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On extendible cardinals and the GCH

Konstantinos Tsaprounis

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Abstract We give a characterization of extendibility in terms of embeddings between the structures H_λ. By that means, we show that the GCH can be forced (by a class forcing) while preserving extendible cardinals. As a corollary, we argue that such cardinals cannot in general be made indestructible by (set) forcing, under a wide variety of forcing notions.

Keywords Extendible cardinals · Indestructible cardinals · Generalized continuum hypothesis (GCH)

Mathematics Subject Classification 03E55 · 03E35

1 Introduction

In 1978, Richard Laver established a landmark result: he showed that the supercompactness of a cardinal κ can be made indestructible under $<\kappa$-directed closed forcing (cf. [15]). For this, he introduced a new kind of partial function on the cardinal κ; such a function is now called a *Laver function*. Subsequently, a vast area of applications has emerged, both extending the concept of a Laver function and, at the same time, producing a variety of indestructibility results.

Indeed, Laver functions are an extremely versatile and fruitful tool in the context of large cardinals and they have enjoyed several generalizations. For example, Corazza has shown the existence of Laver sequences for extendible, super-almost huge, superhuge cardinals, and more (cf. [4]), while Hamkins has introduced the

K. Tsaprounis (✉)
Department of Logic, History and Philosophy of Science, University of Barcelona,
08001 Barcelona, Spain
e-mail: kostas.tsap@gmail.com

lottery preparation (cf. [10]) and has also considered a variety of Laver principles (cf. [11]).

Focusing on the case of extendibility, a natural question, then, is whether extendible cardinals can be made indestructible under appropriate classes of forcing notions. In this note, we argue that such indestructibility results are impossible in general. We actually show that this "deficit" of extendible cardinals applies to any Σ_3-correct cardinal and to a wide range of forcing notions, with the case of $<\kappa$-directed closed posets serving as our motivational example.

We expose this deficit by appealing to the fact that, using Σ_3-statements, we are able to express uniformity in set-theoretic properties which are easily changed by a wide range of forcing notions. Our main example naturally comes from considerations regarding the GCH pattern, although one could very well consider other known set-theoretic principles. As an indication of what to expect, we might already observe the following.

Proposition 1.1 *Suppose that eventual* GCH *holds and that* κ *is* Σ_3*-correct in the universe. Then, for any* $\lambda > \kappa$*, the forcing* $Add(\lambda, \lambda^{++})$ *destroys the* Σ_2*-correctness of* κ.

Proof Note that the sentence "there exists some α such that, for all $\beta > \alpha$, $2^\beta = \beta^+$" is Σ_3 and clearly true in V, by the eventual GCH assumption. Hence it must reflect in V_κ, so fix some $\alpha < \kappa$ such that GCH holds above α in the structure V_κ. But now it is clear that, for any $\lambda > \kappa$, after forcing with the canonical poset $\mathbb{P} = Add(\lambda, \lambda^{++})$, we make GCH fail at λ while preserving all of V_κ (and so, in particular, the GCH pattern below κ as well). Hence, in $V^{\mathbb{P}}$, κ cannot be Σ_2-correct, since the Σ_2-statement "there exists $\lambda > \alpha$ such that GCH fails at λ" does not reflect correctly below κ. □

We shall show in Sect. 2 that, in the case of extendible cardinals, one can actually force the global GCH while preserving such cardinals. The preservation of large cardinals after forcing the GCH is by now part of the set-theoretic folklore: Jensen initially showed that measurables are preserved (cf. [13]), while Menas did it for supercompacts (cf. [16]). Hamkins then followed, dealing with I1 embeddings (cf. [9]), and Friedman accounted for n-superstrong cardinals (cf. [7]). More recently, similar preservation results were obtained by Brooke-Taylor and Friedman for 1-extendible cardinals (cf. [3]), and by Brooke-Taylor for Vopěnka's Principle (cf. [2]).

Here, we extend the above results to include the case of extendibility. To that end, we first obtain a characterization of extendible cardinals in terms of embeddings between the H_λ's, and we then show that the canonical (class-length) iteration forcing the global GCH indeed preserves extendible cardinals. Let us begin.

1.1 Preliminaries

Our notation and terminology is mostly standard.[1] ZFC stands for the usual first-order axiomatization of Zermelo-Fraenkel set theory, together with the Axiom of Choice; we

[1] See [12] for an account of all undefined set-theoretic notions.

write ZFC^- to indicate the absence of the Powerset Axiom.[2] For any set X, $trcl(X)$ is its transitive closure and $|X|$ is its cardinality.

The class of ordinal numbers will be denoted by **ON**. Lower case Greek letters stand for ordinals, with the letters κ, λ and μ typically used in the case of infinite cardinals. $\beth_\alpha$ stands for the αth beth number. If λ is an infinite cardinal, we let H_λ be the collection of all sets whose transitive closure has size less than λ. Ordinal intervals are readily comprehensible; for instance, given $\alpha < \beta$, (α, β) is the set of ordinals which lie strictly between α and β.

Given any function f and any $S \subseteq dom(f)$, we write $f \restriction S$ for the restriction of the function to S and also, we write $f``S$ for the pointwise image, that is, the collection $\{f(x) : x \in S\}$. Moreover, we write $ran(f)$ for the range of f.

Partial orders (a.k.a. posets) which are employed in forcing constructions will be denoted by blackboard bold capital letters such as $\mathbb{P}$ and $\mathbb{Q}$. We write $p < q$ to mean than p is stronger than q or, equivalently, p properly extends q. We denote the greatest element of a poset by $\mathbb{1}$. Given a poset $\mathbb{P}$, the $\mathbb{P}$-names are indicated by "dots" and "checks" as usual; we sometimes supress these in order to ease readability, with the intended meaning being clear from the context. The universe of $\mathbb{P}$-names will be denoted by $V^{\mathbb{P}}$. If $\dot{x}$ is a $\mathbb{P}$-name and G is a $\mathbb{P}$-generic filter (over the relevant model), then $\dot{x}_G$ denotes the interpretation of the name by the filter. Our terminology on chain conditions and closed posets is mostly standard. We are explicit regarding the extent of closure of a given $\mathbb{P}$ by writing, for example, "$\leqslant \kappa$-directed closed" to mean that we may find lower bounds of directed subsets whose cardinality is at most κ.

For any elementary embedding j, we denote by $cp(j)$ its critical point. As far as lifting embeddings through forcing posets is concerned, we have the following important criterion due to Jack Silver.

Lemma 1.2 *Suppose that $j : V \longrightarrow M$ is an elementary embedding with M transitive. Let $\mathbb{P} \in V$ be a poset, let G be $\mathbb{P}$-generic over V and let H be $j(\mathbb{P})$-generic over M. Then, j lifts (uniquely) through $\mathbb{P}$ to $j^* : V[G] \longrightarrow M[H]$ (that is, j^* is an elementary embedding with $j^* \restriction V = j$ and $j^*(G) = H$) if and only if $j``G \subseteq H$.*

We follow the common convention and use the same letter j for the lifted version of the embedding. In practice, we often ensure that "$j``G \subseteq H$" is satisfied by exhibiting a particular condition $q \in H$ which is a lower bound for $j``G$; that is, such that for every $p \in G$, we have that $q \leqslant j(p)$. Such a condition q is called a *master condition*.

Given a regular cardinal κ and some λ, $Add(\kappa, \lambda)$ is the poset consisting of partial functions $p : \lambda \times \kappa \longrightarrow 2$ where $|p| < \kappa$; the ordering is given by reversed inclusion. This poset is $<\kappa$-directed closed and $(2^{<\kappa})^+$-c.c. A special case is the poset $Add(\kappa, 1)$ which adds one Cohen subset to κ via partial functions $p : \kappa \longrightarrow 2$ of size less than κ. If $\kappa = \lambda^+$, then $Add(\kappa, 1)$ forces the GCH at λ, that is, $2^\lambda = \lambda^+$ holds in any generic extension.

[2] See [8] for some nuances related to the theory ZFC^-.

A poset $\mathbb{P}$ is called *weakly homogeneous* if for every $p, q \in \mathbb{P}$ there is an automorphism $\sigma_{p,q} : \mathbb{P} \longrightarrow \mathbb{P}$ such that $\sigma(p)$ and q are compatible. It is basic that, for any cardinals κ and λ, the poset $Add(\kappa, \lambda)$ is weakly homogeneous.

We will be interested in class-length forcing iterations, although at any particular stage of our arguments we will be employing standard set forcing techniques. We refer the reader to [1] or to [3] for more details on the particular well-known iteration of the GCH posets which we shall invoke in Sect. 2.

1.2 Characterizing extendibility

Recall that a cardinal κ is called λ-*extendible*, for some $\lambda > \kappa$, if there exists a θ and an elementary embedding $j : V_\lambda \longrightarrow V_\theta$ such that $cp(j) = \kappa$ and $j(\kappa) > \lambda$. We now use ordinary coding techniques and look at extendibility from the perspective of the H_λ's.

Proposition 1.3 *Let κ be a cardinal and let $\lambda = \beth_\lambda \geqslant \kappa$. Then, κ is $\lambda + 1$-extendible if and only if there is a cardinal μ and an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ with $cp(j) = \kappa$ and $j(\kappa) > \lambda + 1$.*

Proof Suppose that κ is $\lambda + 1$-extendible, for some fixed $\beth_\lambda = \lambda \geqslant \kappa$, and let $h : V_{\lambda+1} \longrightarrow V_{h(\lambda)+1}$ be a witnessing elementary embedding, with $cp(h) = \kappa$ and $h(\kappa) > \lambda + 1$. Let $\mu = h(\lambda)$, which is clearly a cardinal. We shall use h in order to define an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$, such that $cp(j) = \kappa$ and $j \upharpoonright (\lambda + 1) = h \upharpoonright (\lambda + 1)$.

For this, we describe any given set in H_{λ^+} by an appropriate code which lies in $V_{\lambda+1}$; this is a standard coding argument which we include for completeness. Let $x \in H_{\lambda^+}$ and fix some (any) bijection $f_x : |trcl(\{x\})| \longrightarrow trcl(\{x\})$. Consider the binary relation E_x on $dom(f_x)$ which is defined so that, for any $\alpha, \beta \in dom(f_x)$,

$$\langle \alpha, \beta \rangle \in E_x \Longleftrightarrow f_x(\alpha) \in f_x(\beta).$$

Notice that, since $dom(f_x)$ is some cardinal $\leqslant \lambda$, we have that $E_x \in V_{\lambda+1}$ and moreover, if $\langle T_x, \in \rangle$ is the Mostowski collapse of $\langle dom(f_x), E_x \rangle$, it then follows that $T_x \cong trcl(\{x\})$ and, thus, $T_x = trcl(\{x\})$. Clearly, one may easily recover the set x from T_x.

This procedure gives a way of coding sets $x \in H_{\lambda^+}$ by corresponding sets $E_x \in V_{\lambda+1}$. Thus we may define, inside $V_{\lambda+1}$, the class $\mathcal{C}_\lambda \subseteq V_{\lambda+1}$ such that $X \in \mathcal{C}_\lambda$ if and only if $X \subseteq \lambda \times \lambda$ is a well-founded, extensional (binary) relation such that, if $X \neq \varnothing$, then X has a unique maximal element.

Now, for any $X \in \mathcal{C}_\lambda$, let $fld(X)$ denote the field of X, that is, the union of the domain and the range of X and, moreover, let $max(X)$ denote the unique ordinal in $fld(X)$ which is maximal with respect to the relation X. Next, for any X and Y in $\mathcal{C}_\lambda$, define the relation "$=^*$" by declaring that $X =^* Y$ if and only if the structures $\langle fld(X), X \rangle$ and $\langle fld(Y), Y \rangle$ are isomorphic. For any $X \in \mathcal{C}_\lambda$ and any $a \in fld(X)$, we let $X_a = \bigcup_{n \in \omega} A_n$ where $A_0 = \{\langle x, a \rangle \in X : x \in fld(X)\}$ and $A_{n+1} = \{\langle x, z \rangle \in X : z \in dom(A_n)\}$. Now define the relation "$\in^*$" on $\mathcal{C}_\lambda$ by stipulating that $X \in^* Y$ if and only if there exists some $a \in fld(Y)$ such that $\langle a, max(Y)\rangle \in Y$ and $X =^* Y_a$. Clearly, the relations "$=^*$" and "$\in^*$" are definable in $V_{\lambda+1}$. For any $X \in \mathcal{C}_\lambda$, we denote its Mostowski collapsing function by π_X, which gives rise to the collapse $ran(\pi_X)$. One easily checks that, for any X and Y in $\mathcal{C}_\lambda$,

$$X =^* Y \iff ran(\pi_X) = ran(\pi_Y)$$

and

$$X \in^* Y \iff \pi_X(max(X)) \in \pi_Y(max(Y)),$$

where, if $X = \varnothing$, we let $max(X) = \pi_X(max(X)) = \varnothing$.

All of the above makes us capable of translating any first-order formula φ whose parameters range over H_{λ^+}, into an equivalent formula φ^* whose parameters range over $V_{\lambda+1}$: we replace any $x \in H_{\lambda^+}$ by a corresponding code $E_x \in \mathcal{C}_\lambda$ for it; we replace the standard set-theoretic relations "$=$" and "$\in$" by the definable relations "$=^*$" and "$\in^*$", respectively; finally, quantification is taken to range over $\mathcal{C}_\lambda$. That is, for any first-order formula $\varphi(v_1, \ldots, v_n)$ and any $x_i \in H_{\lambda^+}$, for $1 \leqslant i \leqslant n$, we have that

$$H_{\lambda^+} \models \varphi(x_1, \ldots, x_n) \iff V_{\lambda+1} \models \varphi^*(E_{x_1}, \ldots, E_{x_n}),$$

for some (any) codes E_{x_i}, for $1 \leqslant i \leqslant n$. It is also clear that all the above can be done equally well for μ in place of λ. Abusing the notation slightly, we again call φ^* this translation, keeping in mind that the quantification now ranges over the class $\mathcal{C}_\mu$ as defined in $V_{\mu+1}$. At this point, by the elementarity of h, for any $X, Y \in \mathcal{C}_\lambda$, we have that

$$V_{\lambda+1} \models X =^* Y \iff V_{\mu+1} \models h(X) =^* h(Y)$$

and

$$V_{\lambda+1} \models X \in^* Y \iff V_{\mu+1} \models h(X) \in^* h(Y).$$

Then, inductively, for any formula $\varphi(v_1, \ldots, v_n)$ and given any $X_i \in \mathcal{C}_\lambda$, for $1 \leqslant i \leqslant n$, we have that $V_{\lambda+1} \models \varphi^*(X_1, \ldots, X_n)$ if and only if $V_{\mu+1} \models \varphi^*(h(X_1), \ldots, h(X_n))$. We are now in position to define the map $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ by letting, for every $x \in H_{\lambda^+}$,

$$j(x) = \pi_{h(E_x)}(max(h(E_x))),$$

for some (any) bijection $f_x : |trcl(\{x\})| \longrightarrow trcl(\{x\})$, giving rise to the code E_x. We evidently have that $j \restriction (\lambda + 1) = h \restriction (\lambda + 1)$. Let us finally check that j is an elementary embedding. For this, fix any formula $\varphi(v_1, \ldots, v_n)$, any $x_i \in H_{\lambda^+}$ and any corresponding codes $E_{x_i} \in V_{\lambda+1}$, for $1 \leqslant i \leqslant n$. We have the following equivalences:

$$\begin{aligned} H_{\lambda^+} \models \varphi(x_1, \ldots, x_n) &\iff V_{\lambda+1} \models \varphi^*(E_{x_1}, \ldots, E_{x_n}) \\ &\iff V_{\mu+1} \models \varphi^*(h(E_{x_1}), \ldots, h(E_{x_n})) \\ &\iff H_{\mu^+} \models \varphi(j(x_1), \ldots, j(x_n)), \end{aligned}$$

which conclude the proof of the forward direction of the proposition.

Conversely, suppose that for some $\lambda = \beth_\lambda \geqslant \kappa$ and some cardinal μ, we have an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ with $cp(j) = \kappa$ and $j(\kappa) > \lambda + 1$. Clearly, $j(\lambda) = \mu$. Furthermore, as λ is a beth fixed point, $V_{\lambda+1}$ is a definable class in H_{λ^+}, namely,

$$V_{\lambda+1} = \{x \in H_{\lambda^+} : x \subseteq H_\lambda\}.$$

This means that we may relativize any first-order formula to $V_{\lambda+1}$, within H_{λ^+}. Of course, the analogous facts are true for $V_{\mu+1}$ and H_{μ^+} correspondingly. Using these observations, one easily verifies that the restriction $j \upharpoonright V_{\lambda+1} : V_{\lambda+1} \longrightarrow V_{\mu+1}$ is an elementary embedding witnessing the $\lambda + 1$-extendibility of κ. □

As an immediate corollary, we get the following characterization of extendibility in terms of the H_λ's.

Corollary 1.4 *A cardinal κ is extendible if and only if for all $\lambda = \beth_\lambda \geqslant \kappa$, there exists some cardinal μ and an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ with $cp(j) = \kappa$ and $j(\kappa) > \lambda + 1$.*

2 Forcing the GCH

We now use the characterization just obtained in order to show that the global GCH can be forced while preserving extendible cardinals. The following is a well-known definition.

Definition 2.1 The **canonical forcing** $\mathbb{P}$ **for the global** GCH is the class-length reverse Easton iteration of $\langle \dot{\mathbb{Q}}_\alpha : \alpha \in \mathbf{ON} \rangle$, where $\mathbb{P}_0 = \{\mathbb{1}\}$ and, for each α, if α is an infinite cardinal in $V^{\mathbb{P}_\alpha}$, then $\dot{\mathbb{Q}}_\alpha$ is the canonical $\mathbb{P}_\alpha$-name for the poset $Add(\alpha^+, 1)^{V^{\mathbb{P}_\alpha}}$. Finally, $\mathbb{P}$ is the direct limit of the $\mathbb{P}_\alpha$'s, for $\alpha \in \mathbf{ON}$.

The iteration $\mathbb{P}$ preserves ZFC, preserves inaccessible cardinals and forces the GCH everywhere. Moreover, at any inaccessible cardinal α, the iteration factors nicely as $\mathbb{P}_\alpha * \mathbb{P}_{\text{tail}}$, where $|\mathbb{P}_\alpha| = \alpha$ and $\mathbb{P}_{\text{tail}}$ is (forced to be) $\leqslant \alpha$-directed closed. It is also known that the (weak) homogeneity of the individual $Add(\alpha^+, 1)^{V^{\mathbb{P}_\alpha}}$ forcings, holds as well for the whole, class-length iteration and any initial segment of it (see [5] for details). We are now ready to prove the following.

Theorem 2.2 *Every extendible cardinal is preserved by the canonical forcing $\mathbb{P}$ for the global* GCH.

Proof In the context of the earlier results which were mentioned right after Proposition 1.1, the present proof is just an application of standard techniques. In particular, we use very similar arguments to the ones appearing in [3] regarding the case of 1-extendibility. As it will become clear below, the key point in the current setting is the appeal to [5] for the homogeneity property. At the outset, we also recall that extendibility implies the existence of a proper class of inaccessible cardinals.

Fix an extendible cardinal κ and fix an inaccessible $\lambda > \kappa$. By the results of the previous section, let $j : H_{\lambda^+} \longrightarrow H_{j(\lambda)^+}$ be an embedding witnessing the $\lambda + 1$-extendibility of κ in V; that is, $cp(j) = \kappa$, $j(\kappa) > \lambda + 1$ and $j(\lambda)$ inaccessible.

Let G be $\mathbb{P}$-generic over V; it is our aim to lift this ground model embedding j to an embedding of the form $j : H^{V[G]}_{\lambda^+} \longrightarrow H^{V[G]}_{j(\lambda)^+}$, witnessing the $\lambda + 1$-extendibility of κ in $V[G]$, which will be enough in order to conclude the theorem. For this, we factor the whole forcing iteration as

$$\mathbb{P}_\kappa * \dot{\mathbb{P}}_{[\kappa,\lambda)} * \dot{\mathbb{P}}_{[\lambda,\infty)},$$

where, for example, $\dot{\mathbb{P}}_{[\kappa,\lambda)}$ is the ($\mathbb{P}_\kappa$-name for the) partial iteration of forcings which occur at stages between κ and λ; similar interval notation in the subscripts is used for the corresponding projections of G, which are generics for the relevant partial iterations of $\mathbb{P}$. We will lift the ground model embedding in two steps, according to the above factorization.

As our first step, we lift through the initial forcing $\mathbb{P}_\kappa$, where we observe that $\mathbb{P}_\kappa \in H_{\kappa^+}$ and thus, G_κ is certainly $\mathbb{P}_\kappa$-generic over H_{λ^+}. Accordingly, the partial filter $G_{j(\kappa)}$ is $\mathbb{P}_{j(\kappa)}$-generic over $H_{j(\lambda)^+}$, where $\mathbb{P}_{j(\kappa)} = j(\mathbb{P}_\kappa) \in H_{j(\lambda)^+}$. Since the forcing $\mathbb{P}_\kappa$ is a direct limit and $cp(j) = \kappa$, we have that $j``G_\kappa = G_\kappa \subseteq G_{j(\kappa)}$ and hence we may indeed perform the first lift of the embedding as

$$j : H_{\lambda^+}[G_\kappa] \longrightarrow H_{j(\lambda)^+}[G_{j(\kappa)}].$$

For the second step, it is our aim to lift further through the forcing $\mathbb{P}_{[\kappa,\lambda)} = (\dot{\mathbb{P}}_{[\kappa,\lambda)})_{G_\kappa}$. This makes sense since $|\dot{\mathbb{P}}_{[\kappa,\lambda)}| = \lambda$ and thus, $\mathbb{P}_{[\kappa,\lambda)} \in H_{\lambda^+}[G_\kappa]$. Now, it is clear that $G_{[\kappa,\lambda)}$ is $\mathbb{P}_{[\kappa,\lambda)}$-generic over $H_{\lambda^+}[G_\kappa]$. Similarly, since $\mathbb{P}_{[j(\kappa),j(\lambda))} = j(\mathbb{P}_{[\kappa,\lambda)})$, $G_{[j(\kappa),j(\lambda))}$ is $\mathbb{P}_{[j(\kappa),j(\lambda))}$-generic over $H_{j(\lambda)^+}[G_{j(\kappa)}]$. Thus, the only problem is to verify the lifting criterion $j``G_{[\kappa,\lambda)} \subseteq G_{[j(\kappa),j(\lambda))}$. For this, we first find a relevant master condition.

Recall that $\mathbb{P}_{[\kappa,\lambda)}$ has size λ in $H_{\lambda^+}[G_\kappa]$ (and so in $H_{j(\lambda)^+}[G_{j(\kappa)}]$ as well) an also, we clearly have that $j``\lambda \in H_{j(\lambda)^+}[G_{j(\kappa)}]$. Note also that $G_{[\kappa,\lambda)}$ appears explicitly in the partial filter $G_{j(\kappa)}$. Therefore, we may combine $j``\lambda$ with some enumeration of $\mathbb{P}_{[\kappa,\lambda)}$ in order to get that $j \restriction \mathbb{P}_{[\kappa,\lambda)} \in H_{j(\lambda)^+}[G_{j(\kappa)}]$; thus, $j``G_{[\kappa,\lambda)} \in H_{j(\lambda)^+}[G_{j(\kappa)}]$ as well (and has size $\lambda$ there). Now, since $j``G_{[\kappa,\lambda)}$ is a directed subset of $\mathbb{P}_{[j(\kappa),j(\lambda))}$ and the latter is $\leqslant j(\kappa)$-directed closed in $H_{j(\lambda)^+}[G_{j(\kappa)}]$, there is indeed a lower bound for $j``G_{[\kappa,\lambda)}$, that is, there exists some $r \in \mathbb{P}_{[j(\kappa),j(\lambda))}$ such that $r \leqslant j``G_{[\kappa,\lambda)}$; this is the desired master condition. Note that there is no reason to expect that $r \in G_{[j(\kappa),j(\lambda))}$. We now modify the filter $G_{[j(\kappa),j(\lambda))}$ in order to produce an appropriate filter G^* which is $\mathbb{P}_{[j(\kappa),j(\lambda))}$-generic over $H_{j(\lambda)^+}[G_{j(\kappa)}]$ and such that $r \in G^*$.

Working for the moment in the model $H_{j(\lambda)^+}[G_{j(\kappa)}]$, since $\mathbb{P}_{[j(\kappa),j(\lambda))}$ is weakly homogeneous, the set of conditions t for which there is some automorphism $e : \mathbb{P}_{[j(\kappa),j(\lambda))} \longrightarrow \mathbb{P}_{[j(\kappa),j(\lambda))}$ such that $e(t) \leqslant r$ is dense. Since $G_{[j(\kappa),j(\lambda))}$ is generic, we may find such a condition $t \in G_{[j(\kappa),j(\lambda))}$. Note that this condition t cannot be found working in $H_{j(\lambda)^+}[G_{j(\kappa)}]$, because we are appealing to the further generic filter $G_{[j(\kappa),j(\lambda))}$; even so, it indeed exists and it certainly belongs to $H_{j(\lambda)^+}[G_{j(\kappa)}]$, together with the corresponding automorphism e. Then, by standard forcing facts (cf. Chapter VII, Theorem 7.11 in [14]) it follows that, if we let G^* be the filter generated by the pointwise image $e``G_{[j(\kappa),j(\lambda))}$, then G^* is $\mathbb{P}_{[j(\kappa),j(\lambda))}$-generic over $H_{j(\lambda)^+}[G_{j(\kappa)}]$ with $r \in G^*$ and moreover,

$$H_{j(\lambda)^+}[G_{j(\lambda)}] = H_{j(\lambda)^+}[G_{j(\kappa)}][G^*].$$

We may thus conclude the second lift of the embedding, obtaining

$$j : H_{\lambda^+}[G_\kappa][G_{[\kappa,\lambda)}] \longrightarrow H_{j(\lambda)^+}[G_{j(\kappa)}][G^*],$$

or, equivalently,

$$j : H_{\lambda^+}[G_\lambda] \longrightarrow H_{j(\lambda)^+}[G_{j(\lambda)}].$$

As the final part of the argument, we show that the currently lifted embedding is sufficient in order to witness the $\lambda+1$-extendibility of κ in $V[G]$. For this, we argue that, in fact,

$$H_{\lambda^+}^{V[G]} = H_{\lambda^+}^{V[G_\lambda]} = H_{\lambda^+}[G_\lambda].$$

First, notice that the iteration $\dot{\mathbb{P}}_{[\lambda,\infty)}$ is forced to be $\leqslant \lambda$-directed closed and so it does not affect H_{λ^+}; that is, $H_{\lambda^+}^{V[G]} = H_{\lambda^+}^{V[G_\lambda]}$. Let us now check that $H_{\lambda^+}^{V[G_\lambda]} = H_{\lambda^+}[G_\lambda]$ as well. We remark that the latter structure, being a generic extension of the ZFC^- model H_{λ^+} by the generic filter G_λ, is also a ZFC^- model. As the right-to-left inclusion is clear, we fix any element $X \in H_{\lambda^+}^{V[G_\lambda]}$ and we want to find an appropriate $\mathbb{P}_\lambda$-name witnessing that $X \in H_{\lambda^+}[G_\lambda]$. But then, exactly as in the proof of Proposition 1.3, X can be obtained in $V[G_\lambda]$ by the Mostowski collapse of some appropriate code subset of $\lambda \times \lambda$ (where recall that the whole process did not use the Powerset Axiom). Since all subsets of $\lambda \times \lambda$ in $V[G_\lambda]$ have nice names which lie in H_{λ^+}, we get that all such codes belong to $H_{\lambda^+}[G_\lambda]$. Therefore, $X \in H_{\lambda^+}[G_\lambda]$. In a totally analogous manner, $H_{j(\lambda)^+}^{V[G]} = H_{j(\lambda)^+}[G_{j(\lambda)}]$ as well; hence, the lifted embedding is indeed of the form $j : H_{\lambda^+}^{V[G]} \longrightarrow H_{j(\lambda)^+}^{V[G]}$ and the proof is complete. □

We now return to the issue of indestructibility and argue that extendible cardinals do not in general enjoy such niceties. In this setting, we temporarily fix a broad ambient class of posets. Let us declare that a property R of posets is *cofinally sympathetic to non-*GCH, if for all β there exists some (cardinal) $\alpha > \beta$ such that the canonical poset $Add(\alpha, \alpha^{++})$ (which kills the GCH at α) satisfies R.

Typical examples are intended to be the various closure properties, such as being κ-directed closed, for some regular cardinal κ; all these are certainly cofinally sympathetic to non-GCH. On the other hand, chain conditions are not of this sort, as we cannot expect them to hold cofinally in the ordinals. Similarly to Proposition 1.1, we have the following.

Proposition 2.3 *Let R be any property of posets which is cofinally sympathetic to non-*GCH*, let κ be extendible and suppose that the* GCH *holds. Then, no (set) forcing which preserves the Σ_3-correctness of κ can make its Σ_2-correctness indestructible under posets satisfying R.*

Proof Fix some property R which is cofinally sympathetic to non-GCH and assume, towards a contradiction, that there is a (set) forcing notion $\mathbb{P}$ which preserves the Σ_3-correctness of κ and makes its Σ_2-correctness indestructible under R.

We perform the purported forcing $\mathbb{P}$ and we get a model V in which κ is—allegedly—an indestructible Σ_2-correct cardinal. Since $\mathbb{P}$ is a set, the eventual GCH pattern of the universe is not altered; thus, as $\mathbb{P}$ preserves the Σ_3-correctness κ, there exists some $\beta < \kappa$ so that, for every $\alpha \in (\beta, \kappa)$, we have that $2^\alpha = \alpha^+$.

Let $\gamma > \kappa$ be such that $\mathbb{Q} = Add(\gamma, \gamma^{++})$ satisfies property R. Then, forcing with $\mathbb{Q}$ preserves the whole of V_κ, and hence the GCH pattern below κ, while at the same time it kills the GCH at γ. This means that the Σ_2-statement "there exists some $\alpha > \beta$ such that GCH fails at α" is not reflected correctly in $(V_\kappa)^{V^{\mathbb{Q}}}$. This is a contradiction. □

As a corollary it follows that, not only can we not make (in general) an extendible cardinal κ indestructible, but also, forcing the global GCH makes it extremely "destructible": any poset killing GCH above κ kills many of its large cardinal properties (e.g. κ can no longer be supercompact or strong, as it is not even Σ_2-correct). In addition, the same argument shows that any poset killing GCH above κ kills the Σ_2-correctness on a tail of Σ_2-correct cardinals below κ as well.

Note that, although we focus on extendibility, we could have stated Proposition 2.3 by requiring that κ is Σ_3-correct in the universe and that *eventual* GCH holds; in such a case, similar observations would apply.

The following question remains unanswered.

Question 2.4 Can the extendibility of κ be made indestructible to suitable set forcings by some preparatory *class* forcing?

By what we have shown, in the event of a positive answer such a class forcing would necessarily have to make the GCH fail cofinally in the ordinals. As a related comment, we mention that, recently and in the context of HOD-supercompactness, Shoshana Friedman found a way to circumvent the class forcing for the GCH (cf. [6]), although her technique does not seem to apply to our situation. We nevertheless thank the referee for bringing [6] to our attention.

Towards concluding, we give two other ways of killing the extendibility of a cardinal κ, while preserving its inaccessibility. In the present setting, the following examples hopefully do some (partial) justice to chain conditions, which were neglected by properties that are cofinally sympathetic to non-GCH.

On the one hand, we assume that the global GCH holds and we then perform an Easton forcing to kill the GCH at every *regular* cardinal below κ; such a forcing preserves cofinalities and, since κ is Mahlo, it is also κ-c.c. (see Chapter VIII, § 4 in [14]). In the resulting model, the GCH fails at every regular below κ while it continues to hold everywhere above it. Consequently, κ cannot remain Σ_2-correct since the statement "the GCH holds at some regular α" is not reflected correctly.

On the other hand, we may kill the extendibility of κ this time preserving its supercompactness and, thus, its Σ_2-correctness (contrary to the previous examples). We again assume that the global GCH holds in V and we in fact make κ an indestructible supercompact cardinal by the usual Laver preparation $\mathbb{P}$ (alternatively, one may use the lottery preparation; see [10]). Then, in $V^{\mathbb{P}}$, κ cannot be Σ_3-correct anymore since the GCH fails cofinally below κ, whereas it continues to hold everywhere above it.

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