Arch. Math. Logic (2018) 57:141–157
https://doi.org/10.1007/s00153-017-0587-y

Mathematical Logic

The small index property for homogeneous models in AEC's

Zaniar Ghadernezhad[1] · **Andrés Villaveces**[2]

Received: 30 July 2016 / Accepted: 29 January 2017 / Published online: 14 September 2017

Abstract We prove a version of a small index property theorem for strong amalgamation classes. Our result builds on an earlier theorem by Lascar and Shelah (in their case, for saturated models of uncountable first-order theories). We then study versions of the small index property for various non-elementary classes. In particular, we obtain the small index property for quasiminimal pregeometry structures.

Keywords Model theory · Small index property · Quasiminimal pregeometry

Mathematics Subject Classification 03C48 · 22F50

Contents

This research was made possible partially by Colciencias grant Métodos de Estabilidad en Clases No Estables. The first author's research was also partially supported by the Iran National Science Foundation (INSF).

✉ Zaniar Ghadernezhad
zaniar.gh@gmail.com

Andrés Villaveces
avillavecesn@unal.edu.co

[1] School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran

[2] Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá 111321, Colombia

1 Introduction

The role of automorphism groups of mathematical structures and their connections to their theories and *bi-interpretation* classes has been an active research area in model theory in the past two decades. The main theme has been to investigate what kind of information from the structure can be recovered from its group of automorphisms (the *reconstruction problem*). The automorphism groups are also topological groups, in a natural way (pointwise convergence topology). The reconstruction problem is therefore linked to both the purely algebraic aspects of the groups $\mathrm{Aut}(M)$, as well as to their topological aspects. The crucial property linking these two aspects is the "small index property", the center of study of this paper. A structure M of cardinality κ has the *small index property* (SIP) if every subgroup $H \leqslant \mathrm{Aut}(M)$ of small index (that is, such that $[\mathrm{Aut}(M) : H] < |\mathrm{Aut}(M)|$) is open in $\mathrm{Aut}(M)$.

Although in principle the SIP is posed in terms of a (first-order) structure M, previous results have relied heavily on the *first order theory* of M and its bi-interpretability class. The structural properties revealed by $\mathrm{Aut}(M)$ depend somehow more on the theory of M, on the structural properties *around* M, than on M taken in isolation. In this article, we look for the first time at the situation of a structure M that may fail to be saturated yet still have a good reconstruction problem, provided by other structural properties:

- homogeneity,
- being inside an abstract elementary class (AEC) and
- having a strong amalgamation property (for strong embeddings in the abstract elementary class, *and* for automorphisms).

These properties are isolated by carefully checking the case studied before by Lascar and Shelah [7], where they prove the SIP for uncountable, saturated structures, relying on their first order theories. In this paper we show that their techniques may be adapted to our more general setup: saturation is weakened to homogeneity, and we isolate a strong notion of amalgamation in AECs that suffices to replace the technical aspects of the proof from [7] that depend on the first order theory of the structure. So, our result shows that the SIP may be obtained for homogeneous structures, provided they are placed in an AEC that has enough structural properties[1].

[1] Classical model theory deals with mathematical structures using tools of first-order logic; the first-order setting provides many technical tools to study them. However, the first-order setting has some limitations: many classes of mathematical structures are not first-order axiomatizable, and even when they are, there

An important step in reconstruction problems is therefore determining whether a structure has the SIP: this property is a key piece in the recovery of the topological structure of the automorphism group from its pure group structure. The SIP has been proved for a number of countable first-order structures. With the pointwise convergence topology, in the first-order case, the automorphism groups of countable structures are actually *Polish* groups (see [4] for more details). Indeed, the automorphism groups of countable structures being Polish is a very useful fact that can provide many technical tools to prove properties like SIP. For the uncountable first-order structures, we do not have access to such tools, however still nice properties can be proven: In the uncountable case, with some cardinal restrictions, Lascar and Shelah in [7] proved that the automorphism group of an uncountable saturated model has SIP.

In this paper, we prove the following theorem (Theorem 4.1 in this paper, see Sect. 4):

Theorem *Let* M *be a homogeneous model in an abstract elementary class* $(K, \preccurlyeq_{\mathcal{K}})$ *such that* $|M| = \kappa > LS(K)$ *and* $\kappa^{<\kappa} = \kappa$. *Furthermore, assume that* $\mathcal{K}^{<}(M)$ *is a strong amalgamation class. Consider the group* $\mathrm{Aut}(M)$ *with the topology given by* $\mathfrak{T}^{cl}$, *and let* $H \leqslant \mathrm{Aut}(M)$ *be such that* $[\mathrm{Aut}(M) : H] \leqslant \kappa$. *Then,* H *is an open subgroup of* $\mathrm{Aut}(M)$*; i.e., there exists* $A \in \mathcal{K}^{<}(M)$ *such that* $\mathrm{Aut}_A(M) \leqslant H$.

In other words, $\left(\mathrm{Aut}(M), \mathfrak{T}^{cl}\right)$ *has the small index property.*

This theorem provides a purely algebraic framework for a model to have the small index property; we thereby transfer in a rather sharp way the Lascar–Shelah setting [7], where a similar result is proved for saturated uncountable models of a first order theory.

In the last two sections, we study some examples and applications: we prove the SIP for the Zilber field (see [2,6,12]), we study the SIP for the j-mapping (modular invariant) under the light of recent results in its model theory, and for various other $L_{\omega_1,\omega}$-axiomatizable structures.

The authors thank the referee for several questions and remarks that have helped improve the presentation of our results.

2 Setting

We fix two infinite cardinals λ, κ such that $\lambda < \kappa$ and $\kappa^{<\kappa} = \kappa$. We work within a fixed AEC $\mathcal{K} = (K, \prec_{\mathcal{K}})$ with $LS(\mathcal{K}) \leqslant \lambda$. We now provide some notation and definitions.

Footnote 1 continued
are situations in which a "non-elementary" model theoretical analysis results in better regularity properties. Shelah in [9] introduced the notion of *Abstract Elementary Classes* (AECs), where logic and syntax are set aside and the elements of the class are axiomatized using an abstract notion of "strong" embedding (important examples of these classes include Zilber's pseudo-exponentiation, various pseudo-analytic expansions of the complex numbers and several instances of covers—more recently, modular functions have been successfully analyzed using these tools—see [5]). In this general setting, we do not necessarily need to consider the elements of the class in the framework of the first-order logic and work with formulas. However, many important concepts of model theory such as types, forking and other independence notions have been successfully studied in the AEC context.

2.1 Notation

- $\mathrm{Iso}(N_1 \mathcal{I} N_2)$ denotes the set of all isomorphisms from N_1 onto N_2, for $N_1, N_2 \in K$.
- $\mathcal{K}^{<}(M) := \{N : N \preccurlyeq_{\mathcal{K}} M, |N| < \kappa\}$. Clearly, this is not empty, as $LS(\mathcal{K}) \leqslant \lambda < \kappa$.
- For $N_1, N_2 \in \mathcal{K}^{<}(M)$ and $f \in \mathrm{Iso}(N_1 \mathcal{I} N_2)$ we define $\mathcal{O}_f^M := \{g \in \mathrm{Aut}(M) : f \leqslant g\}$, the set of all automorphisms of M extending f. Note that $\mathcal{O}_f^M \neq \emptyset$ means that f can be extended to an automorphism of M.
- As usual, we denote by g^{α} the map $\alpha^{-1} \circ g \circ \alpha$.
- Also, $\mathrm{Aut}_N(M)$ denotes the pointwise stabilizer of N (the subgroup of automorphisms of M that fix N pointwise) and $\mathrm{Aut}_{\{N\}}(M)$ denotes the setwise stabilizer of N, the subgroup of automorphisms of M that fix N setwise.

For the rest of the paper we fix a homogeneous model $M \in K$ of size κ (we use "homogeneous" in the following precise sense: all isomorphisms between small strong substructures of M, $f : N_1 \to N_2$ for $N_1, N_2 \in \mathcal{K}^{<}(M)$, can be extended to automorphisms of M). *All our results* from here on refer to properties of M within the AEC $\mathcal{K}$.

In our context, "small" (subset or submodel) means "of cardinality $< \kappa$".

We may even say that our focus of study is the part of $\mathcal{K}$ "below M", i.e. the class of small $\mathcal{K}$-elementary submodels of M. In this sense the actual arena of our results is the class $(\mathcal{K}^{<}(M), \prec_{\mathcal{K}})$. This is almost an AEC; the only property of an AEC this fails to satisfy is the unions axiom. In recent years, weak AECs have been studied; this class is one of them.

Our notion of homogeneity means precisely that $\mathcal{O}_f^M \neq \emptyset$ for all $f \in \mathrm{Iso}(N_1 \mathcal{I} N_2)$ with $N_1, N_2 \in \mathcal{K}^{<}(M)$ when we translate the concept to this notation.

Remark 2.1 Let $N \in \mathcal{K}^{<}(M)$. Then the restriction map $\pi_N : \mathrm{Aut}_{\{N\}}(M) \to \mathrm{Aut}(N)$ is surjective: if $f \in \mathrm{Aut}(N)$ then there is $f' \in \mathrm{Aut}(M)$ such that $f = f' \restriction N = N$.

Definition 2.2 For $A \subseteq M$, let $\mathrm{cl}^M(A) := \bigcap \{N \in \mathcal{K} \mid A \subseteq N \preccurlyeq_{\mathcal{K}} M\}$.

Remark 2.3 This notion of closure appeared in [1] in connection with the study of the failure of tameness. It is worth noting here that with this notion of closure, unlike in other related constructions, closures *do not necessarily belong* to the class $\mathcal{K}$. Moreover, note that if $A \subseteq M$ and $\mathrm{cl}^M(A) \neq M$, then for every $m \in M \backslash \mathrm{cl}^M(A)$ there is $N \preccurlyeq_{\mathcal{K}} M$ such that $m \notin N$ and $\mathrm{cl}^M(A) \subseteq N$.

It is easy to see that the following holds.

(1) $\mathrm{cl}^M(N) = N$ for all $N \in K$ with $N \preccurlyeq_{\mathcal{K}} M$.
(2) $\mathrm{cl}^M\left(\mathrm{cl}^M(A)\right) = \mathrm{cl}^M(A)$ for all $A \subseteq M$.

Remark 2.4 The definition above of cl^M generalizes the first order definition of the algebraic closure *acl* with $\preccurlyeq_{\mathcal{K}}$ as elementary submodel.

Let $\mathcal{C} := \left\{\mathrm{cl}^M(A) : A \subseteq M \text{ such that } |A| < \kappa\right\}$.

Fact 2.5 (1) *Suppose* $A \subset M$, *then* $\left|cl^M(A)\right| \leqslant |A| + \mathrm{LS}(K)$.
(2) *Small strong submodels of* M *are closed i.e.* $\mathcal{K}^{<}(M) \subseteq \mathcal{C}$.
(3) *If* $A, B \in \mathcal{C}$, *then* $A \cap B \in \mathcal{C}$.
(4) *If* $A, B \in \mathcal{C}$, *then there exists* $N \in \mathcal{K}^{<}(\mathcal{M})$ *such that* $A \cup B \subseteq N$.

Proof (1) and (3) are immediate. (2) and (4) follow from the fact that $|M| = \kappa > \mathrm{LS}(K)$. □

We extend the two notations that we have defined before from elements of $\mathcal{K}^{<}(M)$ to all closed sets.

Definition 2.6 Let $X, Y \in \mathcal{C}$. Define

$$\mathrm{Iso}(X\mathfrak{I}Y) := \{f \restriction X : f \in \mathrm{Iso}(N_1, N_2) \text{ where } N_1, N_2 \in \mathcal{K}^{<}(M), X \subseteq N_1, Y \subseteq N_2 \text{ and } f[X] = Y\}$$

and $\mathrm{Aut}(X) := \mathrm{Iso}(X\mathfrak{I}X)$.

Corollary 2.7 *The set* $\{\mathrm{Aut}_X(M) : X \in \mathcal{C}\}$ *forms a basis of open neighborhoods of identity.*

Let $\mathfrak{T}^{\mathrm{cl}}$ be the topology that is generated by the cosets of stabilizers $\mathtt{Aut}_X(M)$ for $X \in \mathcal{C}$. It is clear that $\left(\mathrm{Aut}(M), \mathfrak{T}^{\mathrm{cl}}\right)$ is a topological group. This is the setup for the proof of our main theorem (see Sect. 4).

Remark 2.8 Similarly to the countable case in first-order logic we have the following property: Suppose M and N are two models of the same uncountable cardinality and we topologize them in the same way. Suppose $\alpha : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ is an embedding as abstract groups. Then the SIP for M implies that α is continuous.

2.2 Some properties of the class of closed sets $\mathcal{C}$

In this subsection we do not require the assumption that M is homogeneous.

Lemma 2.9 *Let* $X, Y \in \mathcal{C}$ *and* $f \in Iso(X\mathfrak{I}Y)$ *be such that* $\mathcal{O}_f^M \neq \emptyset$. *Then, there exists* $Z \in \mathcal{C}$ *and* $f' \in \mathrm{Aut}(Z)$ *such that* $X \cup Y \subseteq Z$, $f \leqslant f'$ *and* $\mathcal{O}_{f'}^M \neq \emptyset$.

Proof Let $\sigma \in \mathcal{O}_f^M$; then $\sigma \in \mathtt{Aut}(M)$ and $\sigma \restriction X = f$. Since $\mathrm{LS}(K) < \kappa$, there exists $N \in \mathcal{K}^{<}(M)$ such that $X \cup Y \subseteq N$. Let $X_1 := N \cup \sigma[N] \cup \sigma^{-1}[N]$ and let $\lambda = |X_1|$. It is clear that $\lambda < \kappa$, hence there exists $N_1 \in \mathcal{K}^{<}(M)$ of cardinality λ such that $X_1 \subseteq N_1$. Inductively, define $X_n := N_{n-1} \cup \sigma[N_{n-1}] \cup \sigma^{-1}[N_{n-1}]$ and $N_n \in \mathcal{K}^{<}(M)$ such that $X_n \subseteq N_n$ and $|N_n| = \lambda$ for all $n \in \mathbb{N}$ (always possible since $\mathrm{LS}(K) < \kappa$). We then have a chain $N_i \prec_{\mathcal{K}} N_j$ for $i \leqslant j \in \mathbb{N}$. Let $N^* := \bigcup_{n \in \mathbb{N}} N_n$. Then clearly $|N^*| = \lambda$ and $N^* \in \mathcal{K}^{<}(M)$. Note that $\sigma \restriction N^* \in \mathrm{Aut}(N^*)$. Let $Z := N^*$ and $f' := \sigma \restriction N^*$. □

Corollary 2.10 *Let* $\sigma \in \mathrm{Aut}(M)$ *and assume* λ *is a cardinal with* $\mathrm{LS}(K) \leqslant \lambda < \kappa$, *then there exists* $N \in \mathcal{K}^{<}(M)$ *with* $|N| = \lambda$ *such that* $\sigma \restriction N \in \mathrm{Aut}(N)$.

Proof We use the same argument as the proof of Lemma 2.9: start with N_0 and build up a $\prec_K$-increasing chain of models N_n such that $N_n \supseteq X_n := N_{n-1} \cup \sigma[N_{n-1}] \cup \sigma^{-1}[N_{n-1}]$. □

Remark 2.11 So far, no special properties of κ as a cardinality have been used. For the next corollary we use $\kappa^{<\omega} = \kappa$.

Corollary 2.12 *Let* I *be an index set with* $|I| < \kappa$. *Assume* $f_i \in \mathrm{Iso}(X_i \mathfrak{I} Y_i)$ *with* $X_i, Y_i \in \mathcal{C}$ *are such that* $\mathcal{O}^M_{f_i} \neq \emptyset$, *for* $i \in I$. *Then there exists* $Z \in \mathcal{C}$ *and* $f'_i \in \mathrm{Aut}_{\{Z\}}(M)$ *such that* $\bigcup_{i\in I} X_i \cup \bigcup_{i\in I} Y_i \subseteq Z$, $f_i \leqslant f'_i$ *and* $\mathcal{O}^M_{f'_i} \neq \emptyset$ *for all* $i \in I$.

Proof Use the facts that $\mathrm{LS}(K) < \kappa$ and $\kappa^{<\kappa} = \kappa$ and repeat the argument in the proof of Lemma 2.9, by closing the sets under all the f'_i and f'^{-1}_is for some $f'_i \in \mathcal{O}^M_f$ with $i \in I$ in each step. □

Corollary 2.13 *Let* I *be an index set with* $|I| = \lambda$ *such that* $\mathrm{LS}(K) \leqslant \lambda < \kappa$. *Assume* $\sigma_i \in \mathrm{Aut}(M)$ *with* $i \in I$. *Then there exists* $N \in \mathcal{K}^<(M)$ *with* $|N| = \lambda$ *such that* $\sigma_i \restriction N \in \mathrm{Aut}(N)$ *for all* $i \in I$.

Proof As in the proof of Lemma 2.9. □

3 Generic sequences of automorphisms

This section sets up the tools for the main proof: generic sequences of automorphisms and strong amalgamation bases. We provide the basic notions (the first one is adapted from [7] to our context, the second one is new) and then derive the existence of large families of generic sequences of automorphisms.

Definition 3.1 Suppose $M_0, M_1, M_2 \in \mathcal{K}^<(M)$ and $M_0 \prec_{\mathcal{K}} M_i$ for $i = 1, 2$. Let γ^0 be a sequence of automorphisms of M_0. Assume γ^1 and γ^2 are sequences of automorphisms of M_1 and M_2; respectively, both extending γ^0. We say γ^1 and γ^2 are *compatible* over M_0 if there exist $M_3 \in \mathcal{K}^<(M)$ and $\alpha_1, \alpha_2 \in \mathrm{Aut}_{M_0}(M)$ such that $\alpha_1[M_1], \alpha_2[M_2] \prec_{\mathcal{K}} M_3$ and $\gamma^3 \in \mathrm{Aut}(M_3)$ extends both $\alpha_1 \circ \gamma^1 \circ \alpha_1^{-1} \restriction \alpha_1[M_1]$ and $\alpha_2 \circ \gamma^2 \circ \alpha_2^{-1} \restriction \alpha_2[M_2]$ (this is equivalent to requiring that $\gamma^3 \circ \alpha_1 \restriction N_1 = \alpha_1 \circ \gamma^1$ and $\gamma^3 \circ \alpha_2 \restriction N_2 = \alpha_2 \circ \gamma^2$).

Notice that in the definition above for a sequence of automorphisms we allow repetitions of automorphisms.

We now define a central notion in the proof of the main theorem: generic sequences of automorphisms (and the particular case of a generic automorphism).

Definition 3.2 Suppose I is an index set with $|I| < \kappa$ and let $\gamma = (g_i : i \in I)$ be a sequence of automorphisms of M. We say γ is a *generic sequence of automorphisms* if whenever $N \in \mathcal{K}^<(M)$ is such that $\gamma \restriction N$ is a sequence of automorphisms of N and $N_1 \in \mathcal{K}^<(M)$ is such that $N \prec_{\mathcal{K}} N_1$ and $\theta = (t_i : i \in I)$ is a sequence of automorphisms of N_1 extending $\gamma \restriction N$ (i.e. $g_i \restriction N \leqslant t_i$ for all $i \in I$), then if $\gamma \restriction N'$ and θ are compatible over N for some $N' \in \mathcal{K}^<(M)$, there exists $\alpha \in \mathrm{Aut}_N(M)$ such that γ extends $\alpha \circ \theta \circ \alpha^{-1}$ (or equivalently $\gamma^\alpha := \left(g_i^\alpha : i \in I\right)$ extends θ).

We abuse language by saying that "g is a *generic automorphism*" when $\gamma = (g)$ is a constant generic sequence of automorphisms.

Remark 3.3 Suppose I is an index set with $|I| < \kappa$ and let $\gamma = (g_i : i \in I)$ be generic. Then γ^α is generic for $\alpha \in \mathrm{Aut}(M)$.

Lemma 3.4 *Let $M_0 \in \mathcal{K}^<(M)$ and I be an index set with cardinality less than κ. Suppose $\gamma = (g_i : i \in I)$ and $\theta = (t_i : i \in I)$ are two generic families of automorphisms of M such that $\gamma \restriction M_0 = \theta \restriction M_0 \in \mathrm{Aut}(M_0)$. Then there exists $\alpha \in \mathrm{Aut}_{M_0}(M)$ such that $\theta^\alpha = \gamma$.*

Proof Similar to the argument in [7], we use back and forth. Since θ and γ are *cofinal* in $\mathcal{K}^<(M)$ (see Corollary 2.13), we claim that we can build a chain of models $N_j^\theta \in \mathcal{K}^<(M)$ and $N_j^\gamma \in \mathcal{K}^<(M)$ for $j < \kappa$, such that $\bigcup_{j<\kappa} N_j^\theta = \bigcup_{j<\kappa} N_j^\gamma = M$; and $\theta \restriction N_j^\theta \in \mathrm{Aut}\left(N_j^\theta\right)$ and $\gamma \restriction N_j^\gamma \in \mathrm{Aut}\left(N_j^\gamma\right)$ and build partial isomorphisms α_i for $i < \kappa$ using the back and forth such that $\alpha := \bigcup_{j<\kappa} \alpha_j$ is the desired automorphism. Let $\alpha_0 = \mathrm{id}_{M_0}$ and $N_0^\theta = N_0^\gamma = M_0$. If i is limit ordinal, let $N_i^\theta := \bigcup_{j<i} N_j^\theta$, $N_i^\gamma := \bigcup_{j<i} N_j^\gamma$ and $\alpha_i := \bigcup_{j<i} \alpha_j$. Suppose $i = j + 1$, without loss of generality we do the "forth" step. Let N_i^θ be an element in $\mathcal{K}^<(M)$ such that $N_j^\theta \preccurlyeq_{\mathcal{K}} N_i^\theta$ and $\theta \restriction N_i^\theta \in \mathrm{Aut}\left(N_i^\theta\right)$; note that existence of such a model follows from Corollary 2.13 or proof of Lemma 2.9. Since γ is generic, then there exists $\beta \in \mathrm{Aut}_{\alpha_j'\left[N_j^\theta\right]}(M)$ such that γ extends $\beta \circ \theta \circ \beta^{-1}$ and $\alpha_j' \in \mathcal{O}^M_{\alpha_j}$. Let $\alpha_i := \alpha_j' \circ \beta \restriction N_i^\theta$. Let then $\alpha := \bigcup \alpha_i$. □

It is an interesting question to determine sufficient conditions for a generic sequence of automorphisms to exist. We now provide one condition that will guarantee precisely that.

Definition 3.5 Assume I is an index set with cardinality less than κ. Suppose $N_0 \in \mathcal{K}^<(M)$ and $\gamma^0 = \left(g_i^0 : i \in I\right)$ is a sequence of automorphisms of N_0. We say $\left(N_0, \gamma^0\right)$ is a *strong amalgamation base* if for all $N_1, N_2 \in \mathcal{K}^<(M)$ with $N_0 \prec_{\mathcal{K}} N_1, N_2$; all γ^1 and γ^2 sequences of automorphisms of N_1 and N_2 that extend γ^0; respectively, are compatible over N_0. We say $\mathcal{K}^<(M)$ is a *strong amalgamation class* if (N, γ) is a strong amalgamation base for every $N \in \mathcal{K}^<(M)$ and $\gamma = (g_i : i \in I)$ a sequence of automorphisms of N.

Remark As we see next, this notion guarantees the existence of generic sequences of automorphisms. Its importance lies in the fact of being able to "amalgamate coherently" sequences of isomorphisms, while *also* doing an amalgam of the domains!

Lemma 3.6 *Suppose $\mathcal{K}^<(M)$ is a strong amalgamation class. Then generic sequences of automorphisms with index set of arbitrary cardinality less than κ exist.*

Proof We write the proof for the case when I is a singleton. The proof for the general case (arbitrary I) is not essentially different, but is more cumbersome; therefore we address here only the case when the sequence consists of a single automorphism.

We will build a generic automorphism f as the union of a tower of automorphisms of some elements of $\mathcal{K}^{<}(M)$. Let $M_0 \in \mathcal{K}^{<}(M)$ and f_0 be an arbitrary automorphism of M_0. Since M is homogeneous, we know that $\mathcal{O}^{M}_{f_0} \neq \emptyset$. Now let $E := \{(N, e) : M_0 \prec_{\mathcal{K}} N \in \mathcal{K}^{<}(M) \text{ and } e \in \mathrm{Aut}(N) \text{ with } f_0 \leqslant e\}$. Since $\kappa^{<\kappa} = \kappa$, using Lemma 2.9, we have that $|E| \leqslant \kappa$. We enumerate E as $\{(N_i, e_i) \,|\, i < \kappa\}$. We now build a sequence of pairs (M_i, f_i) for $i < \kappa$ such that $(M_i, f_i) \in E$, $M_i \prec_{\mathcal{K}} M_j$ and $f_i \leqslant f_j$ for $i \leqslant j < \kappa$. Moreover, we build the sequence (M_i) in such a way that $M = \bigcup_{i\in\kappa} M_i$ (for this, we enumerate M as $\{a_i \mid i < \kappa\}$ and demand that $a_i \in M_{i+1}$ for every $i < \kappa$).

- If i is a limit ordinal then let $M_i = \bigcup_{\beta<i} M_\beta$ and $f_i = \bigcup_{\beta<i} f_\beta$; note that the cardinal assumption $\kappa^{<\kappa} = \kappa$ implies that $M_i \in \mathcal{K}^{<}(M)$.
- Suppose i is successor and $i = j + 1$. Consider $(N_j, e_j) \in E$. Since $\mathcal{K}^{<}(M)$ is a strong amalgamation class, there exist $M_i \in \mathcal{K}^{<}(M)$ and $f_i \in \mathrm{Aut}(M_i)$ such that $M_j \prec_{\mathcal{K}} M_i, f_j \leqslant f_i, \alpha \in \mathrm{Aut}_{M_j}(M)$ and $e_j^{\alpha} := \alpha \circ e_j \circ \alpha^{-1} \restriction \alpha[N_j] \leqslant f_i$.

Then let $f := \bigcup_{i\in\kappa} f_i$.

We claim f is a generic automorphism. There are two cases:

(1) Suppose $M_0 \subseteq N_1, N \in \mathcal{K}^{<}(M)$ and $g \in \mathrm{Aut}(M)$ such that $N_1 \prec_{\mathcal{K}} N$, $g \restriction N \in \mathrm{Aut}(N)$ and $g \restriction N_1 = f \restriction N_1$. Note that because of the coherence axiom of AECs $M_0 \prec_{\mathcal{K}} N_1$. Moreover, by Lemma 2.9 there is $N' \in \mathcal{K}^{<}(M)$ such that $N_1 \prec_{\mathcal{K}} N'$ and $f' := f \restriction N' \in \mathrm{Aut}(N')$. Let i be the smallest index $< \kappa$ such that $N' \subseteq M_i$. Since N_1 is an strong amalgamation base, there is $N_2 \in \mathcal{K}^{<}(M)$ and $e_2 \in \mathrm{Aut}(N_2)$ such that $M_i, \alpha[N] \prec_{\mathcal{K}} N_2$, $f' \leqslant e_2$, $\alpha \in \mathrm{Aut}_{N_1}(M)$ and $\alpha \circ g \circ \alpha^{-1} \leqslant e_2$. It is clear that $(N_2, e_2) \in E$, then the construction of f guarantees that there is $j > i$ such that (N_2, e_2) embeds into (M_j, f_j) that fixes M_i. Hence, and there exists $N^* \in \mathcal{K}^{<}(M)$ and $\alpha \in \mathrm{Aut}_{N_1}(M)$ such that $\alpha[N] = N^*$ and f extends $\alpha \circ g \circ \alpha^{-1} \restriction N^*$.

(2) Suppose $N, N' \in \mathcal{K}^{<}(M)$ and $g \in \mathrm{Aut}(M)$ are such that $N' \prec_{\mathcal{K}} N$, $g \restriction N \in \mathrm{Aut}(N)$, $g \restriction N' = f \restriction N'$ and $N' \cap M_0 \neq M_0$. Using Lemma 2.9, there is $N_1 \in \mathcal{K}^{<}(M)$ such that $N \cup M_0 \subseteq N_1$ and $f \restriction N_1 \in \mathrm{Aut}(N_1)$. Again from the coherence axiom of AECs it follows that $N, M_0 \prec_{\mathcal{K}} N_1$. Since $\mathcal{K}^{<}(M)$ is a strong amalgamation class and $(N, f \restriction N)$ is a strong amalgamation base, $f \restriction N_1$ and $g \restriction N'$ are compatible over N. Let $\alpha \in \mathrm{Aut}_N(M)$, $N_2 \in \mathcal{K}^{<}(M)$ and $h \in \mathrm{Aut}(N_2)$ such that $N_1 \prec_{\mathcal{K}} N_2$ and h extends $f \restriction N_1 \cup g^{\alpha} \restriction N'$. Note that $\alpha[N_1]$ (and hence $\alpha[N]$) is strongly embedded in N_2 (again follows from coherence axiom in AEC's). Now (N_2, h) is an element of E. Then we can again reason as in Case 1.

□

Lemma 3.7 *Fix an index set* I *with* $|I| < \kappa$ *and let* $\mathcal{F}$ *be the set of all* I*-sequences of generic automorphisms. Then* $\mathcal{F}$ *is a dense subset of* $\mathrm{Aut}(M)^{I}$*, with the product topology.*

Proof For the sake of readability, we write the proof again for the case $|I| = 1$. Let $\mathrm{Aut}_A(M)$ be a basic open set with $A \in \mathcal{C}$. Let $g \in \mathrm{Aut}_A(M)$; then, just as in the

proof of Lemma 3.6 (choosing $f_0 := g \restriction A$) we can find a generic automorphism f such that $f_0 \leqslant f$. Therefore $f \in \mathrm{Aut}_A(M)$ and $\mathcal{F} \cap \mathrm{Aut}_A(M)$ is non-empty. □

Remark 3.8 In Definitions 3.1, 3.2 and 3.5 we considered the very natural case where the elements are from $\mathcal{K}^{<}(M)$ and it is enough for us to prove the results of next sections. However, these notions can be considered for the elements from the bigger set $\mathcal{C}$; the required definitions for elements of $\mathcal{C}$ have already been provided in Sect. 2.

3.1 Many generic sequences of automorphisms

We now prove that there are many different generic sequences of automorphisms *relative to the size* of $\mathrm{Aut}(M)$, in the following specific sense:

- If $|\mathrm{Aut}(M)| \geqslant \kappa$, then for any family $(M_i)_{i\in I}$ of $(\leqslant \kappa)$-many models in $\mathcal{K}^{<}(M)$ where $\mathcal{K}^{<}(M)$ is a strong amalgamation base, we obtain a corresponding family $(h_i)_{i\in I}$ of automorphisms of M, each h_i fixing pointwise M_i, and such that the family $(h_i)_{i\in I}$ can be used to build a generic sequence of automorphisms in 2^{κ}-many different ways (the technical details of this are the content of the next lemma).
- We therefore obtain 2^{κ}-many generic sequences of automorphisms of length κ. Moreover, this implies that if $|\mathrm{Aut}(M)| \geqslant \kappa$ then $|\mathrm{Aut}(M)| = 2^{\kappa}$. Our approach in this part has the general structure of Lascar–Shelah [7, Lemma 9] but making explicit the use of strong amalgamation bases.
- Of course, in the other extreme case when the model M is rigid[2], all generic sequences of automorphisms must consist of the identity.

Lemma 3.9 *Suppose* $(g_{i,j} : i \in I, j \in J)$ *is a matrix of automorphisms of* M *and* $|I| = |J| \leqslant \kappa$. *Fix* $(M_i : i \in I)$ *a sequence of elements of* $\mathcal{K}^{<}(M)$ *where* $\mathcal{K}^{<}(M)$ *is a strong amalgamation base. Then there exists* $(h_i : i \in I)$ *such that*

(1) $h_i \in \mathrm{Aut}_{M_i}(M)$ *for all* $i \in I$;
(2) $(h_i \circ g_{i,\delta(i)} : i \in I)$ *is a generic sequence, for all injective functions* $\delta : I \to J$.

Proof Consider the following set

$$\begin{aligned} \mathcal{Y} = \{&(I_0, N_0, N_1, \delta, (k_i : i \in I_0)) : I_0 \subseteq I, |I_0| \\ &< \kappa, \delta \text{ is an injective function from } I_0 \text{ into } J \text{ with} \\ &N_0 \preccurlyeq N_1 \in \mathcal{K}^{<}(M), k_i \in \mathrm{Aut}(N_1) \text{ and } k_i \restriction N_0 \in \mathrm{Aut}(N_0)\}. \end{aligned}$$

Note that $|\mathcal{Y}| = \kappa$. Fix an enumeration $(\mathcal{Y}_\alpha : \alpha < \kappa)$ for $\mathcal{Y}$. We define by induction on $\alpha < \kappa$ a family $(M^i_\alpha : i \in I)$ of elements of $\mathcal{K}^{<}(M)$ and a family $(h^i_\alpha : i \in I)$, where $h^i_\alpha \in \mathrm{Aut}(M^i_\alpha)$ such that

(1) $M^i_0 = M_i$ and $h^i_0 = \mathrm{id}_{M_i}$ for all $i \in I$;

[2] For instance, in the case where the model M is a cardinal and strong elementary submodels of M are just initial segments.

(2) The functions $\left(\alpha \mapsto M^i_\alpha\right)$ and $\left(\alpha \mapsto h^i_\alpha\right)$ are increasing and continuous, for all $i \in I$;
(3) $\bigcup_{\alpha<\kappa} M^i_\alpha = M$ and $h_i := \bigcup_{\alpha<\kappa} h^i_\alpha \in \mathrm{Aut}(M)$ for each $i \in I$;
(4) $\left(h_i \circ g_{i,\delta(i)} : i \in I\right)$ is a generic sequence for all injective functions δ from I to J.

Assume $y_\alpha = (I_0, N_0, N_1, \delta, (k_i : i \in I_0)) \in \mathcal{Y}$. Using Lemma 2.9 and Corollary 2.12 one can show that there is a model $M' \in \mathcal{K}^{<}(M)$ such that $N_0 \prec_{\mathcal{K}} M'$

- $M^i_\alpha \prec_{\mathcal{K}} M'$ for all $i \in I_0$;
- $g_{i,\delta(i)} \restriction M' \in \mathrm{Aut}(M')$ for all $i \in I_0$;
- h^i_α extends to an automorphism $m^i_\alpha \in \mathrm{Aut}(M')$ for each $i \in I_0$.

If $\left(m^i_\alpha \circ g_{i,\delta(i)} : i \in I_0\right)$ and $(k_i : i \in I_0)$ are not compatible over any $\prec_{\mathcal{K}}$-submodel of N_0 (i.e. there is no model and no $\prec_{\mathcal{K}}$-embedding to that model such that we can amalgamate the automorphisms over any $\prec_{\mathcal{K}}$-submodel of N_0), then define $M^i_{\alpha+1} := M'$ and $h^i_{\alpha+1} := m^i_\alpha$ for all $i \in I_0$. Define $M^i_{\alpha+1} = M^i_\alpha$ and $h^i_{\alpha+1} = h^i_\alpha$ for $i \notin I_0$.

Suppose, without loss of generality, $\left(m^i_\alpha \circ g_{i,\delta(i)} : i \in I_0\right)$ and $(k_i : i \in I_0)$ are compatible over N_0. Since $\mathcal{K}^{<}(M)$ is a strong amalgamation class we can find $M^i_{\alpha+1} \succ_{\mathcal{K}} M'$ and $h^i_{\alpha+1} \in \mathrm{Aut}\left(M^i_{\alpha+1}\right)$ such that there exists an $\prec_{\mathcal{K}}$-elementary map f from N_1 into $M^i_{\alpha+1}$ which extends the identity on N_0 in such a way that $h^i_{\alpha+1} \circ g_{i,\delta(i)} \restriction M^i_{\alpha+1}$ extends $f \circ k_i \circ f^{-1}$. If $i \notin I_0$, then define $M^i_{\alpha+1} = M^i_\alpha$ and $h^i_{\alpha+1} = h^i_\alpha$. □

Now for the case where $|\mathrm{Aut}(M)| \geqslant \kappa$ we choose $\left(g_{i,j} : i \in I, j \in J\right)$ to be a $\kappa \times \kappa$-matrix of distinct automorphisms of M. Then we get 2^κ-many generic sequences of automorphisms of length κ.

4 Proof of SIP for strong amalgamation classes

In this section we prove our main theorem.

Theorem 4.1 *Let M be a homogeneous model in an abstract elementary class $(K, \preccurlyeq_{\mathcal{K}})$ such that $|M| = \kappa > \mathrm{LS}(K)$ and $\kappa^{<\kappa} = \kappa$. Furthermore, assume that $\mathcal{K}^{<}(M)$ is a strong amalgamation class. Consider the group $\mathrm{Aut}(M)$ with the topology given by $\mathfrak{T}^{cl}$, and let $H \leqslant \mathrm{Aut}(M)$ be such that $[\mathrm{Aut}(M) : H] \leqslant \kappa$. Then, H is an open subgroup of $\mathrm{Aut}(M)$; i.e., there exists $A \in \mathcal{K}^{<}(M)$ such that $\mathrm{Aut}_A(M) \leqslant H$.*

In other words, $\left(\mathrm{Aut}(M), \mathfrak{T}^{cl}\right)$ has the small index property.

We need only deal with the case $\mathrm{Aut}(M)$ is *rich*[3], that is $|\mathrm{Aut}(M)| = 2^\kappa$ (otherwise, the identity would be isolated and therefore the automorphism group would automatically satisfy the theorem, as in that case all subgroups would be open).

Suppose H is a subgroup of $\mathrm{Aut}(M)$ with small index (i.e. not bigger that κ). Toward a contradiction suppose H is not open.

[3] *With an argument similar to the first order countable case, one can show that if the identity is not isolated, then* $\mathrm{Aut}(M)$ *is a perfect complete topological space.*

Proposition 4.2 (Similar to Proposition 10 in [7]) *There exists a generic sequence* $\gamma = (g_i : i \in I)$ *such that*

(1) *the set* $\{i \in I : g_i \restriction M_0 = h$ *and* $g_i \notin H\}$ *has cardinality* κ *for all* $M_0 \in \mathcal{K}^{<}(M)$ *and* $h \in \mathrm{Aut}(M_0)$*;*
(2) *the set* $\{i \in I : g_i \in H\}$ *has cardinality* κ.

Proof Consider the following set $\mathcal{X} = \{(M_0, f) : M_0 \in \mathcal{K}^{<}(M)$ and $f \in \mathrm{Aut}(M_0)\}$. It is clear that $|\mathcal{X}| = \kappa$.

- Consider I_0 of cardinality κ and a sequence $((M_i, f_i) : i \in I_0)$ of elements of $\mathcal{X}$ such that the set $\{i \in I_0 : (M_i, f_i) = (M_0, f)\}$ has cardinality κ for all $(M_0, f) \in \mathcal{X}$.
- Let I_1 be a set of cardinality κ disjoint from I_0 and let $I = I_0 \cup I_1$.
- Finally, let J be any set of cardinality κ.

For each $i \in I$ and $j \in J$ define $g_{i,j} \in \mathrm{Aut}(M)$ such that the following hold:

(1) $g_{i,j} \restriction M_i = f_i$ for all $i \in I_0$. Moreover, the set $\{g_{i,j} : j \in J\}$ where $i \in I_0$ meets at least two classes modulo H. This is always possible since we assumed H is not open (and hence none of its classes contain a non-empty open set).
(2) The set $\{g_{i,j} \in J\}$ meets all classes modulo H if $i \in I_1$. This is possible because the index of H in $\mathrm{Aut}(M)$ is small (i.e. not bigger that κ).

Now let $(M_i : i \in I)$ a sequence of elements of $\mathcal{K}^{<}(M)$ where M_i is arbitrary for $i \in I_1$. By Lemma 3.9 there is a family $(h_i : i \in I)$ such that satisfying conditions (1) and (2) of this lemma. Then choose a bijective function $\delta : I \to J$ such that:

- $g_{i,\delta(i)}$ is not in the class of h_i^{-1} for $i \in I_0$ (i.e. $g_{i,\delta(i)} \notin h_i^{-1}H$ guaranteed by Condition (1) above);
- $g_{i,\delta(i)}$ is in the class of h_i^{-1} for $i \in I_1$ [i.e. $g_{i,\delta(i)} \in h_i^{-1}H$ guaranteed by Condition (2) above].

Then the sequence $\gamma := \left(h_i \circ g_{i,\delta(i)} : i \in I\right)$ is a generic sequence and satisfies the requirement. □

4.1 Construction of the tree

Let H be as before: a subgroup of small index and we assume, toward a contradiction, that H is not open. Let $\mathcal{S} = 2^{<\kappa}$ be the set of sequences of 0 and 1 of length less than κ, and $\mathcal{S}^* = \{s \in \mathcal{S} :$ the length of s is successor$\}$. Let γ be the generic sequence that is obtained from Proposition 4.2. Fix an enumeration $(a_\alpha : \alpha < \kappa)$ of elements of M.

We construct by induction on $s \in \mathcal{S}$, a model $M_s \in \mathcal{K}^{<}(M)$, an automorphism $g_s \in \mathrm{Aut}(M_s)$, and if $s \in \mathcal{S}^*$, automorphisms h_s and k_s in $\mathrm{Aut}_{\{M_s\}}(M)$ in such a way that the following conditions are satisfied:

(1) $h_{s,0} \in H$ and $h_{s,1} \notin H$ for all $s \in \mathcal{S}^*$;
(2) $k_{s,0} = k_{s,1}$ for all $s \in \mathcal{S}^*$;
(3) $h_t[M_s] = M_s$ (i.e. $h_t \in \mathrm{Aut}_{\{M_s\}}(M)$) for $s \in \mathcal{S}$ and all $t \in \mathcal{S}^*$ with $t \leqslant s$;
(4) $g_s \circ (h_t \restriction M_s) \circ g_s^{-1} = k_t \restriction M_s$ for $s \in \mathcal{S}$ and all $t \in \mathcal{S}^*$ with $t \leqslant s$;

(5) $a_\beta \in M_s$ for $s \in \mathcal{S}$ and $\beta < \text{length}(s)$;
(6) $(h_t : t \leqslant s, t \in \mathcal{S}^*)$ and $(k_t : t \leqslant s, t \in \mathcal{S}^*)$ are sequences elements from γ, for $s \in \mathcal{S}$ (and they are generic as well).

For $s = \emptyset$, define M_s to be an arbitrary element of $\mathcal{K}^<(M)$ and $g_s \in \text{id}(M_s)$. For limit step there is no problem. Suppose everything has been defined up to step s. Write $\mathcal{F}$ for the set of all the automorphisms of the generic sequence γ. First choose $h_{s,0} \in \mathcal{F} \cap H$ not in $\{h_t : t \leqslant s, t \in \mathcal{S}^*\}$. Extend g_s to $g \in \text{Aut}(M)$ such that $g \circ h_t \circ g^{-1} = k_t$ for all $t \in \mathcal{S}^*$ with $t \leqslant s$ in the following manner: First extend g_s to some $g' \in \text{Aut}(M)$. The two families $\{g' \circ h_t \circ g'^{-1} : t \leqslant s, t \in \mathcal{S}^*\}$ and $\{k_t : t \leqslant s, t \in \mathcal{S}^*\}$ are generic and they agree on M_s. Hence, we can find $g'' \in \text{Aut}_{M_s}(M)$ such that $k_t = g'' \circ g' \circ h_t \circ g'^{-1} \circ g''^{-1}$ for all $t \in \mathcal{S}^*$ with $t \leqslant s$. Then let $g = g'' \circ g'$.

Using Lemma Corollary 2.12 we can find $M_{s,0} \in \mathcal{K}^<(M)$ in such a way that:

(1) $M_{s,0}$ contains M_s and a_α, where $\alpha = \text{length}(s)$;
(2) $h_t[M_{s,0}] = M_{s,0}$ for all $t \leqslant (s,0)$ and $g[M_{s,0}] = M_{s,0}$.

Set $M_{s,1} = M_{s,0}$, $g_{s,0} = g_{s,1} = g \restriction M_{s,0}$ and $h_{s,1}$ an element of $\mathcal{F}$ extending $h_{s,0} \restriction M_{s,0}$, not in H and not in $\{h_t : t \leqslant s, t \in \mathcal{S}^*\}$, and $k_{s,0} = k_{s,1}$ an element of $\mathcal{F}$ extending $g_{s,0} \circ (h_{s,0} \restriction M_{s,0}) \circ g_{s,0}^{-1}$ not in $\{k_t : t \leqslant s, t \in \mathcal{S}^*\}$.

For each $\sigma \in 2^\kappa$, let $g_\sigma = \bigcup_{\delta<\sigma} g_\delta$. Then $g_\sigma \in \text{Aut}(M)$ and moreover for all $t < \sigma$ and $t \in \mathcal{S}^*$, $g_\sigma \circ h_t \circ g_\sigma^{-1} = k_t$. Assume τ and σ are two distinct elements of 2^κ; let s be their largest common initial segment and assume, with out loss of generality, that $(s,0) < \sigma$ and $(s,1) < \tau$. Then $g_\sigma \circ h_{s,0} \circ g_\sigma^{-1} = k_{s,0} = k_{s,1} = g_\tau \circ h_{s,1} \circ g_\tau^{-1}$; thus $h_{s,0} = g_\sigma^{-1} \circ g_\tau \circ h_{s,1} \circ g_\tau^{-1} \circ g_\sigma$. Since $h_{s,0} \in H$ and $h_{s,1} \notin H$, $g_\sigma^{-1} \circ g_\tau \notin H$ hence the index of H in $\text{Aut}(M)$ is 2^κ; a contradiction.

5 The SIP for the Zilber field, covers and modular invariants

This section consists of applications of Theorem 4.1. We first provide a general setting (*quasiminimal pregeometry classes*) that both satisfies enough conditions for the theorem to apply and includes several interesting cases.

5.1 The SIP and quasiminimal pregeometry classes

In order to construct an algebraically closed field with the *Schanuel property* categorical in all uncountable cardinals, Zilber introduced the notion of "quasiminimal excellent" classes (see [13])—a combination of Shelah's excellent classes with a weaker variant of strong minimality. Later, the concept was studied in isolation and two major simplifications were brought about: first of all, a notion of quasiminimal classes is enough to imply excellence (see [2]) and a similar notion of quasiminimal classes directly implies categoricity (see [6]). In this chapter we prove, as an application of our Theorem 4.1 that the SIP for homogeneous models holds in the quasiminimal classes setting. Then, we conclude that the automorphism group of Zilber field has the small index property.

We use the setting from [6]. For the sake of completeness (and clarification) we restate the axioms here:

Definition 5.1 Let $\mathfrak{L}$ be a language. A *quasiminimal pregeometry class* $\mathfrak{Q}$ is a class of pairs $\langle H, \mathrm{cl}_H\rangle$ where H is an $\mathfrak{L}$-structure and cl_H is a pregeometry operator on H such that the following conditions hold:

(1) *Closure under isomorphisms* If $\langle H, \mathrm{cl}_H\rangle \in \mathfrak{Q}$ and H' is an $\mathfrak{L}$-structure and $f : H \to H'$ is an isomorphism then $\langle H', \mathrm{cl}_{H'}\rangle \in \mathfrak{Q}$ where $\mathrm{cl}_{H'} := f\left(\mathrm{cl}_H\left(f^{-1}(X')\right)\right)$ for $X' \subseteq H'$.
(2) *Quantifier free theory* The empty function is a partial embedding between any two structures of the class $\mathfrak{Q}$.
(3) *Countable closure* For each $\langle H, \mathrm{cl}_H\rangle \in \mathfrak{Q}$, the closure of any finite set is countable.
(4) *Relativization* If $\langle H, \mathrm{cl}_H\rangle \in \mathfrak{Q}$ and $X \subseteq H$, then $\langle \mathrm{cl}_H(X), \mathrm{cl}_H \restriction \mathrm{cl}_H(X)\rangle \in \mathfrak{Q}$.
(5) *Closure coherence* If $\langle H, \mathrm{cl}_H\rangle, \langle H', \mathrm{cl}_{H'}\rangle \in \mathfrak{Q}$, $X \subseteq H$, $y \in H$ and $f : H \to H'$ is a partial embedding defined on $X \cup \{y\}$, then $y \in \mathrm{cl}_H(X)$ if and only if $f(y) \in \mathrm{cl}_{H'}(f(X))$.
(6) *Homogeneity over countable models/uniqueness of generic types* Let $\langle H, \mathrm{cl}_H\rangle$, $\langle H', \mathrm{cl}_{H'}\rangle \in \mathfrak{Q}$ and suppose $C \subseteq H, C' \subseteq H'$ are countable closed sets and $g : C \to C'$ is an isomorphism:
   (a) If $x \in H \backslash C$ and $x' \in H' \backslash C'$, then $g \cup \{(x, x')\}$ is a partial embedding.
   (b) If $g \cup f : H \rightharpoonup H'$ is a partial embedding such that $X = \mathrm{dom}(f)$ is finite and $y \in \mathrm{cl}_H(X \cup C)$, then there is $y' \in H'$ such that $g \cup f \cup \{(y, y')\}$ is a partial embedding.

Given a structure $H \in \mathfrak{Q}$ and a substructure $G \subseteq H$ also in $\mathfrak{Q}$, denote $G \preccurlyeq H$ when G is closed in H.

Proposition 5.2 (Proposition 4 in [6]) *Let* $H, H' \in \mathfrak{Q}$ *such that* $H \preccurlyeq H'$ *and* $X \subseteq H$. *Then* $cl_H(X) = cl_{H'}(X)$.

Lemma 5.3 *Fix* $M \in \mathfrak{Q}$ *where* $\mathfrak{Q}$ *is a quasiminimal pregeometry class. Then* $cl_M(A) = cl^M(A)$ *(the intersection of all closed sets containing* A*) for all* $A \subseteq M$.

Proof Let $A \subseteq M$ then $\mathrm{cl}_M(A) \in \mathfrak{Q}$ (by point 4 of the definition of $\mathfrak{Q}$). Therefore $\mathrm{cl}^M(A) \subseteq \mathrm{cl}_M(A)$. The other direction is also simple: suppose $a \in \mathrm{cl}_M(A) \backslash \mathrm{cl}^M(A)$, then there is $H \in \mathfrak{Q}$ such that $A \subseteq \mathrm{cl}^M(A) \subseteq H$, $H \preccurlyeq M$ and $a \notin H$. Note that $\mathrm{cl}_H(A) \subseteq H$ does not contain a; but on the other hand from Proposition 5.2 it follows that $\mathrm{cl}_H(A) = \mathrm{cl}_M(A)$ and $a \in \mathrm{cl}_M(A) = \mathrm{cl}_H(A)$ which is a contradiction. □

Recall that in [6], Haykazyan derives uncountable categoricity from the quasiminimal pregeometry axioms. It is worth mentioning that in November 2016, during corrections to an earlier version of this paper, Sebastien Vasey has posted an article in ArXiv [10] where he proves that in AECs admitting intersections (i.e. with a notion of closure defined like our cl^M), the exchange axiom follows from the other axioms of a quasiminimal AEC.

Theorem 5.4 *Suppose* $\mathfrak{Q}$ *is a quasiminimal pregeometry class. Let* $M \in \mathfrak{Q}$ *be the model of size* $\aleph_1$ *and let* $\mathcal{C} := \{cl_M(A) : A \subseteq M, |A| < |M|\}$. *Then the class* $\mathcal{C}$ *is a strong amalgamation class and hence,* $\mathrm{Aut}(M)$ *has SIP.*

Proof Let $N_0 \in \mathcal{C}$ and $\gamma^0 \in \mathrm{Aut}(N_0)$. We want to show (N_0, γ^0) is a strong amalgamation basis. Suppose $N_1, N_2 \in \mathcal{K}^{<}(M)$ are such that $N_0 \preccurlyeq N_1, N_2$; and let γ^1 and γ^2 be sequences of automorphisms of N_1 and N_2 extending γ^0; respectively. We claim one can find $N_2' \in \mathcal{C}$ such that $N_2' \cong_{N_0} N_2$ and $N_2' \cap N_1 = N_0$. Let B be a basis for N_2 (i.e. $\mathrm{cl}_M(B) = N_2$). Choose B' an independent set in M such that there is a bijection between B' and B, and moreover $B' \cap N_1 = B_0 = B \cap N_2$ where $\mathrm{cl}_M(B_0) = N_0$. Now let $N_2' = \mathrm{cl}_M(B')$. Note that we can choose B' in such a way that $N_2' \cap N_1 = N_0$. By Theorem 16 in [6] there is an isomorphism α between N_2 and N_2' and therefore $N_2' \cong_{N_0} N_2$. Now $\gamma^1 \cup (\alpha \circ \gamma^2 \circ \alpha^{-1})$ is an isomorphism of $N_1 \cup \alpha[N_2]$ to itself. Then by Proposition 14 in [6] extends $\gamma^1 \cup (\alpha \circ \gamma^2 \circ \alpha^{-1})$ extends to an automorphism of $N_3 := \mathrm{cl}_M(N_1 \cup \alpha[N_2])$ and then to an automorphism of M. Therefore it follows that $\gamma^1 \cup (\alpha \circ \gamma^2 \circ \alpha^{-1})$ extends to an automorphism of N_3 where $N_3 \in \mathcal{C}$ and $N_1, \alpha[N_2] \preccurlyeq N_3$. Hence, (N_0, γ^0) is an amalgamation base. □

Corollary 5.5 *Let* B *be the Zilber field. Then* $\mathrm{Aut}(B)$ *has SIP for the topology where basic open sets around the identity are given by stabilizers of sets of size less than continuum.*

It is an interesting question to check whether by investigating the automorphism group of the Zilber field (which has the Schanuel property) we would be able to detect any difference between the Zilber field and $(\mathbb{C}, +, \cdot, \exp)$. From [7] it follows that the complex numbers (without $\exp$) have the SIP. The construction of the Zilber field, using the quasiminimal excellence setting, allowed us to prove SIP for its automorphism group. Then the question of whether or not we are able to detect similar properties about the automorphism group of $(\mathbb{C}, +, \cdot, \exp)$ seems plausible.

5.2 More $L_{\omega_1,\omega}$-sentences and the SIP (other directions)

Now that we have established the SIP for classes arising from quasiminimal pregeometry classes (really, for their model of size $\aleph_1$) and we have shown that being a quasiminimal pregeometry class is enough to obtain that the class of small elementary submodels satisfies strong amalgamation, a natural question is for which other classes axiomatizable by $L_{\omega_1,\omega}$-sentences can we guarantee the SIP (at uncountable homogeneous models!). A natural class would be *excellent* $L_{\omega_1,\omega}$-sentences that are not necessarily quasiminimal. The study of the exact conditions that would guarantee that homogeneous models of those classes have strong amalgamation are, however, the subject of possible future work.

5.3 Covers

The fact that all quasiminimal pregeometry classes satisfy our framework for SIP (Theorem 5.4) provides a further source of examples. In Sect. 5.1, we analyzed the case of the "Zilber field" as a quasiminimal pregeometry class. This is indeed one instance of a much more general phenomenon: some covers and some modular invariants (like the classical j-function) have been shown to be quasiminimal pregeometry classes and

therefore their homogeneous models M have automorphism groups $\mathrm{Aut}(M)$ that are SIP, by our results.

Variants of this example have been studied extensively by Zilber and others (Baldwin, Bays, Caycedo, Kirby, Sustretov, etc.). The general setup arises from a short exact sequence

$$0 \to K \xrightarrow{i} G \xrightarrow{f} H \to 1$$

where G, H are groups (typically G is an additive abelian group and H is a multiplicative group) and the map $f : G \to H$ is a homomorphism with kernel K. In particular, for this situation, Bays and Zilber provide in [3] conditions for the quasiminimality of the structure. Any structure axiomatized in $L_{\omega_1,\omega}$ that satisfies the Bays-Zilber conditions for quasiminimality will have the SIP.

The "Zilber field" of our previous section has the added complexity that we are dealing with the field structure and not just with a homomorphism from the additive group structure into the multiplicative group part. A simplified structure often studied (see [11]) in connection with complex exponentiation is sometimes presented as a cover

$$0 \to 2\pi i\mathbb{Z} \xrightarrow{i} (\mathbb{C}, +) \xrightarrow{\exp} \left(\mathbb{C}^\times, \cdot\right) \to 1.$$

5.4 j-Mappings

Harris [8] defines in an $L_{\omega_1,\omega}$-axiomatization of the classical "modular invariant", also called the "j-mapping", an analytic function from the upper half plane $\mathbb{H}$ into the complex numbers,

$$j : \mathbb{H} \to \mathbb{C}.$$

The j-mapping is a crucial component of analytic number theory. It provides surprising connections between properties of extensions of number fields and analytic properties of the mapping—the solution to Hilbert's Twelfth problem for the characteristic zero and complex case hinges on this.

Harris's model-theoretic analysis of j, developed in [5], produces a quasiminimal pregeometry and therefore is a new example of a class with the SIP. We describe briefly the backbone of Harris's analysis, and the connection to quasiminimality and the SIP.

Let L be a language for two-sorted structures of the form

$$\mathfrak{A} = \langle \langle H, \{g_i\}_{i\in\mathbb{N}} \rangle, \langle F, +, \cdot, 0, 1 \rangle, j : H \to F \rangle$$

where $\langle F, +, \cdot, 0, 1\rangle$ is an algebraically closed field of characteristic 0, $\langle H, \{g_i\}_{i\in\mathbb{N}}\rangle$ is a set together with countably many unary function symbols (representing the action of a countable group on the upper half plane), and $j : H \to F$. Really, j is a **cover** from the *action* structure into the field $\mathbb{C}$.

Let then

$$Th_{\omega_1,\omega}(j) := Th(\mathbb{C}_j) \cup \forall x \forall y (j(x) = j(y) \rightarrow \bigvee_{i<\omega} x = g_i(y))$$

where $\mathbb{C}_j$ is the "standard j-model" $(\mathbb{H}, \langle \mathbb{C}, +, \cdot, 0, 1\rangle, j : \mathbb{H} \rightarrow \mathbb{C})$ and $Th(\mathbb{C}_j)$ is its first order theory.

This captures all the first order theory of j (not the analyticity!) plus the fact that fibers are "standard" ("fibers are orbits").

Harris analyzes this structure in order to establish (modulo serious algebraic geometric results[4]) the categoricity of $Th_{\omega_1,\omega}(j) + \text{trdeg}(F) \geqslant \aleph_0$ in all infinite cardinalities. A crucial step in this work consists in proving that the notion of closure given as

$$cl_j(A) := j^{-1}(acl(j(A)))$$

indeed defines a quasiminimal pregeometry structure.

Now, our main theorem implies (modulo Harris's quite long analysis) that the "automorphism group of j", that is, the set of map-pairs φ_H, φ_F such that

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\varphi_{\mathcal{H}}} & \mathcal{H}_2 \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ \mathcal{F}_1 & \xrightarrow[\varphi_{\mathcal{F}}]{} & \mathcal{F}_2 \end{array}$$

commutes, has the SIP (for the topology where basic open sets around the identity are stabilizers of sets of size less than continuum).

References

1. Baldwin, J.T., Shelah, S.: Examples of non-locality. J. Symb. Log. **73**(3), 765–782 (2008)
2. Bays, M., Hart, B., Hyttinen, T., Kesala, M., Kirby, J.: Quasiminimal structures and excellence. Bull. Lond. Math. Soc. **46**(1), 155–163 (2013)
3. Bays, M., Zilber, B.: Covers of multiplicative groups of algebraically closed fields of arbitrary characteristic. Bull. Lond. Math. Soc. **4**(43), 689–702 (2011)
4. Cameron, P.J.: Oligomorphic Permutation Groups. London Mathematical Society Lecture Note Series, vol. 152. Cambridge University Press, Cambridge (1990)
5. Harris, A.: Categoricity of the two sorted j-function. *ArXiv* (2013)
6. Haykazyan, L.: Categoricity in quasiminimal pregeometry classes. *ArXiv* (2014)

[4] In his proof of categoricity, Harris uses an instance of the so called "adelic Mumford-Tate conjecture" (proved by Serre) for products of elliptic curves. The strategy to build an isomorphism between two models M and M' partially consists in proving that the closure given on the "H sort" by $cl_j(A) = j^{-1}(acl(j(A)))$ provides a quasiminimal pregeometry structure. Proving this is not trivial at all: one needs to control the quantifier-free types of "elliptic curves" as represented by the parameter in H in terms of their interactions with other such parameters.

7. Lascar, D., Shelah, S.: Uncountable saturated structures have the small index property. Bull. Lond. Math. Soc. **25**(2), 125–131 (1993)
8. Pillay, A.: Geometric Stability Theory. Oxford Logic Guides, vol. 32. The Clarendon Press Oxford University Press, New York (1996). (Oxford Science Publications)
9. Shelah, S.: Classification of nonelementary classes. II. Abstract elementary classes. In: Baldwin, J.T. (ed.) Classification theory (Chicago, IL, 1985). Lecture Notes in Mathematics, vol. 1292, pp. 419–497. Springer, Berlin (1987)
10. Vasey, S.: Quasiminimal abstract elementary classes. Arch. Math. Logic (2017). doi:10.1007/s00153-017-0570-7
11. Zilber, B.: Model theory, geometry and arithmetic of the universal cover of a semi-abelian variety. In: Bélair, L. et al. (eds.) Model Theory and Applications. Quaterna di Matematica, vol. 11, pp. 427–458. Aracne, Rome (2002)
12. Zilber, B.: A categoricity theorem for quasi-minimal excellent classes. In: Blass, A., Zhang, Y. (eds.) Logic and its Applications. Contemporary Mathematics, vol. 380, pp. 297–306. American Mathematical Society, Providence, RI (2005)
13. Zilber, B.: Pseudo-exponentiation on algebraically closed fields of characteristic zero. Ann. Pure Appl. Log. **132**(1), 67–95 (2005)