{cases}$$

where for $p = \langle p_0, \dots, p_{l-1}\rangle \in \bigcup_{0 \le k \le n+1} A_n^k$, $\pi_\sigma(p) := \langle \pi_\sigma(p_0), \dots, \pi_\sigma(p_l)\rangle$ and $+_2$ denotes addition modulo 2.

Let $A := \bigcup\{A_n : n \in \omega\}$ be the set of atoms and let $\mathrm{Aut}(A)$ be the group of all permutations of A. Then

$$G := \left\{H \in \mathrm{Aut}(A) : \forall n \in \omega\,(H|_{A_n} \in G_n)\right\}$$

is a group of permutations of A. Let $\mathscr{F}$ be the normal filter on G generated by $\{\mathrm{Fix}_G(E) : E \in \mathrm{fin}(A)\}$, and let $\boldsymbol{\mathcal{V}}_S$ be the class of all hereditarily symmetric sets.

Remark In the construction of the permutation model $\boldsymbol{\mathcal{V}}_S$, we can equally well start with an infinite set of atoms A, partitioned into countably many infinite sets A_n for $n \in \omega$. Then, we define for every $n \in \omega$ a bijection between the set of finite sequences of length at most $n+1$ which can be formed with elements of $\bigcup_{i \le n} A_i$ and a set $P_{n+1} \subseteq [A_{n+1}]^2$ of pairwise disjoint 2-element subsets of A_{n+1}. Finally, we define the group G as the group of permutations which swap the elements of P_{n+1} and which respect the bijections.

As an immediate consequence of the definitions we get that for each $n \in \omega$, the set A_n belongs to $\boldsymbol{\mathcal{V}}_S$. In particular, the function

$$\begin{aligned} f : \omega &\to \mathscr{P}(A) \\ n &\longmapsto A_n \end{aligned}$$

is an injective function which belongs to $\boldsymbol{\mathcal{V}}_S$. Moreover, for each atom $a \in A$ there exists a least number $n \in \omega$ such that $a \in A_n$. In particular, there exists a surjection $f : A \to \omega$ which belongs to $\boldsymbol{\mathcal{V}}_S$.

Now, we are ready to prove our main result.

Theorem *Let A be the set of atoms of $\boldsymbol{\mathcal{V}}_S$. Then*

$$\boldsymbol{\mathcal{V}}_S \models \overline{\overline{\mathrm{seq}(A)}} < \overline{\overline{[A]^2}}$$

and no function $F : [A]^2 \to \mathrm{seq}(A)$ in $\boldsymbol{\mathcal{V}}_S$ is finite-to-one.

Proof First we show that $\boldsymbol{\mathcal{V}}_S \models \overline{\overline{\mathrm{seq}(A)}} \le \overline{\overline{[A]^2}}$. For this it is sufficient to find an injective function $f \in \boldsymbol{\mathcal{V}}_S$ from $\mathrm{seq}(A)$ into $[A]^2$. We define such a function as follows. For a finite sequence $s = \langle a_0, \dots, a_{l-1}\rangle \in \mathrm{seq}(A)$ let

$$f(s) := \left\{(m+l, s, 0), \bigl(m+l, s, 1)\right\},$$

where m is the smallest number such that $\{a_0, \dots, a_{l-1}\} \subseteq A_m$. For any $\pi \in G$ and $s = \langle a_0, \dots, a_{l-1}\rangle \in \mathrm{seq}(A)$ we have $\pi f(s) = f\bigl(\pi(s)\bigr)$ and therefore, the function f is as desired and belongs to $\boldsymbol{\mathcal{V}}_S$.

Now, let $g \in \boldsymbol{\mathcal{V}}_S$ be a function from $[A]^2$ to $\mathrm{seq}(A)$ and let E_g be a finite support of g. We show that g is not injective. Since E_g is finite, there is an integer $n_g \in \omega$ such that $E_g \subseteq A_{n_g}$. By extending E_g if necessary, we may assume that if $(n+1, \langle a_0, \dots, a_{l-1}\rangle, \varepsilon) \in E_g$, then also $a_0, \dots, a_{l-1}$ belong to E_g as well as the atom $(n+1, \langle a_0, \dots, a_{l-1}\rangle, 1-\varepsilon)$ (this assumption will be needed later).

Choose two distinct elements $x, y \in A_0 \setminus E_g$ such that $g(\{x, y\}) \neq \langle\,\rangle$. If there are no such elements, then g is not injective and we are done. So, we may assume that for some positive integer $l \in \omega$:

$$g(\{x, y\}) = \langle a_0, \ldots, a_{l-1} \rangle$$

Now, we are in at least one of the following four cases:

(1) $\forall i \in l \, \big(a_i \in E_g\big)$
(2) $\exists i \in l \, \big(a_i \in \{x, y\}\big)$
(3) $\exists i \in l \, \big(a_i \in A_0 \setminus (E_g \cup \{x, y\})\big)$
(4) $\exists i \in l \, \big(a_i \in A \setminus (E_g \cup A_0)\big)$

If we are in Case (1), then let $\pi \in \mathrm{Fix}(E_g)$ be such that $\pi x \notin \{x, y\}$. To see that such a $\pi \in \mathrm{Fix}(E_g)$ exists, recall that by our assumption, E_g has the property that whenever $(n+1, \langle a_0, \ldots, a_{l-1} \rangle, \varepsilon) \in E_g$, also $a_0, \ldots, a_{l-1} \in E_g$. Now, $\pi g(\{x, y\}) = g(\{x, y\})$ (since $\pi \in \mathrm{Fix}(E_g)$), but $\pi\{x, y\} \neq \{x, y\}$. Hence, g is not a injective function.

If we are in Case (2), then let $\pi \in \mathrm{Fix}(E_g)$ be such that $\pi x = y$ and $\pi y = x$. Notice that since $\{x, y\} \subseteq A_0$ and $\{x, y\} \cap E_g = \emptyset$, by condition ($\beta$) in the construction of $\mathcal{V}_S$, such a permutation π exists. Now, by the choice of π, on the one hand we have $\pi\{x, y\} = \{x, y\}$, i.e., $g(\{x, y\}) = g(\pi\{x, y\})$, but on the other hand, for some $i \in l$ we have $a_i \in \{x, y\}$, which implies $a_i \neq \pi a_i$. To see this, notice that for example $a_i = x$ implies $\pi a_i = y$. Therefore, E_g is not a support of g, which contradicts the choice of E_g.

If we are in Case (3), then there is an $i \in l$ such that

$$a_i \in A_0 \setminus (E_g \cup \{x, y\}) .$$

Now, take an arbitrary $b_i \in A_0 \setminus (E_g \cup \{x, y\})$ which is distinct from a_i and let $\pi \in \mathrm{Fix}(E_g \cup \{x, y\})$ be such that $\pi a_i = b_i$ and $\pi b_i = a_i$. Notice that by condition (β) in the construction of $\mathcal{V}_S$, such a permutation π exists. By the choice of π, on the one hand we have $\pi\{x, y\} = \{x, y\}$, i.e., $g(\{x, y\}) = g(\pi\{x, y\})$, but on the other hand, $\pi a_i = b_i$ and $b_i \neq a_i$, i.e., $g(\{x, y\}) \neq \pi g(\{x, y\})$. Therefore, E_g is not a support of g, which contradicts the choice of E_g.

If we are in Case (4), then there is an $i \in l$ such that

$$a_i \in A \setminus (E_g \cup A_0) .$$

In particular, $a_i = (n + 1, p, \varepsilon)$ for some $n \in \omega$, $p \in \mathrm{seq}(A)$, and $\varepsilon \in \{0, 1\}$. Furthermore, let $\pi \in \mathrm{Fix}(E_g \cup \{x, y\})$ be such that

$$\pi(n + 1, p, \varepsilon) = (n + 1, p, 1 - \varepsilon) .$$

To see that such a π exists, recall that by our assumption, E_g has the property that whenever $(n + 1, s, \varepsilon) \in E_g$ for some $s \in \mathrm{seq}(A)$, also $(n + 1, s, 1 - \varepsilon) \in E_g$. Now we have $\pi\{x, y\} = \{x, y\}$ but $\pi g(\{x, y\}) \neq g(\{x, y\})$. Therefore, E_g is not a support of g, which contradicts the choice of E_g.

So, in all four cases, either g is not injective or E_g is not a support of g. In particular, there is no injection in $\mathcal{V}_S$ from $[A]^2$ into $\mathrm{seq}(A)$. Hence,

$$\mathcal{V}_S \models \overline{\overline{\mathrm{seq}(A)}} < \overline{\overline{[A]^2}}.$$

It remains to show that no function from $[A]^2$ to $\mathrm{seq}(A)$ is finite-to-one. For this, let $F : [A]^2 \to \mathrm{seq}(A)$ be a function in $\mathcal{V}_S$ with support E_F. Since E_F is a support of F, for any $\{x, y\} \in [A_0 \setminus E_F]^2$, either $F(\{x, y\}) = \langle\ \rangle$ or $F(\{x, y\}) = \langle a_0, \dots, a_{l-1}\rangle$ for some positive integer l. We consider the following two cases:

(I) *There exists an* $\{x, y\} \in [A_0 \setminus E_F]^2$ *such that* $F(\{x, y\}) \neq \langle\ \rangle$: First, let

$$F(\{x, y\}) = \langle a_0, \dots, a_{l-1}\rangle$$

for some positive integer l. Since F is a function in $\mathcal{V}_S$ with support E_F, we must have that for all $i \in l$, $a_i \in E_F$ (which corresponds to Case (1) above), otherwise, E_F would not be a support of F. Furthermore, for each $\pi \in \mathrm{Fix}(E_F)$ we have

$$\pi\langle a_0, \dots, a_{l-1}\rangle = \langle a_0, \dots, a_{l-1}\rangle,$$

and since E_F is a support of F, we have

$$F\big(\pi\{x, y\}\big) = F\big(\{x, y\}\big) = \langle a_0, \dots, a_{l-1}\rangle.$$

Since A_0 is infinite and E_F is finite, the set $A_0 \setminus E_F$ is infinite. So, by condition (β) in the construction of $\mathcal{V}_S$, there are infinitely many $x' \in A_0 \setminus (E_F \cup \{x, y\})$ for which there exists a $\pi \in \mathrm{Fix}(E_F)$ such that $\pi y = y$ and $\pi x \neq x$. In particular, if $\pi, \pi' \in \mathrm{Fix}(E_F)$ are such that $\pi y = y = \pi' y$ and $\pi x \neq \pi' x$, then $\pi\{x, y\} \neq \pi'\{x, y\}$, but

$$F\big(\pi\{x, y\}\big) = F\big(\pi'\{x, y\}\big),$$

which shows that F is not a finite-to-one function.

(II) *For all* $\{x, y\} \in [A_0 \setminus E_F]^2$ *we have* $F(\{x, y\}) = \langle\ \rangle$: Since A_0 is infinite and E_F is finite, there are infinitely many pairs mapped to the empty sequence, therefore, F is not a finite-to-one function. □

4 Odds and ends

For sets A and B we write $\overline{\overline{A}} \leq^* \overline{\overline{B}}$ if $A = \emptyset$ or if there exists a surjective function $g : B \to A$ (i.e., g is a function from B onto A).

As we have seen above, in the model $\mathcal{V}_S$ there is an injective function $f : \mathrm{seq}(A) \to [A]^2$, where A is the set of atoms. Now, by taking the pre-images of f we get an injective function from a subset of $[A]^2$ onto $\mathrm{seq}(M)$, which can be extended to a surjective function $g : [A]^2 \to \mathrm{seq}(M)$. Hence, in $\mathcal{V}_S$ we have $\overline{\overline{\mathrm{seq}(A)}} \leq^* \overline{\overline{[A]^2}}$. On the other hand, it is easy to see that for any set M we have $\overline{\overline{[M]^2}} \leq^* \overline{\overline{\mathrm{seq}(M)}}$. Furthermore, using again the pre-images of the injective function $f : \mathrm{seq}(A) \to [A]^2$ in $\mathcal{V}_S$, we can construct a surjective function $g : A \to \mathrm{seq}(M)$. To see this, define

$$g(a) = \begin{cases} p & \text{if a} = (n+1, p, \varepsilon) \text{ for some } n \in \omega \text{ and } \varepsilon \in \{0, 1\}, \\ \langle\,\rangle & \text{otherwise.} \end{cases}$$

Hence, in $\mathcal{V}_S$ we have $\overline{\overline{\mathrm{seq}(A)}} \leq^* \overline{\overline{A}}$. On the other hand, it is easy to see that for any set M we have $\overline{\overline{M}} \leq \overline{\overline{\mathrm{seq}(M)}}$ and $\overline{\overline{M}} \leq^* \overline{\overline{\mathrm{seq}(M)}}$. Finally, in $\mathcal{V}_S$ we have $\overline{\overline{A}} < \overline{\overline{\mathrm{seq}(A)}}$. To see this, assume towards a contradiction that there exists an injection $h : \mathrm{seq}(A) \to A$. Let $a \in A$ and consider the sequences $p_0 := \langle\,\rangle$, $p_1 := \langle a \rangle$, $p_2 := \langle a, a \rangle$, and so on. Then $\langle h(p_n) : n \in \omega \rangle$ would be an infinite sequence of pairwise distinct elements of A, which is obviously not a set in $\mathcal{V}_S$.

So, by the Jech- Sochor Embedding Theorem, the existence of an infinite set M for which the following relations hold is consistent with ZF:

$$\overline{\overline{M}} < \overline{\overline{\mathrm{seq}(M)}}, \qquad \overline{\overline{\mathrm{seq}(M)}} < \overline{\overline{[M]^2}},$$
$$\overline{\overline{M}} \leq^* \overline{\overline{\mathrm{seq}(M)}} \leq^* \overline{\overline{M}}, \qquad \overline{\overline{\mathrm{seq}(M)}} \leq^* \overline{\overline{[M]^2}} \leq^* \overline{\overline{\mathrm{seq}(M)}}.$$

Let us now replace $\mathrm{seq}(M)$ with $\mathscr{P}(M)$. By the Cantor Theorem, for all sets M we have $\overline{\overline{M}} < \overline{\overline{\mathscr{P}(M)}}$ and $\overline{\overline{\mathscr{P}(M)}} \not\leq^* \overline{\overline{M}}$. Furthermore, we have $\overline{\overline{[M]^2}} < \overline{\overline{\mathscr{P}(M)}}$, which follows from $\overline{\overline{\mathrm{fin}(M)}} < \overline{\overline{\mathscr{P}(M)}}$ for infinite sets M (see [1] or [3]).

So, for arbitrary sets M, the following relations are provable in ZF:

$$\overline{\overline{M}} < \overline{\overline{\mathscr{P}(M)}}, \qquad \overline{\overline{[M]^2}} < \overline{\overline{\mathscr{P}(M)}},$$
$$\overline{\overline{M}} \leq^* \overline{\overline{\mathscr{P}(M)}} \not\leq^* \overline{\overline{M}}, \qquad \overline{\overline{[M]^2}} \leq^* \overline{\overline{\mathscr{P}(M)}}.$$

However, it is not known whether $\overline{\overline{\mathscr{P}(M)}} \not\leq^* \overline{\overline{[M]^2}}$ is also provable in ZF. In other words, it is not known whether there exists a model of ZF in which there is a set M such that $\overline{\overline{\mathscr{P}(M)}}{}^{*}\overline{\overline{[M]^2}}$ (see [3, Related Result 21] for a similar open problem).

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