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Algebraizable logics with a strong conjunction and their semi-lattice based companions

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Abstract The best known algebraizable logics with a conjunction and an implication have the property that the conjunction defines a meet semi-lattice in the algebras of their algebraic counterpart. This property makes it possible to associate with them a semi-lattice based deductive system as a companion. Moreover, the order of the semi-lattice is also definable using the implication. This makes that the connection between the properties of the logic and the properties of its semi-lattice based companion is strong. We introduce a class of algebraizable deductive systems that includes those systems, and study some of their properties and of their semi-lattice based companions. We also study conditions which, when satisfied by a deductive system in the class, imply that it is strongly algebraizable. This brings some information on the open area of research of Abstract Algebraic Logic which consists in finding interesting characterizations of classes of algebraizable logics that are strongly algebraizable.

Keywords Algebraizable logics · Strongly algebraizable logics · Logics based on semi-lattices · Abstract Algebraic logic

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1 Introduction

One of the central concepts of Abstract Algebraic Logic is algebraizable deductive system. It was introduced by Blok and Pigozzi in their seminal monograph [1]. In current terminology this concept corresponds to the notion of finitary and finitely algebraizable deductive system. Blok and Pigozzi's original concept was broadened by Czelakowski and Hermman to encompass non-finitary deductive systems, and the condition of the equivalence of the deductive system with an equational consequence relation performed by translations from formulas to finite sets of equations and from equations to finite sets of formulas was weakened to translations to sets of arbitrary cardinality.

If a deductive system $\mathcal{S}$ is algebraizable (in the current broad sense), then there exists a greatest class of algebras to the equational consequence relation of which $\mathcal{S}$ is equivalent. This class, known as the equivalent algebraic semantics of $\mathcal{S}$, coincides with the class of algebras that is associated with $\mathcal{S}$ as its canonical algebraic counterpart according to prevailing practice in Abstract Algebraic Logic, and it is a quasivariety when $\mathcal{S}$ is finitary and finitely algebraizable.

Almost all the best known finitary and finitely algebraizable deductive systems have a variety as their equivalent algebraic semantics. But this is not a consequence of the definition of finitary and finitely algebraizable deductive system. Indeed, there are finitary and finitely algebraizable deductive systems whose equivalent algebraic semantics is not a variety: the logic BCK and the 1-assertional logic of implicative algebras are some of the examples. The algebraizable deductive systems whose equivalent algebraic semantics is a variety are called *strongly algebraizable*.

Also, many of the best known finitary and finitely algebraizable deductive systems, like classical and intuitionistic propositional logics, the global consequence relations of normal modal logics, relevance logics R and R_t, intuitionistic and classical linear logics, Łukasiewicz infinite-valued logic, among others, besides being strongly algebraizable have a binary connective $\wedge$ that enjoys the following property:

1. the interpretation of $\wedge$ in every algebra $\mathbf{A}$ of the canonical algebraic counterpart[1] of $\mathcal{S}$ gives a meet semi-lattice $\langle A, \wedge^{\mathbf{A}}\rangle$.

These logics also have a binary connective $\rightarrow$ with the property that the order of the meet semi-lattices is definable using $\rightarrow$ and the set of the equations that perform the translation (involved in being algebraizable) of formulas into sets of equations and, moreover, the formulas $\varphi \rightarrow \psi$ and $(\varphi \wedge \psi) \leftrightarrow \varphi$ are equivalent, in the sense that $(\varphi \rightarrow \psi) \leftrightarrow ((\varphi \wedge \psi) \leftrightarrow \varphi)$ is a theorem.

In [14], to every class of algebras K of a given similarity type with a binary term $\wedge$ such that for every $\mathbf{A} \in \mathsf{K}$ the structure $\langle A, \wedge^{\mathbf{A}}\rangle$ is a meet semi-lattice, is associated a finitary deductive system $\mathcal{S}(\mathsf{K}, \leq)$ defined using the order of the semi-lattice. This deductive system has the congruence property (that is, the mutual entailment relation is a congruence of the algebra of formulas) and its canonical algebraic counterpart is the variety generated by K. In [14] the deductive systems definable in this way are called semi-lattice based.

1 For a precise definition we address the reader to the Preliminaries section.

Property (1) above allows us to associate with each deductive system $\mathcal{S}$ enjoying it the semi-lattice based companion $\mathcal{S}(\mathbf{Alg}\mathcal{S}, \leq)$, where $\mathbf{Alg}\mathcal{S}$ denotes the algebraic counterpart of $\mathcal{S}$. In this paper we study the relations between $\mathcal{S}$ and $\mathcal{S}(\mathbf{Alg}\mathcal{S}, \leq)$ and the properties of these latter deductive systems from an abstract general point of view when $\mathcal{S}$ is algebraizable and has a conjunction that satisfies condition (1) above. Moreover, we do not restrict ourselves to finitary deductive systems. We call the deductive systems with these properties algebraizable deductive systems with a strong conjunction. It is worth noticing that there are strongly algebraizable deductive systems without any binary term that behaves as a conjunction, for example the implicative fragment of intuitionistic logic, whose equivalent algebraic semantics is the variety of Hilbert algebras.

Some of the examples of deductive systems that motivated the introduction in [10] of the strong version of a protoalgebraic logic were pairs of deductive systems in which the first element of the pair is, using the present terminology, algebraizable with a strong conjunction and the second one is its semi-lattice based companion. In [10] we found some concepts and properties that allow for a smooth theory of a class of pairs of deductive systems with very strong links between the members of the pair, but as Theorem 8 of the present paper shows, this theory of the strong version does not seem to capture exactly the relation between an algebraizable deductive system with a strong conjunction and its semi-lattice based companion, when the latter is protoalgebraic. In fact, the first deductive system is the strong version of the second if and only if it is strongly algebraizable and both have the same theorems. Moreover, there are algebraizable deductive systems with a strong conjunction whose semi-lattice based companion is not protoalgebraic.

One of the open problems in AAL is to find interesting characterizations of classes of algebraizable logics which are strongly algebraizable, characterizations that somehow will help to explain the surprising phenomenon that despite the fact that almost all the best known finitary and finitely algebraizable deductive systems are strongly algebraizable, the concept finitary and finitely algebraizable deductive system only implies that the equivalent algebraic semantics is a quasivariety. The present paper can be seen as a contribution in that direction.

The implication fragment of intuitionistic logic, as we mentioned is strongly algebraizable without any conjunction. This is an example of a Fregean deductive system that admits a deduction theorem. All the finitary deductive systems enjoying these two properties are strongly algebraizable. So our contribution to the problem of finding interesting characterizations of classes of algebraizable logics which are strongly algebraizable does not encompass all the strongly algebraizable deductive systems but only a very significant class.

The outline of the paper is as follows. In Sect. 2 we review all the notions of AAL necessary to follow the paper. In Sect. 3 we discuss conjunctions and introduce the concept of strong conjunction. Section 4 is devoted to the study of the main properties of algebraizable logics with a strong conjunction and the properties of their semi-lattice based companion. Finally, in Sect. 5 we study the strongly algebraizable deductive systems with a strong conjunction and properties of their semi-lattice based companions. Several of the results we present are generalizations of similar results for the logics studied in [3] and with more generality in [8].

2 Preliminaries

In this section we review, for the reader's convenience, all the concepts and results of AAL we use in the paper and we fix our notation. For more information on these concepts and facts we address the reader to [5,11].[2] Recall that a *consequence relation* on a set A is a binary relation $\vdash\ \subseteq \mathcal{P}(A) \times A$ such that for every $X, Y \subseteq A$ and every $a \in A$: (1) $X \vdash a$, whenever $a \in X$ and (2) if $X \vdash b$ for every $b \in Y$ and $Y \vdash a$, then $X \vdash a$. A consequence relation $\vdash$ on A is *finitary* if whenever $X \vdash a$ whit $X \cup \{a\} \subseteq A$, there is a finite $Y \subseteq X$ such that $Y \vdash a$. Consequence relations are monotone, in the sense that if $X \subseteq Y$ and $X \vdash a$, then $Y \vdash a$. If $X, Y \subseteq A$, $X \vdash Y$ means that $X \vdash a$ for every $a \in Y$. Moreover, we say that $X, Y \subseteq A$ are $\vdash$-*equivalent* if $X \vdash Y$ and $Y \vdash X$.

Let $\mathcal{L}$ be a set of connectives that we also regard as an algebraic similarity type, and let $\mathbf{Fm}_{\mathcal{L}}$ denote the absolutely free algebra of type $\mathcal{L}$ over a denumerable set Var of generators, which are the variables. The elements of the universe Fm of $\mathbf{Fm}_{\mathcal{L}}$ are the *formulas* of type $\mathcal{L}$. A *substitution* is an endomorphism of $\mathbf{Fm}_{\mathcal{L}}$. An equation $\varphi \approx \psi$ of type $\mathcal{L}$ is identified with the ordered pair of formulas $\langle \varphi, \psi \rangle$. For every substitution σ the map $\sigma^e : Fm \times Fm \to Fm \times Fm$ is defined by $\sigma^e(\langle \varphi, \psi \rangle) = \langle \sigma(\varphi), \sigma(\psi) \rangle$, for every $\varphi, \psi \in Fm$.

A (*finitary*) *deductive system* of type $\mathcal{L}$ is a pair $\mathcal{S} = \langle \mathbf{Fm}_{\mathcal{L}}, \vdash_{\mathcal{S}} \rangle$ where $\vdash_{\mathcal{S}}$ is a (finitary) consequence relation on Fm which is invariant under substitutions, that is, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then for every substitution σ, $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$. The relation $\vdash_{\mathcal{S}}$ is the *consequence, or entailment, relation* of $\mathcal{S}$. If Γ and Γ' are sets of formulas, $\Gamma \vdash_{\mathcal{S}} \Gamma'$ means in this paper that for every $\varphi \in \Gamma'$, $\Gamma \vdash_{\mathcal{S}} \varphi$. A *theorem* of $\mathcal{S}$ is a formula φ such that $\emptyset \vdash_{\mathcal{S}} \varphi$.

We will omit references to the similarity type unless some confusion may arise and we assume that the algebras and deductive systems are of the same type.

A *theory* of a deductive system $\mathcal{S}$, or $\mathcal{S}$-*theory*, is a set of formulas Γ closed under $\vdash_{\mathcal{S}}$, that is, such that for every formula φ, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\varphi \in \Gamma$. The set of $\mathcal{S}$-theories is denoted by $\mathrm{Th}\mathcal{S}$; it forms a complete lattice when it is ordered by inclusion.

Let K be a class of algebras of type $\mathcal{L}$. The *equational consequence* of K, denoted $\models_{\mathsf{K}}$, is the relation between sets of equations and equations defined by

$$\begin{aligned} \{\delta_i \approx \varepsilon_i : i \in I\} \models_{\mathsf{K}} \delta \approx \varepsilon \quad \text{iff} \quad & (\forall \mathbf{A} \in \mathsf{K})(\forall h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})) \\ & (\forall i \in I)\ h(\delta_i) = h(\varepsilon_i) \Longrightarrow h(\delta) = h(\varepsilon). \end{aligned}$$

This relation is a consequence relation on the set of equations of type $\mathcal{L}$ and it is invariant under the set of maps $\{\sigma^e : \sigma : \mathbf{Fm} \to \mathbf{Fm}\}$, that is, if $\Phi \models_{\mathsf{K}} \delta \approx \varepsilon$, then for every substitution σ, $\sigma^e[\Phi] \models_{\mathsf{K}} \sigma^e(\delta \approx \varepsilon)$.

[2] In this Section we do not specify who introduced a concept or proved a result except for concepts and results not covered in [5,11].

Semantics of logical matrices

Let $\mathcal{S}$ be a deductive system and $\mathbf{A}$ an algebra. A set $F \subseteq A$ is an *$\mathcal{S}$-filter* if for every homomorphism $h : \mathbf{Fm} \to \mathbf{A}$, every set of formulas Γ and every formula φ, if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $h[\Gamma] \subseteq F$, then $h(\varphi) \in F$. The set of all $\mathcal{S}$-filters of an algebra $\mathbf{A}$ is denoted by $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$. This set is closed under arbitrary intersections, so it is a complete lattice when it is ordered by inclusion.

A *logical matrix*, a matrix in an abridged form, is a pair $\langle \mathbf{A}, F\rangle$ where $\mathbf{A}$ is an algebra and F is a subset of the universe of $\mathbf{A}$. A matrix $\langle \mathbf{A}, F\rangle$ is a (*matrix*) *model* of a deductive system $\mathcal{S}$ if F is an $\mathcal{S}$-filter of $\mathbf{A}$. Therefore, the matrix models of $\mathcal{S}$ on the algebra of formulas are the matrices of the form $\langle \mathbf{Fm}, T\rangle$ where T is an $\mathcal{S}$-theory.

Let $\mathbf{A}$ be an algebra and $F \subseteq A$. The *Leibniz congruence of F relative to* $\mathbf{A}$, denoted by $\boldsymbol{\Omega}^{\mathbf{A}}(F)$, is the greatest congruence of $\mathbf{A}$ compatible with F, that is, that does not relate elements in F with elements not in F. The map $\boldsymbol{\Omega}^{\mathbf{A}}$ when restricted to the $\mathcal{S}$-filters of $\mathbf{A}$ is called the *Leibniz operator* on $\mathbf{A}$. A matrix $\langle \mathbf{A}, F\rangle$ is *reduced* if the Leibniz congruence $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ is the identity relation Δ_A on A. The class of the algebraic reducts of the reduced matrices which are models of a deductive system $\mathcal{S}$ is denoted by $\mathbf{Alg}^*\mathcal{S}$, that is,

$$\mathbf{Alg}^*\mathcal{S} = \{\mathbf{A} : (\exists F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A})\ \boldsymbol{\Omega}^{\mathbf{A}}(F) = \Delta_A\}.$$

If M is a class of matrices, the relation $\vdash_{\mathsf{M}}$ on Fm defined by

$$\Gamma \vdash_{\mathsf{M}} \varphi \quad \text{iff} \quad (\forall \langle \mathbf{A}, F\rangle \in \mathsf{M})(\forall h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}))(h[\Gamma] \subseteq F \Rightarrow h(\varphi) \in F)$$

is a consequence relation on Fm invariant under substitutions, so $\mathcal{S}_{\mathsf{M}} = \langle \mathbf{Fm}, \vdash_{\mathsf{M}}\rangle$ is a deductive system. When M is closed under ultraproducts, $\vdash_{\mathsf{M}}$ is finitary, and then $\mathcal{S}_{\mathsf{M}}$ is a finitary deductive system. A class of matrices M is a *complete matrix semantics* for a deductive system $\mathcal{S}$ if $\vdash_{\mathcal{S}} = \vdash_{\mathsf{M}}$. For every deductive system $\mathcal{S}$, the classes of matrices $\{\langle \mathbf{A}, F\rangle : F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}\}$ and $\{\langle \mathbf{A}, F\rangle : \mathbf{A} \in \mathbf{Alg}^*\mathcal{S}, F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}, \boldsymbol{\Omega}^{\mathbf{A}}(F) = \Delta_A\}$ are complete matrix semantics for $\mathcal{S}$.

The algebraic counterpart of a deductive system

The class $\mathbf{Alg}^*\mathcal{S}$ is the class of algebras that the semantics of logical matrices associates canonically with a deductive system $\mathcal{S}$. By taking into consideration the properties of non-protoalgebraic logics, it is argued in [9] that this class of algebras is not the right class to take as the canonical algebraic counterpart of an arbitrary deductive system. The class of algebras proposed in [9] as the canonical algebraic counterpart and that is currently considered so in AAL has been defined in more than one way and different authors have arrived at it following diverse research orientations. The most simple way of defining it is by means of the so-called Suszko congruence (cf. [6]).

Let $\mathbf{A}$ be an algebra and F an $\mathcal{S}$-filter of $\mathbf{A}$. The *Suszko congruence* of F is the congruence

$$\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}_{\mathcal{S}}(F) = \bigcap\{\boldsymbol{\Omega}^{\mathbf{A}}(G) : F \subseteq G \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}\}.$$

The class of algebras usually considered nowadays in AAL as the canonical algebraic counterpart of $\mathcal{S}$ is:

$$\mathbf{Alg}\mathcal{S} = \{\mathbf{A} : (\exists F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A})\ \widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}_{\mathcal{S}}(F) = \Delta_A\}.$$

It is easy to show that $\mathbf{Alg}\mathcal{S}$ is the closure of $\mathbf{Alg}^*\mathcal{S}$ under subdirect products.

In this paper it will be also useful to consider the following characterization of $\mathbf{Alg}\mathcal{S}$. Let $\mathbf{A}$ be an algebra. The *Tarski congruence* of $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ (cf. [9]) is the relation

$$\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}(\mathrm{Fi}_{\mathcal{S}}\mathbf{A}) = \bigcap\{\boldsymbol{\Omega}^{\mathbf{A}}(F) : F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}\}.$$

Thus, the Tarski congruence of $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ is the Suszko congruence corresponding to the least $\mathcal{S}$-filter of $\mathbf{A}$. In the particular case of the algebra of formulas, the Tarski congruence of $\mathrm{Th}(\mathcal{S})$ is usually denoted by $\widetilde{\boldsymbol{\Omega}}(\mathcal{S})$. The Tarski congruence of $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ can be characterized as the greatest congruence included in the relation

$$\Lambda(\mathrm{Fi}_{\mathcal{S}}\mathbf{A}) = \{\langle a, b\rangle \in A \times A : (\forall F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A})(a \in F \Leftrightarrow b \in F)\},$$

which is known as the *Frege relation* of $\langle \mathbf{A}, \mathrm{Fi}_{\mathcal{S}}\mathbf{A}\rangle$.

The characterization of $\mathbf{Alg}\mathcal{S}$ using Tarski congruences is as follows:

$$\mathbf{A} \in \mathbf{Alg}\mathcal{S} \quad \text{iff} \quad \widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}(\mathrm{Fi}_{\mathcal{S}}\mathbf{A}) = \Delta_A,$$

for every algebra $\mathbf{A}$. This holds because $\bigcap \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ is an $\mathcal{S}$-filter and, as it is easy to see, this implies that $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}_{\mathcal{S}}(\bigcap \mathrm{Fi}_{\mathcal{S}}\mathbf{A}) = \widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}(\mathrm{Fi}_{\mathcal{S}}\mathbf{A})$.

The intrinsic variety of a deductive system

Another class of algebras associated with a deductive system that plays a prominent role in AAL is the intrinsic variety. The *intrinsic variety* of a deductive system $\mathcal{S}$ (cf. [9,15]) is the variety generated by the algebra $\mathbf{Fm}/\widetilde{\boldsymbol{\Omega}}(\mathcal{S})$, and is denoted by $\mathsf{K}_{\mathcal{S}}$. The congruence $\widetilde{\boldsymbol{\Omega}}(\mathcal{S})$ on $\mathbf{Fm}$ is invariant, and therefore the resuting quotient algebra is free in the class $\mathsf{K}_{\mathcal{S}}$. Thus the class $\mathsf{K}_{\mathcal{S}}$ can be described as the variety axiomatized by the equations $\varphi \approx \psi$ such that $\langle \varphi, \psi\rangle \in \widetilde{\boldsymbol{\Omega}}(\mathcal{S})$.

The relation between the three classes of algebras $\mathbf{Alg}^*\mathcal{S}$, $\mathbf{Alg}\mathcal{S}$ and $\mathsf{K}_{\mathcal{S}}$ associated with a deductive system $\mathcal{S}$ so far is as follows:

1. $\mathbf{Alg}^*\mathcal{S} \subseteq \mathbf{Alg}\mathcal{S} \subseteq \mathsf{K}_{\mathcal{S}}$,
2. $\mathsf{K}_{\mathcal{S}}$ is the variety generated by $\mathbf{Alg}^*\mathcal{S}$ and so also by $\mathbf{Alg}\mathcal{S}$.

This is the best we can obtain in general, because there are deductive systems satisfying each one of the possible combinations of proper inclusions and equalities in expression (1) above.[3]

[3] For example, the algebraizable logic BCK satisfies $\mathbf{Alg}^*\mathrm{BCK} = \mathbf{Alg}\mathrm{BCK} \subsetneq \mathsf{K}_{\mathrm{BCK}}$. The proper inclusion $\mathbf{Alg}\mathrm{BCK} \subsetneq \mathsf{K}_{\mathrm{BCK}}$ holds because $\mathbf{Alg}^*\mathrm{BCK}$ is the class of BCK-algebras, a quasivariety which is not a

Algebraizable deductive systems

Let $\Delta(p, q)$ be a set of formulas in two variables and let $\tau(p)$ be a set of equations in one variable. The set Δ defines a map, that we also denote by Δ, from equations to sets of formulas by letting

$$\Delta(\varphi, \psi) = \{\delta(p/\varphi, q/\psi)) : \delta \in \Delta(p, q)\},$$

for every pair of formulas, i.e. equation, $\langle\varphi, \psi\rangle$. Similarly, τ defines a map, which we denote by τ, from formulas to sets of equations by letting

$$\tau(\varphi) = \{\delta(p/\varphi) \approx \varepsilon(p/\varphi) : \delta \approx \varepsilon \in \tau(p)\},$$

for every formula φ. If Φ is a set of equations and Γ is a set of formulas we let

$$\Delta(\Phi) := \bigcup_{\varphi \approx \psi \in \Phi} \Delta(\varphi, \psi) \quad \text{and} \quad \tau(\Gamma) := \bigcup_{\varphi \in \Gamma} \tau(\varphi).$$

For every substitution σ the maps Δ and τ satisfy

$$\sigma[\Delta(\varphi, \psi)] = \Delta(\sigma(\varphi), \sigma(\psi)) \qquad \sigma^e[\tau(\varphi)] = \tau(\sigma(\varphi)).$$

Maps from formulas to sets of equations and from equations to sets of formulas with these properties will be called *structural transforms* (cf. [5]); they are called transformers in [17]. Every structural transform is definable by a set of formulas Δ or a set of equations τ in the way specified above. A structural transform is *finite* if it is definable by a finite set of formulas or equations.

A deductive system $\mathcal{S}$ is *algebraizable* if there is a class of algebras K, a set of formulas $\Delta(p, q)$ and a set of equations $\tau(p)$ such that for every set of formulas Γ, every formula φ and every equation $\delta \approx \varepsilon$

1. $\Gamma \vdash_{\mathcal{S}} \varphi$ iff $\tau(\Gamma) \models_{\mathsf{K}} \tau(\varphi)$
2. $\delta \approx \varepsilon \models_{\mathsf{K}} \tau(\Delta(\delta, \varepsilon))$ and $\tau(\Delta(\delta, \varepsilon)) \models_{\mathsf{K}} \delta \approx \varepsilon$.

Conditions (1) and (2) imply:

(3) $\Phi \models_{\mathsf{K}} \varphi \approx \psi$ iff $\Delta(\Phi) \vdash_{\mathcal{S}} \Delta(\varphi, \psi)$
(4) $\varphi \vdash_{\mathcal{S}} \Delta(\tau(\varphi))$ and $\Delta(\tau(\varphi)) \vdash_{\mathcal{S}} \varphi$.

Every algebraizable deductive system has theorems. Moreover, if $\mathcal{S}$ is algebraizable by K, τ and Δ and also by K', τ' and Δ', then $\models_{\mathsf{K}} = \models_{\mathsf{K}'}$, τ and τ' are $\models_{\mathsf{K}}$-equivalent and Δ and Δ' are $\vdash_{\mathcal{S}}$-equivalent. Also there is a greatest class of algebras K' such that $\models_{\mathsf{K}} = \models_{\mathsf{K}'}$. This class is known as the *equivalent algebraic semantics* of $\mathcal{S}$ and coincides with $\mathbf{Alg}^*\mathcal{S}$, which for algebraizable deductive systems coincides also with $\mathbf{Alg}\mathcal{S}$.

Footnote 3 continued
variety (cf. [18]). If $\mathcal{S}$ is the $(\wedge, \vee)$-fragment of classical logic we have $\mathbf{Alg}^*\mathcal{S} \subsetneq \mathbf{Alg}\mathcal{S} = \mathsf{K}_{\mathcal{S}}$. Finally, the three classes of algebras are different when $\mathcal{S}$ is the global subintuitionistic logic of the class of all Kripke frames, as it is shown in [2].

An algebraizable deductive system for which the set of formulas Δ may be taken finite is called *finitely algebraizable*. This notion, when applied to finitary deductive systems is the concept of algebraizable logic introduced by Blok and Pigozzi in [1]. In that setting, it implies that the set τ can also be taken as finite. If $\mathcal{S}$ is finitary and finitely algebraizable, then $\models_{\mathsf{K}}$ is finitary and the equivalent algebraic semantics $\mathbf{Alg}^*\mathcal{S}$ is a quasivariety.

Algebraizable deductive systems have been characterized in terms of the definability of the Leibniz congruence of their matrix models and the definability of the filters of their reduced matrix models. Let $\langle \mathbf{A}, F\rangle$ be a matrix. A set of formulas $\Delta(p, q)$ *defines the Leibniz congruence* $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ if

$$\boldsymbol{\Omega}^{\mathbf{A}}(F) = \{\langle a, b\rangle \in A^2 : \Delta^{\mathbf{A}}(a, b) \subseteq F\}.$$

If Δ defines the Leibniz congruences of every matrix model of a deductive system $\mathcal{S}$, then it is called a *set of equivalence formulas* for $\mathcal{S}$. Let $\tau(p)$ be a set of equations. The set of solutions in $\mathbf{A}$ of the equations in τ is denoted by $\tau(\mathbf{A})$, that is:

$$\tau(\mathbf{A}) := \{a \in A : \mathbf{A} \models \tau(p)[a]\}.$$

We say that $\tau(p)$ *defines a set* $F \subseteq A$ on $\mathbf{A}$ if $F = \tau(\mathbf{A})$. If $\mathcal{S}$ is a deductive system and τ defines the $\mathcal{S}$-filters of the reduced matrix models of $\mathcal{S}$, then τ is called a *set of truth-defining equations for* $\mathcal{S}$. Note that if $\mathcal{S}$ has a set of truth-defining equations τ, then in every $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ there is exactly one $\mathcal{S}$-filter F such that $\langle \mathbf{A}, F\rangle$ is reduced; this $\mathcal{S}$-filter is $\tau(\mathbf{A})$.

The characterization of algebraizable deductive system in terms of definability conditions is as follows: a deductive system is algebraizable if and only if it has a set of equivalence formulas and a set of truth-defining equations. These sets define the structural transforms we met in the definition of algebraizable deductive system.

Other classes of deductive systems

Deductive systems may be studied and classified according to the lattice structure properties of the lattices of $\mathcal{S}$-filters that are preserved by the Leibniz operator when passing to the lattices of $\mathbf{Alg}^*\mathcal{S}$-relative congruences. The classification is known as the *Leibniz hierarchy*.

A deductive system $\mathcal{S}$ is *protoalgebraic* if for every algebra $\mathbf{A}$, the Leibniz operator is a monotone map from the lattice $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ to the lattice $\mathrm{Con}\mathbf{A}$ of the congruences of $\mathbf{A}$. There is an important, and useful, syntactic characterization of protoalgebraic deductive systems. A deductive system is protoalgebraic if and only if there is a set of formulas in two variables $\Rightarrow(p, q)$ with the following two properties:

1. $\vdash_{\mathcal{S}} \Rightarrow(p, p)$
2. $p, \Rightarrow(p, q) \vdash_{\mathcal{S}} q$ (MP or detachment)

A set $\Rightarrow(p, q)$ with these two properties will be called in this paper a *set of protoimplication formulas* for $\mathcal{S}$. If $\mathcal{S}$ is finitary and protoalgebraic, then it has a finite set of protoimplication formulas. A protoalgebraic deductive system does not necessarily have theorems, but in each similarity type there is exactly one protoalgebraic deductive system without theorems, the deductive system whose theories are the empty set and the set of all formulas; this deductive system is called the quasi-inconsistent deductive system of the type. For every protoalgebraic deductive system $\mathcal{S}$, the classes of algebras $\mathbf{Alg}^*\mathcal{S}$ and $\mathbf{Alg}\mathcal{S}$ are equal.

A deductive system $\mathcal{S}$ is *equivalential* if it is protoalgebraic and the Leibniz operator commutes with inverse homomorphisms in the sense that if h is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$, then for every $\mathcal{S}$-filter G of $\mathbf{B}$, $h^{-1}[\boldsymbol{\Omega}^{\mathbf{B}}(G)] = \boldsymbol{\Omega}^{\mathbf{A}}(h^{-1}(G))$. It holds that $\mathcal{S}$ is equivalential if and only if $\mathcal{S}$ has a set of equivalence formulas.

A deductive system $\mathcal{S}$ is *weakly algebraizable* if for every algebra $\mathbf{A}$ the Leibniz operator on $\mathbf{A}$ is an isomorphism between the lattice of $\mathcal{S}$-filters of $\mathbf{A}$ and the lattice $\mathrm{Con}_{\mathbf{Alg}^*(\mathcal{S})}\mathbf{A}$ of the congruences θ of $\mathbf{A}$ such that $\mathbf{A}/\theta \in \mathbf{Alg}^*\mathcal{S}$, which are referred to as the $\mathbf{Alg}^*\mathcal{S}$-relative congruences of $\mathbf{A}$. Clearly, every weakly algebraizable deductive system is protoalgebraic. Moreover, every weakly algebraizable deductive system has theorems, otherwise $\emptyset$ and Fm would be different theories with the same Leibniz congruence.

Algebraizable deductive systems can be characterized as the equivalential and weakly algebraizable deductive systems, that is, as the deductive systems such that for every algebra $\mathbf{A}$ the Leibniz operator on $\mathbf{A}$ is an isomorphism between the lattice of $\mathcal{S}$-filters of $\mathbf{A}$ and the lattice of the $\mathbf{Alg}^*\mathcal{S}$-relative congruences of $\mathbf{A}$ that commutes with inverse homomorphisms.

The class of truth-equational deductive systems is introduced in [16]. A deductive system $\mathcal{S}$ is *truth-equational* if it has a set of truth-defining equations. In [16] it is proved that $\mathcal{S}$ is truth-equational if and only if the Leibniz operator is completely order reflecting on every algebra $\mathbf{A}$, that is, if it holds that whenever $\mathcal{F} \cup \{G\} \subseteq \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ and $\bigcap \boldsymbol{\Omega}^{\mathbf{A}}[\mathcal{F}] \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(G)$, then $\bigcap \mathcal{F} \subseteq G$. Every weakly algebraizable deductive system is truth-equational, but the converse does not hold; for instance, there are truth-equational deductive systems which are not protoalgebraic. In fact, a deductive system is weakly algebraizable if and only if it is protoalgebraic and truth-equational.

There are also classes of deductive systems studied in AAL which do not have a characterization by the behaviour of the Leibniz operator w.r.t. the lattice structure properties of the lattices of $\mathcal{S}$-filters and relative congruences. We recall the ones that are the most important to the paper.

A deductive system $\mathcal{S}$ has *the congruence property* (or is selfextensional) if the relation $\dashv_{\mathcal{S}}\vdash$ of mutual entailment between formulas is a congruence of $\mathbf{Fm}$. This is equivalent to saying that $\widetilde{\boldsymbol{\Omega}}(\mathcal{S}) = \dashv_{\mathcal{S}}\vdash$. So, if $\mathcal{S}$ is selfextensional, its intrinsic variety $\mathsf{K}_{\mathcal{S}}$ is axiomatized by the equations $\varphi \approx \psi$ such that $\varphi \dashv_{\mathcal{S}}\vdash \psi$. A notion stronger than having the congruence property is the following. A deductive system $\mathcal{S}$ is *congruential* (or fully selfextensional) if for every algebra $\mathbf{A}$ the Tarski congruence of $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ satisfies that

$$\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}(\mathrm{Fi}_{\mathcal{S}}\mathbf{A}) \quad \text{iff} \quad (\forall F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A})(a \in F \Leftrightarrow b \in F)$$

for every $a, b \in A$, that is, when $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}(\mathrm{Fi}_{\mathcal{S}}\mathbf{A}) = \Lambda(\mathrm{Fi}_{\mathcal{S}}\mathbf{A})$.

Some deductive systems with the congruence property enjoy a stronger property; not only $\dashv_{\mathcal{S}}\vdash$ is a congruence but for every $\mathcal{S}$-theory T the relation

$$\{\langle\varphi;\psi\rangle : T,\varphi \vdash_{\mathcal{S}} \psi, T,\psi \vdash_{\mathcal{S}} \varphi\}$$

is also a congruence. They are called *Fregean*. Similarly, some congruential deductive systems satisfy that for every algebra $\mathbf{A}$ and every $\mathcal{S}$-filter G of $\mathbf{A}$ the relation $\{\langle a,b\rangle \in A^2 : (\forall F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A})(G \subseteq F \Rightarrow (a \in F \Leftrightarrow b \in F))\}$ is a congruence of $\mathbf{A}$. These deductive systems are called *fully Fregean*.

Assertional logics

Let K be a pointed class of algebras with constant term 1. The 1-*assertional logic* of K is the deductive system $\mathcal{S}(\mathsf{K},1)$ defined by

$$\Gamma \vdash_{\mathcal{S}(\mathsf{K},1)} \varphi \quad \text{iff} \quad (\forall \mathbf{A} \in \mathsf{K})(\forall h \in \mathrm{Hom}(\mathbf{Fm},\mathbf{A}))(h[\Gamma] \subseteq \{1^{\mathbf{A}}\} \Rightarrow h(\varphi) = 1^{\mathbf{A}}).$$

If K is a quasivariety, then $\mathcal{S}(\mathsf{K},1)$ is finitary.

A deductive system $\mathcal{S}$ is *regularly algebraizable* if it is algebraizable, the class of algebras $\mathbf{Alg}^*\mathcal{S}$ is pointed, and $\mathcal{S}$ is the 1-assertional logic of $\mathbf{Alg}^*\mathcal{S}$. This is equivalent to say that $\mathcal{S}$ is algebraizable and satisfies the G-rule: $p,q \vdash_{\mathcal{S}} \Delta(p,q)$, where $\Delta(p,q)$ is any set of equivalence formulas for $\mathcal{S}$.

Classes of algebras with semi-lattice reducts

A class of algebras K of a given algebraic similarity type has *semi-lattice reducts* if there is a binary term $\wedge$ such that for every $\mathbf{A} \in \mathsf{K}$ the algebra $\langle A, \wedge^{\mathbf{A}}\rangle$ is a semi-lattice. In this situation we say that K has $\wedge$-*semi-lattice reducts*. Equivalently, K has $\wedge$-semi-lattice reducts if the semi-lattice equations

(L1) $x \wedge x \approx x$ (L2) $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$ (L3) $x \wedge y \approx y \wedge x$

hold in K.

If K is a class of algebras with $\wedge$-semi-lattice reducts, the variety $\mathsf{V}(\mathsf{K})$ generated by K has also $\wedge$-semi-lattice reducts. We will consider in every algebra $\mathbf{A} \in \mathsf{V}(\mathsf{K})$ the partial order $\leq^{\mathbf{A}}$ defined by

$$a \leq^{\mathbf{A}} b \quad \text{iff} \quad a \wedge^{\mathbf{A}} b = a,$$

for every $a, b \in A$. We will omit the superscript in $\leq^{\mathbf{A}}$ and $\wedge^{\mathbf{A}}$ when no confusion is expected.

Let K be a class of algebras with $\wedge$-semi-lattice reducts and let $\mathbf{A} \in \mathsf{K}$. We say that a set $F \subseteq A$ is a *semi-lattice filter* of $\mathbf{A}$ if it is a filter of $\langle A, \wedge^{\mathbf{A}}\rangle$, that is, if it is a nonempty set upper closed under $\leq^{\mathbf{A}}$ and closed under $\wedge^{\mathbf{A}}$.

Semi-lattice based deductive systems

Let K be a class of algebras with $\wedge$-semi-lattice reducts. We associate with K the finitary deductive system $\mathcal{S}(\mathsf{K},\leq)$ defined as follows. First we define the consequences of finite sets by

$$\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{S}(\mathsf{K},\leq)} \varphi$$
$$\text{iff} \quad \forall \mathbf{A} \in \mathsf{K}\, \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})\; v(\varphi_0) \wedge \ldots \wedge v(\varphi_{n-1}) \leq v(\varphi),$$
$$\emptyset \vdash_{\mathcal{S}(\mathsf{K},\leq)} \varphi \quad \text{iff} \quad \forall \mathbf{A} \in \mathsf{K}\, \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}) \forall a \in A\; a \leq v(\varphi),$$

and then we extend the definition to arbitrary sets by:

$$\Gamma \vdash_{\mathcal{S}(\mathsf{K},\leq)} \varphi \quad \text{iff} \quad \text{there is a finite } \Delta \subseteq \Gamma \text{ such that } \Delta \vdash_{\mathcal{S}(\mathsf{K},\leq)} \varphi.$$

It is easily checked that $\langle \mathbf{Fm}, \vdash_{\mathcal{S}(\mathsf{K},\leq)} \rangle$ is a deductive system. Moreover, it is clear that $\mathcal{S}(\mathsf{K}, \leq)$ and $\mathcal{S}(\mathsf{V}(\mathsf{K}), \leq)$ are equal.

We will need the results in [14] which we present in the proposition and lemma below.

Proposition 1 *Let* K *be a class of algebras with* $\wedge$*-semi-lattice reducts. Then*

1. $\mathcal{S}(\mathsf{K}, \leq)$ *is congruential,*
2. $\wedge$ *is a conjunction for* $\mathcal{S}(\mathsf{K}, \leq)$*,*
3. *the variety* $\mathsf{V}(\mathsf{K})$ *is the intrinsic variety of* $\mathcal{S}(\mathsf{K}, \leq)$*,*
4. $\mathbf{Alg}\mathcal{S}(\mathsf{K}, \leq) = \mathsf{V}(\mathsf{K})$*,*
5. $\mathbf{Alg}\mathcal{S}(\mathsf{K}, \leq)$ *is a variety.*
6. *for every* $\mathbf{A} \in \mathbf{Alg}\mathcal{S}(\mathsf{K}, \leq)$*, a nonempty set* $F \subseteq A$ *is an* $\mathcal{S}(\mathsf{K}, \leq)$*-filter iff it is a semi-lattice filter of* $\langle \mathbf{A}, \wedge^{\mathbf{A}} \rangle$*.*

It should be stressed here that to prove that $\mathbf{Alg}\mathcal{S}(\mathsf{K}, \leq)$ is a variety is easy and natural. However, to show that an algebraizable deductive system is strongly algebraizable is not so easy and natural, in general.

Lemma 1 *If* K *is a class of algebras with* $\wedge$*-semi-lattice reducts, then* $\mathcal{S}(\mathsf{K}, \leq)$ *has theorems iff for every* $\mathbf{A} \in \mathsf{K}_{\mathcal{S}(\mathsf{K},\leq)}$ *the semi-lattice* $\langle A, \wedge^{\mathbf{A}} \rangle$ *has a greatest element and there is a formula that is interpreted as the greatest element in every* $\mathbf{A} \in \mathsf{K}_{\mathcal{S}(\mathsf{K},\leq)}$.

We say that a deductive system $\mathcal{S}$ is *semi-lattice based* if there is a binary term $\wedge$ and a class of algebras K with $\wedge$-semi-lattice reducts such that $\mathcal{S} = \mathcal{S}(\mathsf{K}, \leq)$.[4]

Let $\mathcal{S}$ be a deductive system with a binary term $\wedge$ such that $\mathbf{Alg}\mathcal{S}$ has $\wedge$-semi-lattice reducts. The deductive system $\mathcal{S}(\mathbf{Alg}\mathcal{S}, \leq)$ will be called *the semi-lattice based companion* of $\mathcal{S}$. We say that a deductive system $\mathcal{S}$ *has a semi-lattice based companion* if there is a binary term $\wedge$ such that $\mathbf{Alg}\mathcal{S}$ has $\wedge$-semi-lattice reducts. Note that the semi-lattice based companion is (by definition) a finitary deductive system. Note also that since if a class of algebras has semi-lattice reducts, so does the variety it generates, then we have that $\mathsf{K}_{\mathcal{S}}$ has $\wedge$-semi-lattice reducts and $\mathcal{S}(\mathbf{Alg}\mathcal{S}, \leq) = \mathcal{S}(\mathsf{V}(\mathbf{Alg}\mathcal{S}), \leq) = \mathcal{S}(\mathsf{K}_{\mathcal{S}}, \leq)$. Moreover, since $\mathbf{Alg}^*\mathcal{S}$ and $\mathbf{Alg}\mathcal{S}$ generate the variety $\mathsf{K}_{\mathcal{S}}$ we also have $\mathcal{S}(\mathbf{Alg}\mathcal{S}, \leq) = \mathcal{S}(\mathbf{Alg}^*\mathcal{S}, \leq)$.

[4] The definition of semi-lattice based deductive system is slightly different from the definition given in [14]. The present semi-latice based deductive systems are the semi-lattice based and non-pseudo axiomatic deductive systems of [14].

3 Deductive systems with a strong conjunction

Let $\mathcal{S}$ be a deductive system. A binary term $\wedge$ is said to be a *conjunction* of $\mathcal{S}$ if the following conditions hold:

$p, q \vdash_{\mathcal{S}} p \wedge q$ (Adj)
$p \wedge q \vdash_{\mathcal{S}} p$ and $p \wedge q \vdash_{\mathcal{S}} q$.

Notice that if $\wedge$ and $\wedge'$ are conjunctions of $\mathcal{S}$, then for all formulas φ, ψ,

$$\varphi \wedge \psi \dashv_{\mathcal{S}}\vdash \varphi \wedge' \psi.$$

A deductive system $\mathcal{S}$ is said to be *conjunctive* if it has a conjunction.

The concept of conjunction just introduced is well known and it is what it is usually meant when it is said that a term behaves as a conjunction in a given deductive system. But there are deductive systems $\mathcal{S}$ with a conjunction $\wedge$ which enjoys a stronger property, namely, that $\mathbf{Alg}^*\mathcal{S}$ has $\wedge$-semi-lattice reducts. This does not hold for every conjunction. For example, in infinite-valued Łukasiewicz logic L the fusion connective is a conjunction (in the above specific sense) but the fusion reducts of the algebras in $\mathbf{Alg}^*L$ are not necessarily meet semi-lattices. In the next proposition we characterize the conjunctions with this stronger property by means of some syntactic conditions.

Remark 1 Let $\mathcal{S}$ be a deductive system with a conjunction $\wedge$. Then any class of algebras $\mathbf{Alg}^*\mathcal{S}$, $\mathbf{Alg}\mathcal{S}$ or $\mathsf{K}_{\mathcal{S}}$ has $\wedge$-semi-lattice reducts if and only if the other ones do. This follows from the fact that $\mathsf{K}_{\mathcal{S}}$ is the variety generated by $\mathbf{Alg}^*\mathcal{S}$ and $\mathbf{Alg}\mathcal{S}$ and the fact that having $\wedge$-semi-lattice reducts is equivalent to satisfying some specific equations.

Proposition 2 *Let $\mathcal{S}$ be a deductive system and $\wedge$ a conjunction of $\mathcal{S}$. Then $\mathbf{Alg}^*\mathcal{S}$ has $\wedge$-semi-lattice reducts if and only if $\wedge$ satisfies for every formula δ and variable x that:*

1. $\delta(x/p \wedge p) \dashv_{\mathcal{S}}\vdash \delta(x/p)$
2. $\delta(x/p \wedge q) \dashv_{\mathcal{S}}\vdash \delta(r/q \wedge p)$
3. $\delta(x/(p \wedge q) \wedge r) \dashv_{\mathcal{S}}\vdash \delta(x/p \wedge (q \wedge r))$.

Proof Let $\mathcal{S}$ be a deductive system and let $\wedge$ be a a conjunction of $\mathcal{S}$. Assume that $\mathbf{Alg}^*\mathcal{S}$ has $\wedge$-semi-lattice reducts. It is easy to see that $\wedge$ satisfies conditions (1)-(3). For example, let $\delta(x, \overline{y})$ be a formula. Then, that $\delta(x/p \wedge q) \dashv_{\mathcal{S}}\vdash \delta(r/q \wedge p)$ follows from the fact that

$$\{\langle \mathbf{A}, F\rangle : \mathbf{A} \in \mathbf{Alg}^*\mathcal{S}, F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A} \text{ and } \boldsymbol{\Omega}^{\mathbf{A}}(F) = \Delta_A\}$$

is a complete matrix semantics for $\mathcal{S}$ and the fact that for every $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ and every $a, b \in A$, $a \wedge b = b \wedge a$.

Assume now that $\wedge$ satisfies conditions (1)–(3). We need to show that the equations L1-L3 are valid in $\mathbf{Alg}^*\mathcal{S}$. We will prove that equation L3 is valid in $\mathbf{Alg}^*\mathcal{S}$. The proofs that L1 and L2 are valid are similar. Let $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ and $a, b \in A$. To show that $a \wedge b = b \wedge a$, let $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ be such that $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \Delta_A$, which exist because $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$. We show that $\langle a \wedge b, b \wedge a\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}(F)$. We know from the properties of the Leibniz congruences that $\langle a \wedge b, b \wedge a\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}(F)$ iff for every formula $\delta(x, \overline{y})$ and every $\overline{c} \in A$, $\delta(a \wedge b, \overline{c}) \in F$ iff $\delta(b \wedge a, \overline{c}) \in F$. Let $\delta(x, \overline{y})$ be a formula and let $\overline{c} \in A$. Suppose that $\delta(a \wedge b, \overline{c}) \in F$. Since by assumption $\delta(x/p \wedge q) \dashv_{\mathcal{S}}\vdash \delta(x/q \wedge p)$, it follows that $\delta(b \wedge a, \overline{c}) \in F$. Similarly, if $\delta(b \wedge a, \overline{c}) \in F$ then $\delta(a \wedge b, \overline{c}) \in F$. Consequently, $\langle a \wedge b, b \wedge a\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}(F)$ and, therefore, $a \wedge b = b \wedge a$. □

The proposition motivates the following definition.

Definition 1 Let $\mathcal{S}$ be a deductive system. A binary term $\wedge$ is said to be a *strong conjunction* of $\mathcal{S}$ if

1. $\wedge$ is a conjunction of $\mathcal{S}$,
2. for every formula δ and variable x,
   (a) $\delta(x/p \wedge p) \dashv_{\mathcal{S}}\vdash \delta(x/p)$
   (b) $\delta(x/p \wedge q) \dashv_{\mathcal{S}}\vdash \delta(r/q \wedge p)$
   (c) $\delta(x/(p \wedge q) \wedge r) \dashv_{\mathcal{S}}\vdash \delta(x/p \wedge (q \wedge r))$.

A deductive system $\mathcal{S}$ is said to be *strongly conjunctive* if it has a strong conjunction.

From Remark 1 and Proposition 2 it immediately follows:

Proposition 3 *Let $\mathcal{S}$ be a deductive system with a conjunction $\wedge$. The following are equivalent:*

1. $\wedge$ is a strong conjunction,
2. $\mathbf{Alg}^*\mathcal{S}$ has $\wedge$-semi-lattice reducts,
3. $\mathbf{Alg}\mathcal{S}$ has $\wedge$-semi-lattice reducts.

Note that Proposition 3 can be stated by saying that for every deductive system $\mathcal{S}$ with a binary term $\wedge$, $\wedge$ is a strong conjunction of $\mathcal{S}$ if and only if (i) $\wedge$ is a conjunction of $\mathcal{S}$ and (ii) $\mathbf{Alg}^*\mathcal{S}$ has $\wedge$-semi-lattice reducts. Condition (i) is necessary. There are deductive systems with a binary term for which condition (ii) holds and (i) not. For example, let $\mathcal{S}$ be the deductive system in the language with the binary symbol $\vee$ only given by the structural rules and the rules for disjunction of the Gentzen calculus LJ for intuitionistic logic, namely

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}$$

It holds that for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$, $\langle A, \vee^{\mathbf{A}}\rangle$ is a $\vee$-semi-lattice but neither of the rules in the definition of a conjunction hold. Moreover, the concepts conjunction and strong conjunction do not have the same extension; as we already pointed out, in infinite-valued Łukasiewicz logic the fusion connective is a conjunction (in the above sense) but it is not a strong conjunction.

An alternative to Proposition 3 is the next proposition.

Proposition 4 *Let $\mathcal{S}$ be a deductive system with a binary term $\wedge$. The following are equivalent:*

1. *$\wedge$ is a strong conjunction,*
2. $\mathbf{Alg}^*\mathcal{S}$ *has $\wedge$-semi-lattice reducts and for every* $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ *the nonempty $\mathcal{S}$-filters of* $\mathbf{A}$ *are semilattice filters of* $\langle A, \wedge^{\mathbf{A}}\rangle$*,*
3. $\mathbf{Alg}\mathcal{S}$ *has $\wedge$-semi-lattice reducts and for every* $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ *the nonempty $\mathcal{S}$-filters of* $\mathbf{A}$ *are semilattice filters of* $\langle A, \wedge^{\mathbf{A}}\rangle$*.*

As far as we know, the notion of strong conjunction is new; at least as a distinct concept it has not been considered explicitly in AAL before.

Note that for every deductive system $\mathcal{S}$ with the congruence property each conjunction is a strong conjunction. The reason is that for every conjunction $\wedge$ of $\mathcal{S}$, $p \wedge p \dashv_{\mathcal{S}}\vdash p$, $p \wedge q \dashv_{\mathcal{S}}\vdash q \wedge p$ and $(p \wedge q) \wedge r \dashv_{\mathcal{S}}\vdash p \wedge (q \wedge r)$, and since the relation $\dashv_{\mathcal{S}}\vdash$ is a congruence of the formula algebra, it follows that conditions (a), (b), (c) in Definition 1 hold and therefore $\wedge$ is a strong conjunction. So:

Proposition 5 *Every conjunctive deductive system with the congruence property is strongly conjunctive. In particular every semi-lattice based deductive system is strongly conjunctive.*

Remark 2 If $\wedge$ and $\wedge'$ are two strong conjunctions of a deductive system $\mathcal{S}$, then in every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$, $\wedge^{\mathbf{A}} = \wedge'^{\mathbf{A}}$. Therefore, it follows that for every formula δ and every variable r, $\delta(r/p \wedge q) \dashv_{\mathcal{S}}\vdash \delta(r/p \wedge' q)$.

4 Algebraizable deductive systems with a strong conjunction

In this section we first study the algebraizable deductive systems with a strong conjunction and then their semi-lattice based companions.

4.1 Algebraizable deductive systems with a strong conjunction

Let $\mathcal{S}$ be a deductive system with a strong conjunction $\wedge$ and let $\Delta(p, q)$ be a set of formulas in at most the variables p, q. We define the set

$$\Rightarrow_{\Delta}(p, q) := \Delta(p \wedge q, p).$$

Proposition 6 *Let $\mathcal{S}$ be an equivalential deductive system with a strong conjunction $\wedge$ and let $\Delta(x, y)$ be a set of equivalence formulas for $\mathcal{S}$. Then $\Rightarrow_{\Delta}(p, q)$ is a set of protoimplication formulas for $\mathcal{S}$.*

Proof (1) First we prove that $\vdash_{\mathcal{S}} \Rightarrow_{\Delta}(p, p)$, that is, we prove $\vdash_{\mathcal{S}} \Delta(p \wedge p, p)$. Let $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ and $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$. Since $\langle A, \wedge^{\mathbf{A}}\rangle$ is a semi-lattice, for every $a \in A$, $a \wedge^{\mathbf{A}} a = a$. Thus $\langle a \wedge^{\mathbf{A}} a, a\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}(F)$. Since Δ is a set of equivalence formulas it follows that $\Delta^{\mathbf{A}}(a \wedge^{\mathbf{A}} a, a) \subseteq F$. Moreover it holds that $\vdash_{\mathcal{S}} \Delta(p \wedge p, p)$ if and only if for every $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$, every $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ and every $a \in A$, $\Delta^{\mathbf{A}}(a \wedge^{\mathbf{A}} a, a) \subseteq F$. Therefore, $\vdash_{\mathcal{S}} \Delta(p \wedge p, p)$.

Now we prove that $p, \Rightarrow_{\Delta}(p,q) \vdash_{\mathcal{S}} q$. First of all, since Δ is a set of equivalence formulas for $\mathcal{S}$, $p, \Delta(p \wedge q, p) \vdash_{\mathcal{S}} p \wedge q$. Moreover, since $\wedge$ is a conjunction, $p \wedge q \vdash_{\mathcal{S}} q$. Therefore, $p, \Delta(p \wedge q, p) \vdash_{\mathcal{S}} q$. □

Remark 3 Note that to prove that $\Rightarrow_{\Delta}(p,q)$ satisfies detachment we only used the fact that $\wedge$ is a conjunction and Δ satisfies detachment. To prove that $\vdash_{\mathcal{S}} \Rightarrow_{\Delta}(p,p)$ we used the fact that Δ is a set of equivalence formulas and $\mathbf{Alg}^*\mathcal{S}$ has $\wedge$-semi-lattice reducts.

Remark 4 If Δ' is another set of equivalence formulas for $\mathcal{S}$, then $\Rightarrow_{\Delta}(p,q) \dashv_{\mathcal{S}}\vdash \Rightarrow_{\Delta'}(p,q)$. This is because $\Delta(p,q) \dashv_{\mathcal{S}}\vdash \Delta'(p,q)$. In view of this fact we will frequently omit Δ from the expression $\Rightarrow_{\Delta}$.

If $\mathcal{S}$ is algebraizable with a strong conjunction, $\Delta(p,q)$ is a set of equivalence formulas for $\mathcal{S}$ and $\tau(p)$ is any set of truth-defining equations for $\mathcal{S}$, then the set of equations $\tau(\Rightarrow_{\Delta}(p,q))$ defines on every algebra $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ the semi-lattice order and $\Rightarrow_{\Delta}(p,q) \cup \Rightarrow_{\Delta}(p,q)$ is a set of equivalence formulas.

Proposition 7 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction $\wedge$ and let $\Delta(x,y)$ be a set of equivalence formulas for $\mathcal{S}$ and $\tau(p)$ a set of truth-defining equations. Then,*

1. *for every algebra $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ and every $a, b \in A$,*

$$a \leq^{\mathbf{A}} b \ \ \textit{iff} \ \ {\Rightarrow_{\Delta}}^{\mathbf{A}}(a,b) \subseteq \tau(\mathbf{A}),$$

2. $\Delta(p,q) \dashv_{\mathcal{S}}\vdash \Rightarrow_{\Delta}(p,q) \cup \Rightarrow_{\Delta}(p,q)$.

Proof (1) Let $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ and $a, b \in A$. Then $a \leq^{\mathbf{A}} b$ if and only if $a \wedge^{\mathbf{A}} b = a$. Since $\mathcal{S}$ is algebraizable with Δ and τ respectively as a set of equivalence formulas and a set of truth-defining equations, $a \wedge^{\mathbf{A}} b = a$ if an only if $\Delta^{\mathbf{A}}(a \wedge^{\mathbf{A}} b, a) \subseteq \tau(\mathbf{A})$. It follows that $a \leq^{\mathbf{A}} b$ if and only if ${\Rightarrow_{\Delta}}^{\mathbf{A}}(a,b) \subseteq \tau(\mathbf{A})$.

(2) We have that for every $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ and $a, b \in A$, $a = b$ if and only if $a \wedge^{\mathbf{A}} b = a$ and $b \wedge^{\mathbf{A}} a = b$. Thus, since Δ is a set of equivalence formulas, for every $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ and the least $\mathcal{S}$-filter F on $\mathbf{A}$ we have $\Delta^{\mathbf{A}}(a,b) \subseteq F$ if and only if $\Delta^{\mathbf{A}}(a \wedge^{\mathbf{A}} b, a) \cup \Delta^{\mathbf{A}}(b \wedge^{\mathbf{A}} a, b) \subseteq F$, for every $a, b \in A$. It follows that $\Delta(p,q) \dashv_{\mathcal{S}}\vdash \Rightarrow_{\Delta}(p,q) \cup \Rightarrow_{\Delta}(p,q)$. □

The following characterization of algebraizable deductive systems with a strong conjunction uses the existence of a set of truth-defining equations and the existence of a set of protoimplication formulas that together satisfy property (1) in Proposition 7.

Theorem 1 *Let $\wedge$ be a binary term. A deductive system $\mathcal{S}$ is algebraizable with $\wedge$ as a strong conjunction if and only if*

1. *$\mathcal{S}$ has a set $\tau(p)$ of truth-defining equations,*
2. *$\mathbf{Alg}^*\mathcal{S}$ has $\wedge$-semi-lattice reducts,*

3. *there is a set* $\Rightarrow(p, q)$ *of protoimplication formulas for* $\mathcal{S}$ *such that for every* $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$,

$$a \leq^{\mathbf{A}} b \;\; \textit{iff} \;\; \Rightarrow^{\mathbf{A}}(a, b) \subseteq \tau(\mathbf{A}),$$

4. $p, q \vdash_{\mathcal{S}} p \wedge q$.

Proof Assume that $\mathcal{S}$ is algebraizable with a strong conjunction. From the definitions of algebraizable deductive system and strong conjunction and from Proposition 2, conditions (1), (2) and (4) follow. Propositions 6 and 7 give condition (3). Assume now that conditions (1)–(4) hold. To show that $\mathcal{S}$ is algebraizable it is enough to prove that it has a set of equivalence formulas. Let $\Leftrightarrow (p, q) := \Rightarrow(p, q) \cup \Rightarrow(q, p)$. Condition (3) implies that $\Leftrightarrow(p, q)$ defines the identity relation on every $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$. This, in turn, implies that $\Leftrightarrow(p, q)$ is a set of equivalence formulas for $\mathcal{S}$. Now we prove that $\wedge$ is a strong conjunction. Since we have (2), we only need to show that $\wedge$ is a conjunction. Then, from Proposition 2 the result follows. Since we have (4), it remains to show that $p \wedge q \vdash_{\mathcal{S}} p$ and $p \wedge q \vdash_{\mathcal{S}} q$. To prove this it is enough to show that for every $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ and every assignment v on $\mathbf{A}$ such that $v(p \wedge q) \in \tau(\mathbf{A})$, it holds that $v(p), v(q) \in \tau(\mathbf{A})$. Suppose that $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ and v is an assignment on $\mathbf{A}$ such that $v(p \wedge q) \in \tau(\mathbf{A})$. Then, by condition (2), $v(p \wedge q) \leq^{\mathbf{A}} v(p)$, and by condition (3) we have $\Rightarrow(v(p \wedge q), v(p)) \subseteq \tau(\mathbf{A})$. Using detachment for $\Rightarrow(p, q)$, we obtain that $v(p) \in \tau(\mathbf{A})$. Similarly it follows that $v(q) \in \tau(\mathbf{A})$. □

Remark 5 Assuming condition (2) of Theorem 1, the expression on display in condition (3) of this theorem is equivalent to the following condition:

$$p \wedge q \approx p \models_{\mathbf{Alg}^*\mathcal{S}} \tau(\Rightarrow(p, q)) \quad \text{and} \quad \tau(\Rightarrow(p, q)) \models_{\mathbf{Alg}^*\mathcal{S}} p \wedge q \approx p.$$

If $\mathcal{S}$ is algebraizable, it follows that for every set $\Delta(p, q)$ of equivalence formulas for $\mathcal{S}$, (3) is equivalent to

$$\Delta(p \wedge q, p) \dashv_{\mathcal{S}}\vdash \Delta(\tau(\Rightarrow(p, q))),$$

which, using algebraizability, it is equivalent to

$$\Rightarrow_{\Delta}(p, q) \dashv_{\mathcal{S}}\vdash \Rightarrow(p, q).$$

Remark 5 suggests the following characterization of the algebraizable deductive systems inside the class of deductive systems with a strong conjunction.

Proposition 8 *Let* $\mathcal{S}$ *be a deductive system with a strong conjunction* $\wedge$. *Then* $\mathcal{S}$ *is algebraizable if and only if there is a set of formulas* $\Rightarrow(p, q)$ *such that letting* $\Leftrightarrow(p, q) := \Rightarrow(p, p) \cup \Rightarrow(p, p)$:

1. $\mathcal{S}$ *has a set* $\tau(p)$ *of truth-defining equations,*
2. $\Rightarrow(p, q)$ *is a set of protoimplication formulas for* $\mathcal{S}$,
3. $\Leftrightarrow (p, q)$ *is a set of equivalence formulas for* $\mathcal{S}$,
4. $\Leftrightarrow(p \wedge q, p) \dashv_{\mathcal{S}}\vdash \Rightarrow(p, q)$.

Among the many examples of algebraizable logics with a strong conjunction we have Classical logic, Intuitionistic logic, Linear logic with or without exponentials, Relevance logics R and R_t, infinite-valued Łukasiewicz logic, finite-valued Łukasiewicz logics, the global consequences of the normal modal logics, and all the fragments with $\wedge$ and $\rightarrow$ of all of them. In all of them the strong conjunction is the so called additive conjunction.

Remark 6 Let $\wedge$ be a binary term and $\mathcal{S}$ an algebraizable deductive system with a strong conjunction. Let $\tau(p)$ be a set of truth-defining equations and Δ a set of equivalence formulas for $\mathcal{S}$. Then

$$\Rightarrow_{\Delta}(p,q) \cup \Rightarrow_{\Delta}(q,r) \vdash_{\mathcal{S}} \Rightarrow_{\Delta}(q,r).$$

This holds because if $\mathbf{A} \in \mathbf{Alg}^{*}\mathcal{S}$ and $a, b, c \in A$ are such that $\Rightarrow_{\Delta}{}^{\mathbf{A}}(a,b) \cup \Rightarrow_{\Delta}{}^{\mathbf{A}}(b,c) \subseteq \tau(\mathbf{A})$, then $a \leq^{\mathbf{A}} b$ and $b \leq^{\mathbf{A}} c$; therefore $a \leq^{\mathbf{A}} c$ and, hence, $\Rightarrow_{\Delta}{}^{\mathbf{A}}(a,c) \subseteq \tau(\mathbf{A})$.

The following is a purely syntactic characterization of algebraizable deductive systems with a strong conjunction.

Theorem 2 *Let $\wedge$ be a binary term. A deductive system $\mathcal{S}$ is algebraizable with strong conjunction $\wedge$ if and only if there is a set of formulas in two variables $\Rightarrow(p,q)$, and a set of equations in one variable $\tau(p)$ such that letting $\Leftrightarrow(p,q) := \Rightarrow(p,p) \cup \Rightarrow(p,p)$:*

1. $\vdash_{\mathcal{S}} \Rightarrow(p,p)$
2. $p, \Rightarrow(p,q) \vdash_{\mathcal{S}} q$
3. $\Leftrightarrow(p_1,q_1) \cup \ldots \Leftrightarrow(p_n,q_n) \vdash_{\mathcal{S}} \Leftrightarrow(\star p_1 \ldots p_n, \star q_1 \ldots q_n)$, *for every n-ary connective* $\star$
4. $p \dashv_{\mathcal{S}}\vdash \Leftrightarrow(\tau(p))$
5. $\Leftrightarrow(p \wedge q, p) \dashv_{\mathcal{S}}\vdash \Rightarrow(p,q)$
6. *if* $\varphi \approx \psi$ *is one of the semi-lattice equations (L1)-(L3) then* $\vdash_{\mathcal{S}} \Leftrightarrow(\varphi,\psi)$
7. $p, q \vdash_{\mathcal{S}} p \wedge q$

Proof Suppose that $\mathcal{S}$ is algebraizable with strong conjunction $\wedge$. Then by Propositions 7 and 1 we obtain conditions (1)–(7). Suppose now that conditions (1)–(7) hold. First note that conditions (1)–(4) are equivalent to saying that $\Rightarrow(p,q)$ is a set of protoimplication formulas for $\mathcal{S}$ and $\mathcal{S}$ is algebraizable with $\Leftrightarrow(p,q)$ as a set of equivalence formulas and $\tau(p)$ as a set of truth-defining equations. Secondly note that under conditions (1)–(4), condition (5) is equivalent to saying that in every $\mathbf{A} \in \mathbf{Alg}^{*}\mathcal{S}$, for every $a, b \in A$, $a \wedge^{\mathbf{A}} b = a$ iff $\Rightarrow(a,b) \subseteq \tau(\mathbf{A})$. And thirdly note that under conditions (1)–(4), condition (6) is equivalent to saying that in every $\mathbf{A} \in \mathbf{Alg}^{*}\mathcal{S}$, $\langle A, \wedge^{\mathbf{A}}\rangle$ is a meet-semi-lattice. Thus, using (7), it follows from Theorem 1 that $\mathcal{S}$ is algebraizable and $\wedge$ is a strong conjunction. $\square$

Remark 7 In any algebraizable deductive system $\mathcal{S}$ with a strong conjunction the set of formulas $\Rightarrow(p,q)$ is a weak implication in the sense introduced by Cintula and Noguera in [4].

In some of the examples mentioned above, the interpretation of $\rightarrow$ in the algebras of the algebraic counterpart is the residual of the interpretation of $\wedge$. This is frequently related to the existence of the $\to$-deduction theorem, but it is not necessarily so, as the example of the global consequences of normal modal logics shows. Before going on to study the properties of the semi-lattice based companion of an algebraizable deductive system with a strong conjunction, we characterize the algebraizable deductive systems with strong conjunction for which the set $\Rightarrow(p, q)$ is the residual of $\wedge$ in the generalized sense that for every $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ and every $a, b, c \in A$,

$$a \wedge b \leq c \quad \text{iff} \quad a \leq d, \quad \text{for every } d \in \Rightarrow(b, c).$$

Let $\mathcal{S}$ be a deductive system with a protoimplication set of formulas $\Rightarrow(p, q)$. Let $\mathbf{A}$ be an algebra and $a, b, c \in A$. We let

$$\Rightarrow(a, \Rightarrow(b, c)) := \bigcup\{\Rightarrow(a, d) : d \in \Rightarrow(b, c)\}.$$

Proposition 9 *If $\mathcal{S}$ is an algebraizable deductive system with a strong conjunction, then*

$$\Rightarrow(p \wedge q, r) \dashv_{\mathcal{S}}\vdash \Rightarrow(p, \Rightarrow(q, r)),$$

if and only if for every $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ and every $a, b, c \in A$,

$$a \wedge b \leq c \quad \textit{iff} \quad a \leq d, \quad \textit{for every } d \in \Rightarrow(b, c).$$

Proof Assume that $\Rightarrow(p \wedge q, r) \dashv_{\mathcal{S}}\vdash \Rightarrow(p, \Rightarrow(q, r))$. Note that since for every $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$ $\tau(\mathbf{A})$ is an $\mathcal{S}$-filter, for every $a, b, c \in A$,

$$\Rightarrow(a \wedge b, c) \subseteq \tau(\mathbf{A}) \quad \text{iff} \quad \Rightarrow(a, \Rightarrow(b, c)) \subseteq \tau(\mathbf{A}).$$

Now, $a \wedge b \leq c$ iff $\Rightarrow(a \wedge b, c) \subseteq \tau(\mathbf{A})$ iff $\Rightarrow(a, \Rightarrow(b, c)) \subseteq \tau(\mathbf{A})$ iff $\Rightarrow(a, d) \subseteq \tau(\mathbf{A})$ for every $d \in \Rightarrow(b, c)$ iff $a \leq d$ for every $d \in \Rightarrow(b, c)$.

Assume that $a \wedge b \leq c$ iff $a \leq d$, for every $d \in \Rightarrow(b, c)$. We have, $\Rightarrow(a \wedge b, c) \subseteq \tau(\mathbf{A})$ iff $a \wedge b \leq c$ iff $a \leq d$, for every $d \in \Rightarrow(b, c)$ iff $\Rightarrow(a, d) \subseteq \tau(\mathbf{A})$, for every $d \in \Rightarrow(b, c)$ iff $\bigcup\{\Rightarrow(a, d) : d \in \Rightarrow(b, c)\} \subseteq \tau(\mathbf{A})$ iff $\Rightarrow(a, \Rightarrow(b, c)) \subseteq \tau(\mathbf{A})$. It follows that $\Rightarrow(p \wedge q, r) \dashv_{\mathcal{S}}\vdash \Rightarrow(p, \Rightarrow(q, r))$. □

4.2 The semi-lattice based companion of an algebraizable deductive system with a strong conjunction

If $\mathcal{S}$ is an algebraizable deductive system with a strong conjunction we denote by $\mathcal{S}^{\leq}$ its semi-lattice based companion, i.e. the deductive system $\mathcal{S}(\mathbf{Alg}\mathcal{S}, \leq)$.

Proposition 10 establishes some facts on the relation between an algebraizable deductive system with a strong conjunction and its semi-lattice based companion. To prove the proposition we need the following lemma that shows the relation between the consequence relation of $\mathcal{S}^{\leq}$ and the theorems of $\mathcal{S}$. As is apparent, the theorems of $\mathcal{S}$ determine the consequences in $\mathcal{S}^{\leq}$ of empty sets of formulas in the way made explicit in the lemma.

Lemma 2 *If $\mathcal{S}$ is algebraizable with a strong conjunction, then for all formulas $\varphi, \varphi_1, \ldots, \varphi_n$,*

$$\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}^{\leq}} \varphi \quad \textit{iff} \quad \vdash_{\mathcal{S}} \Rightarrow(\varphi_1 \wedge \ldots \wedge \varphi_n, \varphi).$$

Proof $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}^{\leq}} \varphi$ iff $\forall \mathbf{A} \in \mathbf{Alg}\mathcal{S}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})\ v(\varphi_0) \wedge \ldots \wedge v(\varphi_{n-1}) \leq^{\mathbf{A}} v(\varphi)$ iff $\forall \mathbf{A} \in \mathbf{Alg}\mathcal{S}\ \forall v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}) \Rightarrow^{\mathbf{A}}(v(\varphi_0) \wedge \ldots \wedge v(\varphi_{n-1}), v(\varphi)) \subseteq \tau(\mathbf{A})$ iff $\vdash_{\mathcal{S}} \Rightarrow(\varphi_1 \wedge \ldots \wedge \varphi_n, \varphi)$. □

Proposition 10 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction.*

1. *$\mathcal{S}$ is an extension of $\mathcal{S}^{\leq}$;*
2. *$\mathcal{S}^{\leq}$ is congruential with $\wedge$ as a (strong) conjunction;*
3. *$\mathbf{Alg}\mathcal{S}^{\leq}$ is a variety;*
4. *$\mathbf{Alg}\mathcal{S} \subseteq \mathbf{Alg}^*\mathcal{S}^{\leq} \subseteq \mathbf{Alg}\mathcal{S}^{\leq} = \mathsf{K}_{\mathcal{S}^{\leq}}$;*
5. *$\widetilde{\boldsymbol{\Omega}}(\mathcal{S}) = \widetilde{\boldsymbol{\Omega}}(\mathcal{S}^{\leq})$;*
6. *$\mathsf{K}_{\mathcal{S}} = \mathsf{K}_{\mathcal{S}^{\leq}}$, i.e. $\mathcal{S}$ and $\mathcal{S}^{\leq}$ have the same intrinsic variety.*
7. *$\mathbf{Alg}\mathcal{S}^{\leq}$ is the variety generated by $\mathbf{Alg}\mathcal{S}$.*

Proof (2) and (3) follow from the definition of $\mathcal{S}^{\leq}$ and Proposition 1. (6) is an immediate consequence of (5). So, we have only to prove (1), (4), (5) and (7).

(1) First suppose that $\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{S}^{\leq}} \varphi$. Thus, $\vdash_{\mathcal{S}} \Rightarrow(\varphi_0 \wedge \ldots \wedge \varphi_{n-1}, \varphi)$. Then, using (MP) and (Adj), $\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathcal{S}} \varphi$. Now assume that $\vdash_{\mathcal{S}^{\leq}} \varphi$. Then $\mathcal{S}^{\leq}$ has theorems; therefore every algebra $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ has a greatest element and for every $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $v(\varphi)$ is the greatest element of $\mathbf{A}$. Then, for every formula ψ, $v(\psi) \leq v(\varphi)$; thus $\Rightarrow^{\mathbf{A}}(\psi, \varphi) \subseteq \tau(\mathbf{A})$ for every $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$. Hence for every formula ψ, $\vdash_{\mathcal{S}} \Rightarrow(\psi, \varphi)$. In particular, if ψ is a theorem of $\mathcal{S}$, by (MP) we obtain that φ is a theorem of $\mathcal{S}$. But $\mathcal{S}$ has theorems, because it is algebraizable. Therefore, $\vdash_{\mathcal{S}} \varphi$.

(4) Since $\mathcal{S}$ is protoalgebraic, $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}^*\mathcal{S}$, and since $\mathcal{S}$ is an extension of $\mathcal{S}^{\leq}$, $\mathbf{Alg}^*\mathcal{S} \subseteq \mathbf{Alg}^*\mathcal{S}^{\leq}$. The inclusion $\mathbf{Alg}^*\mathcal{S}^{\leq} \subseteq \mathbf{Alg}\mathcal{S}^{\leq}$ holds because it holds for every deductive system. Finally, by Proposition 1, $\mathbf{Alg}\mathcal{S}^{\leq} = \mathsf{K}_{\mathcal{S}^{\leq}}$.

(5) If $\langle \varphi, \psi \rangle \in \widetilde{\boldsymbol{\Omega}}(\mathcal{S}^{\leq})$, then $\varphi \dashv_{\mathcal{S}^{\leq}}\vdash \psi$. Using Lemma 2 we obtain $\vdash_{\mathcal{S}} \Rightarrow(\varphi, \psi) \cup \Rightarrow(\psi, \varphi)$. Then, since $\Rightarrow(p, q) \cup \Rightarrow(q, p)$ is a set of equivalence formulas for $\mathcal{S}$ it follows that for every δ and every p, $\delta(p/\varphi) \dashv_{\mathcal{S}}\vdash \delta(p/\psi)$. So $\langle \varphi, \psi \rangle \in \widetilde{\boldsymbol{\Omega}}(\mathcal{S})$. Suppose now that $\langle \varphi, \psi \rangle \in \widetilde{\boldsymbol{\Omega}}(\mathcal{S})$. We show that $\varphi \dashv_{\mathcal{S}^{\leq}}\vdash \psi$, which implies that $\langle \varphi, \psi \rangle \in \widetilde{\boldsymbol{\Omega}}(\mathcal{S}^{\leq})$ because $\mathcal{S}^{\leq}$ is selfextensional. Since $\Rightarrow(p, q) \cup \Rightarrow(q, p)$ is a set of equivalence formulas for $\mathcal{S}$, $\vdash_{\mathcal{S}} \Rightarrow(\varphi, \psi) \cup \Rightarrow(\psi, \varphi)$. Then Lemma 2 implies that $\varphi \dashv_{\mathcal{S}^{\leq}}\vdash \psi$.

(7) The intrinsic variety $\mathsf{K}_{\mathcal{S}}$ is the variety generated by $\mathbf{Alg}\mathcal{S}$. By (4) and (6), $\mathsf{K}_{\mathcal{S}} = \mathsf{K}_{\mathcal{S}^{\leq}} = \mathbf{Alg}\mathcal{S}^{\leq}$. So, the result follows. □

The deductive system $\mathcal{S}$ and its semi-lattice based companion may have different theorems. For example the system R of relevant logic has theorems but its semi-lattice based companion, which is the logic WR studied in [12], does not. The proposition below gives a condition that characterizes when $\mathcal{S}$ and $\mathcal{S}^{\leq}$ have the same theorems.

Proposition 11 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction. Then $\mathcal{S}$ and $\mathcal{S}^{\leq}$ have the same theorems iff $\mathcal{S}$ has the property that for every φ, ψ, if $\vdash_{\mathcal{S}} \varphi$, then $\vdash_{\mathcal{S}} \Rightarrow(\psi, \varphi)$.*

Proof Assume that $\mathcal{S}$ and $\mathcal{S}^{\leq}$ have the same theorems. Suppose that $\vdash_{\mathcal{S}} \varphi$. Then $\vdash_{\mathcal{S}^{\leq}} \varphi$. Therefore, for every ψ, $\psi \vdash_{\mathcal{S}^{\leq}} \varphi$. This means that for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ and every $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $v(\psi) \leq^{\mathbf{A}} v(\varphi)$; therefore $v[\Rightarrow(\psi, \varphi)] \subseteq \tau(\mathbf{A})$. This implies that $\vdash_{\mathcal{S}} \Rightarrow(\psi, \varphi)$. Assume now that for every φ, if $\vdash_{\mathcal{S}} \varphi$, then for every formula ψ, $\vdash_{\mathcal{S}} \Rightarrow(\psi, \varphi)$. To show that $\mathcal{S}$ and $\mathcal{S}^{\leq}$ have the same theorems it is enough to show that every theorem of $\mathcal{S}$ is a theorem of $\mathcal{S}^{\leq}$. Suppose that $\vdash_{\mathcal{S}} \varphi$. Let ψ be any formula. Then, $\vdash_{\mathcal{S}} \Rightarrow(\psi, \varphi)$. Thus, for any every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ and every $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $\Rightarrow^{\mathbf{A}}(v(\psi), v(\varphi)) \subseteq \tau(\mathbf{A})$, and hence $v(\psi) \leq^{\mathbf{A}} v(\varphi)$. This implies that every algebra in $\mathbf{Alg}\mathcal{S}$ has a greatest element and so $\mathcal{S}^{\leq}$ has theorems. It follows also that $\psi \vdash_{\mathcal{S}^{\leq}} \varphi$ for every ψ. Therefore, $\vdash_{\mathcal{S}^{\leq}} \varphi$. □

If $\mathcal{S}^{\leq}$ has theorems, then the conditions in the proposition are equivalent to $\mathcal{S}$ being regularly algebraizable.

Proposition 12 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction and such that $\mathcal{S}^{\leq}$ has theorems. The following statements are equivalent:*

1. *$\mathcal{S}$ is regularly algebraizable,*
2. *$\mathcal{S}$ and $\mathcal{S}^{\leq}$ have the same theorems.*

Proof Assume that $\mathcal{S}$ is algebraizable with a strong conjunction and $\mathcal{S}^{\leq}$ has theorems. Then for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ $\langle A, \wedge^{\mathbf{A}}\rangle$ has a greatest element. Suppose that $\mathcal{S}$ and $\mathcal{S}^{\leq}$ have the same theorems. Let φ, ψ be theorems of $\mathcal{S}$. Then in every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$, for every valuation v on $\mathbf{A}$, $v(\varphi) = v(\psi) =$ the greatest element of $\mathbf{A}$. Since $\mathcal{S}$ is algebraizable, this implies that $\mathcal{S}$ is regularly algebraizable. Suppose now that $\mathcal{S}$ is regularly algebraizable. Since $\mathcal{S}$ is an extension of $\mathcal{S}^{\leq}$ it is enough to show that every theorem of $\mathcal{S}$ is a theorem of $\mathcal{S}^{\leq}$. To this end assume that $\vdash_{\mathcal{S}} \varphi$. Let ψ be a theorem of $\mathcal{S}^{\leq}$. Then for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ $\langle A, \wedge^{\mathbf{A}}\rangle$ has a greatest element and is the interpretation of ψ in every valuation on $\mathbf{A}$. Since $\mathcal{S}$ is an extension of $\mathcal{S}^{\leq}$, $\vdash_{\mathcal{S}} \psi$. Now using the fact that $\mathcal{S}$ is regularly algebraizable we obtain that for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$, the interpretation of φ is the greatest element of $\langle A, \wedge^{\mathbf{A}}\rangle$. Thus $\vdash_{\mathcal{S}^{\leq}} \varphi$. □

A property that implies that $\mathcal{S}^{\leq}$ has theorems is protoalgebraic.

Proposition 13 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction. If $\mathcal{S}^{\leq}$ is protoalgebraic, then $\mathcal{S}^{\leq}$ has theorems.*

Proof Assume that $\mathcal{S}^{\leq}$ is protoalgebraic and that $\mathcal{S}^{\leq}$ does not have theorems. This implies that $\mathcal{S}$ is consistent, because otherwise every algebra in $\mathbf{Alg}\mathcal{S}$ has only one element and, therefore, has a greatest element which is the interpretation of every formula in every element of $\mathbf{Alg}\mathcal{S}$. Thus, by Lemma 1, $\mathcal{S}^{\leq}$ has theorems, contrary to the assumption. Now note that the consistency of $\mathcal{S}$ implies that $\mathcal{S}$ has more than one nonempty theory. Since $\mathcal{S}$ is an extension of $\mathcal{S}^{\leq}$, it follows that $\mathcal{S}^{\leq}$ also has more than one nonempty theory, and therefore it is not the quasi-inconsistent deductive system. Since this deductive sytem is the only protoalgebraic deductive system without theorems, it follows that if $\mathcal{S}$ is protoalgebraic, then $\mathcal{S}^{\leq}$ has theorems. □

The converse of the implication in the last proposition does not hold. The semi-lattice based companion of Łukasiewicz infinite-valued logic has theorems but it is not protoalgebraic (see [7]).

A corollary to Propositions 12 and 13 is:

Corollary 1 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction and with $\mathcal{S}^{\leq}$ protoalgebraic. Then the following are equivalent:*

1. *$\mathcal{S}$ is regularly algebraizable,*
2. *$\mathcal{S}$ and $\mathcal{S}^{\leq}$ have the same theorems.*

5 Strongly algebraizable deductive systems with a strong conjunction

An algebraizable deductive system $\mathcal{S}$ is strongly algebraizable when $\mathbf{Alg}\mathcal{S}$ is a variety. In this section we discuss some characterizations of strongly algebraizable deductive systems with a strong conjunction, study some of their properties, and analyze conditions on algebraizable deductive systems with a strong conjunction and with a protoalgebraic semi-lattice based companion that imply strong algebraizability.

From (4) in Proposition 10 it follows that for strongly algebraizable deductive systems with a strong conjunction the classes of algebras associated with them and with their semi-lattice based companions are the same.

Proposition 14 *Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction. Then*

$$\mathbf{Alg}^{*}\mathcal{S} = \mathbf{Alg}\mathcal{S} = \mathbf{Alg}^{*}\mathcal{S}^{\leq} = \mathbf{Alg}\mathcal{S}^{\leq} = \mathsf{K}_{\mathcal{S}^{\leq}} = \mathsf{K}_{\mathcal{S}}.$$

This readily implies that the strongly algebraizable deductive systems with a strong conjunction are exactly the deductive systems with the same algebraic counterpart as their semi-lattice based companion. We state this as a theorem.

Theorem 3 *Let $\mathcal{S}$ be a algebraizable deductive system with a strong conjunction. Then $\mathcal{S}$ is strongly algebraizable if and only if $\mathcal{S}$ and its semi-lattice based companion have the same canonical algebraic counterpart, that is, $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}\mathcal{S}^{\leq}$.*

Knowing this fact may help in situations where one knows $\mathbf{Alg}\mathcal{S}$ and $\mathbf{Alg}\mathcal{S}^{\leq}$ one independently of the other to conclude that an algebraizable deductive system $\mathcal{S}$ with a strong conjunction is strongly algebraizable, by showing that $\mathbf{Alg}\mathcal{S}^{\leq} \subseteq \mathbf{Alg}\mathcal{S}$.

Before moving on to discuss properties of strongly algebraizable deductive systems with a strong conjunction we introduce the for this paper auxiliary notion of Leibniz-linked pair of deductive systems. The concept deserves further study, a task that will be pursued elsewhere.

Definition 2 A pair of deductive systems $(\mathcal{S}, \mathcal{S}')$ of the same type is *Leibniz-linked* if

1. $\mathcal{S}$ is an extension of $\mathcal{S}'$,
2. for every $\mathbf{A}$ there is a map $(.)^{\star}_{\mathbf{A}} : \mathrm{Fi}_{\mathcal{S}'}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ such that if $F \in \mathrm{Fi}_{\mathcal{S}'}\mathbf{A}$, then $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \boldsymbol{\Omega}^{\mathbf{A}}(F^{\star})$ and is the identity map when restricted to $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$.

We call the class of maps $(.)^{\star}_{\mathbf{A}}$ a *Leibniz-link* between $\mathcal{S}$ and $\mathcal{S}'$.

Some of the properties of the map $(.)^{\star} : \mathrm{Fi}_{\mathcal{S}'}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ that follow immediately from the definition are gathered in the next lemma.

Lemma 3 *Let $(\mathcal{S}, \mathcal{S}')$ be a Leibniz-linked pair of deductive systems. For every algebra* $\mathbf{A}$

1. *the map* $(.)^{\star} : \mathrm{Fi}_{\mathcal{S}'}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ *is onto,*
2. *the composition of* $(.)^{\star} : \mathrm{Fi}_{\mathcal{S}'}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ *and* $\boldsymbol{\Omega}^{\mathbf{A}} : \mathrm{Fi}_{\mathcal{S}}\mathbf{A} \to \mathrm{Con}\mathbf{A}$ *is the map* $\boldsymbol{\Omega}^{\mathbf{A}} : \mathrm{Fi}_{\mathcal{S}'}\mathbf{A} \to \mathrm{Con}\mathbf{A}$.

Proof Since $\mathcal{S}$ is an extension of $\mathcal{S}'$, $\mathrm{Fi}_{\mathcal{S}}\mathbf{A} \subseteq \mathrm{Fi}_{\mathcal{S}'}\mathbf{A}$, so, since the map $(.)^{\star} : \mathrm{Fi}_{\mathcal{S}'}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ restricted to $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ is the identity, we have (1). (2) is also an immediate consequence of the definition. □

Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction and let $\tau(x)$ be a set of truth-defining equations for $\mathcal{S}$. For every algebra $\mathbf{A}$, and every $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$ let

$$(F)^{\tau}_{\mathbf{A}} = \{a \in A : \tau(a) \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(F)\}.$$

If $\mathcal{S}$ is strongly algebraizable, then for every algebra $\mathbf{A}$ the definition gives a map $(.)^{\tau}_{\mathbf{A}} : \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$, because by Proposition 14 $\mathbf{Alg}^{*}\mathcal{S}^{\leq} = \mathbf{Alg}^{*}\mathcal{S}$ and, therefore, for every $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$ we have $\boldsymbol{\Omega}^{\mathbf{A}}(F) \in \mathrm{Con}_{\mathbf{Alg}^{*}\mathcal{S}^{\leq}}\mathbf{A} = \mathrm{Con}_{\mathbf{Alg}^{*}\mathcal{S}}\mathbf{A}$; this implies that $(F)^{\tau}_{\mathbf{A}}$ is an $\mathcal{S}$-filter and $\boldsymbol{\Omega}^{\mathbf{A}}((F)^{\tau}_{\mathbf{A}}) = \boldsymbol{\Omega}^{\mathbf{A}}(F)$.

Proposition 15 *Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction and let $\tau(x)$ be a set of truth-defining equations for $\mathcal{S}$. The class of maps $(.)^{\tau}_{\mathbf{A}}$ is a Leibniz-link between $\mathcal{S}$ and $\mathcal{S}^{\leq}$.*

Proof Suppose that $\mathcal{S}$ is a strongly algebraizable deductive system with a strong conjunction and $\tau(x)$ a set of truth-defining equations for $\mathcal{S}$. Let us consider for every algebra $\mathbf{A}$ the map $(.)^{\tau}_{\mathbf{A}} : \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ as defined above. To show that the class of maps $(.)^{\tau}_{\mathbf{A}}$ establish a Leibniz-link between $\mathcal{S}$ and $\mathcal{S}^{\leq}$, it remains to show that if $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$, then $(F)^{\tau}_{\mathbf{A}} = F$. Since $\mathcal{S}$ is algebraizable and $\tau(x)$ is a set of truth-defining equations for $\mathcal{S}$, it follows that for every $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$, $F = \{a \in A : \tau(a) \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(F)\}$; so $F = (F)^{\tau}_{\mathbf{A}}$, for every $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$. □

From Proposition 15 the next theorem follows.

Theorem 4 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction. Then $(\mathcal{S}, \mathcal{S}^{\leq})$ is a Leibniz-linked pair if and only if $\mathcal{S}$ is strongly algebraizable.*

Proof Suppose that $(\mathcal{S}, \mathcal{S}^{\leq})$ is a Leibniz-linked pair, and let $\{(.)^{\star}_{\mathbf{A}} : \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A} :$ $\mathbf{A}$ is an algebra$\}$ be a Leibniz-link between $\mathcal{S}$ and $\mathcal{S}^{\leq}$. We show that $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}\mathcal{S}^{\leq}$, which implies that $\mathcal{S}$ is strongly algebraizable because $\mathbf{Alg}\mathcal{S}^{\leq}$ is a variety. The fact that $\mathcal{S}$ is an extension of $\mathcal{S}^{\leq}$ implies that $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}^{*}\mathcal{S} \subseteq \mathbf{Alg}^{*}\mathcal{S}^{\leq}$. We prove the other inclusion. Let $\mathbf{A} \in \mathbf{Alg}^{*}\mathcal{S}^{\leq}$ and let $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$ be such that $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \Delta_{A}$.

Therefore $\boldsymbol{\Omega}^{\mathbf{A}}(F^{\star}) = \Delta_A$. Since $F^{\star}$ is an $\mathcal{S}$-filter, it follows that $\langle \mathbf{A}, F^{\star} \rangle$ is a reduced model of $\mathcal{S}$. Hence, $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$. So we obtain that $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}^*\mathcal{S}^{\leq}$. Since $\mathbf{Alg}\mathcal{S}^{\leq}$ is the closure under subdirect products of $\mathbf{Alg}^*\mathcal{S}^{\leq}$ and $\mathbf{Alg}\mathcal{S}$ is closed under subdirect products, it follows that $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}\mathcal{S}^{\leq}$. Suppose now that $\mathcal{S}$ is strongly algebraizable. By Proposition 15 we obtain that $(\mathcal{S}, \mathcal{S}^{\leq})$ is a Leibniz-linked pair. □

Thus, for every algebraizable deductive system $\mathcal{S}$ with a strong conjunction the following three conditions are equivalent: $(\mathcal{S}, \mathcal{S}^{\leq})$ is a Leibniz-linked pair, $\mathcal{S}$ is strongly algebraizable, $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}\mathcal{S}^{\leq}$.

Proposition 16 *Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction. The following are equivalent:*

1. *$\mathcal{S}^{\leq}$ is truth-equational,*
2. *$\mathcal{S}^{\leq}$ is weakly algebraizable,*
3. *$\mathcal{S}^{\leq}$ is algebraizable,*
4. *$\mathcal{S}^{\leq}$ is strongly algebraizable*
5. *$\mathcal{S} = \mathcal{S}^{\leq}$.*

Proof Obviously the implications from $(n + 1)$ to (n) for all $1 \leq n < 5$ hold. Let us show that (1) implies (5). Suppose that $\mathcal{S}^{\leq}$ is truth-equational. Let $\tau(x)$ be a set of truth-defining equations for $\mathcal{S}$. Then for every $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}^{\leq}$, the map $\boldsymbol{\Omega}^{\mathbf{A}}$ is injective on the $\mathcal{S}^{\leq}$-filters of $\mathbf{A}$. Let $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}^{\leq}$ and let $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$ be such that $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \Delta_A$. Since $(F)^{\tau}_{\mathbf{A}}$ is also an $\mathcal{S}^{\leq}$-filter and $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \boldsymbol{\Omega}^{\mathbf{A}}((F)^{\tau}_{\mathbf{A}})$ it follows that $F = (F)^{\tau}_{\mathbf{A}}$, and therefore that F is an $\mathcal{S}$-filter. So, the class of reduced matrix models of $\mathcal{S}$ is the class of reduced matrix models of $\mathcal{S}^{\leq}$. This implies that $\mathcal{S} = \mathcal{S}^{\leq}$. □

Other properties, but of a different nature, equivalent to the condition that $\mathcal{S} = \mathcal{S}^{\leq}$ when $\mathcal{S}$ is finitary but not necessarily strongly algebraizable are considered in the next proposition.

Proposition 17 *Let $\mathcal{S}$ be a finitary and algebraizable deductive system with a strong conjunction. Then the following are equivalent:*

1. *$\mathcal{S}$ is fully Fregean,*
2. *$\mathcal{S}$ is Fregean,*
3. *$\mathcal{S}$ is congruential,*
4. *$\mathcal{S}$ has the congruence property,*
5. *$\mathcal{S} = \mathcal{S}^{\leq}$.*

Proof Since $\wedge$ is a conjunction for $\mathcal{S}$, from Thm. 4.8 in [9] follows that $\mathcal{S}$ has the congruence property iff $\mathcal{S}$ is congruential. Thus (3) and (4) are equivalent. Moreover, (1) implies (2) and (3), and (2) implies (4). Now, since $\mathcal{S}^{\leq}$ has the congruence property, we have that (5) implies (4). Let us show that (3) implies (5). Suppose that $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}} \varphi$. Then $\varphi_1 \wedge \ldots \wedge \varphi_n \vdash_{\mathcal{S}} \varphi$. Therefore, $\varphi_1 \wedge \ldots \wedge \varphi_n \dashv_{\mathcal{S}}\vdash \varphi_1 \wedge \ldots \wedge \varphi_n \wedge \varphi$. Then for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ and every $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $v(\varphi_1 \wedge \ldots \wedge \varphi_n) = v(\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \varphi)$, because the relation $\Lambda(\mathrm{Fi}_{\mathcal{S}}\mathbf{A})$ is the identity. Therefore, $v(\varphi_1 \wedge \ldots \wedge \varphi_n) \leq^{\mathbf{A}} v(\varphi)$. It follows that $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}^{\leq}} \varphi$. Now let φ be such that $\vdash_{\mathcal{S}} \varphi$ and let p be a propositional variable not in φ. Then $p \wedge \varphi \dashv_{\mathcal{S}}\vdash p$. Therefore, for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ and every $v \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $v(p \wedge \varphi) = v(p)$ and so $v(p) \leq v(\varphi)$. It follows that every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ has a greatest element and that $\vdash_{\mathcal{S}^{\leq}} \varphi$. To conclude the proof we show that (5) implies (1). Suppose that $\mathcal{S} = \mathcal{S}^{\leq}$. Then $\mathcal{S}^{\leq}$ is protoalgebraic and has theorems. Since it has the congruence property, has a conjunction and is algebraizable, Thm. 4.10 in [14] implies that $\mathcal{S}^{\leq}$ is fully Fregean. □

Proposition 18 *Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction and $\tau(x)$ a set of truth-defining equations for $\mathcal{S}$. Then a matrix $\langle \mathbf{A}, F\rangle$ is a reduced model of $\mathcal{S}^{\leq}$ if and only if $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$, F is a semi-lattice filter and $(F)^{\tau}_{\mathbf{A}} = \tau(\mathbf{A})$.*

Proof Let $\langle \mathbf{A}, F\rangle$ be a matrix. Assume that it is a reduced model of $\mathcal{S}^{\leq}$. Then $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \Delta_A$. Moreover $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}^{\leq} = \mathbf{Alg}^*\mathcal{S}$ and F is an $\mathcal{S}^{\leq}$-filter, so a semi-lattice filter of $\mathbf{A}$. Also $(F)^{\tau}_{\mathbf{A}} = \tau(\mathbf{A})$, because $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \Delta_A$ and $\boldsymbol{\Omega}^{\mathbf{A}}(\tau(\mathbf{A})) = \Delta_A$. Assume now that $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}$, F is a semi-lattice filter of $\mathbf{A}$ and $(F)^{\tau}_{\mathbf{A}} = \tau(\mathbf{A})$. Then $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}^{\leq}$ and F is an $\mathcal{S}^{\leq}$-filter of $\mathbf{A}$. Moreover $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \boldsymbol{\Omega}^{\mathbf{A}}((F)^{\tau}_{\mathbf{A}}) = \boldsymbol{\Omega}^{\mathbf{A}}(\tau(\mathbf{A})) = \Delta_A$. So $\langle \mathbf{A}, F\rangle$ is a reduced model of $\mathcal{S}^{\leq}$. □

In [10] the concept of Leibniz-filter for a deductive system is introduced and it is used to define and study the strong version of a protoalgebraic deductive system. The study of the theory of Leibniz filters and of the strong version of a protoalgebraic deductive system is developed in [10,13]. Let $\mathcal{S}$ be a deductive system and $\mathbf{A}$ an algebra. An $\mathcal{S}$-filter F of $\mathbf{A}$ is said to be *Leibniz* if it is included in all the $\mathcal{S}$-filters G of $\mathbf{A}$ such that $\boldsymbol{\Omega}^{\mathbf{A}}(G) = \boldsymbol{\Omega}^{\mathbf{A}}(F)$.

Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction. A natural question is to characterize when the $\mathcal{S}$-filters of every algebra $\mathbf{A}$ are the Leibniz $\mathcal{S}^{\leq}$-filters of $\mathbf{A}$. In other words, when $\mathcal{S}$ is the deductive system whose class of matrix models is

$$\{\langle \mathbf{A}, F\rangle : F \text{ is a Leibniz } \mathcal{S}^{\leq}\text{-filter of } \mathbf{A}\}.$$

The next theorem and corollary answer this question for strongly algebraizable deductive systems.

Theorem 5 *Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction and $\tau(x)$ a set of truth-defining equations. The following statements are equivalent:*

1. $(F)^{\tau}_{\mathbf{A}} \subseteq F$, *for every* $\mathbf{A}$ *and every* $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$,
2. $(F)^{\tau}_{\mathbf{A}}$ *is a Leibniz* $\mathcal{S}^{\leq}$*-filter, for every* $\mathbf{A}$ *and every* $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$,
3. $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$ *is a Leibniz* $\mathcal{S}^{\leq}$*-filter iff* $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$, *for every* $\mathbf{A}$ *and every* $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$.

Proof (1) implies (2). Assume (1) and let $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$. Since $(F)^{\tau}_{\mathbf{A}}$ is an $\mathcal{S}$-filter and $\mathcal{S}$ is an extension of $\mathcal{S}^{\leq}$, $(F)^{\tau}_{\mathbf{A}}$ is an $\mathcal{S}^{\leq}$-filter. Assume that $G \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$ is such that $\boldsymbol{\Omega}^{\mathbf{A}}(G) = \boldsymbol{\Omega}^{\mathbf{A}}((F)^{\tau}_{\mathbf{A}})$. Then by the assumption $(G)^{\tau}_{\mathbf{A}} \subseteq G$. Moreover, $\boldsymbol{\Omega}^{\mathbf{A}}((G)^{\tau}_{\mathbf{A}}) = \boldsymbol{\Omega}^{\mathbf{A}}(G) = \boldsymbol{\Omega}^{\mathbf{A}}((F)^{\tau}_{\mathbf{A}})$. Since $\mathcal{S}$ is algebraizable and $(G)^{\tau}_{\mathbf{A}}, (F)^{\tau}_{\mathbf{A}} \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$, we obtain $(G)^{\tau}_{\mathbf{A}} = (F)^{\tau}_{\mathbf{A}}$. It follows that $(F)^{\tau}_{\mathbf{A}} \subseteq G$. Therefore $(F)^{\tau}_{\mathbf{A}}$ is a Leibniz $\mathcal{S}^{\leq}$-filter. Hence we obtain (2).

(2) implies (1). Suppose (2). Then since $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \boldsymbol{\Omega}^{\mathbf{A}}((F)^{\tau}_{\mathbf{A}})$ and $(F)^{\tau}_{\mathbf{A}}$ is Leibniz, it follows that $(F)^{\tau}_{\mathbf{A}} \subseteq F$. So we obtain (1).

(2) implies (3). Suppose (2). Let $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$. Then $(F)^{\tau}_{\mathbf{A}} = F$. So by (2) F is a Leibniz $\mathcal{S}^{\leq}$-filter. Suppose now that $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$ is a Leibniz $\mathcal{S}^{\leq}$-filter. Since $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \boldsymbol{\Omega}^{\mathbf{A}}((F)^{\tau}_{\mathbf{A}})$ and $(F)^{\tau}_{\mathbf{A}} \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$, it follows that $F \subseteq (F)^{\tau}_{\mathbf{A}}$. So, being $(F)^{\tau}_{\mathbf{A}}$ also Leibniz, $(F)^{\tau}_{\mathbf{A}} \subseteq F$. Therefore, $(F)^{\tau}_{\mathbf{A}} = F$. Hence we have (3).

(3) implies (2). Assume now (3). Let $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$. Since $(F)^{\tau}_{\mathbf{A}} \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$, by (3) we obtain that $(F)^{\tau}_{\mathbf{A}}$ is a Leibniz $\mathcal{S}^{\leq}$-filter. □

Corollary 2 *Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction and $\tau(x)$ a set of truth-defining equations. Then*

$$\mathrm{Mod}\mathcal{S} = \{\langle \mathbf{A}, F\rangle : F \textit{ is a Leibniz } \mathcal{S}^{\leq}\textit{-filter of } \mathbf{A}\}$$

if and only if the equivalent conditions in Proposition 5 hold.

Proof Assume that $(F)^{\tau}_{\mathbf{A}} \subseteq F$, for every $\mathbf{A}$ and every $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$. Then for every algebra $\mathbf{A}$ the set of Leibniz $\mathcal{S}^{\leq}$-filters of $\mathbf{A}$ is the set of $\mathcal{S}$-filters of $\mathbf{A}$, because $\mathrm{Fi}_{\mathcal{S}}\mathbf{A} \subseteq \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$ and $(.)^{\tau}_{\mathbf{A}}$ restricted to $\mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ is the identity. Hence, $\mathrm{Mod}\mathcal{S} = \{\langle \mathbf{A}, F\rangle :$ F is a Leibniz $\mathcal{S}^{\leq}$-filter of $\mathbf{A}\}$. On the other hand, if this last condition holds then obviously (2) in Proposition 5 holds. □

Recall that in [10] it is shown that if a deductive system $\mathcal{S}'$ is protoalgebraic, then for every $\mathcal{S}'$-filter F on an algebra $\mathbf{A}$ there is a unique Leibniz $\mathcal{S}'$-filter on $\mathbf{A}$, denoted by F^{+}, such that $\boldsymbol{\Omega}^{\mathbf{A}}(F^{+}) = \boldsymbol{\Omega}^{\mathbf{A}}(F)$. Moreover, $F^{+} \subseteq F$ and if F is a Leibniz $\mathcal{S}'$-filter, then $F^{+} = F$. We show that if $\mathcal{S}$ is a strongly algebraizable deductive system with a strong conjunction and $\tau(x)$ is a defining set of equations for $\mathcal{S}$ and in addition $\mathcal{S}^{\leq}$ is protoalgebraic, then the maps $(.)^{\tau}_{\mathbf{A}}$ and $(.)^{+}$ are equal if and only if the equivalent conditions in Proposition 5 hold.

Proposition 19 *Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction and $\tau(x)$ a set of truth-defining equations. If $\mathcal{S}^{\leq}$ protoalgebraic, then the following conditions are equivalent:*

1. $(F)^{\tau}_{\mathbf{A}} \subseteq F$, *for every algebra $\mathbf{A}$ and every $\mathcal{S}^{\leq}$-filter F of $\mathbf{A}$.*
2. $(F)^{\tau}_{\mathbf{A}} = F^{+}$, *for every algebra $\mathbf{A}$ and every $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$.*

Proof Assume (1) and let $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$. Consider the Leibniz $\mathcal{S}^{\leq}$-filter F^{+}, that is, the unique Leibniz $\mathcal{S}^{\leq}$-filter with Leibniz congruence $\boldsymbol{\Omega}^{\mathbf{A}}(F)$. This filter is $\bigcap\{G \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A} : \boldsymbol{\Omega}^{\mathbf{A}}(G) = \boldsymbol{\Omega}^{\mathbf{A}}(F)\}$ and is the unique Leibniz $\mathcal{S}^{\leq}$-filter included in F. By Proposition 5 we have $(F)^{\tau}_{\mathbf{A}}$ is Leibniz and by assumption $(F)^{\tau}_{\mathbf{A}} \subseteq F$. So, $(F)^{\tau}_{\mathbf{A}} = F^{+}$.

Suppose now (2). Let $\mathbf{A}$ be an algebra. From the fact that $F^{+} \subseteq F$ for every $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$ it follows that $(F)^{\tau}_{\mathbf{A}} \subseteq F$, for every $F \in \mathrm{Fi}_{\mathcal{S}'}\mathbf{A}$. □

The *strong version* (cf. [10]) of a protoalgebraic deductive system $\mathcal{S}$ with theorems is the deductive system $\mathcal{S}^{+}$ defined by the class of matrices $\{\langle \mathbf{A}, F^{+}\rangle : F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}\}$. Equivalently, it can be defined by the class of matrices $\langle \mathbf{A}, F\rangle$ such that F is a Leibniz $\mathcal{S}$-filter of $\mathbf{A}$ and $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$.

Corollary 3 *Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction and with $\mathcal{S}^{\leq}$ is protoalgebraic, and let $\tau(x)$ be a set of truth-defining equations. If the equivalent conditions in Proposition 5 hold, then $\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$.*

Proof Suppose that the conditions in Proposition 5 hold. Then by Corollary 2 $\mathrm{Mod}\mathcal{S} = \{\langle \mathbf{A}, F\rangle : F$ is a Leibniz $\mathcal{S}^{\leq}$-filter of $\mathbf{A}\}$. Therefore, $\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$. □

In fact the converse of the corollary is true, as we show in Theorem 6. In the way to prove it we consider the following condition

$$\text{for every } \mathbf{A} \in \mathbf{Alg}\mathcal{S}^{\leq} \text{ the least } \mathcal{S}^{\leq}\text{-filter of } \mathbf{A} \text{ is an } \mathcal{S}\text{-filter.} \tag{5.1}$$

In the setting of Corollary 3 this condition is equivalent to $\mathcal{S}$ being the strong version of $\mathcal{S}^{\leq}$. With regard to this it is useful to observe the following general fact:

Lemma 4 *Let $\mathcal{S}$ and $\mathcal{S}'$ be two deductive systems an let $\mathcal{S}'$ be protoalgebraic. Then the following statements are equivalent:*

1. *for every $\mathbf{A}$, the least $\mathcal{S}'$-filter of $\mathbf{A}$ is an $\mathcal{S}$-filter,*
2. *for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}'$, the least $\mathcal{S}'$-filter of $\mathbf{A}$ is an $\mathcal{S}$-filter,*

Proof Obviously (1) implies (2). The Correspondence Theorem for protoalgebraic logics implies that (1) follows from (2). Assume (2) and let $\mathbf{A}$ be an algebra and let F be the least $\mathcal{S}'$-filter of $\mathbf{A}$. Consider the algebra $\mathbf{A}/\boldsymbol{\Omega}^{\mathbf{A}}(F)$. This algebra belongs to $\mathbf{Alg}^*\mathcal{S}'$. Since $\mathcal{S}'$ is protoalgebraic, by the Correspondence Theorem there is an isomorphism between the lattices $\mathrm{Fi}_{\mathcal{S}'}\mathbf{A}$ and $\mathrm{Fi}_{\mathcal{S}'}\mathbf{A}/\boldsymbol{\Omega}^{\mathbf{A}}(F)$, given by the canonical onto homomorphism $\pi : \mathbf{A} \to \mathbf{A}/\boldsymbol{\Omega}^{\mathbf{A}}(F)$. By (2) the least element of $\mathrm{Fi}_{\mathcal{S}'}\mathbf{A}/\boldsymbol{\Omega}^{\mathbf{A}}(F)$, say G, is an $\mathcal{S}$-filter, so the least element of $\mathrm{Fi}_{\mathcal{S}'}\mathbf{A}$, namely F, should be an $\mathcal{S}$ filter because $F = \pi^{-1}[G]$. □

Theorem 6 *Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction and $\tau(x)$ a set of truth-defining equations. If $\mathcal{S}^{\leq}$ is protoalgebraic, then the following statements are equivalent:*

1. *$\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$,*
2. *for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}^{\leq}$, the least $\mathcal{S}^{\leq}$-filter of $\mathbf{A}$ is an $\mathcal{S}$-filter,*
3. *for every $\mathbf{A}$ the least $\mathcal{S}^{\leq}$-filter of $\mathbf{A}$ is an $\mathcal{S}$-filter,*
4. *for every $\mathbf{A}$ and every $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$, $(F)^{\tau}_{\mathbf{A}} \subseteq F$.*

Proof By the lemma above, (2) and (3) are equivalent. To prove that (1) implies (3) suppose that $\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$. Let $\mathbf{A}$ be an algebra. By Prop. 18 in [10], the least $\mathcal{S}$-filter of $\mathbf{A}$ is the least $\mathcal{S}^{\leq}$-filter of $\mathbf{A}$. Thus we have (3). Let us show that (3) implies (4). Let $\mathbf{A}$ be an algebra and let $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$. Consider its Leibniz $\mathcal{S}^{\leq}$-filter F^+. Let $\theta := \boldsymbol{\Omega}^{\mathbf{A}}(F)$. Then $\langle \mathbf{A}/\theta, F^+/\theta\rangle$ is a reduced matrix model of $\mathcal{S}^{\leq}$, because $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \boldsymbol{\Omega}^{\mathbf{A}}(F^+)$. Moreover, by Prop. 10 in [10], F^+/θ is the least $\mathcal{S}^{\leq}$-filter of $\mathbf{A}/\theta$. So from (3) it follows that F^+/θ is an $\mathcal{S}$-filter. So $\langle \mathbf{A}/\theta, F^+/\theta\rangle$ is a reduced matrix model of $\mathcal{S}$. Therefore, $F^+/\theta = \tau(\mathbf{A}/\theta)$. Since $(F)^{\tau}_{\mathbf{A}} = \{a \in A : \tau(a) \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(F)\}$ it follows that $a \in F^+$ iff $a/\theta \in F^+/\theta$ iff $a/\theta \in \tau(\mathbf{A}/\theta)$ iff $\tau(a) \subseteq \theta$ iff $a \in (F)^{\tau}_{\mathbf{A}}$. Now from the fact that $F^+ \subseteq F$ we obtain that $(F)^{\tau}_{\mathbf{A}} \subseteq F$. Finally, from Corollary 3 follows that (4) implies (1). □

Before continuing, let us show that if $\mathcal{S}$ is an algebraizable deductive system with strong conjunction, $\mathcal{S}^{\leq}$ is protoalgebraic and condition (5.1) holds, then $\mathcal{S}$ is strongly algebraizable.

Proposition 20 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction and with $\mathcal{S}^{\leq}$ protoalgebraic. If for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}^{\leq}$ the least $\mathcal{S}^{\leq}$-filter of $\mathbf{A}$ is an $\mathcal{S}$-filter, then*

1. *$\mathcal{S}$ and $\mathcal{S}^{\leq}$ have the same theorems,*
2. *$\mathcal{S}$ is strongly algebraizable,*
3. *$\mathcal{S}$ is regularly algebraizable.*

Proof (1) Every theorem of $\mathcal{S}^{\leq}$ is a theorem of $\mathcal{S}$, because $\mathcal{S}$ is an extension of $\mathcal{S}^{\leq}$. Lemma 4 and the assumption imply that the least $\mathcal{S}^{\leq}$ filter of $\mathbf{Fm}$ is an $\mathcal{S}$-filter because $\mathcal{S}^{\leq}$ is protoalgebraic, and so contains all the theorems of $\mathcal{S}$. Therefore $\mathcal{S}$ and $\mathcal{S}^{\leq}$ have the same theorems.

(2) By Proposition 3 it is enough to show that $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}\mathcal{S}^{\leq}$. The inclusion from left to right follows from Proposition 10. To prove the other inclusion, let $\mathbf{A} \in \mathbf{Alg}\mathcal{S}^{\leq}$ and let F be the least $\mathcal{S}^{\leq}$-filter of $\mathbf{A}$. Since $\mathcal{S}^{\leq}$ is protoalgebraic, $\mathbf{Alg}\mathcal{S}^{\leq} = \mathbf{Alg}^*\mathcal{S}^{\leq}$. So, $\mathbf{A} \in \mathbf{Alg}^*\mathcal{S}^{\leq}$. This and protoalgebraicity implies that $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ is the identity. Therefore, $\langle \mathbf{A}, F\rangle$ is a reduced matrix model of $\mathcal{S}$ and so $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$.

(3) It follows from (1) and Proposition 12. □

A consequence of theorems 6 and 20 is the following characterization of when for an algebraizable deductive system $\mathcal{S}$ with a strong conjunction and a protoalgebraic semi-lattice based companion, $\mathcal{S}$ is the strong version of this companion.

Theorem 7 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction and with $\mathcal{S}^{\leq}$ protoalgebraic. Then, $\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$ if and only if the least $\mathcal{S}^{\leq}$-filter of every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}^{\leq}$ is an $\mathcal{S}$-filter.*

Proof First we prove the implication from right to left. Assume that for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}^{\leq}$, the least $\mathcal{S}^{\leq}$-filter of $\mathbf{A}$ is an $\mathcal{S}$-filter. From Theorem 20 it follows that $\mathcal{S}$ is strongly algebraizable. Then, Theorem 6 implies that $\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$.

To prove the other implication, assume that $\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$. Then by Prop. 18 in [10], $\mathbf{Alg}^*\mathcal{S} = \mathbf{Alg}\mathcal{S} = \mathbf{Alg}\mathcal{S}^{\leq} = \mathbf{Alg}^*\mathcal{S}^{\leq}$. Hence $\mathcal{S}$ is strongly algebraizable. Thus from Theorem 6 we obtain the desired conclusion. □

We can summarize and slightly strengthen our last results as follows:

Theorem 8 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction and with $\mathcal{S}^{\leq}$ protoalgebraic. The following statements are equivalent:*

1. *for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}^{\leq}$, the least $\mathcal{S}^{\leq}$-filter of $\mathbf{A}$ is an $\mathcal{S}$-filter,*
2. *$\mathcal{S}$ is strongly algebraizable and $\mathcal{S}^{\leq}$ and $\mathcal{S}$ have the same theorems,*
3. *$\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$,*
4. *$\mathbf{Alg}\mathcal{S}^{\leq}$ is pointed, the interpretation of the constant term 1 on every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}^{\leq}$ is the $\leq^{\mathbf{A}}$-greatest element, and $\mathcal{S}$ is the 1-assertional logic of $\mathbf{Alg}\mathcal{S}$ and the 1-assertional logic of $\mathbf{Alg}\mathcal{S}^{\leq}$.*

Proof Theorem 7 gives the equivalence between (1) and (3). The implication from (1) to (2) follows from Theorem 20. We show that (2) implies (4). Assume (2). Then from Corollary 1 follows that $\mathcal{S}$ is regularly algebraizable. So, $\mathbf{Alg}\mathcal{S}$ is pointed with constant term, say 1. Moreover, by Proposition 11, since $\mathcal{S}$ and $\mathcal{S}^{\leq}$ have the same theorems, $\vdash_{\mathcal{S}} \Rightarrow(p, 1)$. This implies that for every $\mathbf{A} \in \mathbf{Alg}\mathcal{S}$ and every $a \in A$, $a \leq^{\mathbf{A}} 1^{\mathbf{A}}$. Since $\mathcal{S}$ is the 1-assertional logic of $\mathbf{Alg}\mathcal{S}$ and $\mathbf{Alg}\mathcal{S} = \mathbf{Alg}\mathcal{S}^{\leq}$, (4) follows. To conclude the proof we show that (4) implies (1). If $\mathbf{Alg}\mathcal{S}^{\leq}$ is pointed, since $\mathbf{Alg}\mathcal{S} \subseteq \mathbf{Alg}\mathcal{S}^{\leq}$, $\mathbf{Alg}\mathcal{S}$ is pointed too. Let $\mathcal{S}(\mathbf{Alg}\mathcal{S}^{\leq}, 1)$ and $\mathcal{S}(\mathbf{Alg}\mathcal{S}, 1)$ be the 1-assertional logics of $\mathbf{Alg}\mathcal{S}^{\leq}$ and $\mathbf{Alg}\mathcal{S}$, respectively. By assumption both are $\mathcal{S}$. Since $\mathcal{S}$ is algebraizable, so is $\mathcal{S}(\mathbf{Alg}\mathcal{S}^{\leq}, 1)$, and since $\mathbf{Alg}\mathcal{S}^{\leq}$ is a variety, $\mathbf{Alg}\mathcal{S}^{\leq} = \mathbf{Alg}\mathcal{S}(\mathbf{Alg}\mathcal{S}^{\leq}, 1)$. Therefore the least $\mathcal{S}^{\leq}$-filter of each $\mathbf{A} \in \mathbf{Alg}\mathcal{S}^{\leq}$, which is $\{1^{\mathbf{A}}\}$, is an $\mathcal{S}(\mathbf{Alg}\mathcal{S}^{\leq}, 1)$-filter, thus an $\mathcal{S}$-filter. □

Note that for algebraizable deductive systems $\mathcal{S}$ with a strong conjunction and with a protoalgebraic semi-lattice based companion, Theorem 8 shows that the conjunction of conditions (1)-(2) in Proposition 20 is in fact equivalent to condition (5.1), and therefore imply that $\mathcal{S}$ is regularly algebraizable.

One condition that easily entails condition (5.1) is stated in the next proposition. The proposition captures for example one of the features of normal modal logics.

Proposition 21 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction and a protoalgebraic semi-lattice based companion. Suppose that there is a set of formulas in one variable $\Psi(p)$ such that for every algebra $\mathbf{A}$ and every $\mathcal{S}^{\leq}$-filter F of $\mathbf{A}$ the set*

$$F^{\circ} = \{a \in A : \Psi^{\mathbf{A}}(a) \subseteq F\}$$

is an $\mathcal{S}$-filter of $\mathbf{A}$ and $F^{\circ} \subseteq F$. Then for every algebra $\mathbf{A} \in \mathbf{Alg}\mathcal{S}^{\leq}$ the least $\mathcal{S}^{\leq}$-filter is an $\mathcal{S}$-filter, and therefore $\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$ and $\mathcal{S}$ is strongly algebraizable with the same theorems as $\mathcal{S}^{\leq}$.

Proof We show that for every algebra $\mathbf{A} \in \mathbf{Alg}\mathcal{S}^{\leq}$ the least $\mathcal{S}^{\leq}$-filter is an $\mathcal{S}$-filter. Let G be the least $\mathcal{S}^{\leq}$-filter of $\mathbf{A}$. Then $G = G^{\circ}$ because $G^{\circ} \subseteq G$, G° is an $\mathcal{S}$-filter and every $\mathcal{S}$-filter is an $\mathcal{S}^{\leq}$-filter. Therefore, G is an $\mathcal{S}$-filter. Now we apply Theorem 8 and obtain that $\mathcal{S}$ is strongly algebraizable with the same theorems as $\mathcal{S}^{\leq}$ and is the strong version of $\mathcal{S}^{\leq}$. □

Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction and let τ be a set of defining equations. We say that *the links $(.)^{\tau}_{\mathbf{A}}$ are definable by a set of formulas $\Psi(p)$* if for every algebra $\mathbf{A}$ and every $\mathcal{S}^{\leq}$-filter F, $(F)^{\tau}_{\mathbf{A}} = \{a \in F : \Psi^{\mathbf{A}}(a) \subseteq F\}$.

The proof of the next proposition is almost immediate.

Proposition 22 *Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction an let τ be a set of truth-defining equations. If the links $(.)^{\tau}_{\mathbf{A}}$ are definable by a set of formulas $\Psi(p)$, then $\mathcal{S}^{\leq}$ is equivalential.*

Proof Let $\Delta(x, y)$ be a set of equivalence formulas for $\mathcal{S}$. Consider the set of formulas $\Psi(\Delta(x, y)) := \bigcup\{\Psi(\psi) : \psi \in \Delta(x, y)\}$. We show that this set defines the

Leibniz congruences of the $\mathcal{S}^{\leq}$-filters. Let $\mathbf{A}$ be an algebra and $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$. Then $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}(F)$ iff $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}((F)^{\tau}_{\mathbf{A}})$ iff $\Delta(a, b) \subseteq (F)^{\tau}_{\mathbf{A}}$ iff $\Psi(\Delta(a, b)) \subseteq F$. Thus, $\mathcal{S}^{\leq}$ is equivalential. □

In Proposition 21 it is assumed that the semi-lattice based companion of $\mathcal{S}$ is protoalgebraic and that $F^{\circ} \subseteq F$. Next proposition shows, among other things, that without these assumptions but with an additional assumption about the set $\Psi(p)$, it follows that $\mathcal{S}^{\leq}$ is equivalential, and so also protoalgebraic.

Proposition 23 *Let $\mathcal{S}$ be an algebraizable deductive system with a strong conjunction and $\tau(x)$ a set of truth-defining equations for $\mathcal{S}$. Suppose that there is a set of formulas in one variable $\Psi(p)$ such that for every algebra $\mathbf{A}$ and every $\mathcal{S}^{\leq}$-filter F of $\mathbf{A}$ the set*

$$F^{\circ} = \{a \in A : \Psi^{\mathbf{A}}(a) \subseteq F\}$$

is an $\mathcal{S}$-filter of $\mathbf{A}$. Let us assume in addition that

$$p, \Psi(\Leftrightarrow(p, q)) \vdash_{\mathcal{S}^{\leq}} q.$$

Then, $\mathcal{S}$ is strongly algebraizable, the links $(.)^{\tau}_{\mathbf{A}} : \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ are definable by Ψ and $\mathcal{S}^{\leq}$ is equivalential.

Proof First of all we will show that the map $(.)^{\circ} : \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ is a Leibniz-link from $\mathcal{S}$ to $\mathcal{S}^{\leq}$. From the assumption we have $(.)^{\circ} : \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$. Now we show that for every algebra $\mathbf{A}$ and every $\mathcal{S}^{\leq}$-filter F of $\mathbf{A}$, $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \boldsymbol{\Omega}^{\mathbf{A}}(F^{\circ})$. To this end we prove that $\boldsymbol{\Omega}^{\mathbf{A}}(F)$ is compatible with F°. This implies $\boldsymbol{\Omega}^{\mathbf{A}}(F) \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(F^{\circ})$. Assume that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}(F)$ and that $a \in F^{\circ}$. Then, $\Psi^{\mathbf{A}}(a) \subseteq F$. Thus, $\Psi^{\mathbf{A}}(b) \subseteq F$; hence $b \in F^{\circ}$. To prove the other inclusion we show that $\boldsymbol{\Omega}^{\mathbf{A}}(F^{\circ})$ is compatible with F. Assume that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}}(F^{\circ})$ and that $a \in F$. Since F° is an $\mathcal{S}$-filter, $\Leftrightarrow^{\mathbf{A}}(a, b) \subseteq F^{\circ}$. Therefore, $\Psi^{\mathbf{A}}(\Leftrightarrow^{\mathbf{A}}(a, b)) \subseteq F$. Thus, since $p, \Psi(\Leftrightarrow(p, q)) \vdash_{\mathcal{S}^{\leq}} q$ and F is an $\mathcal{S}^{\leq}$-filter, $b \in F$. Therefore $\boldsymbol{\Omega}^{\mathbf{A}}(F^{\circ}) \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(F)$. It remains to prove that if F is an $\mathcal{S}$-filter of $\mathbf{A}$, then $F^{\circ} = F$. Assume that $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$. Since F° is also an $\mathcal{S}$-filter and $\boldsymbol{\Omega}^{\mathbf{A}}(F^{\circ}) = \boldsymbol{\Omega}^{\mathbf{A}}(F)$, from the algebraizability of $\mathcal{S}$ it follows that $F = F^{\circ}$. From Theorem 4 follows that $\mathcal{S}$ is strongly algebraizable. We show now that for every algebra $\mathbf{A}$, the maps $(.)^{\circ} : \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ and $(.)^{\tau}_{\mathbf{A}} : \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ are equal. Let $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$. Then $F^{\circ} \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ and $\boldsymbol{\Omega}^{\mathbf{A}}(F) = \boldsymbol{\Omega}^{\mathbf{A}}(F^{\circ})$. Since τ defines truth, $F^{\circ} = \{a \in A : \tau(a) \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(F^{\circ})$. Therefore, $F^{\circ} = \{a \in A : \tau(a) \subseteq \boldsymbol{\Omega}^{\mathbf{A}}(F)\} = (F)^{\tau}_{\mathbf{A}}$. Finally, Proposition 22 implies that $\mathcal{S}^{\leq}$ is equivalential. □

When $\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$ we have the converse. Let us recall from [10] that a protoalgebraic logic $\mathcal{S}$ has Leibniz filters explicitly definable if there is a set of formulas in one variable $\Phi(x)$ such that for every algebra $\mathbf{A}$ and every $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$, the Leibniz $\mathcal{S}$-filter F^{+} associated with F is $\{a \in A : \Phi^{\mathbf{A}}(a) \subseteq F\}$.

Proposition 24 *Let $\mathcal{S}$ be a strongly algebraizable deductive system with a strong conjunction and $\mathcal{S}^{\leq}$ protoalgebraic. And let $\tau(p)$ be a set of truth-defining equations for $\mathcal{S}$ Assume that $\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$. Then the following are equivalent:*

1. *the Leibniz-links* $(.)^{\tau}_{\mathbf{A}}$ *are definable by a set of formulas* $\Psi(p)$,
2. $\mathcal{S}^{\leq}$ *has Leibniz filters explicitly definable,*
3. $\mathcal{S}^{\leq}$ *is equivalential.*

Proof Since $\mathcal{S}$ is the strong version of $\mathcal{S}^{\leq}$, by Theorem 6 we have that for every $\mathbf{A}$ and every $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$, $(F)^{\tau}_{\mathbf{A}} \subseteq F$. So, by Proposition 5 follows that the $\mathcal{S}$-filters are the Leibniz $\mathcal{S}^{\leq}$-filters. This implies the equivalence between (1) and (2). Proposition 22 shows that (1) implies (3). Assume now that (3) holds. Then, since $\mathcal{S}$ is algebraizable and is the strong version of $\mathcal{S}^{\leq}$, Thm. 35 in [10] implies that $\mathcal{S}^{\leq}$ has Leibniz filters explicitly definable, that is, we have (2). □

To conclude the paper we prove that if a strongly algebraizable deductive system $\mathcal{S}$ with a strong conjunction is the strong version of $\mathcal{S}^{\leq}$ and satisfies any of the equivalent conditions of the last proposition, then there is a set of formulas $\Psi(x)$ such that $\mathcal{S}$ is the extension of $\mathcal{S}^{\leq}$ obtained by adding the rules in $\{p \vdash \varphi : \varphi \in \Psi\}$.

Proposition 25 *Let* $\mathcal{S}$ *be a strongly algebraizable deductive system with a strong conjunction and let* $\tau(p)$ *be a set of truth-defining equations for* $\mathcal{S}$. *Suppose that* $\Psi(x)$ *is a set of formulas that defines the Leibniz- links* $(.)^{\tau}_{\mathbf{A}} : \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ *and that* $\mathcal{S}$ *is the strong version of* $\mathcal{S}^{\leq}$. *Then* $\mathcal{S}$ *is the least extension of* $\mathcal{S}^{\leq}$ *such that* $p \vdash_{\mathcal{S}} \varphi$, *for every* $\varphi \in \Psi(p)$.

Proof First of all we prove that for every $\mathbf{A}$ and every $F \in \mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A}$, $F = (F)^{\tau}_{\mathbf{A}}$ if and only if for every $a \in A$, if $a \in F$, then $\Psi(a) \subseteq F$. Assume that F is a $\mathcal{S}^{\leq}$-filter of $\mathbf{A}$ such that $F = (F)^{\tau}_{\mathbf{A}}$. Then since $\Psi(x)$ defines the Leibniz-links, we have $F = (F)^{\tau}_{\mathbf{A}} = \{a \in A : \Psi(a) \subseteq F\}$. It follows that for every $a \in F$, $\Psi(a) \subseteq F$. Suppose now that for every $a \in F$, $\Psi(a) \subseteq F$. We show that $F = (F)^{\tau}_{\mathbf{A}}$. That $(F)^{\tau}_{\mathbf{A}} \subseteq F$ follows from Theorem 6. To prove the other inclusion let $a \in F$. Then $\Psi(a) \subseteq F$ and since $(F)^{\tau}_{\mathbf{A}} = \{a \in A : \Psi(a) \subseteq F\}$, it follows that $a \in (F)^{\tau}_{\mathbf{A}}$. So $F \subseteq (F)^{\tau}_{\mathbf{A}}$. Thus, it follows that for every algebra $\mathbf{A}$ and every $F \subseteq A$, $\langle \mathbf{A}, F \rangle$ is a model of $\mathcal{S}$ if and only if it is a model of $\mathcal{S}^{\leq}$ and of the rules in $\{p \vdash \varphi : \varphi \in \Psi\}$. This implies that $\mathcal{S}$ is the least extension of $\mathcal{S}^{\leq}$ such that $p \vdash_{\mathcal{S}} \varphi$, for every $\varphi \in \Psi(p)$. □

Remark 8 Note that if $\mathcal{S}$ is a deductive system that satisfies the conditions of the proposition, then $\Psi(p) \vdash_{\mathcal{S}} p$, because since $\Psi(p)$ defines the Leibniz-links $(.)^{\tau}_{\mathbf{A}}$: $\mathrm{Fi}_{\mathcal{S}^{\leq}}\mathbf{A} \to \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$, it holds that for every algebra $\mathbf{A}$, every $F \in \mathrm{Fi}_{\mathcal{S}}\mathbf{A}$ and every $a \in A$, $a \in A$ if and only if $\Psi^{\mathbf{A}}(a) \subseteq F$.

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