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Monotone operators on Gödel logic

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Abstract We consider an extension of Gödel logic by a unary operator that enables the addition of non-negative reals to truth-values. Although its propositional fragment has a simple proof system, first-order validity is Π_2-hard. We explain the close connection to Scarpellini's result on Π_2-hardness of Łukasiewicz's logic.

Keywords Gödel logic · Modal extension

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1 Introduction

First-order Gödel logic is a superintuitionistic logic, which can be described in different ways and which appears in numerous contexts; see the forthcoming article [3] for a detailed overview and for proofs. In contrast to all other continuous t-norm logics, the valid formulas of first-order Gödel logic over [0, 1] are recursively enumerable. In fact, the validity problem in first-order t-norm logics can exhibit very high complexity, e.g., Łukasiewicz logic is Π_2-complete [11], and first-order product logic and the first-order logic of continuous t-norms even fall outside the arithmetical hierarchy. Therefore first-order Gödel logic is the only one among all t-norm logics that provides a good starting point for extensions, in particular, for modal-like extensions: It has a simple

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sound-complete superintuitionistic proof system, and papers like [6] and [5] furnish evidence that the combination of propositional Gödel logics and of modal logics is accomplishable with decent semantics and uncomplicated proof systems.

In Sect. 2, we will introduce an extension of Gödel logic by a unary operator that adds non-negative constants to truth values. While validity in the propositional fragment of this extension can be plainly characterised by just the three extra axioms

1. $(\bot \prec \circ\bot) \supset (A \prec \circ A)$,
2. $(\bot \leftrightarrow \circ\bot) \supset (A \leftrightarrow \circ A)$,
3. $\circ(A \supset B) \leftrightarrow (\circ A \supset \circ B)$,

we prove the surprising fact that first-order validity is Π_2-hard; see Sect. 3. Thus the valid formulas are not recursively enumerable and neither are they describable by a reasonable proof system.

This extension has two purposes:

(1) We adapt Scarpellini's proof [12] of Π_2-hardness for first-order Łukasiewicz logic to obtain Π_2-hardness also for our extension. Both propositional Łukasiewicz logic and Gödel logic can be characterised by adding a single axiom to Hájek's Basic Logic but as their first-order complexities differ significantly, this immediately raises the question why they show a different behaviour. Our extension gives at least a partial answer: While Gödel logic is the logic of relative comparison, any semantical means (be it in the original language or artificially added in form of an operator) to measure absolute distance between truth values endows the semantics with an expressivity that cannot be effectively captured.

(2) This extension can be regarded as a particular case of a modal logic extension because $A \supset \circ A$ holds and $\circ$ distributes over $\wedge, \vee, \supset$. It serves as an example for the fact that even a seemingly harmless real-valued semantics of such an operator may lead to a complex first-order validity problem. (An extension where the similar formula $\circ A \supset A$ holds was considered by Hájek [10].)

In Sect. 3, we will show that the propositional fragment of our extension stays finitely axiomatisable even when extended further by the Monteiro–Takeuti–Titani Δ operator.

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2 Language and semantics of Gödel logics

The propositional language $\mathcal{L}^{\mathrm{p}}$ of Gödel logics and of classical logics is generated by a denumerable set Var of propositional variables and by the logical connectives $\bot$, $\supset$, $\wedge$, $\vee$ with their usual arities. We understand $\neg A$ as an abbreviation for $A \supset \bot$, and $A \leftrightarrow B$ for $(A \supset B) \wedge (B \supset A)$, and $\top$ for $\bot \supset \bot$, and $A \prec B$ for $(B \supset A) \supset B$. Let $\circ$ and $\triangle$ be two fresh unary connectives, which we will call *operators* for the sake of simplicity. We will use $\mathcal{L}^{\mathrm{p}}_{\circ}$, $\mathcal{L}^{\mathrm{p}}_{\triangle}$, $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$ to denote the extensions of $\mathcal{L}^{\mathrm{p}}$ by these operators. We define $\circ^0 A := A$ and $\circ^{n+1} A := \circ(\circ^n A)$ for all $n \in \mathbb{N}$. The first-order language $\mathcal{L}$ of Gödel logics and of classical logics contains quantifiers $\forall$ and $\exists$, and is constructed from $\mathcal{L}^{\mathrm{p}}$ in the usual way. We will occasionally distinguish free and bound individual variables. A first-order formula is *closed* if no free variable occurs in it. The first-order languages $\mathcal{L}_\circ$, $\mathcal{L}_\triangle$, $\mathcal{L}_{\circ,\triangle}$ correspond to $\mathcal{L}^{\mathrm{p}}_\circ$, $\mathcal{L}^{\mathrm{p}}_\triangle$, $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$.

In order to define Gödel semantics, we first put for all $x, y \in \mathbb{R}$:

$$x \oplus y := \min\{1, x + y\},$$

$$x \trianglelefteq y := \begin{cases} 1 & \text{if } x \le y \\ y & \text{if } x > y, \end{cases}$$

$$x \triangleleft y := \begin{cases} 1 & \text{if } x < y \\ y & \text{if } x \ge y, \end{cases}$$

$$x \bowtie y := \begin{cases} 1 & \text{if } x = y \\ \min\{x, y\} & \text{if } x \ne y, \end{cases}$$

$$\pi_0(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0, \text{ and} \end{cases}$$

$$\pi_1(x) := \begin{cases} 1 & \text{if } x \ge 1 \\ 0 & \text{if } x < 1. \end{cases}$$

It is immediately clear that $\oplus$, $\trianglelefteq$, $\triangleleft$, $\bowtie$ induce functions $[0, 1] \times [0, 1] \to [0, 1]$, and π_0 and π_1 induce functions $[0, 1] \to [0, 1]$.

A *Gödel interpretation* $\mathfrak{I}$ of $\mathcal{L}^{\mathrm{p}}$ assigns a value in $[0, 1]$ to each formula in $\mathcal{L}^{\mathrm{p}}$ such that

$$\mathfrak{I}(\bot) = 0,$$
$$\mathfrak{I}(A \wedge B) = \min\{\mathfrak{I}(A), \mathfrak{I}(B)\},$$
$$\mathfrak{I}(A \vee B) = \max\{\mathfrak{I}(A), \mathfrak{I}(B)\}, \text{ and}$$
$$\mathfrak{I}(A \supset B) = \mathfrak{I}(A) \trianglelefteq \mathfrak{I}(B)$$

for all formulas A and B. Obviously, any function $\mathfrak{I}\colon \mathsf{Var} \to [0, 1]$ can be uniquely extended to a Gödel interpretation. In $\mathcal{L}^{\mathrm{p}}_\triangle$ and $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$, the operator $\triangle$ is interpreted by $\mathfrak{I}(\triangle A) := \pi_1(\mathfrak{I}(A))$. For $\mathcal{L}^{\mathrm{p}}_\circ$ and $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$, a Gödel interpretation $\mathfrak{I}$ consists also of a constant $r_{\mathfrak{I}} \in [0, 1]$, which we use to put $\mathfrak{I}(\circ A) := r_{\mathfrak{I}} \oplus \mathfrak{I}(A)$. For any given $s \in [0, 1]$, every $\mathcal{L}^{\mathrm{p}}_\circ$- or $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$-interpretation $\mathfrak{I}$ with $r_{\mathfrak{I}} = s$ is referred to as an s-interpretation.

A *Gödel interpretation* $\mathfrak{I}$ of $\mathcal{L}$ (or $\mathcal{L}_\circ$, $\mathcal{L}_\triangle$, $\mathcal{L}_{\circ,\triangle}$) interprets the logical connectives and operators like its propositional counterpart and consists, as usual, of a nonempty domain $|\mathfrak{I}|$, a function $P^{\mathfrak{I}}\colon |\mathfrak{I}|^n \to [0, 1]$ for each n-ary predicate symbol P, a function $f^{\mathfrak{I}}\colon |\mathfrak{I}|^n \to |\mathfrak{I}|$ for each n-ary function symbol f, and an element $a^{\mathfrak{I}} \in |\mathfrak{I}|$ for each free variable and for each constant a. The quantifiers are interpreted by $\mathfrak{I}(\forall x A(x)) = \inf\{\mathfrak{I}(A(\underline{u}))\colon u \in |\mathfrak{I}|\}$ and $\mathfrak{I}(\exists x A(x)) = \sup\{\mathfrak{I}(A(\underline{u}))\colon u \in |\mathfrak{I}|\}$; here, we tacitly use the convention that a domain element $u \in |\mathfrak{I}|$ in a formula stands for a fresh constant $\underline{u}$ to be interpreted as u. — A *classical* interpretation of $\mathcal{L}$ is a Gödel interpretation such that $P^{\mathfrak{I}}\colon |\mathfrak{I}|^n \to \{0, 1\}$.

A formula A is *valid* if $\mathfrak{I}(A) = 1$ for every interpretation $\mathfrak{I}$. This leads to definitions that depend on the language, but we easily see that the validity of a formula A does not change when the language is joined with an operator that does not occur in A. For the sake of clarity, we explicitly state that A in $\mathcal{L}_\circ$ is valid if and only if $\mathfrak{I}(A) = 1$ holds for every $r \in [0, 1]$ and every r-interpretation $\mathfrak{I}$.

Let A be a formula in a language L and let Γ be a set of formulas in L. (If L is a first-order language, we stipulate that no free variable occurs in Γ.) We write $\Gamma \vDash A$ and say Γ *entails* A (w.r.t. L) if $\inf\{\mathfrak{I}(B) \colon B \in \Gamma\} \leq \mathfrak{I}(A)$ for every interpretation $\mathfrak{I}$ of L. We write $\Gamma \vDash^1 A$ and say Γ *one-entails* A (w.r.t. L) if $\mathfrak{I}(A) = 1$ for every interpretation $\mathfrak{I}$ of L such that $\mathfrak{I}(B) = 1$ for all $B \in \Gamma$. We say that the entailment relation is *compact* if $\Gamma \vDash A$ implies the existence of a finite subset Γ' of Γ such that $\Gamma' \vDash A$. Compactness for one-entailment is defined analogously.

3 Failure of recursive enumerability of valid formulas in $\mathcal{L}_\circ$

In this section, we will prove that the closed prenex formulas in $\mathcal{L}_\circ$ valid w.r.t. Gödel semantics are not recursively enumerable. We will closely follow the idea of Scarpellini [12] by first defining a faithful translation of the formulas that are classically valid in all finite domains into the prenex fragment of $\mathcal{L}_\circ$ and then applying Trakhtenbrot's theorem.

Definition 1 Let A be a formula in $\mathcal{L}$. The formula $A^{\neg\neg}$ is obtained from A by replacing every occurrence Q of an atom, except $\bot$, by $\neg\neg Q$.

We immediately see that $A^{\neg\neg}$ takes only values 0 and 1 under each Gödel interpretation.

Lemma 1 *There is an effective translation β of the closed prenex formulas in $\mathcal{L}$ to the closed formulas in $\mathcal{L}_\circ$ such that any closed prenex formula A is classically valid in all finite domains if and only if $\beta(A)$ is valid in $\mathcal{L}_\circ$ w.r.t. Gödel semantics.*

Proof Let A be a closed prenex formula in $\mathcal{L}$ so that $A^{\neg\neg}$ has the form $Q_1 w_1 \ldots Q_M w_M$ $U(w_1, \ldots, w_M)$, where $Q_i \in \{\forall, \exists\}$, the w_i are bound variables and U is quantifier free. Let $(P_k)_{k<K}$ be an enumeration of the set of all predicate symbols that occur in A. We will denote the arity of P_k by $\mathrm{ar}(P_k)$. We will construct $\beta(A)$ first and then prove the required properties. Let N be the maximum of 1 and of all arities of the P_k; take a fresh binary predicate E and a fresh unary predicate R. Define

$$
\begin{aligned}
F := {} & \forall x, y, z, a_1, b_1, \ldots, a_N, b_N. \\
& ((\neg\neg E(x, x)) \\
& \wedge(\neg\neg E(x, y) \supset \neg\neg E(y, x)) \\
& \wedge((\neg\neg E(x, y) \wedge \neg\neg E(y, z)) \supset \neg\neg E(x, z)) \\
& \wedge \bigwedge_{k<K} (\neg\neg E(a_1, b_1) \wedge \cdots \wedge \neg\neg E(a_n, b_n)) \\
& \supset(\neg\neg P_k(a_1, \ldots, a_{\mathrm{ar}(P_k)}) \leftrightarrow \neg\neg P_k(b_1, \ldots, b_{\mathrm{ar}(P_k)})) \\
& \wedge\neg\neg\circ\bot \\
& \wedge(\neg E(x, y) \supset ((\circ R(x) \supset R(y)) \vee (\circ R(y) \supset R(x))))).
\end{aligned}
$$

and $\beta(A) := F \supset \exists x Q_1 w_1 \ldots Q_M w_M (U(w_1, \ldots, w_M) \vee R(x))$.

The predicate E obviously serves the purpose of modelling an equivalence relation such that elements are equivalent when they cannot be distinguished by some P_k. To explain the purpose of R, we first remark that it can be easily proved by case distinction that, for all formulas X and Y and for any r-interpretation $\mathfrak{I}$, we have $\mathfrak{I}((\circ X \supset Y) \vee (\circ Y \supset X)) = 1$ whenever $|\mathfrak{I}(X) - \mathfrak{I}(Y)| \geq r$, and that $\mathfrak{I}((\circ X \supset Y) \vee (\circ Y \supset X)) = \max\{\mathfrak{I}(X), \mathfrak{I}(Y)\}$ whenever $|\mathfrak{I}(X) - \mathfrak{I}(Y)| < r$. The last conjunct of F now expresses that $R^{\mathfrak{I}}$ interprets elements in different equivalence classes of E by values in [0, 1] that have at least distance r. By the topological compactness of the [0, 1] interval, this construction ensures the finiteness of the number of equivalence classes but does not impose a bound on their number since $r > 0$ can be arbitrary small. (Observe that the penultimate conjunct in F models the condition $r > 0$.)

To prove the lemma, it obviously suffices to prove that the following conditions are equivalent for every formula A:

(1) There is a classical interpretation $\mathfrak{I}'$ with a finite domain such that $\mathfrak{I}'(A) = 0$.
(2) We have $\mathfrak{I}(\beta(A)) < 1$ for some $r \in [0, 1]$ and some Gödel r-interpretation $\mathfrak{I}$.

We prove (1)→(2): Let $\mathfrak{I}'$ be a classical interpretation with a finite domain $\left|\mathfrak{I}'\right| \neq \emptyset$ such that $\mathfrak{I}'(A) = 0$. Let $d_0, \ldots, d_D$ be an enumeration of $\left|\mathfrak{I}'\right|$ and take $r := \frac{1}{D+2}$. Define a Gödel r-interpretation $\mathfrak{I}$ with the domain $\left|\mathfrak{I}'\right|$ as follows: $E^{\mathfrak{I}}(x, y) := 1$ whenever $x = y$, $E^{\mathfrak{I}}(x, y) := 0$ whenever $x \neq y$, $R^{\mathfrak{I}}(d_i) := \frac{i}{D+2}$, and $P_k^{\mathfrak{I}} := P_k^{\mathfrak{I}'}$. Since $P_k^{\mathfrak{I}}(\overline{x}) = P_k^{\mathfrak{I}'}(\overline{x}) \in \{0, 1\}$ for all arguments $\overline{x}$, we easily see $\mathfrak{I}(A^{\neg\neg}) = \mathfrak{I}'(A)$. In particular, we have $\mathfrak{I}(A^{\neg\neg}) = 0$. We suppose now that (2) would not hold. We find $\mathfrak{I}(\beta(A)) = 1$ then and, since x does not occur in U, also $\mathfrak{I}(F) \leq \max\left\{\sup_x R^{\mathfrak{I}}(x), \mathfrak{I}(Q_1 w_1 \ldots Q_M w_M U(w_1, \ldots, w_M))\right\} = \max\left\{\max\left\{\frac{0}{D+2}, \ldots, \frac{D}{D+2}\right\}, \mathfrak{I}(A^{\neg\neg})\right\} = \frac{D}{D+2}$; here, any index in sup or inf will refer to $|\mathfrak{I}|$. It is readily verified that $1 = \inf_x \mathfrak{I}(\neg\neg E(x, x)) = \inf_{x,y} \mathfrak{I}(\neg\neg E(x, y) \supset \neg\neg E(y, x)) = \inf_{x,y,z} \mathfrak{I}((\neg\neg E(x, y) \wedge \neg\neg E(y, z)) \supset \neg\neg E(x, z)) = \inf_{a_1,b_1,\ldots,a_n,b_n} \mathfrak{I}((\neg\neg E(a_1, b_1) \wedge \cdots \wedge \neg\neg E(a_n, b_n)) \supset (\neg\neg P_k(a_1, \ldots, a_n) \leftrightarrow \neg\neg P_k(b_1, \ldots, b_n)))$ and that $\mathfrak{I}(\neg\neg\circ\bot) = \pi_0(r) = \pi_0\left(\frac{1}{D+2}\right) = 1$. For the remaining conjunct in F, this means that $\inf_{x,y}(\mathfrak{I}(\neg E(x, y)) \trianglelefteq \mathfrak{I}((\circ R(x) \supset R(y)) \vee (\circ R(y) \supset R(x)))) \leq \frac{D}{D+2} < 1$. Thus there exist $d_i, d_j \in |\mathfrak{I}|$ such that $\mathfrak{I}(\neg E(d_i, d_j)) > \mathfrak{I}((\circ R(d_i) \supset R(d_j)) \vee (\circ R(d_j) \supset R(d_i)))$. Hence we have $|\mathfrak{I}(R(d_i)) - \mathfrak{I}(R(d_j))| < r$ and $\mathfrak{I}(\neg E(d_i, d_j)) > \max\left\{\mathfrak{I}(R(d_i)), \mathfrak{I}(R(d_j))\right\} \geq 0$. From the former, we obtain $\frac{1}{D+2} = r > |R^{\mathfrak{I}}(d_i) - R^{\mathfrak{I}}(d_j)| = \left|\frac{i}{D+2} - \frac{j}{D+2}\right|$ and thus $i = j$. This yields $\mathfrak{I}(\neg E(d_i, d_j)) = 0$, which is absurd.

We prove (2)→(1): Suppose we have $\mathfrak{I}(\beta(A)) < 1$ for some $r \in [0, 1]$ and some Gödel r-interpretation $\mathfrak{I}$ so that $\mathfrak{I}(F) > \mathfrak{I}(\exists x Q_1 w_1 \ldots Q_M w_M (U(w_1, \ldots, w_M) \vee R(x)))$. Since U does not contain x, we find $\mathfrak{I}(F) > \sup_x R^{\mathfrak{I}}(x)$ and $1 \geq \mathfrak{I}(F) > \mathfrak{I}(Q_1 w_1 \ldots Q_M w_M U(w_1, \ldots, w_M)) \geq 0$. It can be easily seen that $\mathfrak{I}(F)$ is the minimum of the following expressions:

$$C_1 := \inf_x \pi_0(E^{\mathfrak{I}}(x, x)),$$

$$C_2 := \inf_{x,y} \pi_0(E^{\mathfrak{I}}(x, y)) \trianglelefteq \pi_0(E^{\mathfrak{I}}(x, y)),$$

$$C_3 := \inf_{x,y,z} \min\{\pi_0(E^{\mathfrak{I}}(x, y)), \pi_0(E^{\mathfrak{I}}(y, z))\} \trianglelefteq \pi_0(E^{\mathfrak{I}}(x, z)),$$

$$C_{4,k} := \inf_{a_1,\ldots,a_n,b_1,\ldots,b_n} \min\{\pi_0(E^{\mathfrak{I}}(a_1,b_1)),\ldots,\pi_0(E^{\mathfrak{I}}(a_n,b_n))\} \trianglelefteq$$

$$(\pi_0(P_k^{\mathfrak{I}}(a_1,\ldots,a_{\mathrm{ar}(P_k)})) \bowtie \pi_0(P_k^{\mathfrak{I}}(b_1,\ldots,b_{\mathrm{ar}(P_k)}))), \text{ for all } k < K,$$

$$C_5 := \pi_0(r), \text{ and}$$
$$C_6 := \inf_{x,y}(1 - \pi_0(E^{\mathfrak{I}}(x,y))) \trianglelefteq \mathfrak{I}((\circ R^{\mathfrak{I}}(x) \supset R^{\mathfrak{I}}(y)) \vee (\circ R^{\mathfrak{I}}(y) \supset R^{\mathfrak{I}}(x))).$$

Since $0 < \mathfrak{I}(F)$ we have $0 < C_5$ and hence $0 < r$. Using the definitions of $\trianglelefteq$ and $\bowtie$ and the fact that $\pi_0(x) \in \{0,1\}$ for all $x \in [0,1]$, we see that C_1, C_2, C_3, $C_{4,k}$, $C_5 \in \{0,1\}$ for all $k < K$. Since $0 < \mathfrak{I}(F)$, we find $1 = C_1 = C_2 = C_3 = C_{4,k} = C_5$ for all $k < K$. All atoms in U, except $\bot$, are under double negation, and thus $\mathfrak{I}(A^{\neg\neg}) = \mathfrak{I}(Q_1 w_1 \ldots Q_M w_M U(w_1,\ldots,w_M))$ can only take values 0 and 1, and hence must be 0 because it is less than $\mathfrak{I}(F)$. We thus obtain $C_6 = \mathfrak{I}(F) > \sup_x R^{\mathfrak{I}}(x)$.

We prove now that we have $|R^{\mathfrak{I}}(a) - R^{\mathfrak{I}}(b)| \geq r$ for all $a, b \in |\mathfrak{I}|$ such that $E^{\mathfrak{I}}(a,b) = 0$. Suppose this was not the case so that $|R^{\mathfrak{I}}(a) - R^{\mathfrak{I}}(b)| < r$ and $E^{\mathfrak{I}}(a,b) = 0$ for some $a, b \in |\mathfrak{I}|$. As remarked earlier, we have $\mathfrak{I}((\circ R^{\mathfrak{I}}(a) \supset R^{\mathfrak{I}}(b)) \vee (\circ R^{\mathfrak{I}}(b) \supset R^{\mathfrak{I}}(a))) = \max\{R^{\mathfrak{I}}(a), R^{\mathfrak{I}}(b)\}$. Since $\pi_0(E^{\mathfrak{I}}(a,b)) = 0$, we find from the definition of C_6 that $C_6 = \max\{R^{\mathfrak{I}}(a), R^{\mathfrak{I}}(b)\} \leq \sup_x R^{\mathfrak{I}}(x) < \mathfrak{I}(F) = C_6$, which is a contradiction.

Put $a \sim b :\Leftrightarrow E^{\mathfrak{I}}(a,b) > 0$ for all $a, b \in |\mathfrak{I}|$. Since $1 = C_1 = C_2 = C_3$, this is an equivalence relation. We will prove that there are only finitely many equivalence classes w.r.t. $\sim$. Suppose this was not the case so that we can find a sequence $(x_i)_{i \in \mathbb{N}}$ such that $x_i \sim x_j$ if and only if $i = j$. whenever $i \neq j$, we have $E^{\mathfrak{I}}(x_i, x_j) = 0$ so that then, by $C_6 = 1$ and the remark further above, we see $|R^{\mathfrak{I}}(x_i) - R^{\mathfrak{I}}(x_j)| \geq r$. However this contradicts the topological compactness of $[0,1]$ since, as proved earlier, $r > 0$.

Let P_k be an m-ary predicate symbol. Suppose we have elements in $|\mathfrak{I}|$ such that $a_1 \sim b_1$, …, $a_m \sim b_m$ so that $\pi_0(E^{\mathfrak{I}})(a_i, b_i) = 1$ whenever $1 \leq i \leq m$. It follows from $C_{4,k} = 1$ that $1 = \pi_0(P_k^{\mathfrak{I}}(a_1,\ldots,a_m)) \bowtie \pi_0(P_k^{\mathfrak{I}}(b_1,\ldots,b_m))$. It is easy to see that $x \bowtie y = 1$ if and only if $x = y$. Thus we find $\pi_0(P_k^{\mathfrak{I}}(a_1,\ldots,a_m)) = \pi_0(P_k^{\mathfrak{I}}(a_1,\ldots,a_m))$.

The above paragraph shows that a classical interpretation $\mathfrak{I}'$ whose domain $|\mathfrak{I}'|$ consists of the equivalence classes w.r.t. $\sim$ can be well-defined by $P_k^{\mathfrak{I}'}([x_1],\ldots,[x_m]) := \pi_0(P_k^{\mathfrak{I}}(x_1,\ldots,x_m)) \in \{0,1\}$; here $[x]$ denotes the equivalence classes containing x. Free variables are interpreted by $v^{\mathfrak{I}'} := [v^{\mathfrak{I}}]$. Since π_0 renders double negation, one can readily show by induction on the formula complexity that $\mathfrak{I}'(B[x_1],\ldots,[x_m]) = \mathfrak{I}((B(x_1,\ldots,x_m))^{\neg\neg})$ for all x_1, …, $x_m \in |\mathfrak{I}|$ and all formulas B. In particular, we have $\mathfrak{I}'(A) = 0$ because of $\mathfrak{I}(A^{\neg\neg}) = 0$. This completes the proof. □

The following proposition allows us to sharpen the above lemma by performing a kind of quantifier shift, which goes back to Takeuti and Titani [13], Extra Axiom Schema 6

$$(\forall x A(x) \supset B) \supset \exists x((A(x) \supset C) \vee (C \supset B))$$

and their Theorem 1.1, Formula (13),

$$(A \supset \exists x B(x)) \supset \exists x((A \supset C) \vee (C \supset B(x))),$$

where C does not contain x.

Proposition 1 *Let a_0, b_0, ..., a_n, b_n, c_0, d_0, ..., c_m, d_m be blocks of distinct bound variables, possibly empty, and let $K := K(a_0, b_0, \ldots, a_n, b_n)$ and $L := L(c_0, d_0, \ldots, c_m, d_m)$ be semiformulas in $\mathcal{L}_{\circ,\triangle}$.*

Let U and V be fresh nullary predicate symbols. W.r.t. Gödel semantics, the validity of the following formulas is then equivalent:

(1) $(\exists a_0 \forall b_0 \ldots \exists a_n \forall b_n K) \supset (\exists c_0 \forall d_0 \ldots \exists c_m \forall d_m L)$
(2) $\forall a_0 \exists b_0 \ldots \forall a_n \exists b_n \exists c_0 \forall d_0 \ldots \exists c_m \forall d_m ((K \supset U) \vee (U \supset V) \vee (V \supset L))$.

Proof We only give a sketch of the elementary arguments:

Given $d \in [0, 1]$ and a $[0, 1]$-valued function f on a nonempty set X, we obviously have $d < \inf_x f(x)$ if and only if $\exists c \in [0, 1]. \forall x \in X. d < c < f(x)$. Similarly, we find that $d < \sup_x$ holds if and only if $\exists c \in [0, 1]. \exists x \in X. d < c < f(x)$; this is the case if and only if $\exists x \in X. d < f(x)$.

By applying the above equivalences and by unwinding the definitions of the quantifier interpretations, we see for any $e \in [0, 1]$ that $e < \mathfrak{I}(\exists a_0 \forall b_0 \ldots \exists a_n \forall b_n K)$ holds if and only if there is $r \in [0, 1]$ such that $\exists a_0 \in X. \forall b_0 \in X. \ldots \exists a_n \in X. \forall b_n \in X. e < r < \mathfrak{I}(K)$. Likewise, $\mathfrak{I}(\exists c_0 \forall d_0 \ldots \exists c_m \forall d_m K) < e$ holds if and only if there is $r \in [0, 1]$ such that $\forall c_0 \in X. \exists d_0 \in X. \ldots \forall c_m \in X. \exists d_m \in X. \mathfrak{I}(L) < r < e$.

Clearly, (1) is not valid if and only if there are an interpretation $\mathfrak{I}$ and $e \in [0, 1]$ such that $\mathfrak{I}(\exists c_0 \forall d_0 \ldots \exists c_m \forall d_m L) < e < \mathfrak{I}(\exists a_0 \forall b_0 \ldots \exists a_n \forall b_n K)$. By the above paragraph, this is equivalent to the existence of $r, s \in [0, 1]$ such that $\exists a_0. \forall b_0. \ldots \exists a_n. \forall b_n. \forall c_0. \exists d_0. \ldots \forall c_m. \exists d_m. \mathfrak{I}(L) < s < r < \mathfrak{I}(K)$.

We can readily prove that (2) is not valid if and only if there is some interpretation $\mathfrak{I}'$ and some $g \in [0, 1]$ such that $\exists a_0. \forall b_0. \ldots \exists a_n. \forall b_n. \forall c_0. \exists d_0. \ldots \forall c_m. \exists d_m. F$, where F abbreviates the condition $\mathfrak{I}'((K \supset U) \vee (U \supset V) \vee (V \supset L)) < g < 1$. Here, F can be replaced by $\mathfrak{I}'(L) < \mathfrak{I}'(V) < \mathfrak{I}'(U) < \mathfrak{I}'(K) \wedge \mathfrak{I}'(U) < g < 1$ since it is easy to verify that for every interpretation $\mathfrak{I}'$, every $h \in [0, 1]$ and all propositional atoms A, B, C, D we have $\mathfrak{I}'((A \supset B) \vee (B \supset C) \vee (C \supset D)) < h$ if and only if $\mathfrak{I}'(D) < \mathfrak{I}'(C) < \mathfrak{I}'(B) < \mathfrak{I}'(A) \wedge \mathfrak{I}'(B) < h$.

The above conditions expressing the nonvalidity of (1) and (2) can be seen to be equivalent as follows: For one direction, put $s := \mathfrak{I}'(V)$, $r := \mathfrak{I}'(U)$. For the other, redefine $\mathfrak{I}(U)$ to be r and $\mathfrak{I}(V)$ to be s and put $g := \frac{1+r}{2}$. This provides the required properties and completes the proof. □

Since $\beta(A)$ in the proof of Lemma 1 has the form assumed in Proposition 1, we obtain the following corollary:

Corollary 1 *There is an effective translation α of the closed prenex formulas in $\mathcal{L}$ to the closed prenex formulas in $\mathcal{L}_\circ$ such that any closed prenex formula A is classically valid in all finite domains if and only if $\alpha(A)$ is valid in $\mathcal{L}_\circ$ w.r.t. Gödel semantics.*

Fig. 1 Proof system IPL

(MP)	$\dfrac{A \quad A \supset B}{B}$
(IPL1)	$\bot \supset A$
(IPL2)	$(A \wedge B) \supset A$
(IPL3)	$(A \wedge B) \supset B$
(IPL4)	$A \supset (B \supset (A \wedge B))$
(IPL5)	$A \supset (A \vee B)$
(IPL6)	$B \supset (A \vee B)$
(IPL7)	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
(IPL8)	$A \supset (B \supset A)$
(IPL9)	$(A \supset B \supset C) \supset ((A \supset B) \supset (A \supset C))$

By Trakhtenbrot's theorem [14], the set of closed (prenex) formulas that are classically valid in all finite domains are not r.e. From Corollary 1, we immediately obtain:

Corollary 2 *The set of valid prenex formulas in $\mathcal{L}_\circ$ w.r.t. Gödel semantics is Π_2-hard.*

4 Proof systems for validity in $\mathcal{L}_\circ^{\mathrm{p}}$ and $\mathcal{L}_{\circ,\triangle}^{\mathrm{p}}$

In this section we will prove Theorem 3, where we present (1) proof systems that are sound and (weakly) complete for the Gödel logics in $\mathcal{L}_\circ^{\mathrm{p}}$ and $\mathcal{L}_{\circ,\triangle}^{\mathrm{p}}$ and (2) an algorithm to derive a valid formula in these proof systems. Dummett's paper [7] immediately implies the following theorem, and the method of proof presented therein forms also the basis for our results. After Definition 2, we will describe how we have extended this method. For an exposition of Dummett's result and a discussion for more general operators, see [8, Section 3].

Let IPL denote the proof system in Fig. 1, which is sound and complete for intuitionistic propositional logic.

Theorem 1 *Let* **G** *be the extension of* IPL *by the axiom scheme of linearity*

$$(\mathsf{LIN}):\ (A \supset B) \vee (B \supset A).$$

For any formula A in $\mathcal{L}^p$, the proof system **G** *proves A if and only if A is valid w.r.t. the Gödel semantics for $\mathcal{L}^p$.*

The proof system **G** *is sound and complete w.r.t. Gödel semantics for $\mathcal{L}^p$. Moreover, there is an algorithm that either constructs a* **G***-proof of a formula A or constructs a Gödel evaluation $\mathfrak{I}$ such that $\mathfrak{I}(A) < 1$, i.e. a countermodel to the validity of A.*

The following proposition summarises well-known properties of **G** and will allow us to abbreviate formal proofs in $\mathcal{L}_\circ^{\mathrm{p}}$ and $\mathcal{L}_{\circ,\triangle}^{\mathrm{p}}$.

Proposition 2 *The proof system* **G** *in the language $\mathcal{L}^p$ has the following properties: The deduction theorem holds, i.e.,* $\mathbf{G} + A \vdash B$ *if and only if* $\mathbf{G} \vdash A \supset B$*. The rule* $\dfrac{A \supset B \quad B \supset C}{A \supset C}$ *is derivable. Let $E[\cdot]$ denote an $\mathcal{L}^p$-context, and let A, B, C, D be formulas in $\mathcal{L}^p$. Then* **G** *proves the following:*

(G1) $A \prec \top$
(G2) $(\bot \prec A) \vee (\bot \leftrightarrow A)$
(G3) $(A \prec B) \vee (A \leftrightarrow B) \vee (B \prec A)$
(G4) $(A \leftrightarrow B) \supset (E[A] \leftrightarrow E[B])$
(G5) $(A \Box B) \supset (E[A \wedge B] \leftrightarrow E[A])$ *for* $\Box \in \{\prec, \leftrightarrow\}$
(G6) $(A \Box B) \supset (E[A \vee B] \leftrightarrow E[B])$ *for* $\Box \in \{\prec, \leftrightarrow\}$
(G7) $(A \Box B) \supset (E[A \supset B] \leftrightarrow E[\top])$ *for* $\Box \in \{\prec, \leftrightarrow\}$
(G8) $(A \prec B) \supset (E[B \supset A] \leftrightarrow E[A])$
(G9) $E[A \Box A] \leftrightarrow E[A]$ *for* $\Box \in \{\wedge, \vee\}$
(G10) $E[A \Box B] \leftrightarrow E[B \Box A]$ *for* $\Box \in \{\wedge, \vee, \leftrightarrow\}$
(G11) $E[(A \Box B) \Box C] \leftrightarrow E[A \Box (B \Box C)]$ *for* $\Box \in \{\wedge, \vee\}$
(G12) $E[A \Box (B \Diamond C)] \leftrightarrow E[(A \Box B) \Diamond (A \Box C)]$ *for* $\Box, \Diamond \in \{\wedge, \vee\}$
(G13) $(A \prec A) \leftrightarrow ((A \leftrightarrow A) \wedge (A \leftrightarrow \top))$
(G14) $A \supset A$
(G15) $((A \prec B) \wedge (B \Box A)) \leftrightarrow ((A \leftrightarrow B) \wedge (B \leftrightarrow \top))$ *for* $\Box \in \{\supset, \prec, \leftrightarrow\}$
(G16) $((A \Box B) \wedge (B \Diamond C) \wedge (C \prec A)) \leftrightarrow ((A \leftrightarrow B) \wedge (B \leftrightarrow C) \wedge (C \leftrightarrow \top))$ *for* $\Box, \Diamond \in \{\leftrightarrow, \prec\}$
(G17) $E[\top \vee A] \leftrightarrow E[\top]$
(G18) $E[\bot \vee A] \leftrightarrow E[A]$
(G19) $E[\bot \wedge A] \leftrightarrow E[\bot]$
(G20) $E[A \prec \bot] \leftrightarrow E[\bot]$
(G21) $(\top \prec A) \leftrightarrow (\top \leftrightarrow A)$
(G22) $(B \vee C) \leftrightarrow (((A \leftrightarrow A) \wedge B) \vee C)$
(G23) $(((A \leftrightarrow B) \wedge C) \vee D) \leftrightarrow (((A \leftrightarrow B) \wedge (B \leftrightarrow A) \wedge C) \vee D)$
(G24) $(((A \prec \bot) \wedge B) \vee C) \leftrightarrow C$
(G25) $(A \supset (B \leftrightarrow C)) \supset ((A \wedge (B \prec C)) \leftrightarrow (A \wedge (B \leftrightarrow C) \wedge (C \leftrightarrow \top)))$
(G26) $(A \supset (B \prec C)) \supset ((A \wedge (C \prec B)) \leftrightarrow (A \wedge (B \leftrightarrow C) \wedge (C \leftrightarrow \top)))$
(G27) $(A \supset (B \prec C)) \supset ((A \wedge (B \leftrightarrow C) \wedge (C \prec \top)) \leftrightarrow (A \wedge (B \leftrightarrow C) \wedge (C \leftrightarrow \top)))$
(G28) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$
(G29) $(A \supset B) \supset ((B \prec A) \leftrightarrow ((A \leftrightarrow B) \wedge (B \leftrightarrow \top)))$

Proof The validity of these formulas w.r.t. Gödel semantics can be effortlessly checked. Therefore it suffices to apply Theorem 1 in order to verify the derivability of every instance of the formulas (G1)–(G29). The proofs of the other claims are routine. □

Definition 2 Let $\mathbf{G}_\circ$ denote the proof system in $\mathcal{L}_\circ^{\mathrm{p}}$ that extends **G** by the following axiom schemata:

(R1) $(\bot \prec \circ\bot) \supset (A \prec \circ A)$,
(R2) $(\bot \leftrightarrow \circ\bot) \supset (A \leftrightarrow \circ A)$,
(R3) $\circ(A \supset B) \leftrightarrow (\circ A \supset \circ B)$,

where A and B are any formulas from $\mathcal{L}_\circ^{\mathrm{p}}$. If A is $\mathbf{G}_\circ$-derivable, we write $\mathbf{G}_\circ \vdash A$.

We omit the easy proof of the following lemma.

Lemma 2 $\mathbf{G}_\circ$ *is sound w.r.t. Gödel semantics in* $\mathcal{L}_\circ^{p}$.

The completeness of $\mathbf{G}_\circ$ however requires a lengthy and technical but elementary proof, which occupies the rest of this section. The method of proof goes back to Dummett [7] and is also presented in a more modern way in [4] to investigate interpolation in Gödel logics. For the sake of self-containedness, we shall repeat the main ideas here:

Once one has found a formal derivation of (G3), axiom (IPL7) allows us to make case distinctions in the following sense: In order to prove a given formula D, it suffices to prove $(X \prec Y) \supset D$, $(X \leftrightarrow Y) \supset D$, and $(Y \prec X) \supset D$; here X and Y can be chosen arbitrarily. Let $\Box_i \in \{\prec, \leftrightarrow\}$; since IPL can prove that $(X_1 \Box_1 Y_1) \supset ((X_2 \Box_2 Y_2) \supset \ldots ((X_n \Box_n Y_n) \supset D))$ and $((X_1 \Box_1 Y_1) \wedge \cdots \wedge (X_n \Box_n Y_n)) \supset D$ are equivalent, these case distinctions can be gathered into a conjunction. We now assume that all case distinctions for all unordered pairs $\{X, Y\}$ of variables in D, of $\top$ and of $\bot$ are performed in the above way. The derivable formulas (G13), (G15), (G16) etc. allow us to transform the above conjunction with a branch of these case distinctions into a so-called chain, which is a conjunction of the form $C := (X_1 \Box_1 X_2) \wedge (X_2 \Box_2 X_3) \wedge \cdots \wedge (X_{n-1} \Box_{n-1} X_n)$ with variables X_i. The formulas (G5)–(G8) are then used to "evaluate" D under C, i.e., to find a derivation of $C \supset (D \leftrightarrow D')$, where D' is a variable, or $\bot$, or $\top$. This is done step-by-step by replacing some innermost non-trivial subformula by a variable, $\bot$, or $\top$. Observe that we need two features of Gödel logics for this: In-depth substitution (G4), and the projection property, which says that $\mathfrak{I}(F(A_1, \ldots, A_n)) \in \{\mathfrak{I}(\top), \mathfrak{I}(\bot), \mathfrak{I}(A_1), \ldots, \mathfrak{I}(A_n)\}$ for any interpretation $\mathfrak{I}$ and any formula F in variables $A_1, \ldots, A_n$. If D' is $\bot$ or a variable (that is not in the same $\leftrightarrow$-equivalence class as $\top$ w.r.t. C) one can immediately read off from C a countermodel $\mathfrak{I}$ to the validity of D due to the chain form of C: It is obvious how to construct a "realising" interpretation $\mathfrak{I}$ of C, i.e., with $\mathfrak{I}(X_1) \Diamond_1 \mathfrak{I}(X_2) \Diamond_2 \ldots \Diamond_{n-1} \mathfrak{I}(X_n)$ where $\Diamond_i$ is $=$ whenever $\Box_i = \leftrightarrow$ and where $\Diamond_i = <$ whenever $\Box_i = \prec$. We have $\mathfrak{I}(C) = 1$ but $\mathfrak{I}(D) < 1$ then. But, as D was assumed to be valid, this indirect argument shows that D' can be chosen as $\top$, as required.

For $\mathcal{L}_\circ^{\mathrm{p}}$, this method needs only few modifications, which we sketch briefly: The case distinctions have to be enlarged from all variables to ring powers $\circ^m A$; here A is $\bot$ or a variable, and $m \in \mathbb{N}$ is bound by the maximal nesting level of rings in D. Definition 5 extends the notion of a chain accordingly but also excludes certain orderings to render the fact that the ring axioms (R1)–(R3) can strengthen certain conjunctions of case distinctions further, e.g., $a \prec b$ and $\circ b \prec \circ a$ ought not to occur simultaneously in a chain since there is a $\mathbf{G}_\circ$-derivation of $((a \prec b) \wedge (\circ b \prec \circ a)) \leftrightarrow ((a \prec b) \wedge (\circ a \leftrightarrow \circ b) \wedge (\circ b \leftrightarrow \top))$. The text from Proposition 3 to Definition 6 contains the tedious proof that one can construct, for every chain C, a value $r \in \mathbb{R}$ and a realising r-interpretation $\mathfrak{I}$, with the same properties as in the previous paragraph. As a consequence, Definition 5 can be discerned as an appropriate generalisation of the ringless chain in the sense that it rules out exactly those orderings that can be strengthened.

Let us remark by the way that a typical "algebraical" Lindenbaum-Tarski argument to prove completeness of $\mathbf{G}_\circ$ w.r.t. the given semantics cannot work: A close inspection of this method reveals that it relies on the compactness of the chosen entailment relation but neither 1-entailment nor entailment are compact as the following example shows. For $R := \{\circ^k A \supset B; k \in \mathbb{N}\}$ and $S := \{B \vee \neg\circ\bot\}$, we have $R \vDash S$ and $R \vDash^1 S$ but

$U \not\models S$ and $U \not\models^1 S$ for any finite subset U of R. It would be interesting if any other "algebraical" approach can provide a proof of completeness.

In the following proofs, we will often just state that a certain $\mathcal{L}^{\mathrm{p}}$-formula is **G**-provable but we will leave it to the reader to verify its validity and then apply Theorem 1 to obtain a **G**-proof.

Proposition 3 $\mathbf{G}_\circ$ *proves for any $\mathcal{L}^p_\circ$-formulas A, B, C and any $\mathcal{L}^p_\circ$-context $E[\cdot]$:*

(S1) $A \supset \circ A$

(S2) $\circ(A \,\Box\, B) \leftrightarrow (\circ A \,\Box\, \circ B)$ *for* $\Box \in \{\prec, \wedge, \vee, \leftrightarrow\}$

(S3) $(A \leftrightarrow B) \supset (E[A] \leftrightarrow E[B])$

Proof (S1): Since **G** proves $((\bot \prec C)\supset(D \prec E))\supset(((\bot\leftrightarrow C)\supset(D\leftrightarrow E))\supset(D\supset E))$, $\mathbf{G}_\circ$ proves its instance $((\bot \prec \circ\bot)\supset(A \prec \circ A))\supset(((\bot\leftrightarrow\circ\bot)\supset(A\leftrightarrow\circ A))\supset(A\supset\circ A))$. From (R1) and (R2), we obtain (S1).

First, we consider the case $\Box = \prec$ for (S2). Due to (R3), we have $\mathbf{G}_\circ \vdash \circ((B \supset A) \supset B) \leftrightarrow (\circ(B \supset A) \supset \circ B)$ and $\mathbf{G}_\circ \vdash \circ(B \supset A) \leftrightarrow (\circ B \supset \circ A)$. Since **G** proves $(C \leftrightarrow (D \supset E)) \supset (D \leftrightarrow F) \supset (C \leftrightarrow (F \supset E))$, it follows $\mathbf{G}_\circ \vdash \circ((B \supset A) \supset B) \leftrightarrow ((\circ B \supset \circ A) \supset \circ B)$, as required.

For the case $\Box = \wedge$ of (S2), we observe that **G** proves $(((C\wedge D)\supset C)\supset P)\supset(((E\wedge F)\supset F)\supset Q)\supset((G\supset(H\supset(G\wedge H)))\supset R)\supset(P\leftrightarrow(K\supset N))\supset(Q\leftrightarrow(K\supset M))\supset(R\leftrightarrow(N\supset S))\supset(S\leftrightarrow(M\supset K))\supset(K\leftrightarrow(N\wedge M))$. We apply (MP) to an appropriate instance of this formula and the (S1)-instances $((A \wedge B) \supset A) \supset \circ((A \wedge B) \supset A)$, $((A\wedge B)\supset B)\supset\circ((A\wedge B)\supset B)$, $(A\supset(B\supset(A\wedge B)))\supset\circ(A\supset(B\supset(A\wedge B)))$ and the (R3)-instances $\circ((A \wedge B) \supset A) \leftrightarrow (\circ(A \wedge B) \supset \circ A)$, $\circ((A \wedge B) \supset B) \leftrightarrow (\circ(A \wedge B) \supset \circ B)$, $\circ(A\supset(B\supset(A\wedge B)))\leftrightarrow(\circ A\supset\circ(B\supset(A\wedge B)))$, $\circ(B\supset(A\wedge B))\leftrightarrow(\circ B\supset\circ(A\wedge B))$ to obtain $\circ(A \wedge B) \leftrightarrow (\circ A \wedge \circ B)$.

For the case $\Box = \vee$ of (S2), we take an appropriate instance of the **G**-provable formula $(((U\vee V)\supset V)\supset P)\supset(((U\vee V)\supset U)\supset Q)\supset((C\supset(C\vee D))\supset R)\supset((E\supset(E\vee F))\supset S)\supset(P\leftrightarrow(G\supset E))\supset(Q\leftrightarrow(G\supset F))\supset(R\leftrightarrow(E\supset G))\supset(S\leftrightarrow(F\supset G))\supset(G\leftrightarrow(E\vee F))$ and apply the (S1)-instances $((A\vee B)\supset A)\supset\circ((A\vee B)\supset A)$, $((A\vee B)\supset B)\supset\circ((A\vee B)\supset B)$, $(A\supset(A\vee B))\supset\circ(A\supset(A\vee B))$, $(B\supset(A\vee B))\supset\circ(B\supset(A\vee B))$, the (R3)-instance $\circ((A\vee B)\supset A)\leftrightarrow(\circ(A\vee B)\supset\circ A)$, $\circ((A\vee B)\supset B)\leftrightarrow(\circ(A\vee B)\supset\circ B)$, $\circ(A \supset (A \vee B)) \leftrightarrow (\circ A \supset \circ(A \vee B))$, $\circ(B \supset (A \vee B)) \leftrightarrow (\circ B \supset \circ(A \vee B))$ to obtain the required $\circ(A \vee B) \leftrightarrow (\circ A \vee \circ B)$.

For the case $\Box = \leftrightarrow$ of (S2), we take the **G**-provable formula $(G \leftrightarrow (E \wedge F)) \supset (E\leftrightarrow(C\supset D))\supset(F\leftrightarrow(D\supset C))\supset(G\leftrightarrow(C\leftrightarrow D))$, the (S2)($\wedge$)-instance $\circ((A\supset B)\wedge(B\supset A))\leftrightarrow(\circ(A\supset B)\wedge\circ(B\supset A))$ and the (R3)-instances $\circ(A\supset B)\leftrightarrow(\circ A\supset\circ B)$ and $\circ(B\supset A)\leftrightarrow(\circ B\supset\circ A)$ to obtain $\circ((A\supset B)\wedge(B\supset A))\leftrightarrow((\circ A\supset\circ B)\wedge(\circ B\supset\circ A))$, i.e. $\circ(A \leftrightarrow B) \leftrightarrow (\circ A \leftrightarrow \circ B)$, as required.

We will prove now by induction on formula complexity of $E[\cdot]$: $\mathbf{G}_\circ$ proves $(A \leftrightarrow B) \supset (A \leftrightarrow B)$ and $(A \leftrightarrow B) \supset (C \leftrightarrow C)$ since also **G** does. Given a $\mathbf{G}_\circ$-proof of $(A \leftrightarrow B) \supset (E[A] \leftrightarrow E[B])$, one can obtain $\mathbf{G}_\circ \vdash (A \leftrightarrow B) \supset (\circ E[A] \leftrightarrow \circ E[B])$ from the (S2)($\leftrightarrow$)-instance $\circ(E[A] \leftrightarrow E[B]) \leftrightarrow (\circ E[A] \leftrightarrow \circ E[B])$, the (S1)-instance $(E[A] \leftrightarrow E[B]) \supset \circ(E[A] \leftrightarrow E[B])$ and (IPL2). Given $\mathbf{G}_\circ$-proofs of $(A \leftrightarrow B) \supset (E[A] \leftrightarrow E[B])$ and $(A \leftrightarrow B) \supset (F[A] \leftrightarrow F[B])$, we use the **G**-provable formulas

$(X \supset (P \leftrightarrow Q)) \supset (X \supset (R \leftrightarrow S)) \supset (X \supset ((P \Box R) \leftrightarrow (Q \Box S)))$, for any $\Box \in \{\wedge, \vee, \supset\}$, to find $\mathbf{G}_\circ$-proofs of $(A \leftrightarrow B) \supset ((E[A] \Box F[A]) \leftrightarrow (E[B] \Box F[B]))$. This establishes (S3). □

Definition 3 (*Grid*) Given a finite, non-empty set X, we call $(Y, \ll, \sim)$ a *grid over* X if there is some $N: X \to \mathbb{N}$ such that $Y = \{(x, n); x \in X, n \leq N(x)\}$, $\sim$ is a reflexive, symmetric and transitive relation on Y, $\ll$ is a transitive relation on Y and, moreover, for all $a, b, c \in Y$ we have

(T1) either $a \ll b$ or $a \sim b$ or $b \ll a$;
(T2) $a \sim b \ll c$ implies $a \ll c$,
(T3) $a \ll b \sim c$ implies $a \ll c$,
(T4) $a + 1 \in Y$ implies $a \ll a + 1$,
(T5) $(a + 1 \in Y \wedge b + 1 \in Y)$ implies ($a \ll b$ if and only if $a + 1 \ll b + 1$),
(T6) $(a + 1 \in Y \wedge b + 1 \in Y)$ implies ($a \sim b$ if and only if $a + 1 \sim b + 1$),

here we have put $(x, n) + k := (x, n + k)$ for all $n, k \in \mathbb{N}$ and $x \in X$. We will regard X as a subset of Y by virtue of $x \mapsto (x, 0)$. We put $\lesssim := \ll \cup \sim$.

Lemma 3 *Let $C = (Y, \ll, \sim)$ be a grid over X. Then there is an algorithm to construct a grid $C_* = (Y_*, \ll^*, =)$ over X and a $\sigma: Y \to Y_*$ such that for all $y, y' \in Y$:*

$y \sim y'$ *if and only if* $\sigma(y) = \sigma(y')$,
$y \ll y'$ *if and only if* $\sigma(y) \ll^* \sigma(y')$,
$y + 1 \in Y$ *implies* $\sigma(y + 1) = \sigma(y) + 1 \in Y_*$

Proof We will prove only the following statement: For every grid $C = (Y, \ll, \sim)$ and $p, q \in Y$ such that $p \sim q$, we can specify a grid $C' = (Y_*, \ll^*, \sim^*)$ and $t: Y \to Y_*$ such that $t(p) = t(q)$ and for all $y, y_* \in Y$ holds: (1) $y \sim y_*$ if and only if $t(y) \sim^* t(y_*)$, (2) $y \ll y_*$ if and only if $t(y) \ll^* t(y_*)$, and (3) $t(y + 1) \sim^* t(y) + 1 \in Y_*$ whenever $y \in Y$ such that $y + 1 \in Y$. Observe that, provided that p and q are chosen differently, the number of equivalence classes in C_* is less than in C. By applying the above statement iteratively, we obtain a C_* with only one equivalence class. The lemma is then established by taking σ as the concatenation of the intermediate t's.

Thus, let $C = (Y, \ll, \sim)$ and $p, q \in Y$ such that $p \sim q$. Then there are $a, b \in X$ and $n, m \in \mathbb{N}$ such that $p = (a, n) \sim q = (b, m)$. W.l.o.g. we assume that $n \geq m$. Thus, by (T5), $a + K \sim b$ for $K := n - m \in \mathbb{N}$. Define $N'(x) := N(x)$ for all $x \in X \setminus \{a, b\}$ and let $N'(a) := \max\{N(a), N(b) + K\}$. Put $Y_* := \{(x, i); x \in X \setminus \{b\}, i < N'(x)\}$ and define $t: Y \to Y_*$ by $t(b, i) := (a, i + K)$ and $t(x, i) := (x, i)$ for all $x \in X \setminus \{b\}$.

For all $(x, n), (x', n') \in Y$, we will see by distinguishing four cases that $t(x, n) = t(x', n')$ implies $(x, n) \sim (x', n')$: If both $x, x' \in X \setminus \{b\}$ then $(x, n) = t(x, n) = t(x', n') = (x', n')$. If $x = b = x'$ then $(a, n + K) = t(x, n) = t(x', n') = (a, n' + K)$ so that $n = n'$ and thus $(x, n) \sim (x', n')$. If $x = b$ and $x' \in X \setminus \{b\}$ then $(x, n) = (b, n) \sim (a, n + K) = t(x, n) = t(x', n') = (x', n')$. If $x' = b$ and $x \in X \setminus \{b\}$ then $(x, n) = t(x, n) = t(x', n') = (a, n' + K) \sim (b, n') = (x', n')$.

Since t is surjective, the property just proved enables us to define two relations $\ll^*$ and $\sim^*$ on Y_* by $t(c) \ll^* t(d) :\Leftrightarrow c \ll d$ and by $t(c) \sim^* t(d) :\Leftrightarrow c \sim d$. This establishes properties (1) and (2). Clearly, $t(b, i + 1) = (a, i + K + 1) = t(b, i) + 1$

and $t(x, i+1) = (x, i+1) = (x, i) + 1 = t(x, i) + 1$ for all $x \in X \setminus \{b\}$. Thus property (3) holds, and so it easy to check that $(Y_*, \ll^*, \sim^*)$ is indeed a grid. We also have $t(q) = t(b, m) = (a, m + K) = (a, n) = t(a, n) = t(p)$. □

Given a grid $(Y, \ll, \sim)$, it can be easily seen that every non-empty subset Z of Y has a $\lessapprox$-minimal element, i.e. there is $u \in Z$ such that $u \lessapprox u'$ for all $u' \in Z$; likewise, there is a $\lessapprox$-maximal element $U \in Z$. In particular, any non-empty subset of a grid $(Y, \ll, =)$, where the equivalence relation is the identity, has a unique $\ll$-minimal and a unique $\ll$-maximal element.

Lemma 4 *Let $(Y, \ll, =)$ be a finite grid over X and let s be the $\ll$-minimal element of Y. Then we can construct $f: Y \to [0, \infty) \cap \mathbb{Q}$ such that $f(s) = 0$ and such that for all $y, y' \in Y$:*

$y \ll y'$ implies $f(y) < f(y')$,
$y + 1 \in Y$ implies $f(y + 1) = f(y) + 1$.

Proof We will use the following definitions in the slightly involved iterative construction of f. Let S be the $\ll$-maximal element of Y, cf. the remark before the statement of the lemma. Take $\underline{\ll} := \ll \cup \mathrm{id}_Y$, which is obviously a reflexive and transitive relation on Y. Let $E(a, f)$ abbreviate the condition that $a \in Y$ and f is a function from $\{y \in Y; y \underline{\ll} a\}$ to $[0, \infty) \cap \mathbb{Q}$ such that for all $y, y' \in Y$:

(1) $y \ll y' \underline{\ll} a \supset f(y) < f(y')$,
(2) $(y + 1 \in Y \wedge y + 1 \underline{\ll} a) \supset f(y + 1) = f(y) + 1$,
(3) $(y + 1 \in Y \wedge y \underline{\ll} a \ll y + 1) \supset f(a) < f(y) + 1$.

We start the construction by $f_0(s) := 0$ so that $E(s, f_0)$ holds. The remainder of the proof is dedicated to the demonstration that, for any given $a \in Y$ and any f such that $a \ll S$ and $E(a, f)$, we can construct $a_* \in Y$ and f_* such that $a \ll a_*$ and $E(a_*, f_*)$; in particular, we have $a \neq a_*$ then. This suffices to construct an f such that $E(S, f)$ so that f is total on Y and then conditions (1) and (2) establish the lemma.

Suppose now that $a \in Y$ and f are given such that $a \ll S$ and $E(a, f)$ hold. We will distinguish two cases.

In the first case, we suppose that $\emptyset \neq B := \{b \in Y; b \underline{\ll} a \ll b + 1\}$ holds. Let b be $\underline{\ll}$-minimal in B. By definition of B, we have $b + 1 \in Y$. By (1) and (3), we see $f(b) \leq f(a) < f(b) + 1$, thus $0 < f(b) + 1 - f(a) \leq 1$.

For $C := \{c \in Y; a \ll c \ll b + 1\}$, we have $C \subseteq X$ for otherwise $a \ll c \ll b + 1$ for some $c \in Y \setminus X$; the latter means that $c = d + 1$ for some $d \in Y$, but then $a \ll d + 1 \ll b + 1$ implies $d \ll b$ by (T5), which contradicts the minimality of b.

Let $c_1 \ll c_2 \ll \ldots \ll c_M$ be an enumeration of C. Extend f to f_* by $f_*(c_m) := f(a) + \frac{m}{M+1}(f(b) + 1 - f(a))$ and $f_*(b + 1) := f(b) + 1$ so that f_* is defined for all $y \underline{\ll} b + 1$ and $f(a) = f_*(a) < f_*(c_1) < f_*(c_2) < \cdots < f_*(c_M) < f_*(b + 1)$. Thus, by the definition of C, we see $f_*(a) < f_*(y) < f_*(y') \leq f_*(b + 1)$ whenever $a \underline{\ll} y \ll y' \underline{\ll} b + 1$. Now, we will prove $E(b + 1, f_*)$. The statements (1), (2), (3) will refer to the conditions in $E(a, f)$.

Given $y, y' \in Y$ with $y \ll y' \underline{\ll} b + 1$, we need to show $f_*(y) < f_*(y')$. We may assume $y \ll a$ since $a \underline{\ll} y$ implies $a \underline{\ll} y \ll y' \underline{\ll} b+1$ and this yields, as proved above,

$f_*(y) < f_*(y')$. From $y \ll a$, we see $f_*(y) = f(y) < f(a) = f_*(a)$ by (1). We may assume also $a \ll y'$ since $y' \underline{\ll} a$ implies $y \ll y' \underline{\ll} a$ and then $f_*(y) < f_*(y')$ by (1). As observed earlier, we have $f_*(a) < f_*(y')$ and thus $f_*(y) < f_*(a) < f_*(y')$, as required.

Given $y \in Y$ such that $y+1 \in Y$ and $y+1 \underline{\ll} b+1$, we need to prove $f_*(y+1) = f_*(y)+1$. We may assume $y+1 \ll b+1$ because $b+1 \underline{\ll} y+1$, together with $y+1 \underline{\ll} b+1$, yields $b+1 = y+1$ so that $y = b$ and $f_*(b+1) = f_*(b)+1$ by definition of f_*. We may assume $y+1 \underline{\ll} a$ for otherwise $a \ll y+1 \in Y$ holds and thus $y+1 \in C$ but this contradicts $C \subseteq X$ and $y \in Y$. Now $y \ll y+1 \underline{\ll} a$, and so $f_*(y+1) = f(y+1) = f(y)+1 = f_*(y)+1$ by (2), as required.

The previous three paragraphs complete the proof of $E(b+1, f_*)$ for this first case.

In the second case, we suppose that $\emptyset = \{y \in Y;\ y \underline{\ll} a \ll y+1\}$ holds in addition to $E(a, f)$ and $a \ll S$. Due to $a \ll S$ there is some $c \in Y$ that is $\underline{\ll}$-minimal among the $y \in Y$ with $a \ll y$. We must have $c \in X$ for otherwise there is $y \in Y$ with $a \ll c = y+1$, but since $y \underline{\ll} a \ll y+1$ is impossible, we obtain $a \ll y$ and thus $a \ll y \ll y+1 = c$, which contradicts the minimality of c. Extend f to f_* by $f_*(c) := f(a) + \varepsilon$ for some $\varepsilon > 0$, e. g. $\varepsilon = 1$. We will prove $E(c, f_*)$.

Given $y, y' \in Y$ such that $y \ll y' \underline{\ll} c$, we need to prove $f_*(y) < f_*(y')$. We may assume $a \ll y'$ since otherwise $y' \underline{\ll} a$ holds and then $y \ll y' \underline{\ll} a$ implies $f_*(y) = f(y) < f(y') = f_*(y')$ by (1). From $a \ll y' \underline{\ll} c$, we conclude $y' = c$ by minimality of c. We may assume that $a \neq y$ since $a = y$ implies $f(y) = f(a) < f(a) + \varepsilon = f_*(c) = f_*(y')$. We cannot have $a \ll y$ since then $a \ll y \ll y' = c$ contradicted the minimality of c. Thus $y \ll a$ and now $f_*(y) = f(y) < f(a) < f(a) + \varepsilon = f_*(c) = f_*(y')$ by (1), as required.

Given $y \in Y$ such that $y+1 \in Y$, $y+1 \underline{\ll} c$, we need to prove $f_*(y+1) = f_*(y)+1$. We have $y \underline{\ll} a$ since otherwise $a \ll y$ holds, which implies $a \ll y \ll y+1 \underline{\ll} c$, but this contradicts the minimality of c. Since $y \underline{\ll} a \ll y+1$ is impossible, we have $y+1 \underline{\ll} a$. By (2), we find $f(y+1) = f(y)+1$, as required.

Given $y \in Y$ such that $y+1 \in Y$, $y \underline{\ll} c \ll y+1$, we need to prove $f_*(c) < f_*(y)+1$. We must have $a \ll y+1$ for otherwise we obtain a contradiction from $y+1 \underline{\ll} a$ and $a \ll c \ll y+1$. Since $y \underline{\ll} a \ll y+1$ cannot hold, we must have $a \ll y$. From $y \underline{\ll} c$ and the minimality of c, we conclude that $y = c$. Thus $f_*(c) < f_*(y)+1$ trivially holds.

Thus $E(c, f_*)$ holds, as claimed, also in the second case. This completes the whole proof. □

Definition 4 (*Chain*) Let $K \in \mathbb{N}$, let $X \subseteq \mathsf{Var}$ be finite and choose two distinct fresh formal symbols $\top$ and $\bot$. Put $Z' := \{(x, k);\ x \in X \cup \{\bot\}, k \leq K\}$, $Z := \{\top\} \cup Z'$, and $(x, m) + n := (x, m+n)$ for all $x \in X \cup \{\bot\}$, $m, n \in \mathbb{N}$. We understand X as a subset of Z' by virtue of the embedding $x \mapsto (x, 0)$. We call $(Z, \prec, \leftrightarrow)$ an (X, K)-*chain* if $\leftrightarrow$ is a reflexive, symmetric, transitive relation on Z, $\prec$ is a transitive relation on Z such that for all $a, b, c \in Z$ and for all $\alpha, \beta \in Z'$:

(U1) $a \leftrightarrow b \prec c \ \supset\ a \prec c$,
(U2) $a \prec b \leftrightarrow c \ \supset\ a \prec c$,
(U3) either $a \prec \top$ or $a \leftrightarrow \top$,
(U4) either $\bot \prec a$ or $\bot \leftrightarrow a$,

(U5) either $a \prec b$ or $a \leftrightarrow b$ or $b \prec a$,
(U6) $(\alpha + 1 \in Z \wedge \bot \leftrightarrow \bot + 1) \supset \alpha \leftrightarrow \alpha + 1$.
(U7) $(\alpha + 1 \in Z \wedge \bot \prec \bot + 1) \supset (\alpha \prec \alpha + 1 \vee \alpha \leftrightarrow \alpha + 1 \leftrightarrow \top)$,
(U8) $(\alpha + 1 \in Z \wedge \beta + 1 \in Z \wedge \alpha \leftrightarrow \beta) \supset \alpha + 1 \leftrightarrow \beta + 1$,
(U9) $(\alpha + 1 \in Z \wedge \beta + 1 \in Z \wedge \bot \prec \bot + 1 \wedge \alpha \prec \beta)$
$\supset(\alpha + 1 \prec \beta + 1 \vee \alpha + 1 \leftrightarrow \beta + 1 \leftrightarrow \top)$,

Lemma 5 *Let $(Z, \prec, \leftrightarrow)$ be an (X, K)-chain. Put $Z' := Z \setminus \{\top\}$. Then we can construct $r \in [0, 1] \cap \mathbb{Q}$ and $g\colon Z \to [0, 1] \cap \mathbb{Q}$ such that $g(\bot) = 0$, $g(\top) = 1$ and such that for all $a, b \in Z$ and all $\alpha \in Z'$:*

$a \prec b$ implies $g(a) < g(b)$,
$a \leftrightarrow b$ implies $g(a) = g(b)$,
$\alpha + 1 \in Z$ implies $g(\alpha + 1) = g(\alpha) \oplus r$.

Proof By (U4), either $\bot \leftrightarrow \bot+1$ or $\bot \prec \bot+1$ holds. In the case $\bot \leftrightarrow \bot+1$, condition (U6) yields $x \leftrightarrow x + 1 \leftrightarrow \cdots \leftrightarrow x + K$ for every $x \in X \cup \{\bot\}$. Due to (U1)–(U5), the equivalence classes of X w.r.t. $\leftrightarrow$ are linearly ordered and thus it is easy to find some $g\colon X \cup \{\bot, \top\} \to [0, 1] \cap \mathbb{Q}$ such that $g(\bot) = 0$, $g(\top) = 1$ and such that for all $x, y \in X$: $x \prec y$ implies $g(x) < g(y)$, and $x \leftrightarrow y$ implies $g(x) = g(y)$. Putting $r := 0$ and extending g to Z' by $g((x, k)) := g(x)$, where $k \leq K$ and $x \in X$, now obviously yields the desired properties. Therefore, we assume $\bot \prec \bot + 1$ w.l.o.g. in the remainder of the proof.

The subsets $Y_0 := \{z \in Z; z \prec \top\}$ and $Y_1 := \{z + 1; z \in Z', z \prec \top \leftrightarrow z + 1\}$ of Z' are disjoint by (U3). For any relation R, let R^T denote the transposed relation. By (U5), the sets $L_0 := \prec \restriction (Y_0 \times Y_0)$, ${L_0}^T$ and $Q_0 := \leftrightarrow \restriction (Y_0 \times Y_0)$ form a pairwise disjoint partition of $Y_0 \times Y_0$. Employing all properties of a chain, it is easily seen that the sets $L_1 := \{(z + 1, z' + 1); z, z' \in Z', z \prec z' \prec \top \leftrightarrow z + 1 \leftrightarrow z' + 1\}$, ${L_1}^T$ and $Q_1 := \{(z+1, z'+1); z, z' \in Z', z \leftrightarrow z' \prec \top \leftrightarrow z+1 \leftrightarrow z'+1\}$ form a pairwise disjoint partition of $Y_1 \times Y_1$. For $Y := Y_0 \cup Y_1$ and $L_2 := Y_0 \times Y_1$, we conclude therefore that the sets L_0, ${L_0}^T$, Q_0, L_1, ${L_1}^T$, Q_1, L_2, ${L_2}^T$ comprise a pairwise disjoint partition of $Y \times Y$. In the following paragraphs, we will prove that $(Y, \ll, \sim)$ is a grid over $X \cup \{\bot\}$ for $\ll := L_0 \cup L_1 \cup L_2$ and $\sim := Q_0 \cup Q_1$. Since $\ll^T = {L_0}^T \cup {L_1}^T \cup {L_2}^T$, it follows that the sets $\ll$, $\sim$, $\ll^T$ comprise a pairwise disjoint partition of $Y \times Y$. Thus (T1) holds for Y. Since Q_0 is reflexive w. r. t. Y_0 and Q_1 w. r. t. Y_1, also $\sim$ is reflexive w. r. t. Y. Since Q_0 and Q_1 are symmetric, so is $\sim$. We see that $\sim$ is transitive since $a \sim b \sim c \in Y_0$ implies $a Q_0 b Q_0 c$ and $a Q_0 c$, and $a \sim b \sim c \in Y_1$ implies $a Q_1 b Q_1 c$ and $a Q_1 c$.

We define $\precsim := \prec \cup \leftrightarrow$ and $\lesssim := \ll \cup \sim$. Since $\lesssim = L_0 \cup L_1 \cup L_2 \cup Q_0 \cup Q_1 \subseteq (Y_0 \times Y_0) \cup (Y_0 \times Y_1) \cup (Y_1 \times Y_1)$, we remark for use in the next paragraph that $d \lesssim e \in Y_0$ implies $d \in Y_0$ and that $Y_1 \ni d \lesssim e$ implies $e \in Y_1$.

We will now prove that $a \lesssim b \ll c$ implies $a \ll c$. Since $Y_0 \times Y_1 \subseteq \ll$, we only need to distinguish the case $a \in Y_1$ and the case $c \in Y_0$. First, suppose $c \in Y_0$ so that $b \in Y_0$ and, in turn, $a \in Y_0$; thus $(a, b) \in \lesssim \restriction (Y_0 \times Y_0) = L_0 \cup Q_0$, i.e. $a \precsim b$, and $(b, c) \in \ll \restriction (Y_0 \times Y_0) = L_0$, i.e. $b \prec c$, and therefore $a \prec c$ is established by (U1) or by transitivity of $\prec$. Second, suppose $a \in Y_1$ so that $b \in Y_1$ and, in turn, $c \in Y_1$; thus $(a, b) \in \lesssim \restriction (Y_1 \times Y_1) = L_1 \cup Q_1$ and $(b, c) \in \ll \restriction (Y_1 \times Y_1) = L_1$;

hence there are $z_a, z_b, z_c \in Z'$ such that $a = z_a + 1$, $b = z_b + 1$, $c = z_c + 1$ and $z_a \mathrel{\overset{\prec}{\leftrightarrow}} z_b \prec z_c \prec \top \leftrightarrow z_a + 1 \leftrightarrow z_b + 1 \leftrightarrow z_c + 1$, which yields $a \ll c$ by (U1) or by transitivity of $\prec$.

In a completely symmetrical way, we can prove that $a \ll b \lessapprox c$ implies $a \ll c$. This yields that $(Y, \ll, \sim)$ satisfies (T2) and (T3) and that $\ll$ is transitive.

We will prove for later use that any $a \in Y$ such that $a + 1 \in Y$ satisfies $a \in Y_0$ and $a \prec a + 1$. Since $a + 1 \in Y_1$ implies $a \prec \top \sim a + 1$ and, in turn, $a \in Y_0$ and $a \prec a + 1$, we may assume that $a + 1 \notin Y_1$ so that $a + 1 \in Y_0$, i.e. $a + 1 \prec \top$. By (U7), we have either $a \prec a + 1$ or $a \leftrightarrow a + 1 \leftrightarrow \top$. From $a + 1 \prec \top$ and (U5), we conclude $a \prec a + 1 \prec \top$ and hence $a \in Y_0$.

To prove (T4), we suppose $a \in Y$ such that $a+1 \in Y$; we need to show $a \ll a+1$. By the above, we have $a \in Y_0$ and $a \prec a + 1$. Since $Y_0 \times Y_1 \subseteq \ll$, we may assume $a + 1 \notin Y_1$ so that $a + 1 \in Y_0$. Since $L_0 \subseteq \ll$, we have $a \ll a + 1$.

The next two paragraphs prepare to prove (T5) and (T6).

Let $a, b \in Y$ such that $a + 1, b + 1 \in Y$ and $a \ll b$; we will prove $a + 1 \ll b + 1$. As observed above, we have $a, b \in Y_0$, thus $a \prec b$ by $L_0 \subseteq \ll$. From (U9) and (U5), we conclude that either $a + 1 \prec b + 1$ or $a + 1 \sim b + 1 \sim \top$. If $b + 1 \in Y_0$, then $a + 1 \prec b + 1 \prec \top$ so that also $a + 1 \in Y_0$ and thus $(a + 1, b + 1) \in L_0 \subseteq \ll$. Therefore, we may assume $b+1 \in Y_1$. If $a+1 \in Y_1$, then $(a+1, b+1) \in L_1 \subseteq \ll$. Therefore, we may assume $a + 1 \in Y_0$. Hence $(a + 1, b + 1) \in Y_0 \times Y_1 \subseteq \ll$ as required.

Let $a, b \in Y$ such that $a + 1, b + 1 \in Y$ and $a \sim b$; we will prove $a + 1 \sim b + 1$. As observed above, we have $a, b \in Y_0$, thus $a \leftrightarrow b$ by $Q_0 \subseteq \sim$ and therefore $a + 1 \leftrightarrow b + 1$ by (U8). If $a + 1 \in Y_0$ or $b + 1 \in Y_0$, then $\{a + 1, b + 1\} \subseteq Y_0$ and thus $(a + 1, b + 1) \in Q_0 \subseteq \sim$. Thus we may assume $\{a + 1, b + 1\} \subseteq Y_1$. Therefore $(a + 1, b + 1) \in Q_1 \subseteq \sim$ as required.

For any $a, b \in Y$ such that $a + 1, b + 1 \in Y$, the two preceding paragraphs have shown the implications $a \ll b \Rightarrow a + 1 \ll b + 1$ and $a \sim b \Rightarrow a + 1 \sim b + 1$ and $b \ll a \Rightarrow b + 1 \ll a + 1$. By (U5), we see that $a + 1 \ll b + 1$ implies that neither $a \sim b$ nor $b \ll a$ can hold and thus, again by (U5), we must have $a \ll b$. Similarly, $a+1 \sim b+1$ implies that neither $a \ll b$ nor $b \ll a$ can hold, and thus $a \sim b$ follows. This establishes (T5) and (T6).

Now, we have proved that $(Y, \ll, \sim)$ is indeed a grid over $X \cup \{\bot\}$. By Lemma 3 and Lemma 4, we can construct $f\colon Y \to [0, \infty) \cap \mathbb{Q}$ such that $f(\bot) = 0$, $\forall a, b \in Y\big(a \ll b \supset f(a) < f(b)\big)$, $\forall a, b \in Y\big(a \sim b \supset f(a) = f(b)\big)$, and $\forall a \in Y \big(a+1 \in Y \supset f(a+1) = f(a)+1\big)$. Since $(Y, \ll, \sim)$ is a grid, Y_0 has a $\lessapprox$-maximal element u, i.e. $u \in Y_0$ such that $a \lessapprox u$ for all $a \in Y_0$, in particular, $f(a) \leq f(u)$. Likewise, there is $U \in Y_1$ such that $U \lessapprox b$ for all $b \in Y_1$, in particular $f(U) \leq f(b)$. Since $(u, U) \in Y_0 \times Y_1 \subseteq \ll$, we have $f(u) < f(U)$. Put $r := \frac{2}{f(u)+f(U)}$ and $g(y) := \min\{1, r \cdot f(y)\}$ for all $y \in Y$. Clearly, $g(\bot) = 0$. We observe the two following facts: For all $a \in Y_0$ and $b \in Y_1$, we conclude from $0 \leq f(a) \leq f(u) < \frac{f(u)+f(U)}{2} = \frac{1}{r} < f(U) \leq f(b)$ that $0 \leq g(a) < 1 = g(b)$. For all $a, b \in Y_0$ such that $a \prec b$, we conclude from $(a, b) \in L_0 \subseteq \ll$ that $0 \leq f(a) < f(b) \leq f(u) < \frac{1}{r}$, therefore $0 \leq g(a) < g(b) < 1$.

We extend the domain of g from Y to Z by $g(z) := 1$ for all $z \in Z \setminus Y$. In particular, $g(\top) = 1$. Since $(Z \setminus Y) \cup Y_1 \subseteq (Z \setminus Y_0) \cup Y_1 \subseteq \{z \in Z; z \leftrightarrow \top\}$, we see for every $z \in Z$ that $g(z) < 1$ holds if and only if $z \prec \top$.

We need to prove $g(a) < g(b)$ for all $a, b \in Z$ with $a \prec b$. If $b \prec \top$, then $a \prec b \prec \top$, thus $a, b \in Y_0$ and, as observed earlier, $g(a) < g(b)$. Thus we may assume $b \leftrightarrow \top$ so that now $g(b) = 1$. We have $a \prec \top$ for otherwise $a \leftrightarrow \top \leftrightarrow b$, which contradicts $a \prec b$. As observed earlier, we have $g(a) < 1 = g(b)$, as required.

We need to prove $g(a) = g(b)$ for all $a, b \in Z$ with $a \leftrightarrow b$. If $a \prec \top$, we find $b \prec \top$ and thus $(a, b) \in Q_0 \subseteq \sim$ so that $f(a) = f(b)$ and $g(a) = g(b)$. Thus we may assume $a \leftrightarrow \top$. Now, we see $b \leftrightarrow \top$ and $g(a) = 1 = g(b)$ as required.

We need to prove that $g(a + 1) = \min\{1, g(a) + r\}$ for all $a \in Z'$ such that $a + 1 \in Z$. In the case of $a \prec a + 1 \sim \top$, we find $a \in Y_0$ and $a + 1 \in Y_1$ so that $g(a) < 1 = g(a + 1)$ as observed earlier; since $\min\{1, r \cdot f(a)\} = g(a) < 1$ and $f(a+1) = f(a)+1$, we see $r \cdot f(a) = g(a)$ and $g(a+1) = \min\{1, r \cdot (f(a)+1)\} = \min\{1, r \cdot f(a)+r\} = \min\{1, g(a)+r\}$, as required. In the case of $a \prec a+1 \prec \top$, we find $0 \leq g(a) < g(a+1) < 1$ as observed earlier; since $\min\{1, r \cdot f(a)\} = g(a) < 1$, $\min\{1, r \cdot f(a+1)\} = g(a+1) < 1$ and $f(a+1) = f(a)+1$, we see $1 > g(a+1) = r \cdot f(a+1) = r \cdot f(a)+r = g(a)+r$, thus $g(a+1) = \min\{1, g(a)+r\}$, as required. The remaining case is $a \leftrightarrow \top \leftrightarrow a+1$, due to (U7). We now have $g(a) = 1 = g(a+1)$ and thus $g(a + 1) = \min\{1, g(a) + r\}$, as required.

This completes the proof of all claimed properties. □

Definition 5 Let $X \subseteq \mathsf{Var}$ be finite and $K \in \mathbb{N}$. Let $h_{uv} \in \{\prec, \leftrightarrow\}$ be given for all $u, v \in Z$; here $Z := \{\top\} \cup \{(x, k); x \in X \cup \{\bot\}, k \leq K\}$ as in the above definition of a chain. Define a map ι from Z to formulas in $\mathcal{L}^{\mathrm{p}}_{\circ}$ by $(x, k) \mapsto \circ^k x$, $\top \mapsto \top$, $\bot \mapsto \bot$. Let $R_{\prec} := \{(u, v); h_{uv} = \prec\}$ and $R_{\leftrightarrow} := \{(u, v); h_{uv} = \leftrightarrow\}$. If $(Z, R_{\prec}, R_{\leftrightarrow})$ is an (X, K)-chain, an (X, K)-*chain formula* is an $\mathcal{L}^{\mathrm{p}}_{\circ}$-conjunction $\bigwedge_{u \in Z,\, v \in Z} \iota(u)\, h_{uv}\, \iota(v)$, regardless of parenthesisation and order in the conjunction. In this case, we define $\iota(u) \prec_C \iota(v)$ whenever $h_{uv} = \prec$, and $\iota(u) \leftrightarrow_C \iota(v)$ whenever $h_{uv} = \leftrightarrow$.

Theorem 2 *Suppose C is an (X, K)-*chain formula. *Let $Z_* := \{\top\} \cup \{\circ^k x; x \in X \cup \{\bot\}, k \leq K\}$. Then there is a Gödel r-interpretation $\mathfrak{I}\colon \mathsf{Var} \to [0, 1]$ such that $\mathfrak{I}(C) = 1$ and for all $a, b \in Z_*$ we have: $\mathfrak{I}(a) < \mathfrak{I}(b)$ whenever $a \prec_C b$; and $\mathfrak{I}(a) = \mathfrak{I}(b)$ whenever $a \leftrightarrow_C b$.*

Proof We use the same notation as in the of the (X, K)-chain formula. By Lemma 5, we can construct $r \in [0, 1]$ and $g\colon Z \to [0, 1]$ such that (1) $g(\bot) = 0$, (2) $g(\top) = 1$, (3) $\forall u, v \in Z.(h_{uv} = \prec) \supset g(u) < g(v)$, (4) $\forall u, v \in Z.(h_{uv} = \leftrightarrow) \supset g(u) = g(v)$, (5) $\forall u \in Z'.u + 1 \in Z \supset g(u + 1) = r \oplus g(u)$.

Let $\mathfrak{I}(x) := g(x)$ for all $x \in X$ and $\mathfrak{I}(x) := 0$ for all $x \in \mathsf{Var} \setminus X$, and extend $\mathfrak{I}$ to all formulas in $\mathcal{L}^{\mathrm{p}}_{\circ}$ such that $\mathfrak{I}$ is a Gödel r-interpretation.

We claim $\mathfrak{I}(\iota(u)) = g(u)$ for all $u \in Z$. For $u \in \{\bot, \top\}$, this follows from (1) and (2). It remains to check $\mathfrak{I}(\iota(x, k)) = \mathfrak{I}(\circ^k x) = g(x, k)$ for all $x \in X$ and $k \leq K$. By elementary arithmetics, we see $\mathfrak{I}(\circ^k x) = (k \cdot r) \oplus \mathfrak{I}(x)$. Using (5) for $k - 1$ times, we find $g(x + k) = (k \cdot r) \oplus g(x) = (k \cdot r) \oplus \mathfrak{I}(x)$. This establishes the claim.

We claim $\mathfrak{I}(\iota(u)\,h_{uv}\,\iota(v)) = 1$ for all $u, v \in Z$. We have to distinguish two cases: If $(h_{uv} = \prec)$, then $\mathfrak{I}(\iota(u) \prec \iota(v)) = \mathfrak{I}(\iota(u)) \lhd \mathfrak{I}(\iota(v)) = 1$ by (3). If $(h_{uv} = \leftrightarrow)$, then $\mathfrak{I}(\iota(u) \prec \iota(v)) = \mathfrak{I}(\iota(u)) \bowtie \mathfrak{I}(\iota(v)) = 1$ by (4). This proves that $\mathfrak{I}(C) = 1$.

The other properties are immediate consequences of (3) and (4). □

Example 1 Since the relations $\prec$ and $\leftrightarrow$ of a chain must fulfil transitivity, (U1), and (U2), we need not specify h_{uv} in detail. As is done in the following example, it suffices to string the elements of a chain and insert $\prec$ and $\leftrightarrow$ between them. Still, the result needs to be checked to be a chain; however, this is easy for the following $(\{d, e, f, g, h\}, 3)$-chain C given by $(\bot, 0) \leftrightarrow_C (d, 0) \prec_C (e, 0) \prec_C (f, 0) \prec_C (\bot, 1) \leftrightarrow_C (d, 1) \prec_C (e, 1) \prec_C (g, 0) \prec_C (f, 1) \prec_C (\bot, 2) \leftrightarrow_C (d, 2) \prec_C (h, 0) \prec_C (e, 2) \prec_C (g, 1) \prec_C (f, 2) \prec_C (\bot, 3) \leftrightarrow_C (d, 3) \prec_C (h, 1) \prec_C (e, 3) \prec_C (g, 2) \leftrightarrow_C (f, 3) \leftrightarrow_C (h, 2) \leftrightarrow_C (g, 3) \leftrightarrow_C (h, 3) \leftrightarrow_C \top$. Then Corollary 2 says that there is an r-Gödel interpretation $\mathfrak{I}$ such that $\mathfrak{I}(C) = 1$ and $0 = \mathfrak{I}(\bot) = \mathfrak{I}(d) < \mathfrak{I}(e) < \mathfrak{I}(f) < \mathfrak{I}(\circ\bot) = \mathfrak{I}(\circ d) < \mathfrak{I}(\circ e) < \mathfrak{I}(g) < \mathfrak{I}(\circ f) < \mathfrak{I}(\circ\circ\bot) = \mathfrak{I}(\circ\circ d) < \mathfrak{I}(h) < \mathfrak{I}(\circ\circ e) < \mathfrak{I}(\circ g) < \mathfrak{I}(\circ\circ f) < \mathfrak{I}(\circ\circ\circ\bot) = \mathfrak{I}(\circ\circ\circ d) < \mathfrak{I}(\circ h) < \mathfrak{I}(\circ\circ\circ e) < \mathfrak{I}(\circ\circ g) = \mathfrak{I}(\circ\circ\circ f) = \mathfrak{I}(\circ\circ h) = \mathfrak{I}(\circ\circ\circ g) = \mathfrak{I}(\circ\circ\circ h) = 1$.

Definition 6 Let the ring depth rdp of a formula in $\mathcal{L}^{\mathrm{p}}_{\circ}$ or $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$ be recursively defined by $\mathsf{rdp}(A) := 0$ for all $A \in \mathsf{Var} \cup \{\bot\}$, and by $\mathsf{rdp}(\circ A) := 1 + \mathsf{rdp}(A)$, $\mathsf{rdp}(A \Box B) := \max\{\mathsf{rdp}(A), \mathsf{rdp}(B)\}$, for $\Box \in \{\wedge, \vee, \supset\}$ and $\mathsf{rdp}(\triangle A) := \mathsf{rdp}(A)$; here A and B are $\mathcal{L}^{\mathrm{p}}_{\circ}$- and $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$-formulas.

Item (c) of the following theorem establishes the weak completeness of $\mathbf{G}_\circ$ for validity in $\mathcal{L}^{\mathrm{p}}_{\circ}$ w. r. t. Gödel $\circ$-semantics.

Theorem 3 *Suppose $X \subseteq \mathsf{Var}$ is finite and $K \in \mathbb{N}$. Let $Z_* := \{\top\} \cup \{\circ^k x; x \in X \cup \{\bot\}, k \leq K\}$.*

(a) *Then we can construct a set $\mathcal{C}$ of (X, K)-chain formulas and a $\mathbf{G}_\circ$-proof of $\bigvee_{C\in\mathcal{C}} C$.*
(b) *For any (X, K)-chain C and any formula F with $\mathsf{Var}(F) \subseteq X$ and $\mathsf{rdp}(F) \leq K$, we can construct a (not necessarily unique) $z \in Z_*$ and a $\mathbf{G}_\circ$-proof of $C \supset (F \leftrightarrow z)$. We will say that C* evaluates *F to z.*
(c) *If F in $\mathcal{L}^{p}_{\circ}$ is valid, we can construct a $\mathbf{G}_\circ$-proof of F; thus F is valid if and only if $\mathbf{G}_\circ \vdash F$.*

Proof Since the case of $K = 0$, i. e. without rings, is contained in [7], we will stipulate $K \neq 0$ to avoid trivialities.

(a) We will often tacitly treat the abbreviations $\prec$, $\leftrightarrow$ and $\top$ as if they were connectives in their own right, e. g., a formula presented as $a \leftrightarrow b$ is not meant to undergo a transformation applied to all formulas with top symbol $\wedge$.

By (G3), we have $\mathbf{G}_\circ \vdash (a \prec b) \vee (a \leftrightarrow b) \vee (b \prec a)$ for all $a, b \in Z_*$. The conjunction of these formulas is $\mathbf{G}_\circ$-provable by (IPL4). Applying (G12) and (S3) repeatedly, we obtain a $\mathbf{G}_\circ$-proof of a disjunctive normal form $\bigvee_m C^0_m$. Now, each disjunct C^0_m has the property $(*)$: it is a conjunction that consists only of conjuncts $a \Box b$ with $a, b \in Z_*$, $\Box \in \{\leftrightarrow, \prec\}$ and that, moreover, contains for each pair $a, b \in Z_*$ at least one conjunct of the form $a \prec b$, $a \leftrightarrow b$, $b \leftrightarrow a$ or $b \prec a$.

In the next paragraph, we will specify an iterative procedure that turns $\bigvee_m C_m^0$ into the required disjunction of chains. We will leave the easy task to the reader to verify that property $(*)$ is retained in the intermediate disjunctions. We will neglect parenthesisation and ordering in conjunctions and disjunctions due to (G10), (G11), and (S3).

Take $\bigvee_m C_m^0$ and repeatedly apply the first matching rule of the following list until none of the rules matches:

1. Contract equal conjuncts, i. e. replace $e \wedge e$ by e.
2. Contract disjuncts that are equal up to the order of their contained conjuncts, i. e. replace $C \vee C$ by C.
3. Remove some disjunct that contains a conjunct $a \prec \bot$.
4. Replace $\top \prec a$ by $\top \leftrightarrow a$.
5. Replace a conjunct $a \prec a$ by $(a \leftrightarrow a) \wedge (a \leftrightarrow \top)$.
6. Replace $(a \prec b) \wedge (b \mathbin{\Box} a)$, where $\Box \in \{\leftrightarrow, \prec\}$, by $(a \leftrightarrow b) \wedge (b \leftrightarrow \top)$.
7. Replace $(a \mathbin{\Box} b) \wedge (b \mathbin{\Diamond} c) \wedge (c \prec a)$, where $\Box, \Diamond \in \{\leftrightarrow, \prec\}$, by $(a \leftrightarrow b) \wedge (b \leftrightarrow c) \wedge (c \leftrightarrow a) \wedge (a \leftrightarrow \top)$.
8. Replace $(a \leftrightarrow b) \wedge (\circ a \prec \circ b)$ by $(a \leftrightarrow b) \wedge (\circ a \leftrightarrow \circ b) \wedge (\circ b \leftrightarrow \top)$.
9. Replace $(a \prec b) \wedge (\circ b \prec \circ a)$ by $(a \prec b) \wedge (\circ a \leftrightarrow \circ b) \wedge (\circ b \leftrightarrow \top)$.
10. Replace $(a \prec b) \wedge (\circ a \leftrightarrow \circ b) \wedge (\circ b \prec \top)$ by $(a \prec b) \wedge (\circ a \leftrightarrow \circ b) \wedge (\circ b \leftrightarrow \top)$.
11. Replace $(\bot \leftrightarrow \circ\bot) \wedge (a \prec \circ a)$ by $(\bot \leftrightarrow \circ\bot) \wedge (a \leftrightarrow \circ a) \wedge (\circ a \leftrightarrow \top)$.
12. Replace $(\circ a \prec a)$ by $(a \leftrightarrow \circ a) \wedge (\circ a \leftrightarrow \top)$.
13. Replace $(\bot \prec \circ\bot) \wedge (a \leftrightarrow \circ a) \wedge (a \prec \top)$ by $(\bot \prec \circ\bot) \wedge (a \leftrightarrow \circ a) \wedge (a \leftrightarrow \top)$.
14. If a disjunct does not contain the conjunct $a \leftrightarrow a$, $a \in Z$, add it.
15. If a disjunct contains the conjunct $a \leftrightarrow b$ but not the conjunct $b \leftrightarrow a$, add $b \leftrightarrow a$.

For termination, observe the following: The number of disjuncts cannot increase. Property $(*)$ and rule 1 provide a quadratic upper bound of the number of conjuncts of any disjunct in the size of Z_*. The number of Z_*-pairs joined by $\prec$ properly decreases in the rules 3–13 and does not increase in the other rules so that the rules 3–13 succeed only a bounded number of times. The system with the rules 3–13 removed is easily seen to be terminating.

Let $\bigvee_m C_m^1$ denote the result of the procedure. Given a $\mathbf{G}_\circ$-proof of $\bigvee_m C_m^0$, we will iteratively construct a $\mathbf{G}_\circ$-proof of $\bigvee_m C_m^1$: If rule 1 transforms the disjunctive normal form $E[e \wedge e]$ to $E[e]$, extend the $\mathbf{G}_\circ$-proof of $E[e \wedge e]$ by the (G9)-instance $(e \wedge e) \leftrightarrow e$ and the (S3)-instance $((e \wedge e) \leftrightarrow e) \supset (E[e \wedge e] \leftrightarrow E[e])$ to obtain a proof of $E[e]$; proceed similarly for rule 2. For the other rules, tacitly apply (S3) and the following: Use (G24) for rule 3, (G21) for 4, (G13) for 5, (G15) for 6, (G16) for 7. Use (G25) and the (S3)-instance $(a \leftrightarrow b) \supset (\circ a \leftrightarrow \circ b)$ for rule 8. For rule 9, first use the (S1) and (S2) to prove $(a \prec b) \supset (\circ a \prec \circ b)$; then use (G26). For rule 10, use (G27) instead of (G26). Use (R2) and (G25) for 11; (G29) for 12; (R1) and (G27) for 13; use (G22) for 14; use (G23) for 15.

Observe that rule 3 can never yield an empty disjunction so that there is at least one disjunct in $\bigvee_m C_m^1$.

We will now prove that each disjunct C_m^1 of $\bigvee_m C_m^1$ is an (X, K)-formula. We use the notation of Definition 5, in particular, $Z = \{\top\} \cup \{(x, k); x \in X \cup \{\bot\}, k \leq K\}$. For $u, v \in Z$, let $h_{uv} := \prec$ if $\iota(u) \prec \iota(v)$ is contained in C_m^1 and $h_{uv} := \leftrightarrow$ if

$\iota(u) \leftrightarrow \iota(v)$ is contained in C^1_m. It remains to show that $(Z, R_\prec, R_\leftrightarrow)$ fulfills properties (U1)–(U9).

By construction, none of the above rules is applicable to $\bigvee_m C^1_m$. From property $(*)$ and the fact that rules 6 and 15 do not apply, we see that either $h_{uv} = \prec$ or $h_{uv} = \leftrightarrow$ or $h_{vu} = \prec$ holds for any $u, v \in Z$; this proves property (U5). In particular, $h_{uu} = \prec$ or $h_{uu} = \leftrightarrow$ holds for any $u \in Z$. Since rule 5 does not apply but $h_{uu} = \prec$ would trigger it, $R_\leftrightarrow$ is reflexive. Similarly, rule 15 causes the symmetry of $R_\leftrightarrow$. Transitivity of $R_\leftrightarrow$ can be attributed to rule 7. In a similar way, transitivity of $R_\prec$ and properties (U1), (U2) follow from 7; (U3) from rule 4; (U4) from 3; (U6) from 11 and 12; (U7) from 12, 13 and 4; (U8) from 8; (U9) from 9, 10 and 4.

(b) The abbreviations $\prec$ and $\leftrightarrow$ are meant to have been unwound in F. In contrast, we will always treat any occurrence of $\bot \supset \bot$ as a nullary connective $\top$, which is contained in Z_*. We will construct a finite sequence $(F_n)_{n \le N}$ of formulas with strictly decreasing formula complexity such that $F_N \in Z_*$ and such that $\mathbf{G}_\circ \vdash C \supset (F \leftrightarrow F_n)$ and $\mathsf{rdp}(F_n) \le K$ for all n.

Take $F_0 := F$ for the induction basis so that $\mathsf{rdp}(F_0) \le K$ holds by assumption and, clearly, we have $\mathbf{G}_\circ \vdash C \supset (F \leftrightarrow F_0)$. For the induction step, assume $\mathbf{G}_\circ \vdash C \supset (F \leftrightarrow F_n)$ and $\mathsf{rdp}(F_n) \le K$. We may assume $F_n \notin Z_*$ for otherwise the construction of the sequence is completed. Since $\top \in Z_*$, we have $F_n \ne \top = (\bot \supset \bot)$ and then there exist some context $E[\cdot]$, $a, b \in Z_*$ and $\Box \in \{\wedge, \vee, \supset\}$. such that $F_n = E[a \Box b]$.

For the case of $\Box = \supset$, we distinguish three sub-cases: First, we consider the case $b \prec_C a$: We see that C contains the conjunct $b \prec a$ because of $\mathsf{rdp}(a) \le \mathsf{rdp}(F_n) \le K$ and $\mathsf{rdp}(b) \le \mathsf{rdp}(F_n) \le K$. Now $\mathbf{G}_\circ \vdash C \supset (b \prec a)$ follows from (IPL2) and (IPL3). We use this together with the (G8)-instance $\mathbf{G}_\circ \vdash (b \prec a) \supset ((a \supset b) \leftrightarrow b)$, the (S3)-instance $\mathbf{G}_\circ \vdash ((a \wedge b) \leftrightarrow b) \supset (F_n \leftrightarrow E[b])$ and the assumption $\mathbf{G}_\circ \vdash C \supset (F \leftrightarrow F_n)$ to obtain $\mathbf{G}_\circ \vdash C \supset (F \leftrightarrow E[b])$; hence we put $F_{n+1} := E[b]$ then. In the case of $a \prec_C b$, we use $\mathbf{G}_\circ \vdash C \supset (a \prec b)$, the (G7)-instance $\mathbf{G}_\circ \vdash (a \prec b) \supset ((a \supset b) \leftrightarrow \top)$, the (S3)-instance $\mathbf{G}_\circ \vdash ((a \supset b) \leftrightarrow \top) \supset (F_n \leftrightarrow E[\top])$ and the assumption $\mathbf{G}_\circ \vdash C \supset (F \leftrightarrow F_n)$ to obtain $\mathbf{G}_\circ \vdash C \supset (F \leftrightarrow E[\top])$; hence we put $F_{n+1} := E[\top]$ then. In the case of $a \leftrightarrow_C b$, we use $\mathbf{G}_\circ \vdash (a \leftrightarrow b) \supset ((a \supset b) \leftrightarrow \top)$ to conclude in a similar way that $\mathbf{G}_\circ \vdash C \supset (F \leftrightarrow E[\top])$; hence we put $F_{n+1} := E[\top]$ also then. In all of these sub-cases, we find $\mathbf{G}_\circ \vdash C \supset (F \leftrightarrow F_{n+1})$ and $\mathsf{rdp}(F_{n+1}) \le \mathsf{rdp}(F_n) \le K$ hold and that F_{n+1} has a strictly lower formula complexity than F_n, as required. The other cases of $\Box = \wedge$ and $\Box = \vee$ can be treated similarly by (G5), (G6) and (G7).

(c) Soundness has been proved in Lemma 2. For the converse direction, let now F be valid, put $X := \mathsf{Var}(F)$ and $K := \mathsf{rdp}(F)$ and stipulate that $\mathcal{C}$ has the properties as described in (a); we have to construct a $\mathbf{G}_\circ$-proof of F.

First, we will construct a $\mathbf{G}_\circ$-proof of $C \supset F$ for every $C \in \mathcal{C}$. By (b), C evaluates F to some $z \in Z_*$, i. e. $\mathbf{G}_\circ \vdash C \supset (F \leftrightarrow z)$. In particular, $C \supset (F \leftrightarrow z)$ is valid by soundness. If we had $z \prec_C \top$, then Corollary 2 provided a Gödel r-interpretation $\mathfrak{I}\colon X \to [0, 1]$ with $\mathfrak{I}(C) = 1$ and $\mathfrak{I}(z) < \mathfrak{I}(\top) = 1$ so that $\mathfrak{I}(C \supset (z \leftrightarrow \top)) = \mathfrak{I}(C) \trianglelefteq (\mathfrak{I}(z) \bowtie 1) = 1 \trianglelefteq \mathfrak{I}(z) = \mathfrak{I}(z) < 1$, but this contradicts the validity of $C \supset (F \leftrightarrow z)$. By (U3), we conclude $z \leftrightarrow_C \top$ and therefore we can construct a $\mathbf{G}_\circ$-proof of $C \supset (z \leftrightarrow \top)$ by (IPL2) and (IPL3). Since $\mathbf{G} \vdash (U \supset (V \leftrightarrow W)) \supset (U \supset W) \supset (U \supset V)$, we conclude from $\mathbf{G}_\circ \vdash C \supset (F \leftrightarrow z)$ and $\mathbf{G}_\circ \vdash C \supset (z \leftrightarrow \top)$ that $\mathbf{G}_\circ \vdash C \supset F$.

Having constructed $\mathbf{G}_\circ$-proofs of $C \supset F$ for every $C \in \mathcal{C}$, we can join them by multiple use of (IPL7) to obtain $\mathbf{G}_\circ \vdash (\bigvee_{C \in \mathcal{C}} C) \supset F$. Since $\mathbf{G}_\circ \vdash \bigvee_{C \in \mathcal{C}} C$ by (a), we find $\mathbf{G}_\circ \vdash F$, as claimed. □

We will now generalise Theorem 3 to $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$. The $\triangle$-operator has a long history and is known by different names in different branches of research. A sound and complete proof system for Gödel logic with $\triangle$ is given in [1]; we will extend it for our purposes.

Definition 7 Let $\mathbf{G}_{\circ,\triangle}$ denote the proof system of $\mathbf{G}_\circ$ extended by the axiom schemata

($\triangle$1) $\triangle A \supset A$
($\triangle$2) $\triangle A \supset \triangle\triangle A$
($\triangle$3) $\triangle A \vee \neg\triangle A$
($\triangle$4) $\triangle(A \vee B) \supset (\triangle A \vee \triangle B)$
($\triangle$5) $\triangle(A \supset B) \supset (\triangle A \supset \triangle B)$

and the rule ($\triangle$N) $\frac{A}{\triangle A}$.

We will prove in Theorem 3 that $\mathbf{G}_{\circ,\triangle}$ characterises validity in $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$. Clearly, $\mathbf{G}_{\circ,\triangle}$ is substitutive. By [1, Theorem 3.1], $\mathbf{G}_\triangle$ proves all validities in $\mathcal{L}^{\mathrm{p}}_\triangle$.

Proposition 4 $\mathbf{G}_{\circ,\triangle}$ *proves*

(D1) $\triangle A \supset \triangle\circ A$,
(D2) $\triangle(A \supset B) \supset \triangle(\circ A \supset \circ B)$
(D3) $(\triangle A \wedge \triangle B) \leftrightarrow \triangle(A \wedge B)$
(D4) $(\triangle A \vee \triangle B) \leftrightarrow \triangle(A \vee B)$
(D5) $\triangle(A \supset B) \supset \triangle(\triangle A \supset \triangle B)$
(D6) $\triangle(A \leftrightarrow B) \supset \triangle(\triangle A \leftrightarrow \triangle B)$
(D7) $\triangle(A \leftrightarrow B) \supset \triangle(E[A] \leftrightarrow E[B])$
(D8) $\triangle(A \leftrightarrow B) \supset (E[A] \leftrightarrow E[B])$

here $E[\cdot]$ *denotes an* $\mathcal{L}^{p}_{\circ,\triangle}$*-context.*

Proof (D1) follows from $\mathbf{G}_{\circ,\triangle} \vdash A \supset \circ A$, ($\triangle$N) and ($\triangle$5).

Apply ($\triangle$N) and ($\triangle$5) to the $\mathbf{G}_\circ$-provable $(A \supset B) \supset (\circ A \supset \circ B)$ to obtain (D2).

(D3): Apply ($\triangle$N) and ($\triangle$5) to $\mathbf{G}_{\circ,\triangle} \vdash (A \wedge B) \supset A$ to obtain $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \wedge B) \supset \triangle A$. Similarly, $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \wedge B) \supset \triangle B$ holds and thus $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \wedge B) \supset (\triangle A \wedge \triangle B)$. To show the converse direction, apply ($\triangle$N) and ($\triangle$5) to $\mathbf{G}_{\circ,\triangle} \vdash A \supset B \supset (A \wedge B)$ so that $\mathbf{G}_{\circ,\triangle} \vdash \triangle A \supset \triangle(B \supset (A \wedge B)))$. Since $\triangle(B \supset (A \wedge B)) \supset \triangle B \supset \triangle(A \wedge B)$ by ($\triangle$5), we have $\mathbf{G}_{\circ,\triangle} \vdash \triangle A \supset \triangle B \supset \triangle(A \wedge B)$. Thus (D3) follows.

(D4): One direction is ($\triangle$4). Apply ($\triangle$N) and ($\triangle$5) to $A \supset (A \vee B)$ to obtain $\mathbf{G}_{\circ,\triangle} \vdash \triangle A \supset \triangle(A \vee B)$. Similarly, $\mathbf{G}_{\circ,\triangle} \vdash \triangle B \supset \triangle(A \vee B)$. Now, (D4) follows by (IPL3).

(D5): By ($\triangle$5), $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \supset B) \supset \triangle A \supset \triangle B$ holds. Applying ($\triangle$N) and ($\triangle$5), we see $\mathbf{G}_{\circ,\triangle} \vdash \triangle\triangle(A \supset B) \supset \triangle(\triangle A \supset \triangle B)$. Since $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \supset B) \supset \triangle\triangle(A \supset B)$ by ($\triangle$2), we obtain (D5).

(D6) follows from instances of (D5) and (D3).

(D7) is proved by induction on the complexity of the context. Clearly, $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(A \leftrightarrow B)$ and $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(C \leftrightarrow C)$. Suppose we already have $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(E[A] \leftrightarrow E[B])$, then $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(\triangle E[A] \leftrightarrow \triangle E[B])$ by (D6) and also $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(\circ E[A] \leftrightarrow \circ E[B])$ from instances of (D2) and (D3). Suppose we already have $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(E[A] \leftrightarrow E[B])$ and $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(F[A] \leftrightarrow F[B])$; by applying ($\triangle$N) and ($\triangle$5) several times to $\mathbf{G} \vdash (E[A] \leftrightarrow E[B]) \supset (F[A] \leftrightarrow F[B]) \supset ((E[A] \square F[A]) \leftrightarrow (E[B] \square F[B]))$ for any $\square \in \{\wedge, \vee, \supset\}$ and by using the assumptions, we find $\mathbf{G}_{\circ,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle((E[A] \square F[A]) \leftrightarrow (E[B] \square F[B]))$.

(D8) follows from (D7) and ($\triangle$1). □

Definition 8 An (X, K)-*chain formula* in $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$ has the form $\triangle S \wedge \neg\triangle N$ where S is an (X, K)-chain formula in $\mathcal{L}^{\mathrm{p}}_{\circ}$ and N is a conjunction $\bigwedge_i A_i$ with $A_i \in Y := \{\circ^k x;\, x \in X \cup \{\bot\}, k \leq K\}$ such that either S contains $A \leftrightarrow \top$ or N contains A.

Theorem 4 *Let F be a formula in $\mathcal{L}^{p}_{\circ,\triangle}$. Then F is valid if and only if $\mathbf{G}_{\circ,\triangle} \vdash F$.*

Proof Soundness is a routine matter. For completeness, we only sketch how to prove statements analogous to (a), (b) and (c) of Theorem 3 because the reasoning is very similar. In addition to the notation of Definition 8, we put $X := \mathsf{Var}(F)$, $K := \mathsf{rdp}(F)$ and $Z := Y \cup \{\top\}$.

For part (a), we start by constructing a set $\mathcal{C}$ of (X, K)-chain formulae in $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$ such that $\mathbf{G}_{\circ,\triangle}$ proves $\bigvee_{C \in \mathcal{C}} C$. Clearly, $\mathbf{G}_{\circ,\triangle}$ proves the formulas $\triangle(a \prec b) \vee \triangle(a \leftrightarrow b) \vee \triangle(b \prec a)$ and $\triangle(a \leftrightarrow \top) \vee \neg\triangle a$ for all $a, b \in Z$ so that $\mathbf{G}_{\circ,\triangle}$ proves also their conjunction, which we transform into a disjunctive normal form D by (G12). For each $a \in Z$, each conjunction in D contains $\triangle(a \leftrightarrow \top)$, or $\neg\triangle a$, or both by construction. As ($\triangle$N) and (D8) allow us to replace subformulas by provably equivalent ones, the algorithm in part (a) of Theorem 3 needs only small modifications to work. The $\mathbf{G}_{\circ,\triangle}$-provable formula $(\triangle(a \leftrightarrow \top) \wedge \neg\triangle a) \leftrightarrow \bot$ eliminates conjunctions in D that simultaneously contain $\triangle(a \leftrightarrow \top)$ and $\neg\triangle a$ so that D becomes a disjunction of chains in the sense of Definition 8.

For part (b), it suffices to remark that we can evaluate a given formula step-by-step by the $\mathbf{G}_{\circ,\triangle}$-provable formulas $\triangle(a \prec b) \supset (E[a \wedge b] \leftrightarrow E[a])$, $\triangle(a \leftrightarrow b) \supset (E[a \wedge b] \leftrightarrow E[b])$, $\triangle(b \prec a) \supset (E[a \wedge b] \leftrightarrow E[b])$ and formulas similar in fashion to (G5)–(G8).

Part (c) can be almost literally carried over to $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$ because Theorem 2 does not only hold for $\mathcal{L}^{\mathrm{p}}_{\circ}$ but also for $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$ as we will show now. Using the notation from Definition 8, let $C := \triangle S \wedge \neg\triangle N$ be an (X, K)-chain formula in $\mathcal{L}^{\mathrm{p}}_{\circ,\triangle}$. The application of Theorem 2 for $\mathcal{L}^{\mathrm{p}}_{\circ}$ already yields the desired interpretation $\mathfrak{I}$ with $\mathfrak{I}(C) = 1$, $\mathfrak{I}(a) < \mathfrak{I}(b)$ whenever $a \prec_S b$, and $\mathfrak{I}(a) = \mathfrak{I}(b)$ whenever $a \leftrightarrow_S b$ for all $a, b \in Z$ because Definition 8 rules out all chains that would require both $\triangle A$ and $\neg\triangle A$, for any $A \in Z$, to receive the value 1 under $\mathfrak{I}$. □

It is astonishing that the axioms we added for $\circ$ and for $\triangle$ do not interfere with each other. One of the reasons for this is that the countermodels in the construction for the fragment with $\circ$ alone can be used as countermodels for the fragment with $\circ$ and $\triangle$.

5 Final remarks

The exact complexity class of the validity problem in $\mathcal{L}_\circ$ remains an open problem. Another unanswered question from a semantical point of view is the existence of a feasible truth-preserving embedding from $\mathcal{L}_\circ$ into Łukasiewicz logic.

The paper [8] presents, e.g., a weakly complete proof system for propositional Gödel semantics where the interpretation of $\circ$ is generalised from all functions $x \mapsto x + r$ with $r \in [0, 1]$ to all functions $f : [0, 1] \to [0, 1]$ such that $f(1) = 1, \forall x, y.\ x < y \Rightarrow f(x) < f(y)$. These functions, which preserve relative order, are more natural for Gödel semantics than the addition of constants because Gödel semantics does not perform arithmetical operations but merely compares truth values. These semantics reveal which $\circ$-axioms are forced by the use of the $[0, 1]$-interval (in contrast of using an interpretation with values in an algebra). Also here the complexity of first-order validity remains an open problem.

In [9], Theorem 3.4.23, it was proved that satisfiability in $\mathcal{L}_\circ^{\mathrm{p}}$ is NP-complete. As suggested by a reviewer, we show here coNP-completeness of the validity problem in $\mathcal{L}_\circ^{\mathrm{p}}$ by embedding it into the validity problem in $\mathcal{L}^{\mathrm{p}}$ with Gödel semantics. This is accomplished by translating axioms with $\circ$ into a conjunction of $\circ$-free formulas in an antecedent of an implication. The same method of proof can also be found in Theorem 4.6 of [2] to show the weak completeness of a proof system for a related semantics of a $\circ$-operator. We follow the outline of the reviewer closely in the following proof sketch.

Suppose we are given a formula A in propositional variables a_n, $1 \le n \le N$, with a maximal nesting level $K \ge 1$ of rings in A; we define $a_0 := \bot$. Introduce fresh propositional variables $(a_{n,k})_{1\le n\le N,\,0\le k\le K}$ and $(a_{0,k})_{1\le k\le K}$. Moreover, define $a_{0,0} := \bot$. Now construct A' from A by removing all $\circ$ and by replacing each occurrence of an a_i by $a_{i,n}$, where n is the number of $\circ$-operators in whose scope this occurrence of a_i in A is contained. (Thus, if $\bot$ is not in the scope of any $\circ$, it is left unchanged.) We claim that A is valid w.r.t. Gödel semantics in $\mathcal{L}_\circ^{\mathrm{p}}$ if and only if $\Gamma_A \supset A'$ is valid w.r.t. Gödel semantics in $\mathcal{L}^{\mathrm{p}}$; here Γ_A is the conjunction of the following formulas, for all combinations of meaningful indices:

$$
\begin{aligned}
&(a_{0,0} \prec a_{0,1}) \supset (a_{n,k} \prec a_{n,k+1})\\
&(a_{0,0} \leftrightarrow a_{0,1}) \supset (a_{n,0} \leftrightarrow a_{n,1})\\
&(a_{n,k} \prec a_{m,\ell}) \supset (a_{n,k+1} \prec a_{m,\ell+1})\\
&(a_{n,k} \leftrightarrow a_{m,\ell}) \supset (a_{n,k+1} \leftrightarrow a_{m,\ell+1})
\end{aligned}
$$

First observe that, by (R3), that any $\circ$ in A can be moved without altering validity across binary connectives towards propositional variables and $\bot$. (For a detailed proof in an even weaker proof system, see Proposition 3.11 in [8].) Thus we may assume w.l.o.g. that no scope of any $\circ$ in A contains a binary connective, i.e., the $\circ$-operator occurs in A only in the form of $\circ^k a_n$, $0 \le n \le N$, $1 \le k \le K$ so that A can be obtained from A' by substituting $a_{n,k}$ by $\circ^k a_n$, $0 \le n \le N, 0 \le k \le K$.

If A is not valid w.r.t. Gödel semantics in $\mathcal{L}_\circ^{\mathrm{p}}$, there is an r-interpretation $\mathfrak{I}$ such that $\mathfrak{I}(A) < 1$. By putting $\mathfrak{I}'(a_{n,k}) := \mathfrak{I}(\circ^k a_n)$, one obtains a well-defined Gödel interpretation $\mathfrak{I}'$. It is easy to check that $\mathfrak{I}'(\Gamma_A) = 1$ and $\mathfrak{I}'(A') = \mathfrak{I}(A) < 1$, thus $\Gamma_A \supset A'$ is not valid w.r.t. Gödel semantics in $\mathcal{L}^{\mathrm{p}}$, as claimed.

Conversely, if $\Gamma_A \supset A'$ is not valid w.r.t. Gödel semantics in $\mathcal{L}^{\mathrm{p}}$, there is $I'(\Gamma_A \supset A') < 1$ for some Gödel interpretation $\mathfrak{I}'$. We can even assume w.l.o.g. that $\mathfrak{I}'(A') < I'(\Gamma_A) = 1$, due to the Lifting Lemma for Gödel logic (The propositional Lifting Lemma is included in its first-order variant, cf. e.g. [2], Proposition 4.4.). Using $I'(\Gamma_A) = 1$, it is easy to check that the $(a_{n,k})_{n,k}$ together with the transitive relation $a_{n,k} \prec a_{m,\ell} :\Leftrightarrow \mathfrak{I}'(a_{n,k}) < \mathfrak{I}'(a_{m,\ell})$ and the equivalence relation $a_{n,k} \leftrightarrow a_{m,\ell} :\Leftrightarrow \mathfrak{I}'(a_{n,k}) = \mathfrak{I}'(a_{m,\ell})$ form a chain according to Definition 4. Theorem 2 thus provides an $r \in [0, 1]$ and a Gödel r-interpretation $\mathfrak{I}$ such that $\mathfrak{I}(\circ^k a_n) < \mathfrak{I}(\circ^\ell a_m)$ if and only if $\mathfrak{I}'(a_{n,k}) < \mathfrak{I}'(a_{m,\ell})$ and as well $\mathfrak{I}(\circ^k a_n) = \mathfrak{I}(\circ^\ell a_m)$ if and only if $\mathfrak{I}'(a_{n,k}) = \mathfrak{I}'(a_{m,\ell})$. It is now easy to obtain $\mathfrak{I}(A) < 1$ from the assumed condition $\mathfrak{I}'(A') < 1$. Thus A is not valid w.r.t. Gödel semantics in $\mathcal{L}^{\mathrm{p}}_\circ$, which concludes the proof.

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