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# Collapsing $\omega_2$ with semi-proper forcing

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**Abstract** We examine the differences between three standard classes of forcing notions relative to the way they collapse the continuum. It turns out that proper and semi-proper posets behave differently in that respect from the class of posets that preserve stationary subsets of $\omega_1$.

**Keywords** Oscillation · Forcing · Stationary set

**Mathematics Subject Classification** Primary 03E20

## 1 Introduction

The classes of proper, semi-proper, and stationary preserving forcing notions are well established and the corresponding forcing axioms PFA, SPFA and MM are important set-theoretic principles with many applications (see, for example, [14]). While it is still open if these forcing axioms are of different consistency strengths analysis of their consequence reveals that while SPFA and MM are equivalent forcing axioms (see [8]) some of their consequences are not consequences of PFA. For example, while SPFA implies the strong reflection of stationary subsets of $[\theta]^{\aleph_0}$ (see [12,15]) the corresponding axiom PFA for proper forcing notions does not even imply reflection of stationary subsets of $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$ (see [2]). In this note we shall show that

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there is also a difference between the classes of semi-proper and stationary preserving forcing notions but not at the level of the corresponding forcing axioms but on the way they could collapse the cardinal $\aleph_2$. More precisely, we show that one may collapse $\omega_2$ with a forcing notion that preserves stationary subsets of $\omega_1$ without collapsing the continuum to $\aleph_1$ and that on the other hand if a semi-proper forcing notion collapses $\aleph_2$ then it also forces $|\check{\mathbb{R}}| = \aleph_1$. This improves a bit a result from [7] who proved this statement for proper forcing notions. However, the purpose of this paper is also to expose an old oscillation theory for closed and unbounded subsets of $[\omega_2]^{\aleph_0}$ which plays an important role in the proofs of these results, an oscillation theory that is of independent interest and that could have some other applications.

We shall follow standard notation and terminology about forcing and stationary subsets of the structures of the form $[\theta]^{\aleph_0}$ that could be found in standard sources such as, for example, [5]. In Sect. 2 we show that there is a poset that collapses $\omega_2$ but preserves all other cardinals and all stationary subset of $\omega_1$. This poset shows up first in our note [10] as an example of application of the side condition method introduced there. In Sect. 3 we show that in general with semi-proper poset one can't collapse $\omega_2$ without collapsing the continuum. In the final section we summarize results of the oscillation theory used in the proof from Sect. 3 as we think that it might be of independent interest.

## 2 Collapsing $\omega_2$ with stationary preserving forcing

In this section we recall the following result from [10].

**Theorem 2.1** *There is a forcing notion $\mathcal{P}$ of cardinality $\aleph_2$ which preserves stationary subsets of $\omega_1$ but collapses $\omega_2$.*

The proof of this result is based on the following standard fact.

**Lemma 2.2** *There is a set $\mathcal{S} \subseteq [\omega_2]^{\aleph_0}$ of cardinality $\aleph_2$ such that $\mathcal{S}(E) = \{A \in \mathcal{S} : A \cap \omega_1 \in E\}$ is stationary for all stationary $E \subseteq \omega_1$.*

*Proof* For each $\delta < \omega_2$ pick a countable-to-one mapping $e_\delta : \delta \to |\delta|$ and define

$$\mathcal{S} = \{A \in [\omega_2]^{\aleph_0} : (\exists \nu < \omega_1)(\exists \delta < \omega_2)\ A = \{\gamma < \delta : e_\delta(\gamma) \in \nu\}\}.$$

Given a stationary subset $E$ of $\omega_1$ and closed and unbounded $\mathcal{C} \subseteq [\omega_2]^{\aleph_0}$, find $\delta < \omega_2$ such that $\mathcal{C} \cap [\delta]^{\aleph_0}$ contains a closed and unbounded subset of $[\delta]^{\aleph_0}$. It follows that if for $\nu < \omega_1$, we set

$$A_\nu(\delta) = \{\gamma < \delta : e_\delta(\gamma) \in \nu\},$$

then $D = \{\nu < \omega_1 : A_\nu(\delta) \cap \omega_1 = \nu \text{ and } A_\nu(\delta) \in \mathcal{C}\}$ contains a closed and unbounded subset of $\omega_1$. Pick $\nu \in D \cap E$. Then $A_\nu(\delta) \in \mathcal{S}(E) \cap \mathcal{C}$. This shows that $\mathcal{S}(E)$ intersects all closed and unbounded subsets of $[\omega_2]^{\aleph_0}$, and so it is stationary. □

We can now give the proof of Theorem 2.1. Fix a stationary set $\mathcal{S}$ as in Lemma 2.2. Fix a one-to-one map $i : \mathcal{S} \to \omega_2$. Let $\mathcal{P}$ the collection of all finite subsets $p$ of $\mathcal{S}$ such that for all $A \neq B$ in $\mathcal{P}$ either $A \cup \{i(A)\} \subseteq B$ or vice versa $B \cup \{i(B)\} \subseteq A$. We order $\mathcal{P}$ by the inclusion. To show that $\mathcal{P}$ reserves stationary subsets of $\omega_1$ fix such a subset $E$ of $\omega_1$. To show that $E$ remains stationary in any forcing extension of $\mathcal{P}$, pick $p_0 \in \mathcal{P}$ and a $\mathcal{P}$-name $\dot{C}$ for a closed and unbounded subset of $\omega_1$. Choose a countable elementary submodel $M$ of $H_{\aleph_3}$ containing $p_0$, $\dot{C}$, $S$ and $i$ such that $M \cap \omega_2 \in \mathcal{S}(E)$. Let $p = p_0 \cup \{M \cap \omega_2\}$. We claim that $p$ forces that $\delta = M \cap \omega_1$ belongs to $\dot{C}$ which will finish the proof since this ordinal by definition belongs to $E$. It suffices to show that $p$ forces that $\dot{C} \cap \delta$ is unbounded in $\delta$. To see this pick an extension $q$ of $p$ and $\gamma < \delta$. Then since $i \in M$, we know that $q_0 = \{A \in q : A \subset M \cap \omega_2\}$ belongs to $\mathcal{P} \cap M$. Using elementarily of $M$, we can find ordinal $\alpha > \gamma$ in $M \cap \omega_1 = \delta$ and $r \in \mathcal{P} \cap M$ which forces $\alpha \in \dot{C}$. Then again since $i \in M$, we conclude that $r \cup q$ is a condition of $\mathcal{P}$ extending $q$ and forcing that $\dot{C} \cap [\gamma, \delta) \neq \emptyset$.

To finish the proof of Theorem 2.1 it remains to show that $\omega_2$ is collapsed in the forcing extension of $\mathcal{P}$. To see this note that the generic object of $\mathcal{P}$ is an $\subseteq$-chain of order type $\omega_1$ of countable subset of $(\omega_2)^V$ that cover $(\omega_2)^V$ since clearly every $p \in \mathcal{P}$ can be extended to a $q \in \mathcal{P}$ such that $\bigcup q$ includes any given ordinal $\alpha < \omega_2$.

**Corollary 2.3** *If the continuum is bigger than $\aleph_2$ then a stationary-preserving forcing notion may collapse $\omega_2$ without forcing $|\check{\mathbb{R}}| = \aleph_1$.*

## 3 Collapsing $\omega_2$ with semi-proper forcing

In this section we prove the following result which, according to Theorem 2.1, extends a result from [7] to the optimal class of forcing notions.

**Theorem 3.1** *If a semi-proper forcing notion $\mathcal{P}$ collapses $\omega_2$ then it forces $|\check{\mathbb{R}}| = \aleph_1$.*

Note that the conclusion of the theorem is immediate if the continuum is not bigger than $\aleph_2$. So we concentrate to proving this theorem assuming $2^{\aleph_0} > \aleph_2$. We shall apply the tools from the oscillation theory over a club-guessing sequences (see [3]) and a particular sequence of stationary subsets of $[\omega_2]^{\aleph_0}$ indexed by reals (see Lemma 43 of [12]). Recall, that a sequence of the form $\{C_\alpha : \alpha < \omega_2, \mathrm{cf}(\alpha) = \omega\}$ is a *club-guessing* on $\omega_2$ if for every $\alpha < \omega_2$ with $\mathrm{cf}(\alpha) = \omega$, the set $C_\alpha$ is an unbounded subset of $\alpha$ such that every closed and unbounded subset $C$ of $\omega_2$ contains one of $C_\alpha$'s (see [9]). So from now one we fix a club guessing sequence $\{C_\alpha : \alpha < \omega_2, \mathrm{cf}(\alpha) = \omega\}$ and let $\{C_\alpha(n) : n < \omega\}$ be the increasing enumeration of $C_\alpha$ for every $\alpha < \omega_2$ such that $\mathrm{cf}(\alpha) = \omega$. For a set $A \in [\omega_2]^{\aleph_0}$ of limit order type, set

$$\mathrm{osc}(A, C_{\sup(A)}) = \{n < \omega : A \cap [C_{\sup(A)}(n), C_{\sup(A)}(n+1)) \neq \emptyset\}.$$

We shorten the notation and write $\mathrm{osc}(A)$ instead of $\mathrm{osc}(A, C_{\sup(A)})$ and implicitly consider only countable subsets $A$ of $\omega_2$ of limit order type, i.e., subsets for which this definition makes sense. For a real $r \in [\omega]^\omega$, set

$$\mathcal{S}_r = \{A \in [\omega_2]^{\aleph_0} : \mathrm{osc}(A) = r\}.$$

It is known (see [3,12]) that for every $r \in [\omega]^\omega$ and stationary $E \subseteq \omega_1$, the set

$$\mathcal{S}_r(E) = \{A \in \mathcal{S} : A \cap \omega_1 \in E\}$$

is stationary in $[\omega_2]^{\aleph_0}$.

Theorem 3.1 will follow from the following result.

**Proposition 3.2** *If $2^{\aleph_0} > \aleph_2$ then for every semi-proper forcing notion $\mathcal{P}$ and every real $r \in [\omega]^\omega$, the set $\mathcal{S}_r$ remains stationary after forcing by $\mathcal{P}$.*

The proof of this results requires a finer analysis of the proof that the sets $\mathcal{S}_r$ are stationary. Recall that the main combinatorial tool in that proof is a lemma about *Namba trees*, downwards closed subtrees $T$ of the tree of all finite sets ordinals in $\omega_2$ ordered by the relation $\sqsubseteq$ of end-extension such that $T$ has a *stem*, $\mathrm{st}(T)$, the maximal node that is comparable to all elements of $T$, and such that for every $t \in T$, $t \sqsupseteq \mathrm{st}(T)$, the set

$$I_T(t) = \{\alpha < \omega_2 : t \cup \{\alpha\} \in T\}$$

of immediate successors is stationary subset of $\omega_2$. We shall actually restrict ourselves to Namba subtrees of the tree of all finite subsets of the set $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1\}$. We shall use the following basic lemma about these trees from [6].

**Lemma 3.3** *If the collection $[T]$ of all infinite branches of a Namba tree $T$ is coloured into at most $\aleph_1$ Borel colors then there is a Namba subtree $U$ of $T$ such that $[U]$ is monochromatic.*

It will also be useful to have some notation about Namba trees. Given a Namba tree $T$ and integer $n < \omega$, the *nth level* of $T$ is the set $T(n)$ of all nodes of $T$ of length $n$. For Two Namba trees $T$ and $U$ set and $n < \omega$, set $U \subseteq_n T$ if $U$ is a subtree of $T$, if $\mathrm{st}(U) = \mathrm{st}(T)$, and if $U(n) = T(n)$. A *fusion sequence* of Namba trees is a sequence $T_n$ $(n < \omega)$ of Namba trees such that $T_{n+1} \subseteq_n T_n$ for all $n$. Note that if $T_n$ $(n < \omega)$ is a fusion sequence of Namba trees then its *fusion* $\bigcap_{n<\omega} T_n$ is a Namba tree as well. Finally, for a Namba tree $T$ and a $t \in T$, set

$$T^t = \{s \in T : s \sqsubseteq t \text{ or } t \sqsubseteq s\}.$$

Note that if $t \sqsubseteq \mathrm{st}(T)$ then $T^t = T$ and that otherwise $T^t$ is a Namba subtree of $T$ with stem equal to $t$.

Recall that for every closed and unbounded subset $\mathcal{C}$ of $[\omega_2]^{\aleph_0}$ there is $f : [\omega_2]^{<\omega} \to \omega_2$ such that for every countable subset $X$ of $\omega_2$, the $f$-closure of $X$

$$\mathrm{CL}_f(X) = \{f(s) : s \in [X]^{<\omega}\}$$

belongs to $\mathcal{C}$ (see [5]) . It is also well-known that we can choose such an $f$ such that $\mathrm{CL}_f(\mathrm{CL}_f(X)) = \mathrm{CL}_f(X)$ for every countable subset $X$ of $\omega_2$, or in other words sets $Y$ of the form $\mathrm{CL}_f(X)$ are all *$f$-closed*, i.e., have the property that $\mathrm{CL}_f(Y) = Y$. So we shall always implicitly choose such $f$ with this extra property and will put some additional information into $f$ such as, for example, particular functions of the form $e : [\omega_2]^2 \to \omega_1$.

**Lemma 3.4** *For every $f : [\omega_2]^{<\omega} \to \omega_2$ there is a Namba tree $T$ with stem $\emptyset$ and functions $A : T \to [\omega_2]^{\leq\aleph_0}$ and $a : T \to \omega_2$ such that*[1]

(1) $A(\emptyset) = \emptyset$,
(2) $\max(t) < a(t) < \alpha$ *for* $t \in T$ *and* $\alpha < \omega_2$ *such that* $\alpha \in I_T(t)$,
(3) $A(t \cup \{\alpha\}) \subseteq [\max(t), a(t))$ *for all* $t \in T$ *all* $\alpha < \omega_2$ *such that* $\alpha \in I_T(t)$,
(4) *for every infinite branch $b$ of $T$, the set $A(b) = \bigcup_{n<\omega} A(b \upharpoonright n)$ is $f$-closed.*

*Proof* Let $T_0$ be the Namba tree of all finite subsets of the set $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1\}$ ordered by the relation $\sqsubseteq$ of end-extension. For an infinite branch $b$ of $T_0$ which we identify with its union, the infinite set of order type $\omega$ whose finite initial segments form the branch $b$. Let $A(b) = \mathrm{CL}_f(b)$. As explained above we are assuming that we have modified our given mapping $f$ so that the sets $A(b)$ are all $f$-closed. In fact, we assume that $f$ is expanded so that it incorporates a map $e : [\omega_2]^2 \to \omega_1$ such that for all $\alpha < \omega_2$, the section $e(\cdot, \alpha)$ is a one-to-one mapping from $\alpha$ into $|\alpha|$. This in particular means that for all $b \in [T_0]$, the intersection $\nu_b = A(b) \cap \omega_1$ is a countable ordinal and that for every $\alpha \in A(b)$,

$$A(b) \cap \alpha = \{\xi < \alpha : e(\xi, \alpha) < \nu_b\}.$$

Starting from $T_0$ we build a fusion sequence $T_n$ of Namba trees in order that the fusion $T = \bigcap_{n<\omega} T_n$ satisfies the conclusion of the Lemma.

To find the subtree $T_1 \subseteq T_0$, for every $\alpha \in I_{T_0}(\emptyset)$, using the assumption that $\alpha$ has uncountable cofinality, we apply Lemma 3.3 and find a Namba subtree $U_\alpha$ of $T_0^{\{\alpha\}}$ with stem $\{\alpha\}$ and ordinals $\nu_\alpha < \omega_1$ and $\xi_\alpha < \alpha$ such that $\beta > \xi_\alpha$ for all $\beta \in I_{U_\alpha}(\{\alpha\})$ and

$$A(b) \cap \omega_1 = \nu_\alpha \text{ and } \sup(A(b)) \cap \alpha < \xi_\alpha \text{ for all } b \in [U_\alpha].$$

Find a stationary subset $E_\emptyset \subseteq I_{T_0}(\emptyset)$ and ordinals $\nu < \omega_1$ and $a(\emptyset) < \omega_2$ such that $\nu_\alpha = \nu$ and $\xi_\alpha = a(\emptyset)$ for all $\alpha \in E_\emptyset$. Let

$$T_1 = \bigcup_{\alpha \in E_\emptyset} U_\alpha.$$

Then $T_1$ is a Namba subtree of $T_0$ with the same root $\emptyset$ such that $I_{T_1}(\emptyset) = E_\emptyset$ and such that for all $\alpha \in I_{T_1}(\emptyset)$, we have that $T_1^{\{\alpha\}} = U_\alpha$ and therefore

$$A(b) \cap \alpha = \{\xi < \alpha : e(\xi, \alpha) < \nu\} \text{ for all } b \in [T_1^{\{\alpha\}}].$$

[1] Here, we let $\max(\emptyset) = 0$.

It follows that $b \mapsto A(b) \cap \alpha$ is a constant function on $[T_1^{\{\alpha\}}]$, so we denote its constant value by $A(\{\alpha\})$. This completes the initial step of the recursive construction.

The recursive step from the tree $T_n$ to $T_{n+1}$ is done similarly. Fix $t \in T_n$ of length $n$. Working as above we can find an ordinal $a(t) < \omega_2$ and a stationary subset $E_t$ of $I_{T_n}(t)$ and for each $\alpha \in E_t$ a Namba subtree $U_{t\alpha}$ of $T_n^{t \cup \{\alpha\}}$ such that

$$\sup(A(b) \cap \alpha) < a(t) \text{ for all } b \in [U_{t\alpha}].$$

Then as before we know that the function $b \mapsto A(b) \cap \alpha$ has the constant value $\{\xi < \alpha : e(\xi, \alpha) < \nu\}$ on $[U_{t\alpha}]$. For $\alpha \in E_t$, let

$$A(t \cup \{\alpha\}) = \{\xi \in [\max(t), \alpha) : e(\xi, \alpha) < \nu\}.$$

Note that $A(t \cup \{\alpha\}) \subseteq [\max(t), a(t))$ for all $\alpha \in E_t$. Let

$$T_{n+1} = \bigcup_{t \in T_n(n), \alpha \in E_t} U_{t\alpha}.$$

Then $T_{n+1}$ is a Namba tree such that $T_{n+1} \subseteq_n T_n$ and such that for every $t \cup \{\alpha\} \in T_{n+1}(n+1)$,

$$A(b) \cap [\max(t), \alpha) = A(t \cup \{\alpha\}) \subseteq [\max(t), a(t)) \text{ for all } b \in [T_{n+1}^{t \cup \{\alpha\}}].$$

This finishes the inductive step. It is clear that the fusion $T = \bigcap_{n<\omega} T_n$ satisfies the conclusion of the Lemma. □

Fix a map $e : [\omega_2]^2 \to \omega_1$ such that for all $\alpha < \omega_2$, the section $e(\cdot, \alpha)$ is a one-to-one mapping from $\alpha$ into $|\alpha|$. The definition below would feel more natural if we assume that $e$ satisfies the two subadditivity condition of a $\varrho$-fuction, or equivalently, that for every $\nu < \omega_1$ the relation $\alpha <_\nu \beta$ iff $\alpha < \beta$ and $e(\alpha, \beta) < \nu$ defines a tree ordering on $\omega_2$ (see [13]). Given such $e : [\omega_2]^2 \to \omega_1$, we recall the corresponding notation

$$A_\nu(\alpha) = \{\xi < \alpha : e(\xi, \alpha) < \nu\} \text{ for } \nu < \omega_1 \text{ and } \alpha < \omega_2.$$

Finally, we are ready to define sets that will be relevant to completing the proof of Proposition 3.2. For a real $r \in [\omega]^\omega$, we consider the following subset of $\mathcal{S}_r$,

$$\mathcal{S}_r^* = \{A \in \mathcal{S}_r : A \cap \omega_1 \in \omega_1 \text{ implies } (\forall \alpha < \omega_2)\ A \neq A_{A \cap \omega_1}(\alpha)\}.$$

Let us prove the following version of Proposition 3.2.

**Proposition 3.5** *If for some real $r \in [\omega]^\omega$ the set $\mathcal{S}_r^*$ is stationary then $\mathcal{S}_r^*$ remains stationary after forcing by any semi-proper forcing notion $\mathcal{P}$.*

*Proof* Pick a name $\dot{C}$ for a closed and unbounded set and pick a condition $p_0$ in $\mathcal{P}$. We need to find an extension of $p_0$ forcing that $\mathcal{S}_r^*$ and $\dot{C}$ intersect. Pick a name $\dot{f}$ for a function from $[\check{\omega_2}]^{<\check{\omega}} \to \check{\omega_1}$ such that $p_0$ forces that countable sets closed under $\dot{f}$ all belong to $\dot{C}$ and that $\dot{f}$ incorporates our function $\check{e} : [\check{\omega_2}]^2 \to \check{\omega_1}$. Choose a countable elementary submodel $M$ of some large-enough $H_\theta$ such that $M$ contains all these objects and such that $A = M \cap \omega_2$ belongs to $\mathcal{S}_r^*$. Let $p$ be an extension of $p_0$ that is $(M, \mathcal{P})$-semigeneric, i.e, that forces that $M[\dot{G}] \cap \omega_1 = \check{M} \cap \omega_1$. Let $\nu = M \cap \omega_1$. Note that $p$ forces that $M[\dot{G}] \cap \check{\omega_2}$ is $\dot{f}$-closed and that therefore it belongs to $\dot{C}$. So it suffices to show that $p$ forces that $M[\dot{G}] \cap \check{\omega_2} = \check{M} \cap \check{\omega_2} = \check{A}$. Otherwise, we can find an extension $q$ of $p$ and an ordinal $\alpha < \omega_2$ such that $q$ forces that $\check{\alpha}$ is the minimal ordinal of $M[\dot{G}] \cap \check{\omega_2}$ which does not belong to the set $\check{A}$. Note that since $q$ forces that $M[\dot{G}]$ is an elementary submodel of $H_{\check{\theta}}[\dot{G}]$ it must force that its intersection with $\check{\omega_2}$ end-extends $\check{A}$ and that, therefore, forces that $\check{A} = \check{\alpha} \cap M[\dot{G}]$. Since $q$ forces that $M[\dot{G}] \cap \check{\omega_2}$ is an $\dot{f}$-closed set and that $\dot{f}$ incorporates $\check{e}$, we have that $q$ forces the equality $\check{A} = A_{\check{\nu}}(\check{\alpha}) = \check{A_\nu(\alpha)}$. It follows that $A = A_\nu(\alpha)$, a contradiction. □

The proof of Theorem 3.1 is based on the following result that is of independent interest.

**Proposition 3.6** *If* $2^{\aleph_0} > \aleph_2$ *then every real* $r \in [\omega]^\omega$, *the set* $\mathcal{S}_r^*$ *is a stationary subset of* $[\omega_2]^{\aleph_0}$.

*Proof* This really follows from the standard proof, using Lemma 3.4, that the sets $\mathcal{S}_r$ are stationary. To see this, fix a real $r \in [\omega]^\omega$ and a closed and unbounded subset $\mathcal{C}$ of $[\omega_2]^{\aleph_0}$. We identify $r$ with the strictly increasing map from $\omega$ into $\omega$ that enumerates it. Thus $r(n)$ is the $n$th element of $r$ in that enumeration. Choose $f : [\omega_2]^{<\omega} \to \omega_2$ such that $f$-closed countable subsets of $\omega_2$ are all elements of $\mathcal{C}$. Applying Lemma 3.4 to this $f$ we get a Namba tree $T$ and mappings $A : T \to [\omega_2]^{\leq\aleph_0}$ and $a : T \to \omega_2$ satisfying the conditions (1) to (4) from the conclusion of this lemma. Let $\theta = (2^{\aleph_0})^+$. Choose a continuous $\in$-chain $M_\xi$ $(\xi < \omega_2)$ of elementary submodel of $H_\theta$ of cardinality $\aleph_1$ containing $\omega_1$ and all these object such that $\delta_\xi = M_\xi \cap \omega_2 \in \omega_2$. Let

$$C = \{\xi < \omega_2 : \delta_\xi = \xi\}.$$

Then $C$ is a closed and unbounded subset of $\omega_2$. Since $\{C_\gamma : \gamma < \omega_2, \mathrm{cf}(\gamma) = \omega\}$ is club-guessing, we can choose $\gamma < \omega_2$ of cofinality $\omega$ such that $C_\gamma \subseteq C$. It follows that

$$M_{C_\gamma(n)} \cap \omega_2 = C_\gamma(n) \text{ for all } n < \omega.$$

Using elementarity of the model $M_{C_\gamma(r(0)+1)}$ we can choose $\alpha_0 \in I_T(\emptyset)$ belonging to that submodel such that $\alpha_0 \geq C_\gamma(r(0))$. Note that by the elementarily of $M_{C_\gamma(0)}$ and $M_{C_\gamma(r(0)+1)}$, we know that $a(\emptyset) < C_\gamma(0)$ while $a(\{\alpha_0\}) < C_\gamma(r(0)+1)$. Choose now $\alpha_1$ in $I_T(\{\alpha_0\}) \cap M_{C_\gamma(r(1)+1)}$ such that $\alpha_1 \geq C_\gamma(r(1))$. Then again by elementarity $a(\{\alpha_0, \alpha_1\}) < C_\gamma(r(1)+1)$, and so on. It is clear that going this way we can choose an infinite branch $b = \{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\}$ of $T$ such that for all $n < \omega$

$$C_\gamma(r(n)) \leq \alpha_n < a(\{\alpha_0, \ldots, \alpha_n\}) < C_\gamma(r(n)+1)$$

It follows that

$$A(b) = \bigcup_{n<\omega} A(b \upharpoonright n) \subseteq [0, C_\gamma(0)) \cup \bigcup_{n<\omega} [C_\gamma(r(n), C_\gamma(r(n)+1))$$

and therefore $\mathrm{osc}(A(b)) = r$ and so $A(b) \in \mathcal{C} \cap \mathcal{S}_r$. Note that that we can in fact obtain continuum many such sets $A(b)$, i.e., that we have established that

$$|\mathcal{C} \cap \mathcal{S}_r| \geq 2^{\aleph_0} > \aleph_2.$$

Since the difference $\mathcal{S}_r \setminus \mathcal{S}_r^*$ has cardinality at most $\aleph_2$, this shows that $\mathcal{C} \cap \mathcal{S}_r^* \neq \emptyset$, finishing the proof. □

To see how Theorem 3.1 follows from Proposition 3.2, consider a semiproper poset $\mathcal{P}$ that forces that the ordinal $\omega_2^V$ has cardinality $\aleph_1$. It follows that there is a $\mathcal{P}$-name $\dot{\mathcal{C}}$ for a closed and unbounded subset of $[\omega_2^V]^{\aleph_0}$ of cardinality $\aleph_1$. From Propositions 3.5 and 3.6 it follows that $\mathcal{P}$ forces that the pairwise disjoint sets $\mathcal{S}_r^*$ $(r \in ([\omega]^\omega)^V)$ all intersect $\dot{\mathcal{C}}$. It follows that $\mathcal{P}$ forces $|\check{\mathbb{R}}| = \aleph_1$, as required.

## 4 More of the oscillation theory of $[\omega_2]^{\aleph_0}$

Let us recall that the oscillation theory (in dimension 2) for $[\omega]^\omega$ was developed by the author in the early 1980's (see [11]). The 3-dimensional oscillation theory for subsets of $[\theta]^{\aleph_0}$ for regular $\theta \geq \omega_2$ was first introduced by Gitik in [4]. Around the same time, motivated by the result of Baumgartner and Taylor [1] that $[\omega_2]^{\aleph_0}$ can be split into $2^{\aleph_0}$ pairwise disjoint stationary sets, the author was studying the 2-dimensional oscillation theory for such $[\theta]^{\aleph_0}$. The 2-dimensional oscillation theory at that time used an arbitrary sequence $\{C_\alpha : \alpha < \theta, \mathrm{cf}(\alpha) = \omega\}$ with the property $\sup(C_\alpha) = \alpha$ and $\mathrm{otp}(C_\alpha) = \omega$ rather than a club-guessing one. Soon after Shelah's discovery of the club-guessing sequence the 2-dimensional oscillation theory came to its present form.[2]

Fix a surjection $c : [\omega]^\omega \to [\omega_1]^{\leq\aleph_0}$. We shall say that a real $r \in [\omega]^\omega$ *codes a countable subset* $A$ *of* $\omega_1$ whenever $c(r) = A$. This gives us a way to extend the functor $r \mapsto \mathcal{S}_r$ to the power-set of $\omega_1$. For $X \subseteq \omega_1$, set

$$\mathcal{S}_X = \{A \in [\omega_2]^{\aleph_0} : \mathrm{osc}(A) \text{ codes } X \cap A \cap \omega_1\}.$$

The following statement about this set appears explicitly in the proof of Lemma 43 of [12].[3]

**Proposition 4.1** *For every stationary subset* $E$ *of* $\omega_1$ *and every subset* $X$ *of* $\omega_1$, *the restriction* $\mathcal{S}_X(E)$ *is a stationary subset of* $[\omega_2]^{\aleph_0}$.

[2] The original 2-dimensional oscillation theory over an arbitrary $C$-sequence is, however, of independent interest and we shall present it in a separate paper.

[3] Recall that for $\mathcal{S} \subseteq [\omega_2]^{\aleph_0}$ and $E \subseteq \omega_1$, we let $\mathcal{S}(E) = \{A \in \mathcal{S} : A \cap \omega_1 \in E\}$.

*Proof* This is really what the proof of Proposition 3.2 above shows. The Namba tree $T$ given by Lemma 3.4 has the property that $b \mapsto A(b) \cap \omega_1$ is constant on $[T]$. So if $\nu$ is the constant value (which we can also make sure that it belongs to the stationary set $E$) and if $r \in [\omega]^\omega$ is a real that codes $X \cap \nu$, the branch $b$ of $T$ such that $\mathrm{osc}(A(b)) = r$ gives us the set in the intersection $\mathcal{C} \cap \mathcal{S}_X(E)$. □

It is also clear that the functor $r \mapsto \mathcal{S}_r$ extend to the power-set of $\omega_2$ as well. To see this, for every countable ordinal $\delta$ we fix an onto map

$$\Phi_\delta : 2^\omega \to \mathcal{P}(\delta)$$

that takes the same value on reals that have only finitely many disagreements. For $X \subseteq \omega_2$, set

$$\mathcal{S}_X = \{A \in [\omega_2]^{\aleph_0} : \mathrm{osc}(A) \text{ codes } X \cap A \text{ inside } A\}.$$

It is rest to specify what we mean when we say that a real[4] $r \in [\omega]^\omega$ is *coding* a subset of a given set $A \in [\omega_2]^{\aleph_0}$ so that we can rely on the proof of Proposition 3.2 and show that the coding can indeed be realized. We assume that $A$ is of a limit order type so we have a natural decomposition of $A$ into its initial parts $A_n = A \cap C_{\sup(A)}(n)$. Let $\pi_{A_n} : A_n \to \nu_n(A)$ be the corresponding maps that collapse these sets to countable ordinals. We shall say that a map $r$ from $\omega$ into 2 *codes a subset* $X$ *of the set* $A$ if for every $n < \omega$,

$$\Phi_{\nu_n(A)}((r)_n) = \pi_{A_n}[X \cap A_n],$$

where the section $(x)_n$ of an $x \in 2^\omega$ is defined by

$$(x)_n(k) = x(2^n(2k+1)).$$

**Proposition 4.2** *For every stationary subset $E$ of $\omega_1$ and every subset $X$ of $\omega_2$, the restriction $\mathcal{S}_X(E)$ is a stationary subset of* $[\omega_2]^{\aleph_0}$.

*Proof* To see that the proof of Proposition 3.2 shows this, fixing the Namba tree $T$ satisfying Lemma 3.4 with the additional property that $b \mapsto A(b) \cap \omega_1$ is constant on $[T]$ with the constant value in the set $E$, we show that we can choose its branch $b = \{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\}$ so that $r = \mathrm{osc}(A(b))$ codes $X \cap A(b)$ inside $A(b)$. We start with choosing $\alpha_0$ in $M_{C_\gamma(1)} \cap I_T(\emptyset)$ such that $\alpha_0 \geq C_\gamma(0)$. This implies that $r(0) = 1$ and will give us opportunity to fix the section $(r)_0$ as any real such that $\Phi_{\nu_{A(\emptyset)}}((r)_0) = \pi_{A(\emptyset)}[X \cap A(\emptyset)]$ and such that $(r)_0(k) = 1$ for some $k$. Let $k_1$ be the minimal $k$ such that $r(2k+1) = (r)_0(k) = 1$, we can choose $\alpha_1$ in $I_T(\{\alpha_0\}) \cap M_{C_\gamma(2k_1+2)}$ such that $\alpha_1 \geq C_\gamma(2k_1+1)$. Now we can choose the section $(r)_1$ as any real such that $\Phi_{A(b)_1}((r)_1) = \pi_{A(b)_1}[X \cap A(b)_1]$, where $A(b)_1 = (A(\emptyset) \cup A(\{\alpha_0\})) \cap C_\gamma(1)$. Moreover, we assume that $(r)_1(k) = 1$ for some $k$ but that $(r)_1(k) = 0$ for all $k$ such

[4] identified with its characteristic function, an element of $2^\omega$.

that $2(2k+1) \leq 2k_1 + 1$. Knowing the sections $(r)_0$ and $(r)_1$ we now know $n_2$, the minimal integer $n > 2k_1 + 1$ such that $(r)_0(n) = 1$ or $(r)_1(n) = 1$. This integer $n_2$ will therefore have the form $2k_2 + 1$ or $2(2k_2 + 1)$ for some $k_2$, so we can proceed to picking $\alpha_2 \in I_T(\{\alpha_0, \alpha_1\}) \cap M_{C_\gamma(n_2+1)}$, and so on. It is clear that this recursion will give us the desired branch $b$ finishing the proof. □

It should be clear that similar results hold for $[\theta]^{\aleph_0}$ where $\theta$ is any regular cardinal greater or equal to $\omega_2$. In fact one can have similar oscillation results for many regular cardinals simultaneously (see [3]).

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