Arch. Math. Logic (2018) 57:195–201
https://doi.org/10.1007/s00153-017-0563-6

Mathematical Logic

Some remarks on Baire's grand theorem

Riccardo Camerlo[1] · **Jacques Duparc**[2]

Received: 8 November 2016 / Accepted: 22 May 2017 / Published online: 1 June 2017

Abstract We provide a game theoretical proof of the fact that if f is a function from a zero-dimensional Polish space to $\mathbb{N}^{\mathbb{N}}$ that has a point of continuity when restricted to any non-empty compact subset, then f is of Baire class 1. We use this property of the restrictions to compact sets to give a generalisation of Baire's grand theorem for functions of any Baire class.

Keywords Baire class ξ function · Wadge game · Eraser game · Polish zero · Dimensional space · Compact set

Mathematics Subject Classification 03E15

1 Introduction and generalities

The first formulation of Baire's grand theorem appeared in 1904, in the written version of the Cours Peccot, taught by Baire in 1903-04. The original formulation stated that, given a function $f : \mathbb{R}^n \to \mathbb{R}$, then f is a pointwise limit of continuous functions if and only if for every non-empty closed set $F \subseteq \mathbb{R}^n$ the restriction $f|_F$ has at least one point of continuity.

✉ Riccardo Camerlo
riccardo.camerlo@polito.it

Jacques Duparc
Jacques.Duparc@unil.ch

1 Department of Mathematical Sciences Joseph-Louis Lagrange, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

2 Département des systèmes d'information, Université de Lausanne, Quartier UNIL-Dorigny, Bâtiment Internef, Lausanne 1015, Switzerland

The notions involved in this statement are related to the *Baire hierarchy* of functions (see also [3, §24]). Here and in the sequel we follow the standard descriptive set theoretic notations; in particular, we use $\boldsymbol{\Sigma}^0_\xi, \boldsymbol{\Pi}^0_\xi$ to denote the levels of the Borel hierarchy, and $\boldsymbol{\Delta}^0_\xi$ for the ambiguous classes.

Definition 1 Given topological spaces X, Y and a function $f : X \to Y$, say that f is of *Baire class* 1 if $\forall V \in \boldsymbol{\Sigma}^0_1(Y)\ f^{-1}(V) \in \boldsymbol{\Sigma}^0_2(X)$. Inductively, given an ordinal $\xi \geq 2$, say that f is of Baire class ξ if f is the pointwise limit of functions f_n, where f_n is of Baire class $\xi_n < \xi$.

The connection is clarified by the following facts.

Theorem 1 1. *Let X, Y be separable metrisable spaces and $f : X \to Y$. Suppose that either X is zero-dimensional or that Y is homeomorphic to some $\mathbb{R}^m$ ($m > 0$). Then f is of Baire class* 1 *if and only if f is the pointwise limit of a sequence of continuous functions.*
2. *If X, Y are metrisable spaces and Y is separable, then for $1 \leq \xi < \omega_1$ the function $f : X \to Y$ is Baire class ξ if and only if $\forall V \in \boldsymbol{\Sigma}^0_1(Y)\ f^{-1}(V) \in \boldsymbol{\Sigma}^0_{\xi+1}(X)$.*

Nowadays, Baire's grand theorem is usually stated with more generality than the original formulation, as follows (see for example [3, Theorem 24.15]).

Theorem 2 *Let X be a Polish space, Y be a separable metrisable space, and let $f : X \to Y$. Then the following are equivalent:*

1. *f is of Baire class* 1
2. *for every non-empty closed $F \subseteq X$, the restriction $f|_F$ has a point of continuity*
3. *for every non-empty compact $K \subseteq X$, the restriction $f|_K$ has a point of continuity*

The purpose of this note is first to point out a game theoretic argument to prove the implication (3) $\Rightarrow$ (1) for functions $f : X \to \mathbb{N}^\mathbb{N}$, where X is a zero-dimensional Polish space; secondly, to show how condition (3) can be used to generalise Baire's grand theorem to higher Baire classes.

2 Games for continuous and Baire class 1 functions

Given a function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, we shall consider two games associated with f. The first one is the classical *Wadge game* $G_W(f)$: players *I* and *II* alternate their rounds, playing elements of $\mathbb{N}$; Player *II* can skip her turn. If x, y are the sequences of moves played by *I*, *II*, respectively, let *II* win this run of the game if and only if $f(x) = y$ (in particular, $y \in \mathbb{N}^\mathbb{N}$). Then f is continuous if and only if player *II* has a winning strategy in $G_W(f)$.

To deal with Baire class 1 functions, we shall use the so-called *eraser game* introduced in [2] (for a different presentation, see [1] and the references contained there). We recall here the details, since we need them for our proof.

If $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, let the game $G_{\mathcal{B}_1}(f)$ be defined as follows. Players *I* and *II* alternate their rounds. Player *I* plays elements of $\mathbb{N}$; Player *II* can skip her turn, or play elements from $\mathbb{N} \cup \{\leftarrow\}$, where $\leftarrow$ is a new symbol, called an *eraser*. When player *II* plays $\leftarrow$, we say that she is *erasing*. Indeed, given $z \in (\mathbb{N} \cup \{\leftarrow\})^{\mathbb{N}}$, let $z^{\leftarrow} = \lim(z|_n)^{\leftarrow} = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid \exists m \forall n > m\ (i, j) \in (z|_n)^{\leftarrow}\}$, where $\emptyset^{\leftarrow} = \emptyset$ and, for any s such that $s^{\leftarrow} \in \mathbb{N}^k$ and $a \in \mathbb{N}$,

$$
\begin{aligned}
(sa)^{\leftarrow} &= s^{\leftarrow} a \\
(s \leftarrow)^{\leftarrow} &= \begin{cases} s^{\leftarrow}|_{k-1} & \text{if} \quad k > 0 \\ \emptyset & \text{if} \quad k = 0 \end{cases}
\end{aligned}
$$

If x, y are the sequences played by *I*, *II*, respectively, let *II* win this run of the game if and only if $f(x) = y^{\leftarrow}$ (in particular, $y^{\leftarrow} \in \mathbb{N}^{\mathbb{N}}$).

Lemma 1 *If f is of Baire class* 1*, then Player II has a winning strategy in $G_{\mathcal{B}_1}(f)$.*

Proof Suppose f is of Baire class 1, so that there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ of continuous functions that converges pointwise to f. Let σ_n be a winning strategy for *II* in $G_W(f_n)$ – the Wadge game for f_n.

Define a strategy σ for *II* in $G_{\mathcal{B}_1}(f)$ as follows:

- σ coincides with σ_0 until, and including, when σ_0 plays some $y_0^0 \in \mathbb{N}$
- then σ passes until σ_1 has produced two replies $y_0^1, y_1^1 \in \mathbb{N}$; if $y_0^0 = y_0^1$ then player *II* plays y_1^1, otherwise she plays consecutively $\leftarrow, y_0^1, y_1^1$
- in general, suppose σ has produced a sequence s such that $s^{\leftarrow} = (b_0, \ldots, b_n)$ using σ_n. Then σ passes until σ_{n+1} has played a sequence of integers $s' = (y_0^{n+1}, \ldots, y_{n+1}^{n+1})$ of length $n + 2$; letting $k \leq n + 1$ be the length of $s^{\leftarrow} \cap s'$, strategy σ makes Player *II* play consecutively

$$
\underbrace{\leftarrow, \ldots, \leftarrow}_{n+1-k}, y_k^{n+1}, \ldots, y_{n+1}^{n+1}
$$

Given any $x \in \mathbb{N}^{\mathbb{N}}$, let $y \in (\mathbb{N} \cup \{\leftarrow\})^{\mathbb{N}}$ be the element produced by σ in response to x, and let $y^{\leftarrow} = (y_0, y_1, \ldots)$. Since $(f_n(x))_{n \in \mathbb{N}}$ converges pointwise to $f(x)$, given $N \in \mathbb{N}$ for large enough n all y_N^n will coincide with $f(x)(N)$. Consequently, the play y_N^n according to σ will eventually remain unchanged, that is, it will not be erased by the effect of subsequent plays of the symbol $\leftarrow$. So σ is winning. □

Lemma 2 *If f is not of Baire class* 1*, then Player I has a winning strategy in $G_{\mathcal{B}_1}(f)$.*

Proof Given $s \in \mathbb{N}^{<\omega}$, let $N_s = \{x \in \mathbb{N}^{\mathbb{N}} \mid s \subseteq x\}$. The sets N_s form a basis of the Baire space.

By the direction (2) $\Rightarrow$ (1) of Baire's grand theorem, let C be a non-empty closed subset of $\mathbb{N}^{\mathbb{N}}$ such that $f|_C$ has no continuity point. So for every $x \in C$ there is a least $o(x) \in \mathbb{N}$ such that in each neighbourhood, relative to C, of x there is a point x' with $f(x)(o(x)) \neq f(x')(o(x))$. By the minimality of $o(x)$, there exists a least $p(x) \in \mathbb{N}$ such that $f(N_{x|_{p(x)}} \cap C) \subseteq N_{f(x)|_{o(x)}}$.

Define a strategy τ for *I* in $G_{\mathcal{B}_1}(f)$ as follows. Pick any $x_0 \in C$. The strategy τ begins by enumerating $x_0|_{p(x_0)}$; after that, it continues the enumeration of x_0 until Player *II* produces, if ever, a position $b_0 \in (\mathbb{N} \cup \{\leftarrow\})^{<\omega}$ such that length$(b_0^{\leftarrow}) \geq o(x_0) + 1$ and such that $f(x_0)|_{o(x_0)+1} = b_0^{\leftarrow}|_{o(x_0)+1}$. In general, assume that τ has enumerated an initial segment s of x_n of length at least $p(x_n)$, and Player *II* is producing a position $b_n \in (\mathbb{N} \cup \{\leftarrow\})^{<\omega}$ such that length$(b_n^{\leftarrow}) \geq o(x_n)+1$ and such that $f(x_n)|_{o(x_n)+1} = b_n^{\leftarrow}|_{o(x_n)+1}$. Pick an element $x_{n+1} \in N_s \cap C$ such that $f(x_{n+1})(o(x_n)) \neq f(x_n)(o(x_n))$ (we apply dependent choices, here). Then τ continues by enumerating x_{n+1}. Note that $o(x_n) \leq o(x_{n+1})$ and $p(x_n) \leq p(x_{n+1})$ both hold.

Let $x \in \mathbb{N}^{\mathbb{N}}$, $y \in (\mathbb{N} \cup \{\leftarrow\})^{\leq\omega}$ be the elements played by players *I*, *II*, respectively, with *I* following τ. Suppose, towards a contradiction, that $y^{\leftarrow} \in \mathbb{N}^{\mathbb{N}}$ and that $f(x) = y^{\leftarrow}$. Then the sequence x_n is defined for all $n \in \mathbb{N}$, and $x = \lim_{n\to\infty} x_n$. There are two cases to consider.

Assume first $\lim_{n\to\infty} o(x_n) = +\infty$. Since $\forall n \in \mathbb{N}\ f(C \cap N_{x|_{p(x_n)}}) = f(C \cap N_{x_n|_{p(x_n)}}) \subseteq N_{f(x_n)|_{o(x_n)}}$, the infimum of the diameters of the images of the neighbourhoods of x, the so-called *oscillation* of $f|_C$ at x, is null. So x is a point of continuity of $f|_C$, a contradiction.

Otherwise there is some $o \in \mathbb{N}$ such that eventually $o(x_n) = o$. But this means that at infinitely many of *II*'s positions z, the sequence $z^{\leftarrow}$ has length o, so that $y^{\leftarrow} \in \mathbb{N}^{<\omega}$, a contradiction again. □

From Lemmas 1 and 2, one obtains once again the result of [1, Theorem 4.1].

Corollary 1 *For any $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, the game $G_{\mathcal{B}_1}(f)$ is determined.*

For later use, we denote by S the set of all $x \in \mathbb{N}^{\mathbb{N}}$ that in the proof of Lemma 2 are of the form x_n, that is those elements used to define strategy τ by enumerating their initial segments.

3 Extensions of Baire's grand theorem

In this section we give in Theorem 3 the announced proof of Baire's grand theorem for compact sets, and in Theorem 4 the extension to functions of higher Baire classes.

Theorem 3 *Let X be a zero-dimensional Polish space. A function $f : X \to \mathbb{N}^{\mathbb{N}}$ is Baire class 1 if and only if, for any non-empty compact subset $K \subseteq X$, the restriction $f|_K$ has a point of continuity.*

Proof Assume first $X = \mathbb{N}^{\mathbb{N}}$. We present only the portion of the proof that is new, which is the backward direction.

Assume f is not Baire class 1, and let τ be the winning strategy for *I* in $G_{\mathcal{B}_1}(f)$ constructed in the proof of Lemma 2. Let T be the tree of *I*'s positions according to τ. Then, using the notation introduced at the end of section 2, $[T] = \bar{S}$. Indeed, following the argument of the proof of Lemma 2, $\forall s \in T\ \exists x \in S\ s \subseteq x$, showing $[T] \subseteq \bar{S}$. Conversely, given $x \in S$, let $s \in T$ be a position built by τ while enumerating x; if we suppose that, as soon as s is produced, player *II* skips until the end of the game, then τ continues by enumerating all of x, so that $x \in [T]$; consequently $S \subseteq [T]$ holds, and subsequently the claim.

Consider now any $s \in T$. Then Player I played s while enumerating an initial segment of some $x \in S$. Notice that, by induction on the length of s, only a finite subset $S_s \subseteq S$ of such x could lead to s, since whenever a position $t \in T$ is reached while enumerating an initial segment of some y, there are at most two possible next moves for I according to τ: either he continues enumerating y, or he plays the next value of some y' that depends only on y and the position t. The same argument applied to s shows that, for any $x \in S_s$, there are at most two possible next moves for I, depending on whether or not II's reply to s leads to a position z for II such that $f(x)|_{o(x)+1} \subseteq z^{\twoheadleftarrow}$. Consequently, tree T is finitely branching, so $[T]$ is compact.

It remains to prove that $f|_{[T]}$ has no continuity point. So let $x \in [T]$. Assume first that $x \in S$ (so that, in the notation of the proof of Lemma 2, x is of the form x_n) and fix any $N \geq p(x)$. Suppose that, when Player I is enumerating x, Player II passes a sufficient number of times so that I reaches as his position a prefix of x of length some $M \geq N$. Then, let II play in order to reach a position z such that $z^{\twoheadleftarrow} = f(x)|_{o(x)+1}$ and then pass for the rest of the game. Then I will finally produce an element $y \in [T]$ extending $x|_M$ and such that $f(y)(o(x)) \neq f(x)(o(x))$ ($y = x_{n+1}$ in the notation of Lemma 2). By the arbitrary choice of N, this shows that x is a point of discontinuity of $f|_{[T]}$.

If $x \notin S$ then, once again by the argument in the proof of Lemma 2, the initial segments of x are initial segments of the terms of a sequence of elements of S that were denoted x_n, so that $x = \lim_{n\to\infty} x_n$. Recall that $\forall n \in \mathbb{N}\ o(x_n) \leq o(x_{n+1})$. If infinitely many of the $o(x_n)$ coincide, then the sequence of the x_n witnesses that $f|_{[T]}$ is discontinuous at x. Finally, if $\lim_{n\to\infty} o(x_n) = +\infty$, we show how II can win against τ, which contradicts the fact that τ is winning. The element x is the sequence played by I following τ in an actual run of the game $G_{\mathcal{B}_1}(f)$ where, for $n \in \mathbb{N}$, he switched from enumerating x_n to enumerating x_{n+1} when he was in position $s_n \subseteq x_n$, of length length$(s_n) \geq p(x_n)$, and player II was in position q_n (so that, in particular, $q_n^{\twoheadleftarrow}|_{o(x_n)+1} = f(x_n)|_{o(x_n)+1}$). The corresponding run of player II, though guessing longer and longer initial segments of $f(x)$, may not be winning as she could have erased too much, leading to infinitely many of II's positions u_i such that all $u_i^{\twoheadleftarrow}$ have the same length. To fix this, we describe by induction a new, winning run for player II. Let first II reach position $b_0 = q_0$ as above. So I is in position s_0, trying to enumerate x_0, and starting from his next move he will switch to x_1 (that may still have some values in common with x_0). In general, suppose that, following τ, player I is in position s_n, while player II has reached a position b_n such that $b_n^{\twoheadleftarrow} = q_n^{\twoheadleftarrow}$, so that from his next move player I will continue with the enumeration of x_{n+1} (that may still have some values in common with x_n). It is enough to show that player II can play to make I, following strategy τ, reach position s_{n+1}, while II arrives in a position b_{n+1} such that $b_{n+1}^{\twoheadleftarrow} = q_{n+1}^{\twoheadleftarrow}$ and such that for every intermediate position t (that is, $b_n \subseteq t \subseteq b_{n+1}$), length$(t^{\twoheadleftarrow}) \geq o(x_n)$. For this, it is enough for II to play as in the run that led from q_n to q_{n+1} except that every time this run produced a position t such that length$(t^{\twoheadleftarrow}) < o(x_n)$, player II now skips her turn.

In the general case of X a zero-dimensional Polish space, one can assume that X is a closed subspace of $\mathbb{N}^{\mathbb{N}}$. Let $\hat{f} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ extend f and be constant on $\mathbb{N}^{\mathbb{N}} \setminus X$; then $\hat{f}$ is Baire class 1 if and only if f is Baire class 1. So if f is Baire class 1, for any non-empty compact $K \subseteq X$, the function $\hat{f}|_K$ has a point of continuity, and consequently the same holds for $f|_K$. If $f|_K$ has a point of continuity for every non-empty K compact in X, let H be non-empty compact in $\mathbb{N}^{\mathbb{N}}$. If $H \subseteq X$, then $\hat{f}|_H = f|_H$ has a point of continuity; otherwise, any point of $H \setminus X$ is a point of continuity of $\hat{f}|_H$. In any case, $\hat{f}$ is Baire class 1, and the same holds for f. □

We can now point out yet another equivalence condition for Baire class 1 functions.

Corollary 2 *Let X be a zero-dimensional Polish space. A function $f : X \to \mathbb{N}^{\mathbb{N}}$ is Baire class 1 if and only if for any compact subspace $K \subseteq X$ the restriction $f|_K$ is Baire class 1.*

Proof It is enough to prove the backward implication. For this, observe that if every restriction $f|_K$ to a compact subspace K of X is Baire class 1, then each such restriction has a point of continuity. Then apply Theorem 3. □

We conclude the paper with the generalisation of Baire's grand theorem to higher Baire classes.

Theorem 4 *Let $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and let $2 \leq \xi < \omega_1$. Then the following are equivalent:*

1. *f is Baire class ξ*
2. *there is a sequence of sets $A_m \in \bigcup_{1 \leq \rho < \xi} \boldsymbol{\Pi}^0_\rho(\mathbb{N}^{\mathbb{N}})$ such that $f|_C$ has a point of continuity for every non-empty $C \in \boldsymbol{\Pi}^0_1(\mathbb{N}^{\mathbb{N}})$ such that $C \cap A_m$ is clopen in C for all $m \in \mathbb{N}$*
3. *there is a sequence of sets $A_m \in \bigcup_{1 \leq \rho < \xi} \boldsymbol{\Pi}^0_\rho(\mathbb{N}^{\mathbb{N}})$ such that $f|_K$ has a point of continuity for every non-empty compact subset $K \subseteq \mathbb{N}^{\mathbb{N}}$ such that $K \cap A_m$ is clopen in K for all $m \in \mathbb{N}$*

By using the complements of the sets A_m, in statements 2 and 3 one can replace $\boldsymbol{\Pi}^0_\rho$ with $\boldsymbol{\Sigma}^0_\rho$.

Proof (1) ⇒ (2). Assume f is a Baire class ξ function with respect to the usual Baire space topology $\mathcal{T}$, so that the preimage of any $\boldsymbol{\Delta}^0_1$ set is $\boldsymbol{\Delta}^0_{\xi+1}$. For any $s \in \mathbb{N}^{<\omega}$, let $M_s = f^{-1}(N_s) \in \boldsymbol{\Delta}^0_{\xi+1}(\mathcal{T})$. So, by [3, Theorem 22.27], $M_s = D_\theta(C_{s\alpha})$ is a θ difference of the $\boldsymbol{\Sigma}^0_\xi$ sets $C_{s\alpha}$, for some $\theta \in \omega_1$. Let $C_{s\alpha} = \bigcup_{h \in \mathbb{N}} D_{s\alpha h}$, where $D_{s\alpha h} \in \bigcup_{1 \leq \rho < \xi} \boldsymbol{\Pi}^0_\rho(\mathcal{T})$. Let τ be a zero-dimensional Polish topology refining $\mathcal{T}$ and making all $D_{s\alpha h}$ clopen. Notice that, by the constructions of [3, §13.A], one can assume that τ has a basis of clopen sets $\mathcal{B} = \{V_m\}_{m \in \mathbb{N}} \subseteq \bigcup_{1 \leq \rho < \xi}(\boldsymbol{\Sigma}^0_\rho(\mathcal{T}) \cup \boldsymbol{\Pi}^0_\rho(\mathcal{T})) \subseteq \boldsymbol{\Delta}^0_\xi(\mathcal{T})$, that can be assumed to be closed under complementation. In particular, $\boldsymbol{\Delta}^0_1(\tau) \subseteq \boldsymbol{\Delta}^0_\xi(\mathcal{T})$. Let $\{A_m\}_{m \in \mathbb{N}}$ enumerate $\mathcal{B} \cap \bigcup_{1 \leq \rho < \xi} \boldsymbol{\Pi}^0_\rho$.

Since, $\forall s \in \mathbb{N}^{<\omega}$ $M_s \in \boldsymbol{\Delta}^0_2(\tau)$, the function $f : (\mathbb{N}^{\mathbb{N}}, \tau) \to (\mathbb{N}^{\mathbb{N}}, \mathcal{T})$ is Baire class 1. By Baire's grand theorem, for every non-empty $C \in \boldsymbol{\Pi}^0_1(\tau)$, the function $f|_C$ has a point of $(\tau|_C, \mathcal{T})$-continuity, where $\tau|_C$ is the topology induced by τ on C. Now, if $C \in \boldsymbol{\Pi}^0_1(\mathbb{N}^{\mathbb{N}})$ is such that $C \cap A_m$ is clopen in $\mathcal{T}|_C$ for every $m \in \mathbb{N}$, then $\tau|_C = \mathcal{T}|_C$. Moreover C is also τ-closed, and any point of $(\tau|_C, \mathcal{T})$-continuity witnesses condition 2.

(2) ⇒ (3) is immediate.

(3) $\Rightarrow$ (1). Assume condition 3. As above, let $\mathcal{T}$ be the usual Baire space topology, and τ be a zero-dimensional Polish topology refining $\mathcal{T}$ and such that all sets A_m are clopen. Again, we may assume that τ has a countable basis of clopen sets $\mathcal{B} \subseteq \bigcup_{1\leq\rho<\xi}(\boldsymbol{\Sigma}^0_\rho(\mathcal{T}) \cup \boldsymbol{\Pi}^0_\rho(\mathcal{T}))$. Let K be τ-compact, so $\mathcal{T}$-compact as well. Since the identity $id : (\mathbb{N}^\mathbb{N}, \tau) \to (\mathbb{N}^\mathbb{N}, \mathcal{T})$ is continuous, it induces a homeomorphism $id : (K, \tau|_K) \to (K, \mathcal{T}|_K)$. This implies that $K \cap A_m$ is $\mathcal{T}|_K$ clopen, for every $m \in \mathbb{N}$. By the assumption, f has a $(\tau|_K, \mathcal{T})$ point of continuity. It follows that $f : (\mathbb{N}^\mathbb{N}, \tau) \to (\mathbb{N}^\mathbb{N}, \mathcal{T})$ is Baire class 1. As $\boldsymbol{\Delta}^0_1(\tau) \subseteq \boldsymbol{\Delta}^0_\xi(\mathcal{T})$, so that $\boldsymbol{\Delta}^0_2(\tau) \subseteq \boldsymbol{\Delta}^0_{\xi+1}(\mathcal{T})$, one can conclude that $f : (\mathbb{N}^\mathbb{N}, \mathcal{T}) \to (\mathbb{N}^\mathbb{N}, \mathcal{T})$ is Baire class ξ. □

Acknowledgements The research presented in this paper has been done while the first author was visiting the Department of information systems of the University of Lausanne. He wishes to thank the Équipe de logique for providing such a friendly environment for research.

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