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Generic Vopěnka's Principle, remarkable cardinals, and the weak Proper Forcing Axiom

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Abstract We introduce and study the first-order *Generic Vopěnka's Principle*, which states that for every definable proper class of structures $\mathcal{C}$ of the same type, there exist $B \neq A$ in $\mathcal{C}$ such that B elementarily embeds into A in some set-forcing

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extension. We show that, for $n \geq 1$, the Generic Vopěnka's Principle fragment for Π_n-definable classes is equiconsistent with a proper class of n-remarkable cardinals. The n-remarkable cardinals hierarchy for $n \in \omega$, which we introduce here, is a natural generic analogue for the $C^{(n)}$-extendible cardinals that Bagaria used to calibrate the strength of the first-order Vopěnka's Principle in Bagaria (Arch Math Logic 51(3–4):213–240, 2012). Expanding on the theme of studying set theoretic properties which assert the existence of elementary embeddings in some set-forcing extension, we introduce and study the *weak Proper Forcing Axiom*, wPFA. The axiom wPFA states that for every transitive model $\mathcal{M}$ in the language of set theory with some ω_1-many additional relations, if it is forced by a proper forcing $\mathbb{P}$ that $\mathcal{M}$ satisfies some Σ_1-property, then V has a transitive model $\bar{\mathcal{M}}$, satisfying the same Σ_1-property, and in some set-forcing extension there is an elementary embedding from $\bar{\mathcal{M}}$ into $\mathcal{M}$. This is a weakening of a formulation of PFA due to Claverie and Schindler (J Symb Logic 77(2):475–498, 2012), which asserts that the embedding from $\bar{\mathcal{M}}$ to $\mathcal{M}$ exists in V. We show that wPFA is equiconsistent with a remarkable cardinal. Furthermore, the axiom wPFA implies $\mathrm{PFA}_{\aleph_2}$, the Proper Forcing Axiom for antichains of size at most ω_2, but it is consistent with $\square_\kappa$ for all $\kappa \geq \omega_2$, and therefore does not imply $\mathrm{PFA}_{\aleph_3}$.

Keywords Large cardinals · Vopěnka's Principle · Generic Vopěnka's Principle · Remarkable cardinals · Proper Forcing Axiom

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1 Introduction

Vopěnka's *Principle* is a large cardinal principle which states that for every proper class $\mathcal{C}$ of structures of the same type there are $B \neq A$, both in $\mathcal{C}$, such that B elementarily embeds into A. It can be formalized in first-order set theory[1] as a schema, where for each natural number n in the meta-theory there is a formula expressing that Vopěnka's Principle holds for all Σ_n-definable (with parameters) classes. Following [1], we call $\mathrm{VP}(\boldsymbol{\Sigma}_n)$ the fragment of Vopěnka's Principle for Σ_n-definable classes and let $\mathrm{VP}(\Sigma_n)$ be the weaker principle, where parameters are not allowed in the definition of the class (with analogous definitions for Π_n). Bagaria introduced in [1] a family of Vopěnka-like principles $\mathrm{VP}(\kappa, \boldsymbol{\Sigma}_n)$, where κ is a cardinal, which state that for every proper class $\mathcal{C}$ of structures of the same type that is Σ_n-definable with parameters in H_κ (the collection of all sets of hereditary size less than κ), $\mathcal{C}$ reflects below κ, namely for every $A \in \mathcal{C}$ there is $B \in H_\kappa \cap \mathcal{C}$ that elementarily embeds into A. Bagaria established a relationship between Vopěnka's Principle fragments and his family of principles $\mathrm{VP}(\kappa, \boldsymbol{\Sigma}_n)$, and provided a complete characterization of Vopěnka's Principle fragments $\mathrm{VP}(\boldsymbol{\Pi}_n)$, as well as the weaker principles $\mathrm{VP}(\Pi_n)$, in terms of the existence of supercompact and $C^{(n)}$-extendible cardinals [1].

[1] The first-order and second-order versions of Vopěnka's Principle are equiconsistent over the Gödel-Bernays set theory GBC, but there are models of GBC in which Vopěnka's Principle holds for all definable classes, but fails for some class [8].

Recall that $C^{(n)}$ denotes the class club of ordinals δ such that $V_\delta \prec_{\Sigma_n} V$. A cardinal κ is called *$C^{(n)}$-extendible* if for every $\alpha > \kappa$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ and with $j(\kappa) \in C^{(n)}$. Note that every extendible cardinal is 1-extendible. Bagaria [1] showed that the weaker principle $\mathrm{VP}(\Pi_1)$ holds if and only if $\mathrm{VP}(\kappa, \boldsymbol{\Sigma}_2)$ holds for some κ, and if and only if there is a supercompact cardinal. For $n \geq 1$, $\mathrm{VP}(\Pi_{n+1})$ holds if and only if $\mathrm{VP}(\kappa, \boldsymbol{\Sigma}_{n+2})$ holds for some κ, and if and only if there is a $C^{(n)}$-extendible cardinal. The results generalize to show that the Vopěnka's Principle fragment $\mathrm{VP}(\boldsymbol{\Pi}_1)$ holds if and only if $\mathrm{VP}(\kappa, \boldsymbol{\Sigma}_2)$ holds for a proper class of κ, and if and only if there is a proper class of supercompact cardinals. For $n \geq 1$, $\mathrm{VP}(\boldsymbol{\Pi}_{n+1})$ holds if and only if $\mathrm{VP}(\kappa, \boldsymbol{\Sigma}_{n+2})$ holds for a proper class of κ, and if and only if there is a proper class of $C^{(n)}$-extendible cardinals. Thus, Vopěnka's Principle holds precisely when, for every $n \in \omega$, there is a proper class of $C^{(n)}$-extendible cardinals.

In this article, we introduce and study generic versions of Vopěnka's Principle and its variants. The *Generic Vopěnka's Principle* states that for every proper class $\mathcal{C}$ of structures of the same type there are $B \neq A$, both in $\mathcal{C}$, such that B elementarily embeds into A in some set-forcing extension. We call $\mathrm{gVP}(\boldsymbol{\Sigma}_n)$ the Generic Vopěnka's Principle fragment for Σ_n-definable (with parameters) classes and we let $\mathrm{gVP}(\Sigma_n)$ be the weaker principle where parameters are not allowed in the definition of the class (with analogous definitions for Π_n). We also call $\mathrm{gVP}(\kappa, \boldsymbol{\Sigma}_n)$ the analogous generic version of $\mathrm{VP}(\kappa, \boldsymbol{\Sigma}_n)$.

It turns out that an elementary embedding $j : B \to A$ between first-order structures exists in some set-forcing extension if and only if it already exists in $V^{\mathrm{Coll}(\omega, B)}$ (Proposition 2.7). We show that to every pair of structures B and A of the same type, we can associate a closed game $G(B, A)$ such that B elementarily embeds into A in $V^{\mathrm{Coll}(\omega, B)}$ precisely when a particular player has a winning strategy in that game. The game $G(B, A)$ is a variant of an Ehrenfeucht–Fraïssé game of length ω, where player I starts out by playing some $b_0 \in B$ and player II responds by playing $a_0 \in A$. Players I and II continue to alternate, choosing elements b_n and a_n from their respective structures at stage n of the game. Player II wins if for every formula $\varphi(x_0, \dots, x_n)$,

$$B \models \varphi(b_0, \dots, b_n) \leftrightarrow A \models \varphi(a_0, \dots, a_n),$$

and otherwise player I wins. Since if player II loses she must do so at some finite stage of the game, the game $G(B, A)$ is closed and hence determined by the Gale–Stewart theorem [5]. Thus, either player I or player II has a winning strategy. We show that player II has a winning strategy precisely when B elementarily embeds into A in $V^{\mathrm{Coll}(\omega, B)}$ (Proposition 4.1). It follows that each first-order fragment of Generic Vopěnka's Principle is characterized by the existence of certain winning strategies in its associated class of closed games.

The consistency strength of Generic Vopěnka's Principle fragments is measured by a hierarchy of cardinals, the *n-remarkable* cardinals (Definition 3.1) we introduce here, which generalize Schindler's remarkable cardinals analogously to how $C^{(n)}$-extendible cardinals generalize extendible cardinals. A remarkable cardinal (which is 1-remarkable by our definition) is a type of generic supercompact cardinal (see Sect. 2) and, correspondingly, an n-remarkable cardinal (for $n > 1$) is a type of generic $C^{(n)}$-

extendible cardinal (see Sect. 3). The n-remarkable cardinals sit relatively low in the large cardinal hierarchy. Call a large cardinal *completely remarkable* if it is n-remarkable for every $n \in \omega$. Completely remarkable cardinals can exist in L and the consistency of a completely remarkable cardinal follows from a 2-iterable cardinal (Theorem 3.6). We show that the Generic Vopěnka's Principle fragment $\mathrm{gVP}(\Pi_n)$ is equiconsistent with an n-remarkable cardinal.

Theorem 1.1 *The following are equiconsistent.*

(1) $\mathrm{gVP}(\Pi_n)$.
(2) $\mathrm{gVP}(\kappa, \boldsymbol{\Sigma}_{n+1})$ *for some* κ.
(3) *There is an* n*-remarkable cardinal.*

The result generalizes to the bold-face $\mathrm{gVP}(\boldsymbol{\Pi}_n)$ principles.

Theorem 1.2 *The following are equiconsistent.*

(1) $\mathrm{gVP}(\boldsymbol{\Pi}_n)$.
(2) $\mathrm{gVP}(\kappa, \boldsymbol{\Sigma}_{n+1})$ *for a proper class of* κ.
(3) *There is a proper class of* n*-remarkable cardinals.*

See Sect. 5 for proofs.

The notion of a generic embedding existing in some forcing extension leads naturally to a weak version of the Proper Forcing Axiom PFA, which we introduce and study here. Schindler and Claverie showed in [2] that PFA has the following equivalent formulation.

Theorem 1.3 *The following are equivalent.*

(1) PFA
(2) *If* $\mathcal{M} = (M; \in, (R_i \mid i < \omega_1))$ *is a transitive model,* $\varphi(x)$ *is a* Σ_1*-formula, and* $\mathbb{Q}$ *is a proper forcing such that*

$$\Vdash_{\mathbb{Q}} \varphi(\mathcal{M}),$$

then there is in V *some transitive* $\bar{\mathcal{M}} = (\bar{M}; \in, (\bar{R}_i \mid i < \omega_1))$ *together with some elementary embedding*

$$j : \bar{\mathcal{M}} \to \mathcal{M}$$

such that $\varphi(\bar{\mathcal{M}})$ *holds.*

By weakening this formulation of PFA to say that the embedding j exists in $V^{\mathrm{Coll}(\omega, \bar{M})}$, we obtain the *weak Proper Forcing Axiom* wPFA. We show that wPFA is equiconsistent with a remarkable cardinal.

Theorem 1.4

(1) *If* κ *is remarkable, then there is a forcing extension in which* wPFA *holds.*
(2) *If* wPFA *holds, then* ω_2^V *is remarkable in* L.

The principle wPFA implies $\mathrm{PFA}_{\aleph_2}$, the Proper Forcing Axiom for meeting antichains of size $\leq \aleph_2$, but it does not imply $\mathrm{PFA}_{\aleph_3}$. For proofs see Sect. 6.

2 Remarkable cardinals

Remarkable cardinals were introduced by Schindler, who showed that the assertion that the theory of $L(\mathbb{R})$ cannot be changed by proper forcing is equiconsistent with the existence of a remarkable cardinal [11]. Remarkable cardinals have also found applications in other settings: recently Cheng and Schindler showed that third-order arithmetic together with Harrington's principle is equiconsistent with the existence of a remarkable cardinal [4].

Definition 2.1 (Schindler [11,12]) *A cardinal κ is* remarkable *if for every regular cardinal $\lambda > \kappa$, there is a regular cardinal $\bar{\lambda} < \kappa$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : H^V_{\bar{\lambda}} \to H^V_\lambda$ with $j(crit(j)) = \kappa$.*

We can view a remarkable cardinal as a type of generic supercompact cardinal using the following theorem of Magidor.

Theorem 2.2 (Magidor [10]) *A cardinal κ is supercompact if and only if for every regular cardinal $\lambda > \kappa$ there is a regular cardinal $\bar{\lambda} < \kappa$ and an elementary embedding $j : H_{\bar{\lambda}} \to H_\lambda$ with $j(crit(j)) = \kappa$.*

Remarkable cardinals are much weaker than supercompact cardinals. Remarkable cardinals are downward absolute to L and the consistency of a remarkable cardinal follows from a 2-iterable cardinal, which is much weaker than an ω-Erdős cardinal. It is not difficult to see that remarkable cardinals are totally indescribable and ineffable. (See [7,11].)

For the rest of the article, we will make the convention that structures of the form H_λ or V_λ always refer to ground model objects, so that we don't have to use superscripts.

If κ is remarkable, then every set a can be put into the range of some remarkability embedding $j : H_{\bar{\lambda}} \to H_\lambda$ in $V^{\mathrm{Coll}(\omega,<\kappa)}$ with λ arbitrarily large. See Proposition 3.2 for a proof of a more general statement.

Proposition 2.3 (Schindler [12]) *If κ is remarkable, then for every set a and regular λ such that $a \in H_\lambda$, there is a regular $\bar{\lambda} < \kappa$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : H_{\bar{\lambda}} \to H_\lambda$ with $j(crit(j)) = \kappa$ and $a \in range(j)$.*

Recall that $C^{(n)}$ is the Π_n-definable club proper class of ordinals δ such that $V_\delta \prec_{\Sigma_n} V$. In particular, $C^{(1)}$ is the class of uncountable strong limit cardinals δ such that $V_\delta = H_\delta$ (see [1] for details). Note, more generally, that for every uncountable cardinal δ, $H_\delta \prec_{\Sigma_1} V$.

Proposition 2.4 *The following are equivalent for a cardinal κ.*

(1) *κ is remarkable.*
(2) *For every $\lambda > \kappa$ and every $a \in V_\lambda$, there is $\bar{\lambda} < \kappa$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(crit(j)) = \kappa$ and $a \in range(j)$.*
(3) *For every $\lambda > \kappa$ in $C^{(1)}$ and every $a \in V_\lambda$, there is $\bar{\lambda} < \kappa$ also in $C^{(1)}$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(crit(j)) = \kappa$ and $a \in range(j)$.*
(4) *There is a proper class of $\lambda > \kappa$ such that for every λ in the class, there is $\bar{\lambda} < \kappa$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(crit(j)) = \kappa$.*

Proof Clearly, (3) implies (4).

Let us show (1) implies (2). So, assume κ is remarkable. Fix $\lambda > \kappa$ and $a \in V_\lambda$. Choose a regular δ large enough so that $V_\lambda \in H_\delta$. By Proposition 2.3, there is a regular $\bar{\delta} < \kappa$ such that in $V^{\text{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : H_{\bar{\delta}} \to H_\delta$ with $j(\text{crit}(j)) = \kappa$ and $a, \lambda \in \text{range}(j)$. Let $j(\bar{\lambda}) = \lambda$. Suppose x is the pre-image of V_λ under j. $H_{\bar{\delta}}$ thinks that x is $V_{\bar{\lambda}}$ by elementarity and it must be correct about this since "$x = V_{\bar{\lambda}}$" is Π_1 expressible, with $\bar{\lambda}$ as a parameter, and $H_{\bar{\delta}} \prec_{\Sigma_1} V$. Thus, we can restrict j to $j : V_{\bar{\lambda}} \to V_\lambda$ and the restriction has all the required properties.

For (2) implies (3), it suffices to observe that if $j : V_{\bar{\lambda}} \to V_\lambda$ is elementary and λ is in $C^{(1)}$, then $\bar{\lambda}$ must be in $C^{(1)}$ as well. Since $\lambda \in C^{(1)}$, λ is an uncountable limit cardinal and $V_\lambda = H_\lambda$. Thus, by elementarity, $\bar{\lambda}$ is a limit of cardinals and hence a limit cardinal, and then, by elementarity, it must be the case that $V_{\bar{\lambda}} = H_{\bar{\lambda}}$.

It only remains to show that (4) implies (1). So, assume that for every λ in some proper class $\mathcal{C}$, there is $\bar{\lambda}$ such that in $V^{\text{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\text{crit}(j)) = \kappa$. Suppose towards a contradiction that κ is not remarkable and let $\lambda > \kappa$ be the least V-regular cardinal witnessing the non-remarkability of κ. By (4), there is some $\delta > \lambda$ in $\mathcal{C}$ and $\bar{\delta} < \kappa$ such that in $V^{\text{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : V_{\bar{\delta}} \to V_\delta$ with $j(\text{crit}(j)) = \kappa$. Note that λ is definable in V_δ as the least regular cardinal witnessing the non-remarkability of κ. So λ is in the image of j and we can let $j(\bar{\lambda}) = \lambda$, noting that $\bar{\lambda}$ must be regular by elementarity. Now we restrict j to $j : H_{\bar{\lambda}} \to H_\lambda$ and note that the restriction has all the desired properties, thus contradicting our assumption there was no such embedding for λ. □

Proposition 2.5 *Every remarkable cardinal is in* $C^{(2)}$.

Proof Suppose κ is remarkable, $\varphi(x, y)$ is a Π_1-formula, $a \in V_\kappa$, and $\exists x\, \varphi(x, a)$ holds in V. Then $V \models \varphi(a, b)$ for some witness b. We must find some witness $\bar{b} \in V_\kappa$. Let $\delta > \kappa$ be regular such that $b \in H_\delta$ and let $\alpha < \kappa$ be some ordinal above the rank of a. By Proposition 2.3, there is a regular $\bar{\delta} < \kappa$ such that in $V^{\text{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : H_{\bar{\delta}} \to H_\delta$ with $j(\text{crit}(j)) = \kappa$ and $\alpha \in \text{range}(j)$. It follows that $\text{crit}(j)$ is above the rank of a and hence $j(a) = a$. Since $H_\delta \models \varphi(a, b)$, there is some $\bar{b} \in H_{\bar{\delta}}$ such that $H_{\bar{\delta}} \models \varphi(a, \bar{b})$, but then $V \models \varphi(a, \bar{b})$ as well and $\bar{b} \in V_\kappa$. □

When working with remarkable cardinals we often appeal to the following folklore result, which asserts that the existence of an embedding of a countable model into another fixed model is absolute.

Lemma 2.6 *Suppose M is a countable first-order structure and $j : M \to N$ is an elementary embedding. If $W \subseteq V$ is a transitive (set or class) model of (some sufficiently large fragment of)* ZFC *such that M is countable in W and $N \in W$, then W has some elementary embedding $j^* : M \to N$. Moreover, if both M and N are transitive $\in$-structures, we can additionally assume that $\text{crit}(j^*) = \text{crit}(j)$. Also, we can assume that j and j^* agree on some fixed finite number of values.*

The proof proceeds by fixing an enumeration $\{a_i \mid i < \omega\}$ of M in W and constructing in W the tree of all finite partial isomorphisms between M to N with domain some

$\{a_i : i < n\}$ for some n. This tree is ill-founded in V, and hence must be ill-founded in W (for details see Lemma 2.7 in [3]). The absoluteness lemma 2.6 immediately gives the equivalence between the assertion that an embedding $j : B \to A$ exists in some set-forcing extension and the assertion that such an embedding exists in $V^{\mathrm{Coll}(\omega,B)}$.

Proposition 2.7 *The following are equivalent for structures B and A in the same language.*

(1) *There is a complete Boolean algebra $\mathbb{B}$ such*

$$V^{\mathbb{B}} \models \text{“There exists an elementary embedding } j : B \to A.\text{”}$$

(2) *In $V^{\mathrm{Coll}(\omega,B)}$ there is an elementary embedding $j : B \to A$.*

(3) *For every complete Boolean algebra $\mathbb{B}$,*

$$V^{\mathbb{B}} \models \text{“}|B| = \aleph_0 \to \text{There exists an elementary embedding } j : B \to A.\text{”}$$

Moreover, if B and A are transitive $\in$-structures, we can assume that the embeddings have the same critical point and agree on finitely many fixed values.

Proof Clearly (2) implies (1) and (3) implies (2).

Let's show (1) implies (2). So suppose a forcing extension $V[G]$ has an elementary embedding $j : B \to A$ and let $|B|^V = \delta$. Let $H \subseteq \mathrm{Coll}(\omega, \delta)$ be $V[G]$-generic. Since j exists in $V[G][H]$ and B is countable in $V[H] \subseteq V[G][H]$, by Lemma 2.6, there is some elementary embedding $j^* : B \to A$ in $V[H]$ satisfying the “moreover” conditions.

Finally, let's show (2) implies (3). So suppose a forcing extension $V[G]$ satisfies $|B| = \aleph_0$ and let $|B|^V = \delta$. Let $H \subseteq \mathrm{Coll}(\omega, \delta)$ be $V[G]$-generic. Then, by (2), $V[H]$ has an elementary embedding $j : B \to A$, and hence so does $V[G][H]$. But then by Lemma 2.6, since B is countable in $V[G]$, it must have some $j^* : B \to A$ as desired. □

In particular, we can rephrase the definition of a remarkable cardinal κ to say that for every regular $\lambda > \kappa$, there is some regular $\bar{\lambda} < \kappa$ such that some set-forcing extension has an elementary embedding $j : H_{\bar{\lambda}} \to H_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$.

3 *n*-Remarkable cardinals and virtual large cardinals

We generalize remarkable cardinals to obtain the notion of n-remarkable cardinal, for $n > 0$. We show that the n-remarkable cardinals form a hierarchy of strength and, for $n \geq 2$, they can be viewed as a type of generic $C^{(n-1)}$-extendible cardinal.

Definition 3.1 *A cardinal κ is n-*remarkable*, for $n > 0$, if for every $\lambda > \kappa$ in $C^{(n)}$, there is $\bar{\lambda} < \kappa$ also in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$, there is an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(crit(j)) = \kappa$. A cardinal κ is* completely remarkable *if it is n-remarkable for every $n > 0$.*

By Proposition 2.4, remarkable cardinals are precisely the 1-remarkable cardinals.

Proposition 3.2 *If κ is n-remarkable, then for every $\lambda > \kappa$ in $C^{(n)}$ and $a \in V_\lambda$, there is $\bar{\lambda} < \kappa$ also in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(crit(j)) = \kappa$ and $a \in range(j)$.*

Proof Suppose towards a contradiction that there is some $\lambda > \kappa$ in $C^{(n)}$ and $a \in V_\lambda$ for which the apparently stronger notion of n-remarkability of κ fails to hold. Let $\delta \in C^{(n)}$ be large enough so that V_δ sees that this is the case. By n-remarkability, there is $\bar{\delta} < \kappa$ in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is $j : V_{\bar{\delta}} \to V_\delta$ with $j(\mathrm{crit}(j)) = \kappa$. Let $\mathrm{crit}(j) = \bar{\kappa}$. By elementarity, $V_{\bar{\delta}}$ satisfies that the apparently stronger notion of n-remarkability of $\bar{\kappa}$ fails to hold for some $\eta > \bar{\kappa}$ in $C^{(n)}$ and $b \in V_\eta$. Thus, by elementarity upward, V_δ satisfies that the apparently stronger notion of n-remarkability of κ fails to hold for $j(\eta) \in C^{(n)}$ and $j(b) \in V_{j(\eta)}$. But this is impossible because j restricts to $j : V_\eta \to V_{j(\eta)}$ and obviously $j(b) \in \mathrm{range}(j)$. □

Next, we give an analogue of Proposition 2.5.

Proposition 3.3 *Every n-remarkable cardinal is in $C^{(n+1)}$.*

Proof Suppose that κ is n-remarkable. It suffices to show that whenever $\varphi(x, y)$ is a Π_n-formula, $a \in V_\kappa$, and $\exists x\, \varphi(x, a)$ holds, then there is $b \in V_\kappa$ such that $\varphi(b, a)$ holds. So suppose the hypothesis to be true. Let $\lambda > \kappa$ be in $C^{(n+1)}$, so that in particular, $V_\lambda \models \exists x\, \varphi(x, a)$. By n-remarkability and Proposition 3.2, there is $\bar{\lambda} < \kappa$ in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$ and $\mathrm{crit}(j)$ above the rank of a. It follows, by elementarity, that $V_{\bar{\lambda}} \models \exists x\, \varphi(x, a)$. So there is $b \in V_{\bar{\lambda}}$ such that $V_{\bar{\lambda}} \models \varphi(b, a)$. Since $\bar{\lambda} \in C^{(n)}$, $\varphi(b, a)$ holds in the universe and $b \in V_\kappa$. □

Theorem 3.4 *Every $n+1$-remarkable cardinal is a limit of n-remarkable cardinals.*

Proof First, observe that being n-remarkable is a Π_{n+1}-property. Suppose that κ is $n+1$-remarkable and fix $\alpha < \kappa$. We will show that there is an n-remarkable cardinal between α and κ. In $V^{\mathrm{Coll}(\omega,<\kappa)}$ fix some elementary $j : V_{\bar{\lambda}} \to V_\lambda$ with $\lambda > \kappa, \bar{\lambda} < \kappa$ both in $C^{(n+1)}$ and $j(\mathrm{crit}(j)) = \kappa$. Let $j(\bar{\kappa}) = \kappa$, and note that by putting a large enough ordinal into the range of j we can assume that $\bar{\kappa} > \alpha$. Since $\lambda \in C^{(n+1)}$, V_λ satisfies that κ is n-remarkable, and so by elementarity $V_{\bar{\lambda}}$ satisfies that $\bar{\kappa}$ is n-remarkable. But $\bar{\lambda}$ is also in $C^{(n+1)}$, and so $\bar{\kappa}$ is truly n-remarkable in V. □

It follows that the n-remarkable cardinals form a hierarchy of strength bounded above by completely remarkable cardinals.

Theorem 3.5 *If $0^\#$ exists, then every Silver indiscernible is completely remarkable in L.*

Proof Suppose $0^\#$ exists and let κ be a Silver indiscernible. Fix $\alpha > \kappa$ such that $L_\alpha \prec_{\Sigma_n} L$. Let $\delta > \alpha$ be a Silver indiscernible and let $j : L \to L$ be an elementary embedding generated by a shift of indiscernibles such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) = \delta$. The embedding j restricts to $j : L_\alpha \to L_{j(\alpha)}$. It follows, by Lemma 2.6, that there is $\bar{\alpha} < j(\kappa)$ (namely $\bar{\alpha} = \alpha$) such that in $L^{\mathrm{Coll}(\omega,<j(\kappa))}$ there is an elementary embedding $j^* : L_{\bar{\alpha}} \to L_{j(\alpha)}$ with $j^*(\mathrm{crit}(j^*)) = j(\kappa)$ and $L_{\bar{\alpha}} \prec_{\Sigma_n} L$. So by elementarity via j, L satisfies that in $L^{\mathrm{Coll}(\omega,<\kappa)}$ there is $\bar{\alpha} < \kappa$ and an elementary embedding $j^* : L_{\bar{\alpha}} \to L_\alpha$ such that $j^*(\mathrm{crit}(j^*)) = \kappa$ and $L_{\bar{\alpha}} \prec_{\Sigma_n} L$. □

Thus, the consistency of a completely remarkable cardinal follows from $0^\#$, but in fact the assertion is much weaker, and already follows from a 2-iterable cardinal. A cardinal κ is said to be *α-iterable*, for some $1 \leq \alpha \leq \omega_1$, if every $A \subseteq \kappa$ can be put into a weak κ-model[2] M for which there is a weakly amenable M-ultrafilter[3] on κ with α-many well-founded iterated ultrapowers. For a finite n, an n-iterable cardinal is stronger than a completely ineffable cardinal but weaker than an ω-Erdős cardinal. If κ is (at least) 2-iterable, it can be shown that every $A \subseteq \kappa$ can be put into a weak κ-model $M \models$ ZFC with an elementary embedding $j : M \to N$ such that N is well-founded, $\text{crit}(j) = \kappa$, $M \prec N$, and $M = V^N_{j(\kappa)}$. (See [7] for details.)

Theorem 3.6 *If κ is 2-iterable, then V_κ is a model of proper class many completely remarkable cardinals.*

Proof Suppose κ is 2-iterable. Fix a weak κ-model $M \models$ ZFC containing V_κ for which there is an elementary embedding $j : M \to N$ such that N is well-founded, $\text{crit}(j) = \kappa$, $M \prec N$, and $M = V^N_{j(\kappa)}$. To show that V_κ is a model of proper class many completely remarkable cardinals, it suffices to show that κ is completely remarkable in $M = V^N_{j(\kappa)}$. So, fix n and fix $\alpha > \kappa$ in M such that $V^M_\alpha \prec_{\Sigma_n} M$. Note that, since $M \prec N$ and $M = V^N_{j(\kappa)}$, $V^M_\alpha = V^N_\alpha \prec_{\Sigma_n} N$ as well. Consider the restriction $j : V^M_\alpha \to V^N_{j(\alpha)}$. By Lemma 2.6, N satisfies that in $V^{\text{Coll}(\omega,<j(\kappa))}$ there is $\bar{\alpha} < j(\kappa)$ and an elementary embedding $j^* : V_{\bar{\alpha}} \to V_{j(\alpha)}$ such that $j^*(\text{crit}(j^*)) = j(\kappa)$ and $V_{\bar{\alpha}} \prec_{\Sigma_n} N$. By elementarity, M satisfies that in $V^{\text{Coll}(\omega,<\kappa)}$ there is $\bar{\alpha} < \kappa$ and an elementary embedding $j^* : V_{\bar{\alpha}} \to V_\alpha$ such that $j^*(\text{crit}(j^*)) = \kappa$ and $V_{\bar{\alpha}} \prec_{\Sigma_n} M$. Thus, κ is n-remarkable in M, for every $n \in \omega$. ◻

It is also not difficult to see that n-remarkable cardinals are downward absolute to L.

If we assume, for a cardinal κ, that the embeddings characterizing a supercompact cardinal given by Magidor's theorem exist in some set-forcing extension, then we get a remarkable cardinal. In [6], Gitman and Schindler apply this procedure to obtain generic variants of other large cardinals including extendible, huge, and rank-into-rank. We will show that 2-remarkable cardinals are precisely the *virtually extendible* cardinals defined in this manner and that more generally, the n-remarkable cardinals, for $n > 1$ correspond to virtually $C^{(n-1)}$-extendible cardinals.

Definition 3.7 ([6]) *A cardinal κ is* virtually extendible *if for every $\alpha > \kappa$, in some set-forcing extension (equivalently in $V^{\text{Coll}(\omega,V_\alpha)}$) there is $j : V_\alpha \to V_\beta$ such that $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$. A cardinal κ is* virtually $C^{(n)}$-extendible *if additionally $j(\kappa) \in C^{(n)}$.*

Note that virtually extendible cardinals are $C^{(1)}$-extendible because $j(\kappa)$ must be inaccessible in V.

2 A transitive model $M \models \text{ZFC}^-$ is called a *weak κ-model* if it has size κ and height above κ.

3 Suppose M is a transitive model of ZFC^- and κ is a cardinal in M. Then $U \subseteq \mathcal{P}^M(\kappa)$ is called an *M-ultrafilter* if the structure $\langle M, \in, U\rangle$ with a predicate for U satisfies that U is a normal ultrafilter. An M-ultrafilter is *weakly amenable* if for every $A \in M$ with $|A|^M = \kappa$, $A \cap U \in M$. Weak amenability makes it possible to carry out the iterated ultrapowers construction with an external ultrafilter.

Theorem 3.8 *A cardinal κ is virtually extendible if and only if it is 2-remarkable. More generally, κ is virtually $C^{(n)}$-extendible if and only if it is $n+1$-remarkable.*

Proof Let us first show that if κ is virtually extendible, then it is 2-remarkable. Fix $\lambda > \kappa$ in $C^{(2)}$ and let $\alpha > \lambda$ also be in $C^{(2)}$. By virtual extendibility, in $V^{\mathrm{Coll}(\omega, V_\alpha)}$ there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$. Consider the restriction of j to $j : V_\lambda \to V_{j(\lambda)}$. Let's argue that $V_\lambda \prec_{\Sigma_2} V_{j(\lambda)}$. Since $\lambda \in C^{(1)}$, and j is elementary, $j(\lambda) \in C^{(1)}$ as well. So suppose $V_{j(\lambda)}$ satisfies $\exists x\, \varphi(x, a)$, where φ is Π_1 and $a \in V_\lambda$. Then $V_{j(\lambda)}$ satisfies $\varphi(b, a)$ for some witness b. So V satisfies $\varphi(b, a)$ as well. Hence V satisfies $\exists x\, \varphi(x, a)$ and V_λ must agree because $\lambda \in C^{(2)}$.

So V_β satisfies that there is $\bar\lambda < j(\kappa)$ such that $V_{\bar\lambda} \prec_{\Sigma_2} V_{j(\lambda)}$, and in $V^{\mathrm{Coll}(\omega, <j(\kappa))}$ there is an elementary embedding $j^* : V_{\bar\lambda} \to V_{j(\lambda)}$ with $j^*(\mathrm{crit}(j^*)) = j(\kappa)$. So V_α satisfies that there is $\bar\lambda < \kappa$ such that $V_{\bar\lambda} \prec_{\Sigma_2} V_\lambda$, and in $V^{\mathrm{Coll}(\omega, <\kappa)}$ there is $j^* : V_{\bar\lambda} \to V_\lambda$ such that $j^*(\mathrm{crit}(j^*)) = \kappa$. Since $\lambda \in C^{(2)}$, it follows that $\bar\lambda \in C^{(2)}$ as well, completing the argument.

Let us now show that every 2-remarkable κ is virtually extendible. It follows from 2-remarkability, that there must be some $\bar\lambda < \kappa$ in $C^{(2)}$ such that for cofinally many $\lambda > \kappa$ in $C^{(2)}$, in $V^{\mathrm{Coll}(\omega, <\kappa)}$ there is $j_\lambda : V_{\bar\lambda} \to V_\lambda$ with $j_\lambda(\mathrm{crit}(j_\lambda)) = \kappa$. We can also assume that $\mathrm{crit}(j_\lambda)$ is some fixed $\bar\kappa$ for all λ. Let us argue that $\bar\kappa$ is virtually extendible in $V_{\bar\lambda}$. Fix $\beta < \bar\lambda$ above $\bar\kappa$. Fix any λ and consider the restriction $j_\lambda : V_\beta \to V_{j_\lambda(\beta)}$. Thus, in $V^{\mathrm{Coll}(\omega, V_\beta)}$ there is some ξ and an elementary embedding $j^* : V_\beta \to V_\xi$ such that $\mathrm{crit}(j^*) = \bar\kappa$ and $j^*(\bar\kappa) > \beta$. But this is a Σ_2 fact. So it holds true in $V_{\bar\lambda}$. So $\bar\kappa$ is virtually extendible in $V_{\bar\lambda}$, but then since there are embeddings from $V_{\bar\lambda}$ into cofinally many V_λ, with $\lambda \in C^{(2)}$, we have that $\kappa = j_\lambda(\bar\kappa)$ is virtually extendible in V.

Now let's suppose that $n > 1$ and the equivalence holds for all $m < n$.

First, suppose that κ is virtually $C^{(n)}$-extendible. Fix $\lambda > \kappa$ in C^{n+1} and let $\alpha > \lambda$ also be in $C^{(n+1)}$. By virtual $C^{(n)}$-extendibility, in $V^{\mathrm{Coll}(\omega, V_\alpha)}$ there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$, and $j(\kappa) \in C^{(n)}$. Consider the restriction $j : V_\lambda \to V_{j(\lambda)}$. Let's argue that $V_\lambda \prec_{\Sigma_{n+1}} V_{j(\lambda)}$. If suffices to show that whenever $\varphi(x, y)$ is a Π_n-formula, $a \in V_\lambda$, and $\exists x\, \varphi(x, a)$ holds in $V_{j(\lambda)}$, then there is a witness in V_λ. So suppose that the hypothesis holds. Since V_α knows that $\lambda \in C^{(n+1)}$, V_β satisfies that $j(\lambda)$ is in $C^{(n+1)}$. Thus, $V_\beta \models \exists x\, \varphi(x, a)$. By the assumption that the equivalence holds for $n-1$ and Proposition 3.3, $\kappa \in C^{(n+1)}$, and so V_β satisfies that $j(\kappa) \in C^{(n+1)}$. It now follows that $V_{j(\kappa)} \models \exists x\, \varphi(x, a)$, and so $V_{j(\kappa)} \models \varphi(b, a)$ for some b. Since $j(\kappa) \in C^{(n)}$, $\varphi(b, a)$ holds in V, from which it finally follows that $V_\lambda \models \exists x\, \varphi(x, a)$. The rest of the argument proceeds identically to the case $n = 1$ above.

Next, suppose that κ is $n+1$-remarkable. As in the case $n = 1$, we fix $\bar\lambda < \kappa$ in $C^{(n+1)}$ such that for cofinally many $\lambda > \kappa$ in $C^{(n+1)}$, in $V^{\mathrm{Coll}(\omega, <\kappa)}$ there is $j_\lambda : V_{\bar\lambda} \to V_\lambda$ with $\mathrm{crit}(j_\lambda) = \bar\kappa$ and $j_\lambda(\bar\kappa) = \kappa$. As in that case, we will now argue that $\bar\kappa$ is virtually $C^{(n)}$-extendible in $V_{\bar\lambda}$. Fix $\beta < \bar\lambda$ above $\bar\kappa$. Fix any λ and consider the restriction $j_\lambda : V_\beta \to V_{j_\lambda(\beta)}$. By our assumption that the equivalence holds for all $m < n$, it follows that $j(\bar\kappa) = \kappa \in C^{(n+2)}$, and so in particular, $j(\bar\kappa) \in C^{(n)}$. Thus, in $V^{\mathrm{Coll}(\omega, V_\beta)}$ there is some ξ and elementary embedding $j^* : V_\beta \to V_\xi$ such that $\mathrm{crit}(j^*) = \bar\kappa$, $j^*(\bar\kappa) > \beta$, and $j^*(\bar\kappa) \in C^{(n)}$, but this is a Σ_{n+1} fact, and the rest of the argument proceeds as before. □

Another virtual large cardinal which shows up in the context of generic Vopěnka-like principles is the virtually rank-into-rank cardinal.

Definition 3.9 ([6]) *A cardinal κ is* virtually rank-into-rank *if there is $\alpha > \kappa$ such that in $V^{\mathrm{Coll}(\omega, V_\alpha)}$, there is an elementary embedding $j : V_\alpha \to V_\alpha$ with $crit(j) = \kappa$.*

If an elementary embedding $j : V_\alpha \to V_\alpha$ exists in V, then α can be at most $\lambda + 1$, where λ is the supremum of the critical sequence of j, by Kunen's Inconsistency [9]. But Kunen's Inconsistency does not hold for embeddings existing in a forcing extension of V. If $j : V_\alpha \to V_\alpha$ exists in some forcing extension, then we can have α be as large as desired above the supremum λ of the critical sequence. It is not difficult to see that virtually rank-into-rank cardinals are consistent with $V = L$, and indeed are downward absolute to L. A virtually rank-into-rank cardinal is stronger than a completely remarkable cardinal. The following proposition follows from the arguments in [6].

Proposition 3.10 ([6]) *If κ is virtually rank-into-rank, then V_κ is a model of proper class many completely remarkable cardinals.*

4 Remarkable games

The main theme of this article is studying assertions of the form that an elementary embedding between two structures exists in some set-forcing extension. It turns out that such assertions can be reformulated in terms of the existence of winning strategies in a class of Ehrenfeucht–Fraïssé-like games.

Let B and A be two structures in the same language. We consider a two-player game, denoted by $G(B, A)$, where in the n-th move player I chooses $b_n \in B$ and player II chooses $a_n \in A$. Player II wins the game if for every formula $\varphi(x_0, \ldots, x_n)$,

$$B \models \varphi(b_0, \ldots, b_n) \leftrightarrow A \models \varphi(a_0, \ldots, a_n),$$

and otherwise player I wins. Since if player II loses she has to lose by some finite stage, the game is closed and hence determined by the Gale–Stewart Theorem [5].

Proposition 4.1 *The following are equivalent for structures B and A in the same language.*

(1) *Player II has a winning strategy in $G(B, A)$.*
(2) *In $V^{\mathrm{Coll}(\omega, B)}$, there is an elementary embedding $j : B \to A$.*
(3) *There is a complete Boolean algebra $\mathbb{B}$ such that*

$$V^{\mathbb{B}} \models \text{“There exists an elementary embedding } j : B \to A.\text{”}$$

(4) *For every complete Boolean algebra $\mathbb{B}$,*

$$V^{\mathbb{B}} \models \text{“}|B| = \aleph_0 \to \text{There is an elementary embedding } j : B \to A.\text{”}$$

Proof By Proposition 2.7, it suffices to show only that (1) and (2) are equivalent.

Let's show (1) implies (2). So, suppose σ is a winning strategy for player II and G is $\mathrm{Coll}(\omega, B)$-generic over V. In $V[G]$, we fix an enumeration $\{b_i \mid i < \omega\}$ of the universe of B. Notice that, in $V[G]$, σ is still a winning strategy for player II, because the game is a closed game and there are no new finite sets in $V[G]$. So, by playing according to σ against the moves b_n of player I given by the fixed enumeration, player II obtains the desired elementary embedding $j : B \to A$.

Next, we show (2) implies (1). So, suppose that in $V^{\mathrm{Coll}(\omega,B)}$ there is an elementary embedding $j : B \to A$. Let τ be a $\mathrm{Coll}(\omega, B)$-name for j. The following is a winning strategy for player II: When player I plays some b_0 at stage $n = 0$, choose some $p_{\langle b_0\rangle}$ which forces $\tau(b_0) = a_0$ and play a_0. When player I plays b_1 at stage $n = 1$, choose some $p_{\langle b_0,b_1\rangle} \leq p_{\langle b_0\rangle}$ which forces $\tau(b_1) = a_1$ and play a_1. Continuing in this manner, at stage $n + 1$ of the game, to every sequence of plays $\langle b_0, \ldots, b_n\rangle$ of player I, we have associated a condition $p_{\langle b_0,\ldots,b_n\rangle}$ which forces $\tau(b_i) = a_i$ and the a_i are the plays according to the strategy. So when player I plays b_{n+1} at stage $n + 1$, we choose a condition $p_{\langle b_0,\ldots,b_n,b_{n+1}\rangle} \leq p_{\langle b_0,\ldots,b_n\rangle}$ which forces $\tau(b_{n+1}) = a_{n+1}$ and play a_{n+1}. □

5 Generic Vopěnka's Principle

We introduce the *Generic Vopěnka's Principle* which states that for every proper class $\mathcal{C}$ of structures of the same type, there are $B \neq A$, both in $\mathcal{C}$, such that B elementarily embeds into A in some set-forcing extension; equivalently, by Proposition 4.1, player II has a winning strategy in $G(B, A)$. We will study fragments of Generic Vopěnka's Principle for Σ_n-definable classes, as well as similarly defined generic variants of other Vopěnka-like principles, such as $\mathrm{VP}(\kappa, \mathbf{\Sigma}_n)$.

Definition 5.1

(1) *The principle* $\mathrm{gVP}(\mathbf{\Sigma}_n)$ *asserts that for every* Σ_n*-definable with parameters proper class* $\mathcal{C}$ *of structures of the same type, there are* $B \neq A$*, both in* $\mathcal{C}$*, such that* B *elementarily embeds into* A *in some set-forcing extension. The principle* $\mathrm{gVP}(\Sigma_n)$ *is defined analogously but does not allow parameters in the definition of the class. The principles* $\mathrm{gVP}(\Pi_n)$ *and* $\mathrm{gVP}(\mathbf{\Pi}_n)$ *are defined analogously.*

(1) *The principle* $\mathrm{gVP}(\kappa, \mathbf{\Sigma}_n)$*, where* κ *is a cardinal, asserts that every* Σ_n*-definable with parameters from* H_κ *class* $\mathcal{C}$ *of structures of the same type* $\tau \in H_\kappa$ generically reflects below κ*, meaning that for every* $A \in \mathcal{C}$*, there is* $B \in H_\kappa$ *such that* B *elementarily embeds into* A *in some set-forcing extension.*

Theorem 5.2 *Suppose* κ *is* n*-remarkable. Then* $\mathrm{gVP}(\kappa, \mathbf{\Sigma}_{n+1})$ *holds, and hence* $\mathrm{gVP}(\Sigma_{n+1})$ *holds.*

Proof Let $\mathcal{C}$ be a proper class of structures of type $\tau \in H_\kappa$ that is Σ_{n+1}-definable from $a \in H_\kappa$. Fix $\alpha < \kappa$ such that $\tau, a \in V_\alpha$. Fix $A \in \mathcal{C}$ and fix a cardinal $\lambda > \kappa$ in $C^{(n+1)}$ with $A \in V_\lambda$. By Proposition 3.2, there is $\bar{\lambda} < \kappa$ in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $j(\mathrm{crit}(j)) = \kappa$ and $A \in \mathrm{range}(j)$. By also putting α into the range of j, we can assume that τ is fixed point-wise by j and

$j(a) = a$. Let $j(B) = A$. Since $\mathcal{C}$ is Σ_{n+1}-definable and $\lambda \in C^{(n+1)}$, we have that $V_\lambda \models$ "$A \in \mathcal{C}$". Since $j(a) = a$ by assumption, it follows that $V_{\bar{\lambda}} \models$ "$B \in \mathcal{C}$", and it must be correct about because $\bar{\lambda} \in C^{(n)}$. Thus, truly $B \in \mathcal{C}$. Thus, since we made sure that τ is fixed point-wise by j, the restriction $j : B \to A$ is the desired elementary embedding. □

Corollary 5.3 *Geneneric Vopěnka's Principle can hold in L.*

Proof For instance, if $0^\#$ exists, then by Theorem 3.5, L has a proper class of n-remarkable cardinals for every $n \in \omega$. □

Theorem 5.4

(1) *Suppose* $\mathrm{gVP}(\Pi_n)$ *holds. Then either there is an n-remarkable cardinal or there is a virtually rank-into-rank cardinal.*
(2) *Suppose* $\mathrm{gVP}(\mathbf{\Pi}_n)$ *holds. Then either there is a proper class of n-remarkable remarkable cardinals or there is a proper class of virtually rank-into-rank cardinals.*

Proof We will say that a cardinal κ is *n-remarkable up to* $\lambda > \kappa$ if for every $\kappa < \eta < \lambda$ in $C^{(n)}$, there is $\bar{\eta} < \kappa$ in $C^{(n)}$ and such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : V_{\bar{\eta}} \to V_\eta$ with $j(\mathrm{crit}(j)) = \kappa$.

First, we prove (1). If there is an n-remarkable cardinal, we are done. So let's assume that there are no n-remarkable cardinals. We follow the proof of Theorem 4.3(1) in [1]. Let $\mathcal{C}$ be the class of structures of the form $\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle$ such that $\alpha \in C^{(n)}$ and λ is the least ordinal in $C^{(n)}$ above α such that no $\kappa \leq \alpha$ is n-remarkable up to λ. It is not difficult to see that $\mathcal{C}$ is Π_n-definable without parameters. It follows from our assumption that there are no n-remarkable cardinals that $\mathcal{C}$ is a proper class. By $\mathrm{gVP}(\Pi_n)$ applied to $\mathcal{C}$, there exist structures

$$\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \neq \langle V_{\mu+2}, \in, \beta, \mu\rangle$$

such that in $V^{\mathrm{Coll}(\omega, V_{\lambda+2})}$ there is an elementary embedding

$$j : \langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \to \langle V_{\mu+2}, \in, \beta, \mu\rangle.$$

If j was the identity, then we would have $\lambda = \mu$ and $\alpha = \beta$, which is impossible since we assumed $\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \neq \langle V_{\mu+2}, \in, \beta, \mu\rangle$. So j has a critical point, call it κ.

Let's argue that $\alpha < \beta$, and hence $\kappa \leq \alpha$. If $\lambda = \mu$, then this must be the case because j is not the identity. If $\lambda < \mu$, then it must also be the case because no $\xi \leq \alpha$ is n-remarkable up to λ (by definition of $\mathcal{C}$) and there is some $\xi \leq \beta$ which is n-remarkable up to λ, by minimality of μ. Let's argue next that $j(\kappa) < \lambda$. If not, then we claim κ is n-remarkable up to λ, which is impossible. Fix some $\delta > \kappa$ in $C^{(n)}$ below λ. Consider the restriction $j : V_\delta \to V_{j(\delta)}$, which has $j(\mathrm{crit}(j)) = j(\kappa)$. By Proposition 2.7 and our assumption that $\delta < j(\kappa)$, there is some $j^* : V_\delta \to V_{j(\delta)}$ with $j^*(\mathrm{crit}(j^*)) = j(\kappa)$ in $V^{\mathrm{Coll}(\omega,<j(\kappa))}$ and $V_{\mu+2}$ sees this. Thus, by elementarity, $V_{\lambda+2}$ satisfies that there is some $\bar{\delta} < \kappa$ in $C^{(n)}$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j^* : V_{\bar{\delta}} \to V_\delta$ with $j^*(\mathrm{crit}(j^*)) = \kappa$. So we verified that $j(\kappa) < \lambda$. The above given argument now shows that κ is n-remarkable up to $j(\kappa)$, and of course, $V_{\mu+2}$ see this. So by elementarity, there is some $\alpha < \kappa$ which is n-remarkable up to κ. It follows by elementarity upward that α is n-remarkable up to $j(\kappa)$ and beyond. So let ρ be the largest ordinal such that α is n-remarkable up to ρ. Since ρ is definable from α and α is fixed by j, it follows that $j(\rho) = \rho$. Thus, the restriction $j : V_{\rho+2} \to V_{\rho+2}$, say, witnesses that κ is virtually rank-into-rank.

Next, we prove (2). We follow the proof of Theorem 4.3(2) in [1]. We will show that above every ordinal ξ, there is either an n-remarkbale cardinal or a virtually rank-into-rank cardinal. Let $\mathcal{C}$ be the class of structures of the form $\langle V_{\lambda+2}, \in, \alpha, \lambda, \{\gamma\}_{\gamma\leq\xi}\rangle$ such that $\alpha > \xi$ is in $C^{(n)}$ and λ is the least ordinal in $C^{(n)}$ greater than α such that no $\kappa \leq \alpha$ and above ξ is remarkable up to λ. The class $\mathcal{C}$ is Π_n-definable in the parameter ξ and if there are no n-remarkable cardinals above ξ, then $\mathcal{C}$ is a proper class. An analogous argument to (1) now shows that, in this case, there is a virtually rank-into-rank cardinal above ξ. □

Note that the virtually rank-into-rank cardinal arises from the possibility which is eliminated in the analogous argument about Vopěnka's Principle fragments by Kunen's Inconsistency.

Putting together the above results, we get that the principles gVP(Π_n) and gVP(κ, $\boldsymbol{\Sigma}_{n+1}$) are equiconsistent with an n-remarkable cardinal.

Theorem 5.5 *The following are equiconsistent.*

(1) gVP(Π_n).
(2) gVP(κ, $\boldsymbol{\Sigma}_{n+1}$) *for some* κ.
(3) *There is an* n*-remarkable cardinal.*

Proof Suppose there is a model with gVP(Π_n). By Theorem 5.4 (1), either there is an n-remarkable cardinal or there is a virtually rank-into-rank cardinal, say κ, in which case V_κ is a model of proper class many n-remarkable cardinals by Proposition 3.10.

Next, suppose that gVP(κ, $\boldsymbol{\Sigma}_{n+1}$) holds for some κ. Then gVP(Σ_{n+1}) holds, and so again there is a model with an n-remarkable cardinal.

Finally, by Theorem 5.2, if there is an n-remarkable cardinal, then gVP(κ, $\boldsymbol{\Sigma}_{n+1}$) holds. □

Theorem 5.6 *The following are equiconsistent.*

(1) gVP($\boldsymbol{\Pi}_n$).
(2) gVP(κ, $\boldsymbol{\Sigma}_{n+1}$) *for a proper class of* κ.
(3) *There is a proper class of* n*-remarkable cardinals.*

Proof Suppose there is a model with gVP($\boldsymbol{\Pi}_n$). By Theorem 5.4 (2), either there is a proper class of n-remarkable cardinals or there is a virtually rank-into-rank cardinal (in fact, proper class many of them), say κ, in which case V_κ is a model of proper class many n-remarkable cardinals by Proposition 3.10.

Next, suppose that gVP(κ, $\boldsymbol{\Sigma}_{n+1}$) holds for a proper class of κ. Then gVP($\boldsymbol{\Sigma}_{n+1}$) holds, and so again there is a model with proper class many n-remarkable cardinals.

Finally, by Theorem 5.2, if there is a proper class of n-remarkable cardinals κ, then gVP(κ, $\boldsymbol{\Sigma}_{n+1}$) holds for each of them. □

We don't know whether equiconsistency can be replaced by direct implication in Theorems 5.5 and 5.6, as in the case of Vopěnka's Principle fragments and supercompact and $C^{(n)}$-extendible cardinals. The chief obstacle to obtaining direct implications seems to be that the "virtual" version of Kunen's Inconsistency does not hold (as discussed in Sect. 3).

Question 5.7

(1) *If* $\mathrm{gVP}(\kappa, \boldsymbol{\Sigma}_{n+1})$ *holds for some* κ*, holds does it follows that there is an* n*-remarkable cardinal?*
(2) *If* $\mathrm{gVP}(\Pi_n)$ *holds, does it follow that there is an* n*-remarkable cardinal?*

Bagaria showed in [1] that the least κ for which $\mathrm{VP}(\kappa, \boldsymbol{\Sigma}_2)$ holds is the least supercompact and, for $n > 1$, the least κ for which $\mathrm{VP}(\kappa, \boldsymbol{\Sigma}_{n+1})$ holds is the least $C^{(n)}$-extendible. We can obtain analogous results for a potentially stronger variant of Generic Vopěnka's Principle and an analogous strengthening of $\mathrm{gVP}(\kappa, \boldsymbol{\Sigma}_n)$.

Definition 5.8 *Suppose* B *and* A *are transitive* $\in$*-structures and* $j : B \to A$ *is an elementary embedding. We say that* j *is* overspilling *if* j *has a critical point and* $j(crit(j)) > rank(B)$.

Definition 5.9 *The principle* $\mathrm{gVP}^*(\Sigma_n)$ *asserts for every* Σ_n*-definable, without parameters, proper class* $\mathcal{C}$ *of transitive* $\in$*-structures, that there are* $B \neq A$ *in* $\mathcal{C}$ *such that there is an overspilling elementary embedding* $j : B \to A$ *in some set-forcing extension. The principles* $\mathrm{gVP}^*(\Pi_n)$, $\mathrm{gVP}^*(\boldsymbol{\Pi}_n)$*, and* $\mathrm{gVP}^*(\kappa, \boldsymbol{\Sigma}_n)$ *are defined analogously.*

Theorem 5.10 *The following are equivalent for a cardinal* κ*.*

(1) κ *is the least for which* $\mathrm{gVP}^*(\kappa, \boldsymbol{\Sigma}_{n+1})$ *holds.*
(2) κ *is the least* n*-remarkable cardinal.*

Proof Since $\mathrm{gVP}^*(\kappa, \boldsymbol{\Sigma}_{n+1})$ holds for n-remarkable κ by the proof of Theorem 5.2, it follows that the least κ of (2) is at least as large as the least κ of (1). Thus, it suffices to show that the least κ in (1) is n-remarkable. So suppose now that κ is least such that $\mathrm{gVP}^*(\kappa, \boldsymbol{\Sigma}_{n+1})$ holds. Let $\mathcal{C}$ be the Π_n-definable proper class of structures of the form $\langle V_{\alpha+2}, \in\rangle$ with $\alpha \in C^{(n)}$. Fix some $\lambda \in C^{(n+1)}$ that is not inaccessible. By $\mathrm{gVP}^*(\kappa, \boldsymbol{\Sigma}_{n+1})$, there is $\bar{\lambda} < \kappa$ such that in $V^{\mathrm{Coll}(\omega, V_{\bar{\lambda}+2})}$ there is an elementary embedding $j : V_{\bar{\lambda}+2} \to V_{\lambda+2}$ with $\mathrm{crit}(j) = \bar{\alpha}$ and $j(\bar{\alpha}) > \bar{\lambda}$. Observe that $\bar{\alpha} < \bar{\lambda}$ since the critical point of j must be inaccessible and $\bar{\lambda}$ is not inaccessible by elementarity. By the argument given in the proof of Theorem 5.4, $\bar{\alpha}$ is n-remarkable up to $\bar{\lambda}$, and so by elementarity $\alpha = j(\bar{\alpha})$ is n-remarkable up to λ. But since being n-remarkable is Π_{n+1} and we chose our $\lambda \in C^{(n+1)}$, it follows that α is actually n-remarkable. Clearly α cannot be smaller than κ. So it remains to argue that α cannot be larger than κ.

Recall that $\alpha \in C^{(n+1)}$ by Proposition 3.3. It follows that V_α satisfies that $\mathrm{gVP}(\boldsymbol{\Sigma}_{n+1}, \kappa)$ holds, and hence by elementarity, there must be some $\gamma < \bar{\alpha}$ such that $V_{\bar{\alpha}}$ satisfies that $\mathrm{gVP}(\boldsymbol{\Sigma}_{n+1}, \gamma)$ holds. So now by elementarity upward, V_α satisfies that $\mathrm{gVP}(\boldsymbol{\Sigma}_{n+1}, \gamma)$ holds (since $j(\gamma) = \gamma$) and it must be correct about it. But this is impossible since $\gamma < \kappa$. Thus, $\alpha = \kappa$, and so κ is n-remarkable. □

Theorem 5.11

(1) *If* $\mathrm{gVP}^*(\Pi_n)$ *holds, then there is an* n*-remarkable cardinal.*
(2) *If* $\mathrm{gVP}^*(\boldsymbol{\Pi}_n)$ *holds, then there is a proper class of* n*-remarkable cardinals.*

6 A weak version of the Proper Forcing Axiom

In the spirit of investigating principles which assert the existence of elementary embeddings in a set-forcing extension, we introduce and study a weakening of PFA, the Proper Forcing Axiom, based on the notion that the embeddings arising from PFA exist in a set-forcing extension. As we noted in the introduction, the proof of [2, Theorem 1.3] produces the following characterization of PFA.

Theorem 6.1 *The following are equivalent.*

(1) PFA
(2) *If* $\mathcal{M} = (M; \in, (R_i \mid i < \omega_1))$ *is a transitive model,* $\varphi(x)$ *is a* Σ_1*-formula, and* $\mathbb{Q}$ *is a proper forcing such that*

$$\Vdash_{\mathbb{Q}} \varphi(\mathcal{M}),$$

then there is in V *some transitive* $\bar{\mathcal{M}} = (\bar{M}; \in, (\bar{R}_i \mid i < \omega_1))$ *together with some elementary embedding*

$$j : \bar{\mathcal{M}} \to \mathcal{M}$$

such that $\varphi(\bar{\mathcal{M}})$ *holds.*

For instance, to see that (2) implies PFA, suppose that $\mathbb{Q}$ is a proper poset and $\langle D_\alpha \mid \alpha < \omega_1\rangle$ is a sequence of dense sets of $\mathbb{Q}$. Let $\mathcal{M}$ have the form $\langle H_\lambda, \in, \mathbb{Q}, (D_\alpha \mid \alpha < \omega_1)\rangle$, where H_λ is sufficiently large that it contains all subsets of $\mathbb{Q}$. Clearly $\mathbb{Q}$ forces the Σ_1-assertion about $\mathcal{M}$ that there is a filter for $\mathbb{Q}$ meeting all the D_α. So by (2), V has an elementary embedding $j : \bar{\mathcal{M}} \to \mathcal{M}$ for some transitive model $\bar{\mathcal{M}} = (\bar{M}; \in, \bar{\mathbb{Q}}, (\bar{D}_\alpha \mid \alpha < \omega_1))$ and V has a filter $\bar{G}$ for $\bar{\mathbb{Q}}$ meeting all the $\bar{D}_\alpha$. Let G be the point-wise image of $\bar{G}$ under j. Clearly G is a filter on $\mathbb{Q}$ meeting all the D_α as required by PFA.

PFA is weakened to the *weak Proper Forcing Axiom*, wPFA, by asserting that the embedding $j : \bar{\mathcal{M}} \to \mathcal{M}$ exists in some set-forcing extension.

Definition 6.2 *The* weak Proper Forcing Axiom wPFA *asserts that if* $\mathcal{M} = (M; \in, (R_i \mid \xi < \omega_1))$ *is a transitive model,* $\varphi(x)$ *is a* Σ_1*-formula, and* $\mathbb{Q}$ *is a proper forcing such that*

$$\Vdash_{\mathbb{Q}} \varphi(\mathcal{M}),$$

then there is in V *some transitive* $\bar{\mathcal{M}} = (\bar{M}; \in, (\bar{R}_i \mid i < \omega_1))$ *such that* $\varphi(\bar{\mathcal{M}})$ *holds and inside some set-forcing extension (equivalently in* $V^{\mathrm{Coll}(\omega, \bar{M})}$*) there is an elementary embedding*

$$j : \bar{\mathcal{M}} \to \mathcal{M}.$$

We will show that wPFA is equiconsistent with a remarkable cardinal.

Let us first show that wPFA is consistent relative to a remarkable cardinal. The proof uses a remarkable Laver function which is the analogue of a Laver function on a supercompact cardinal.

Suppose κ is a cardinal and $\ell : \kappa \to V_\kappa$ is a partial function. We say that a set $x \in V_\lambda$ with $\lambda > \kappa$ is λ-*anticipated* by ℓ if there is $\bar{\lambda} < \kappa$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $\mathrm{crit}(j) = \xi$ and $j(\xi) = \kappa$ so that $\ell \restriction \xi + 1 \in V_{\bar{\lambda}}$, $j(\ell \restriction \xi) = \ell$, and $j(\ell(\xi)) = x$. The function ℓ is called a *remarkable Laver function* if whenever $x \in V_\lambda$ with $\lambda > \kappa$, then x is λ-anticipated by ℓ. Gitman showed in [3] that every remarkable cardinal has a remarkable Laver function.

Theorem 6.3 *Let κ be remarkable. Then* wPFA *holds in a forcing extension by a proper poset.*

Proof We imitate the standard argument which produces PFA in a forcing extension of a ground model with a supercompact cardinal.

Let $\ell : \kappa \to V_\kappa$ be a remarkable Laver function. We define a countable support κ-length iteration $\mathbb{P}$, where at stage ξ, if $\ell(\xi) = (\dot{\mathbb{Q}}, M)$ for some set M and $\mathbb{P}_\xi$-name $\dot{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}_\xi}$ "$\dot{\mathbb{Q}}$ is proper", then we force with $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{Q}}$, and with the trivial forcing otherwise. The iteration $\mathbb{P}$ is proper and therefore preserves ω_1.

Let G be $\mathbb{P}$-generic over V. We claim that wPFA holds in $V[G]$. To this end, let $\mathcal{M} = (M; \in, (R_i \mid i < \omega_1)) \in V[G]$ be a transitive model, let $\varphi(x)$ be a Σ_1-formula, and let $\mathbb{Q} \in V[G]$ be a proper forcing such that, in $V[G]$, $\Vdash_{\mathbb{Q}} \varphi(\mathcal{M})$ holds.

Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for $\mathbb{Q}$ such that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper", let τ be a $\mathbb{P}$-name for $\mathcal{M}$, and let $x = (\dot{\mathbb{Q}}, \tau)$. Let $\lambda > \kappa$ be sufficiently large such that $x \in V_\lambda$, V_λ satisfies that $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is proper", and $V_\lambda[G]$ satisfies that $\Vdash_{\mathbb{Q}} \varphi(\mathcal{M})$. This is possible because if λ is large enough, then $V_\lambda[G]$ has a sufficiently large $H_\delta^{V[G]}$ and a club of models in $[H_\delta^{V[G]}]^\omega$ witnessing the properness of $\mathbb{Q}$ and $V_\lambda[G]$ has a $\mathbb{Q}$-name witnessing the Σ_1-formula $\varphi(\mathcal{M})$. By the properties of ℓ, there is some $\bar{\lambda} < \kappa$ such that in $V^{\mathrm{Coll}(\omega,<\kappa)}$ there is an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with $\mathrm{crit}(j) = \xi$ and $j(\xi) = \kappa$ so that $\ell \restriction \xi + 1 \in V_{\bar{\lambda}}$, $j(\ell \restriction \xi) = \ell$, and $j(\ell(\xi)) = (\dot{\mathbb{Q}}, \tau)$. It follows by elementarity that $\ell(\xi) = (Q, \bar{\tau})$ and $V_{\bar{\lambda}}$ satisfies that $\Vdash_{\mathbb{P}_\xi}$ "Q is proper". But then it must truly be the case that $\Vdash_{\mathbb{P}_\xi}$ "Q is proper" because any $\mathbb{P}_\xi$-generic extension of $V_{\bar{\lambda}}$ would provide the necessary witnessing club of models. By the definition of $\mathbb{P}$, it follows that $\dot{\mathbb{Q}}_\xi = Q$.

Note that j fixes all elements of $\mathbb{P}_\xi \subseteq V_\xi$ and, since $j(\ell \restriction \xi) = \ell$, it follows that $j(\mathbb{P}_\xi) = \mathbb{P}$. Thus, inside $V[G]^{\mathrm{Coll}(\omega,<\kappa)}$, we may lift j to an elementary embedding $j : V_{\bar{\lambda}}[G \restriction \xi] \to V_\lambda[G]$ by setting $j(\sigma_{G\restriction\xi}) = j(\sigma)_G$ for every $\mathbb{P}_\xi$-name $\sigma \in V_{\bar{\lambda}}$. In particular, setting $\bar{\mathcal{M}} = (\bar{M}; \in, (\bar{R}_i \mid i < \omega_1)) = \bar{\tau}_{G\restriction\xi}$, $\bar{\mathcal{M}} \in V[G \restriction \xi] \subseteq V[G]$ and

$$j \restriction \bar{M} : \bar{\mathcal{M}} \to \mathcal{M}$$

is an elementary embedding. By Lemma 2.6 such an embedding then also exists in $V[G]^{\mathrm{Coll}(\omega,\bar{M})}$. Note that we used the preservation of ω_1 to conclude that $\bar{\mathcal{M}}$ has ω_1-many relations.

Since $V_\lambda[G]$ satisfies that $\Vdash_{\mathbb{Q}} \varphi(\mathcal{M})$, we will now clearly have that

$$V_\lambda[G_{\xi+1}] \models \varphi(\bar{\mathcal{M}}),$$

so that because $\varphi(x)$ is Σ_1,

$$V[G] \models \varphi(\bar{\mathcal{M}}).$$

We have verified that wPFA holds true in $V[G]$. □

Next, we show that if wPFA holds, then ω_2^V is remarkable in L.

Theorem 6.4 *Assume* wPFA. *Then ω_2^V is remarkable in L.*

Proof We may assume without loss of generality that $0^\#$ does not exist, as otherwise all cardinals of V are remarkable in L. We shall exploit an argument of Todorčević from [15], which shows that $\Box_\kappa$ fails under PFA for all uncountable κ. In what follows, we shall make references to the proof of [13, Theorem 11.64].

Let us write $\kappa = \omega_2^V$. Let $\alpha > \kappa$ be an L-cardinal. It suffices to find some L-cardinal $\beta < \kappa$ such that in $V^{\mathrm{Coll}(\omega,\beta)}$ there is some elementary embedding $j\colon J_\beta \to J_\alpha$ with $j(\mathrm{crit}(j)) = \kappa$. This suffices by Proposition 2.3, because for any infinite L-cardinal γ, $J_\gamma = L_\gamma = H_\gamma^L$.

By way of notation, if J_γ is a model of ZFC^- with the largest cardinal, say γ', then by $(C_\xi^\gamma : \xi < \gamma)$ we mean the canonical $\Box_{\gamma'}$-sequence as being constructed in J_γ as in the proof of [13, Theorem 11.64]. In particular, if γ is an L-cardinal, then γ' will also be an L-cardinal and $(C_\xi^\gamma : \xi < \gamma)$ is the canonical $\Box_{\gamma'}$-sequence of L.

Let us assume that $\alpha = (\alpha')^{+L}$. By the Jensen Covering Lemma, $\mathrm{cf}(\alpha) \geq |\alpha'| \geq \omega_2$ in V. There is then by [15] some proper forcing $\mathbb{P}$ such that if g is $\mathbb{P}$-generic over V, then in $V[g]$, $|\alpha| = \aleph_1$ and there is a pair (C, F) such that

(1) $C \subseteq \alpha$ is a club subset of α of order type ω_1,
(2) $F : C \to \omega$ is such that if $\eta < \xi$ are both in C and η is a limit point of C_ξ^α, then $F(\xi) \neq F(\eta)$.

Let $\mathcal{M}$ have the form $\langle H_\lambda, \in, \mathbb{P}, \alpha, (\xi \mid \xi < \omega_1)\rangle$ for a sufficiently large $\lambda > \alpha$ and consider the Σ_1-assertion about $\mathcal{M}$ that there exists a pair (C, F) as above. By wPFA, there is in V some transitive model $\bar{\mathcal{M}} = \langle \bar{M}, \in, \bar{\mathbb{P}}, \beta, (\xi \mid \xi < \omega_1)\rangle$ and a pair (c, f) such that

(1) $c \subseteq \beta$ is a club subset of β of order type ω_1,
(2) $f : c \to \omega$ is such that if $\eta < \xi$ are both in c and η is a limit point of C_ξ^β, then $f(\xi) \neq f(\eta)$,

and inside $V^{\mathrm{Coll}(\omega,\bar{M})}$ there is an elementary embedding $j : \bar{\mathcal{M}} \to \mathcal{M}$.

We must have $j \restriction (\omega_1^V) + 1 = \mathrm{id}$ because we included constants for countable ordinals of V in our language. Let's consider the restriction $j \restriction J_\beta : J_\beta \to J_\alpha$ and let β' be the largest cardinal of J_β, which exists by elementarity, since α' was the largest cardinal of J_α.

It remains to verify that β is an L-cardinal. If not, then let $\gamma > \beta$ be least such that $\rho_\omega(J_\gamma) \leq \beta'$. Let $\rho_{n+1}(J_\gamma) \leq \beta' < \rho_n(J_\gamma)$. Let $d \subset \beta$ be the set of all $\xi < \beta$ such that $J_\xi \prec J_\beta$ and if $\nu > \xi$ is least with $\rho_\omega(J_\nu) = \beta'$, then $\rho_{n+1}(J_\nu) = \beta' < \rho_n(J_\nu)$ and there is a weakly $r\Sigma_n$ elementary embedding

$$\sigma : J_\nu \to J_\gamma$$

with $\sigma \restriction \xi = \text{id}$ and $\sigma(\xi) = \bar{\alpha}$. By the proof of [13, Theorem 11.64], there is a club $e \subset d \cap c$ in β such that if $\xi \in e$, then $C^\beta_\xi \cap e = e \cap \xi$. Let e' be the set of limit points of e. We now have that if $\eta < \xi$ are both in e', then η is a limit point of C^β_ξ, so that $f(\eta) \neq f(\xi)$. This gives that $f \restriction e' \to \omega$ is injective, which contradicts $\text{cf}(\beta) = \omega_1$.

We note that we now must have $\beta < \omega_2^V = \kappa$ by the Jensen Covering Lemma. It follows, since $j(\beta) = \alpha$, that the critical point of j is below ω_2^V, and hence $j(\text{crit}(j)) = \omega_2^V = \kappa$ as desired. □

For a cardinal κ, let us write PFA_κ for the statement that if $\mathbb{B}$ is any proper complete Boolean algebra and if $\langle A_\xi \mid \xi < \omega_1 \rangle$ is any family of maximal antichains in $\mathbb{B}$ with $|A_\xi| \leq \kappa$ for each $\xi < \omega_1$, then there is some filter $G \subseteq \mathbb{B}$ such that $G \cap A_\xi \neq \emptyset$ for all $\xi < \omega_1$. $\text{PFA}_{\aleph_1}$ is then BPFA, the Bounded Proper Forcing Axiom. The proof of [2, Theorem 1.3] easily shows that PFA_κ can be characterized analogously to PFA as in Theorem 6.1 with the restriction that $|M| = \kappa$, where M is the universe of $\mathcal{M}$.

The axiom wPFA implies $\text{PFA}_{\aleph_2}$, but it does not imply $\text{PFA}_{\aleph_3}$.

Theorem 6.5

(1) wPFA *implies* $\text{PFA}_{\aleph_2}$.
(2) *The assertion* "wPFA $\wedge \forall \kappa \geq \aleph_2 \square_\kappa$" *is consistent relative to a remarkable cardinal.*
(3) wPFA *does not imply* $\text{PFA}_{\aleph_3}$.

Proof Let's prove (1). So assume that $\mathbb{P}$ is a proper poset and $\langle A_\xi \mid \xi < \omega_1 \rangle$ is a sequence of maximal antichains of $\mathbb{P}$ such that each A_ξ has size at most ω_2. Let $\mathbb{Q}$ be a subposet of $\mathbb{P}$ of size ω_2 containing all the A_ξ and preserving compatibility from $\mathbb{P}$, so that if p and q are compatible in $\mathbb{P}$, they remain compatible in $\mathbb{Q}$. By taking an isomorphic copy, we can assume without loss of generality that $\mathbb{Q}$ has universe ω_2. Let $\mathcal{M}$ be the structure $\langle H_{\omega_2}, \mathbb{Q}, (A_\xi \mid \xi < \omega_1) \rangle$. In any forcing extension by $\mathbb{P}$ there is a filter for $\mathbb{Q}$ meeting all the A_ξ. Thus, by wPFA, there is a transitive model $\bar{\mathcal{M}} = (\bar{M}; \bar{\mathbb{Q}}, (\bar{A}_\xi \mid \xi < \omega_1))$ and a filter for $\bar{\mathbb{Q}}$ meeting all the $\bar{A}_\xi$ so that in $V^{\text{Coll}(\omega, M)}$ there is an elementary embedding $j : \bar{\mathcal{M}} \to \mathcal{M}$. By including constants for all countable ordinals in $\mathcal{M}$, we can assume without loss of generality that j fixes ω_1. Since $\bar{\mathcal{M}}$ satisfies by elementarity that all ordinals are bijective with ω_1, it follows that j must be the identity map, and so we have a filter for $\mathbb{Q}$ meeting all the A_ξ. Closing this filter downwards gives a filter for $\mathbb{P}$ meeting all the A_ξ.

Assertion (2) follows from the proof of Theorem 6.3 by starting with a remarkable κ in L. Recall that κ is the ω_2 of the forcing extension, and so because the forcing iteration does not collapse any cardinals above κ, for $\delta \geq \kappa$, the old square sequences from L witness that $\square_\delta$ holds.

Assertion (3) follows by the proof of [14, Theorem 1], which shows that $\text{PFA}_{\aleph_3}$ implies the failure of $\square_{\omega_2}$, whereas by (2), wPFA is compatible with $\square_{\omega_2}$. □

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