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Some consequences of Rado's selection lemma

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Abstract We prove in set theory without the Axiom of Choice, that Rado's selection lemma (**RL**) implies the Hahn-Banach axiom. We also prove that **RL** is equivalent to several consequences of the Tychonov theorem for compact Hausdorff spaces: in particular, **RL** implies that every filter on a well orderable set is included in a ultrafilter. In set theory with atoms, the "Multiple Choice" axiom implies **RL**.

Keywords Axiom of choice · Product topology · Compactness · Rado's selection lemma · Hahn-Banach

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1 Introduction

We work in set theory without the Axiom of Choice **ZF**. For every set I we denote by $fin(I)$ the set of finite subsets of I. The following statement was introduced by Rado ([9]):

> **RL**: *(Rado's selection lemma, form 99 of* [3]*) "Given an infinite family of non-empty finite sets $(X_i)_{i\in I}$, and given any family $(\sigma_J)_{J\in fin(I)}$ such that for every $J \in fin(I)$, $\sigma_J \in \prod_{i\in J} X_i$, there exists $f \in \prod_{i\in I} X_i$ such that for every $J \in fin(I)$, there exists $L \in fin(I)$ satisfying $J \subseteq L$ and $f\restriction J = \sigma_L\restriction J$."*

Remark 1 Under the hypotheses of Rado's lemma, the set $X := \prod_{i\in I} X_i$ is non-empty (in **ZF**): indeed, for every $i \in I$, $\sigma_{\{i\}}(i) \in X_i$.

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Rado used this statement and the Axiom of Choice (**AC**) to prove (see ([9]) that two bases of an (infinite) finitary matroid are equipotent. Rado's proof of **RL** relied on **AC** but **RL** is a consequence (see [1,10]) of the following "Tychonov axiom", a consequence of **AC** which does not imply **AC** ([2]):

$\mathbf{T}_2$: *"Every family of compact Hausdorff topological spaces has a compact product."*

Notice that $\mathbf{T}_2$ has the two following consequences (which are not provable in **ZF**, see [3]):

$\mathbf{AC}^{\mathbf{fin}}$: *(Finite Axiom of Choice, form 62 of* [3]*) "Given an infinite family* $(X_i)_{i\in I}$ *of non-empty finite sets, the set* $\prod_{i\in I} X_i$ *is non-empty."*

$\mathbf{AC}^2$: *(Axiom of Choice for pairs, form 88 of* [3]*) "Given an infinite family* $(X_i)_{i\in I}$ *of two-element sets, the set* $\prod_{i\in I} X_i$ *is non-empty."*

In this paper, we prove (see Sect. 2) that **RL** implies the following statement, which is equivalent (see [6]) to the Hahn-Banach axiom **HB** (form 52 of [3]):

M: *(Measure axiom, form 52A of* [3]*) "For every non trivial boolean algebra* $\mathbb{B}$*, there exists a measure* $m : \mathbb{B} \to [0, 1]$ *such that* $m(1_{\mathbb{B}}) = 1$*."*

Here, a *boolean algebra* is a (commutative) unitary ring $(\mathbb{B}, \oplus, \times, 0_{\mathbb{B}}, 1_{\mathbb{B}})$ which is *idempotent* (for every $x \in \mathbb{B}$, $x.x = x$). If $0_{\mathbb{B}} = 1_{\mathbb{B}}$, then the boolean algebra $\mathbb{B}$ is said to be *trivial*. A (real valued) *measure* on $\mathbb{B}$ is a mapping $m : \mathbb{B} \to [0, 1]$ which is *finitely additive* (for every $x, y \in \mathbb{B}$ satisfying $x.y = 0_{\mathbb{B}}$, $m(x \oplus y) = m(x) + m(y)$). A measure m on $\mathbb{B}$ is *unitary* if $m(1_{\mathbb{B}}) = 1$. Recall (see [3]) that the Tychonov axiom $\mathbf{T}_2$ is equivalent to the *Boolean Prime Ideal*:

(**BPI**, form 14 of [3]): *"Every non trivial boolean algebra has a prime ideal."*

Remark 2 The statement **HB** is a consequence of $\mathbf{T}_2$ (see [3]) which does not imply $\mathbf{T}_2$ (see [7]).

In Sect. 3, we prove in $\mathbf{ZF}^0$ (set-theory with atoms, a theory weaker than **ZF**—see [3, pp. 1–2]—), that the following "Multiple Choice" axiom implies **RL**:

MC: (form 67 of [3]) *"Given an infinite family* $(X_i)_{i\in I}$ *of infinite sets, there exists a family of non-empty finite sets* $(F_i)_{i\in I}$ *such that for every* $i \in I$, $F_i \subseteq X_i$*."*

In **ZF**, the statement **MC** is equivalent to **AC**, but in $\mathbf{ZF}^0$, **MC** does not imply $\mathbf{AC}^2$ (see [5]). Thus our result implies that every model of $\mathbf{ZF}^0$+**MC** satisfies **RL**: this enlightens a result due to Howard (see [4]) who built a model of $\mathbf{ZF}^0$+**MC**+$\neg\mathbf{AC}^2$ in which he proved **RL**. However, the following questions seem to stay open:

Question 1 Does **RL** imply $\mathbf{T}_2$?

Remark 3 Blass (see [3, Note 33]) noticed that ($\mathbf{RL}$+$\mathbf{AC}^{\mathbf{fin}}$) $\Leftrightarrow$ $\mathbf{T}_2$, so Question 1 is equivalent to ask whether **RL** implies $\mathbf{AC}^{\mathbf{fin}}$.

Even the following Question seems to be open:

Question 2 Does **RL** imply $\mathbf{AC}^2$?

We finally provide various equivalents of **RL** in **ZF** (see the recapitulating Diagram in Sect. 4).

2 Rado's selection lemma implies Hahn-Banach

2.1 Boolean algebras

Given a boolean algebra $(\mathbb{B}, \oplus, \times, 0_{\mathbb{B}}, 1_{\mathbb{B}})$, the binary relation $\leq$ on $\mathbb{B}$ defined for every $x, y \in \mathbb{B}$ by the formula "$x \leq y$ if and only if $x.y = x$" is a partial order; the *poset* $(\mathbb{B}, \leq)$ is a complemented distributive lattice with smallest element $0_{\mathbb{B}}$ and greatest element $1_{\mathbb{B}}$: notice that the *infimum law* $\wedge$ is the multiplicative law of the ring $(\mathbb{B}, \oplus, \times, 0_{\mathbb{B}}, 1_{\mathbb{B}})$, the *supremum* law $\vee$ is defined for every $x, y \in \mathbb{B}$ by $x \vee y := (x \oplus y) \oplus x.y$, and for every $x \in \mathbb{B}$, the complement of x (i.e., the unique element y of $\mathbb{B}$ satisfying $x \wedge y = 0_{\mathbb{B}}$ and $x \vee y = 1_{\mathbb{B}}$) is $x \oplus 1$.

Remark 4 Conversely, one checks (in **ZF**) that given a complemented distributive lattice $(\mathbb{B}, \vee, \wedge)$ with first element $0_{\mathbb{B}}$, last element $1_{\mathbb{B}}$ and complementation function $.^c$, then one can define the binary law $\oplus$ on $\mathbb{B}$ by $x \oplus y := (x \wedge y^c) \vee (y \wedge x^c)$ for every $x, y \in \mathbb{B}$, and the structure $(\mathbb{B}, \oplus, \wedge, 0_{\mathbb{B}}, 1_{\mathbb{B}})$ is a (commutative) unitary idempotent ring.

Given a boolean algebra $\mathbb{B}$, an *atom* of $\mathbb{B}$ is a non-null element a of $\mathbb{B}$ which is minimal in the *poset* $(\mathbb{B}\backslash\{0_{\mathbb{B}}\}, \leq)$: for every $x \in \mathbb{B}$, if $x \leq a$ then ($x = 0_{\mathbb{B}}$ or $x = a$). If $\mathbb{B}$ is a *finite* boolean algebra, if A is the set of atoms of $\mathbb{B}$, then the canonical mapping $can_{\mathbb{B}} : \mathbb{B} \to \mathcal{P}(A)$ associating to each $x \in \mathbb{B}$ the set of atoms minorating x is an isomorphism of boolean algebras.

Recall that a measure (see Sect. 1) on a boolean algebra $\mathbb{B}$ is a mapping $m : \mathbb{B} \to [0, 1]$ satisfying $\big(x \wedge y = 0_{\mathbb{B}} \Rightarrow m(x \vee y) = m(x) + m(y)\big)$ for every $x, y \in \mathbb{B}$. Notice that every measure $m : \mathbb{B} \to [0, 1]$ is *ascending*: given $x, y \in \mathbb{B}$ satisfying $x \leq y$ then $m(x) \leq m(y)$).

For every *finite* set F we denote by $|F|$ the number of elements of F (i.e., the cardinal of the finite set F).

Proposition 1 (uniform probability on a finite boolean algebra) *If $\mathbb{B}$ is a finite non-trivial boolean algebra, there exists a unique unitary measure $\mathcal{U}$ on $\mathbb{B}$ associating the same real number to all the atoms of $\mathbb{B}$. This measure is called the* uniform probability *on the finite boolean algebra $\mathbb{B}$.*

Proof Let A be the (finite) set of atoms of $\mathbb{B}$. The uniform probability on $\mathcal{P}(A)$ is the unique measure on $\mathcal{P}(A)$ associating to each subset B of A the (rational) number $\frac{|B|}{|A|}$. This unitary measure can be carried into a unitary measure $\mathcal{U} : \mathbb{B} \to [0, 1]$ using the canonical isomorphism between the boolean algebras $\mathbb{B}$ and $\mathcal{P}(A)$. Notice that for every $x \in \mathbb{B}, \mathcal{U}(x) = \frac{|\{a \in A : a \leq x\}|}{|A|}$.

2.2 **RL** implies Hahn-Banach

Definition 1 Let $(X_i)_{i\in I}$ an infinite family of non-empty finite sets. Let $\mathcal{C}$ be a subset of $fin(I)$. Assume that $(\sigma_F)_{F\in\mathcal{C}}$ is a family such that for every $J\in\mathcal{C}$, $\sigma_J\in\prod_{i\in J}X_i$. Say that an element $f\in\prod_{i\in I}X_i$ is *Rado-compatible* with $(\sigma_F)_{F\in\mathcal{C}}$ if for every $F\in fin(I)$, there exists $G\in\mathcal{C}$ such that $F\subseteq G$ and $f\restriction F=\sigma_G\restriction F$.

Rado's lemma says that given an infinite family $(X_i)_{i\in I}$ of non-empty finite sets, and given a family $(\sigma_F)_{F\in fin(I)}$ such that for every $J\in fin(I)$, $\sigma_J\in\prod_{i\in J}X_i$, there exists a mapping $f\in\prod_{i\in I}X_i$ which is Rado-compatible with $(\sigma_F)_{F\in fin(I)}$.

For every $n\in\mathbb{N}$, we denote by D_n the set $\{\frac{k}{2^n}:k\in\mathbb{N}\}\cap[0,1]$. Given some $n\in\mathbb{N}$, for every $x\in[0,1]$, there is a unique $k\in\{0,\dots,2^n\}$ such that $\frac{k}{2^n}\le x<\frac{k+1}{2^n}$; call the number $\frac{k}{2^n}$ the (default) *n-approximation of* x. Notice that $\cup_{n\in\mathbb{N}}D_n$ is countable and dense in $[0,1]$.

Given a subset A of a boolean algebra $\mathbb{B}$, we denote by $bool_{\mathbb{B}}(A)$ the boolean algebra generated by A in $\mathbb{B}$.

Theorem 1 **RL** *implies* **HB**.

Proof Let $\mathbb{B}$ be an infinite boolean algebra. For every $n\in\mathbb{N}$, let $\mathbb{B}_n:=\mathbb{B}\times\{n\}$, and let $\mathbb{B}_\omega:=\cup_{n\in\mathbb{N}}\mathbb{B}_n$. Thus $\mathbb{B}_\omega$ is the union of ω pairwise disjoint copies of $\mathbb{B}$. For every non-empty finite subset F of $\mathbb{B}_\omega$, we define σ_F as follows: since F is of the form $\cup_{0\le i\le n}(F_i\times\{i\})$ where $n\in\mathbb{N}$ and F_n is non-empty, consider the uniform probability P_F on $bool_{\mathbb{B}}(\cup_{0\le i\le n}F_i)$, and, for every $(x,i)\in F_i\times\{i\}$, let $\sigma_F((x,i))$ be the i-approximation of $P_F(x)$: thus $|\sigma_F((x,i))-P_F(x)|\le\frac{1}{2^i}$. For every $F\in fin(\mathbb{B}_\omega)$, for every $x,y\in\mathbb{B}$ and every $i,j,k,l\in\mathbb{N}$:

1. $(1_{\mathbb{B}},i)\in F\Rightarrow\sigma_F((1_{\mathbb{B}},i))=1-\frac{1}{2^i}$
2. $(0_{\mathbb{B}},i)\in F\Rightarrow\sigma_F((0_{\mathbb{B}},i))=0$
3. If $(x,i),(x,j)\in F$ then $|\sigma_F((x,i))-\sigma_F((x,j))|\le\frac{1}{2^i}+\frac{1}{2^j}$
4. If $x\wedge y=0_{\mathbb{B}}$ and if $(x,i),(y,k),(x\vee y,l)\in F$ then:
   $|\sigma_F((x\vee y,l))-\sigma_F((x,i))-\sigma_F((y,k))|\le\frac{1}{2^l}+\frac{1}{2^i}+\frac{1}{2^k}$.

Using **RL**, let $f=(f_n)_{n\in\mathbb{N}}\in\prod_{n\in\mathbb{N}}D_n^{\mathbb{B}_n}$ be a mapping which is Rado-compatible with the family $(\sigma_F)_{F\in fin(\mathbb{B}_\omega)}$. Given $x\in\mathbb{B}$, Condition (3) implies that the sequence $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy: indeed, given some real number $\varepsilon>0$, let $N\in\mathbb{N}$ such that $\frac{1}{2^N}<\frac{\varepsilon}{2}$; given $p,q\in\mathbb{N}$ such that $p,q\ge N$, let $F:=\{(p,x),(q,x)\}$ and let $G\in fin(\mathbb{B}_\omega)$ such that $F\subseteq G$ and $f\restriction F=\sigma_G\restriction F$; then $|f_p(x)-f_q(x)|=|\sigma_G(p,x)-\sigma_G(q,x)|\le\frac{1}{2^p}+\frac{1}{2^q}\le\frac{2}{2^N}<\varepsilon$. Thus the Cauchy sequence $(f_n(x))_{n\in\mathbb{N}}$ of real numbers converges to a real number $m(x)\in[0,1]$. Conditions (1) and (2) imply that $m(1_{\mathbb{B}})=1$ and $m(0_{\mathbb{B}})=0$. Condition (4) implies that m is finitely additive: let $x,y\in\mathcal{B}$ such that $x\wedge y=0_{\mathbb{B}}$ and let $\varepsilon>0$. Let $N\in\mathbb{N}$ satisfying $\frac{3}{2^N}<\frac{\varepsilon}{2}$ such that $|m(x)-f_N(x)|$, $|m(y)-f_N(y)|$ and $|m(x\vee y)-f_N(x\vee y)|$ are all less that $\frac{\varepsilon}{6}$; let $F:=\{(x,N),(y,N),(x\vee y,N)\}$; let G be a finite subset of $\mathbb{B}_\omega$ such that $F\subseteq G$ and $f\restriction F=\sigma_G\restriction F$; then $|m(x)+m(y)-m(x\vee y)|\le|m(x)-\sigma_G(x,N)|+|m(y)-\sigma_G(y,N)|+|m(x\vee y)-\sigma_G(N,x\vee y)|+|\sigma_G(x,N)+\sigma_G(y,N)-\sigma_G(x\vee y,N)|=|m(x)-f_N(x)|+|m(y)-f_N(y)|+|m(x\vee y)-f_N(x\vee y)|+$

$|\sigma_G(x, N) + \sigma_G(y, N) - \sigma_G(x \vee y, N)| \leq \frac{3}{2^N} + 3\frac{\varepsilon}{6} \leq \varepsilon$ (using 4); this is true for every $\varepsilon > 0$ so $|m(x) + m(y) - m(x \vee y)| = 0$. It follows that the mapping $m : \mathbb{B} \to [0, 1]$ is a unitary measure on $\mathbb{B}$.

3 In $\mathbf{ZF^0}$, MC implies RL

3.1 Rado-compatibility and compactness

Remark 5 Let $(X_i)_{i \in I}$ an infinite family of non-empty finite sets. Let $\mathcal{C}$ be a *cofinal* subset of the *poset* $(fin(I), \subseteq)$. If $f \in \prod_{i \in I} X_i$ is Rado-compatible with $(\sigma_F)_{F \in \mathcal{C}}$, then f is also Rado-compatible with $(\sigma_F)_{F \in fin(I)}$.

Given a family $(X_i)_{i \in I}$ of topological spaces, and denoting by X the product set $\prod_{i \in I} X_i$, for every nonempty subset J of I we denote by $p_J : X \to \prod_{i \in J} X_i$ the canonical mapping $(x_i)_{i \in I} \mapsto (x_i)_{i \in J}$. The *product topology* on the set $X := \prod_{i \in I} X_i$ is the coarsest topology for which all canonical projections $p_{\{i\}} : X \to X_i$ are continuous. Let $\mathcal{L}_X$ be the lattice generated by closed subsets of the form $p_{\{i\}}^{-1}[F]$ where $i \in I$ and F is a closed subset of X_i: elements of $\mathcal{L}_X$ are called *elementary closed* subsets of X. Since every closed subset of X is the intersection of elements of $\mathcal{L}_X$, the product space X is compact if and only if every filter of the lattice $\mathcal{L}_X$ has a non-empty intersection. More generally, the following (easy) Proposition holds:

Proposition 2 *Let X be a topological space. Let $\mathcal{C}$ be a family of closed subsets of X, such that every closed subset of X is an intersection of elements of $\mathcal{C}$. If every sub-family of $\mathcal{C}$ satisfying the finite intersection property has a non-empty intersection, then X is compact.*

Given a set $\mathcal{F}$ of non-empty subsets of a set X, we say that $\mathcal{F}$ is *inf-directed* if for every $F_1, \ldots, F_n \in \mathcal{F}$, there exists $F \in \mathcal{F}$ such that $F \subseteq \cap_{1 \leq i \leq n} F_i$.

Theorem 2 *Consider an infinite family $(X_i)_{i \in I}$ of finite sets and assume that $\mathcal{C}$ is a cofinal subset of $fin(I)$, and that $(\sigma_F)_{F \in \mathcal{C}}$ is a family such that for every $F \in \mathcal{C}$, $\sigma_F \in \prod_{i \in F} X_i$. For every finite subset J of I, consider the following set:*

$$F_J^{\mathcal{C}} := \left\{ \sigma \in \prod_{i \in J} X_i \,:\, \forall K \in fin(I)\ \exists L \in \mathcal{C}\ K \cup J \subseteq L \text{ and } \sigma_L \upharpoonright J = \sigma \right\}$$

1. *For each $J \in fin(I)$, $F_J^{\mathcal{C}}$ is non-empty;*
2. *If $\mathcal{C}$ admits a well order $\preceq$, then there is a mapping associating to each $J \in fin(I)$ and each non-empty subset A of $F_J^{\mathcal{C}}$ an element z_J^A of A: such a choice function is definable from $(\sigma_F)_{F \in \mathcal{C}}$ and the well order $\preceq$ (and in particular the product $X := \prod_{i \in I} X_i$ is non-empty);*
3. *If $X \neq \varnothing$, then for every $J \in fin(I)$, the (closed) subset $\tilde{F}_J^{\mathcal{C}} := F_J^{\mathcal{C}} \times \prod_{i \in I \setminus J} X_i$ of X is non-empty, and for every finite subsets J, K of I satisfying $J \subseteq K$, then $\tilde{F}_K^{\mathcal{C}} \subseteq \tilde{F}_J^{\mathcal{C}}$; in particular, the family of non-empty sets $\{\tilde{F}_J^{\mathcal{C}} : J \in fin(I)\}$ is inf-directed: in this case we denote by $\mathcal{F}^{\mathcal{C}}$ the filter of the lattice $\mathcal{L}_X$ generated by this family;*

4. *Given an element f of X, the following statements are equivalent:*
   (a) *f is Rado-compatible with $(\sigma_J)_{J \in \mathcal{C}}$;*
   (b) *$f \in \cap_{J \in \mathcal{C}} \tilde{F}_J^{\mathcal{C}}$;*
5. *If $\mathcal{C}$ admits a well order $\preceq$ then there exists $f \in X$ which is Rado-compatible with $(\sigma_F)_{F \in \mathcal{C}}$ and which is definable from $\preceq$ and $(\sigma_F)_{F \in \mathcal{C}}$.*

Proof (1) Given some $J \in fin(I)$, the set $F_J^{\mathcal{C}}$ is non-empty: seeking for a contradiction, assume that for each $\sigma \in \prod_{i \in J} X_i$, there exists $K_\sigma \in fin(I)$ such that for every $L \in \mathcal{C}$, $(K_\sigma \cup J \subseteq L \Rightarrow \sigma_L \upharpoonright J \neq \sigma)$; then consider the *finite* set $K := \cup_{\sigma \in \prod_{i \in J} X_i} K_\sigma$; since $\mathcal{C}$ is cofinal in $fin(I)$, consider some element $L \in \mathcal{C}$ such that $(J \cup K) \subseteq L$; then for every $\sigma \in \prod_{i \in J} X_i$, $\sigma_L \upharpoonright J \neq \sigma$: since $\sigma_L \upharpoonright J \in \prod_{i \in J} X_i$ this is contradictory!
(2) Assume that $\mathcal{C}$ is well-orderable. Given $J \in fin(I)$ and some non-empty subset A of $F_J^{\mathcal{C}}$, let C be the first element of $\mathcal{C}$ (*w.r.t.* this well-order of $\mathcal{C}$) such that $z_J^A := \sigma_C \upharpoonright J \in A$: then $(z_J^A)_{J \in fin(I), \varnothing \neq A \subseteq F_J^{\mathcal{C}}}$ is a choice function satisfying the required conditions.
(3) and (4): the proofs are easy.
(5) Since $\mathcal{C}$ is well-orderable, X is non-empty: we shall show that there is an element of $\cap \mathcal{F}^{\mathcal{C}}$ which is definable from $(\sigma_F)_{F \in \mathcal{C}}$ and the well-order $\preceq$ on $\mathcal{C}$. Let $J \in fin(I)$. Since J is finite, $\prod_{i \in J} X_i$ is compact thus the set $A_J := \cap_{Z \in \mathcal{F}^{\mathcal{C}}} p_J[Z]$ is non-empty. We shall define by recursion on the well order $\preceq$ a family $(\tau_J)_{J \in \mathcal{C}}$ of pairwise compatible finite functions, such that for every $J \in \mathcal{C}$, $\tau_J \in A_J$: it will follow that $\tau := \cup_{J \in \mathcal{C}} \tau_J$ is a mapping defined on I and that $\tau \in \cap \mathcal{F}^{\mathcal{C}}$. Given $J \in \mathcal{C}$, τ_J is defined from $(\tau_F)_{F \prec J}$ as follows: consider the filter $\mathcal{G}_J$ of $\mathcal{L}_X$ generated by $\mathcal{F}^{\mathcal{C}}$ and the elementary closed sets $p_F^{-1}[\{\tau_F\}]$ for $F \prec J$; then $A := \cap_{Z \in \mathcal{G}_J} p_J[Z]$ is a non-empty subset of A_J; we define $\tau_J := z_J^A$ (see point (2)). The finite functions τ_J are pairwise compatible because given distinct elements $J, K \in \mathcal{C}$ such that $J \preceq K$, $\tau_K \in p_K[\{f \in X : f \upharpoonright J = \tau_J\}]$. Then $\tau := \cup_{J \in \mathcal{C}} \tau_J \in \cap \mathcal{F}^{\mathcal{C}}$.

3.2 **RL** implies a weak form of $\mathbf{T}_2$

Consider the following consequence of $\mathbf{T}_2$:

> $\mathbf{RL}_0$: *"Given a set I and a non-empty inf-directed set $\mathcal{F}$ of non-empty elementary closed subsets of $\{0, 1\}^I$, if $\mathcal{F}$ has a choice function then $\cap \mathcal{F}$ is non-empty."*

Proposition 3 $\mathbf{RL} \Rightarrow \mathbf{RL}_0$.

Proof Let I be an infinite set. Let X be the topological product $\{0, 1\}^I$. Let $\mathcal{F}$ be a non-empty inf-directed set of non-empty elementary closed subsets of X, having a choice function $(\tau_Z)_{Z \in \mathcal{F}}$. Without loss of generality, we may assume that $\{0, 1\}^I \in \mathcal{F}$. Since $\mathcal{F}$ is inf-directed, for every finite subset F of I, the set of subsets A of $\{0, 1\}^F$ such that $A \times \{0, 1\}^{I \setminus F} \in \mathcal{F}$ has a smallest element, that we denote by A_F. For every $F \in fin(I)$, let $\sigma_F := \tau_{A_F \times \{0,1\}^{I \setminus F}}$. Using **RL**, let $f \in \{0, 1\}^I$ which is Rado-compatible with $(\sigma_F)_{F \in fin(I)}$. Let us show that $f \in \cap \mathcal{F}$. Given some $Z \in \mathcal{F}$, let $F \in fin(I)$ and let $A \subseteq \{0, 1\}^F$ such that $A \times \{0, 1\}^{I \setminus F} = Z$; since f is Rado-compatible with $(\sigma_F)_{F \in fin(I)}$, let $G \in fin(I)$ such that $F \subseteq G$ and $f \upharpoonright F = (\sigma_G) \upharpoonright F$; then $A_G \subseteq A_F \times \{0, 1\}^{G \setminus F}$ thus $(\sigma_G) \upharpoonright F \in A_F \subseteq A$ so $f \upharpoonright F \in A$ i.e., $f \in Z$.

3.3 A statement which is intermediate between $\mathbf{T}_2$ and **RL**

It is known (see [3]) that $\mathbf{T}_2$ is equivalent to the fact that for every set I, the product space $\{0, 1\}^I$ is compact, or, equivalently, to the following fact: *"Given a set I and a non-empty set $\mathcal{F}$ of non-empty elementary closed subsets of $\{0, 1\}^I$ satisfying the finite intersection property, $\cap\mathcal{F}$ is non-empty."* Consider now the following statement:

> **RLT**: *"Given a set I and a non-empty set $\mathcal{F}$ of non-empty elementary closed subsets of $\{0, 1\}^I$ satisfying the finite intersection property, if $\mathcal{F}$ has a choice function then $\cap\mathcal{F}$ is non-empty."*

Then $\mathbf{T}_2 \Rightarrow \mathbf{RLT} \Rightarrow \mathbf{RL}_0$.

Question 3 Is **RLT** equivalent to $\mathbf{T}_2$ or to $\mathbf{RL}_0$?

3.4 $\mathbf{RL}_0$ implies **RL**, and in $\mathbf{ZF}_0$, **MC** implies **RL**

In this Section, we shall show that conversely, $\mathbf{RL}_0$ implies **RL**; the same idea allows to prove (in $\mathbf{ZF}^0$) that **MC** implies **RL**.

Corollary 1 *1.* $\mathbf{RL}_0 \Leftrightarrow \mathbf{RL}$.
2. In $\mathbf{ZF}^0$, **MC** *implies* **RL**.

Proof The implication $\mathbf{RL} \Rightarrow \mathbf{RL}_0$ has been proved in Proposition 3. We now prove $\mathbf{RL}_0 \Rightarrow \mathbf{RL}$ and $\mathbf{MC} \Rightarrow \mathbf{RL}$: in both cases, we consider an infinite family $(X_i)_{i\in I}$ of non-empty finite sets, and we assume that for every finite subset J of I, $\sigma_J \in \prod_{i\in J} X_i$. In particular, $X := \prod_{i\in I} X_i$ is non-empty: let $a = (a_i)_{i\in I} \in X$. As in Theorem 2, for each $J \in fin(I)$ we define the non-empty set $F_J := \{\sigma \in \prod_{i\in J} X_i \ : \ \forall K \in fin(I)\ \exists L \in fin(I)\ (K \cup J \subseteq L \text{ and } \sigma_L\restriction J = \sigma)\}$, and the closed subset $\tilde{F}_J := F_J \times \prod_{i\in I\setminus J} X_i$ of X. The family $(\tilde{F}_J)_{J\in fin(I)}$ of non-empty elementary closed subsets of $\{0, 1\}^I$ is inf-directed; moreover, $(\tilde{F}_J)_{J\in fin(I)}$ has a choice function: indeed for every $J \in fin(I)$, $\sigma_J \frown (a_i)_{i\in I\setminus J} \in \tilde{F}_J$.
$\mathbf{RL}_0 \Rightarrow \mathbf{RL}$: using $\mathbf{RL}_0$, consider some $f \in \cap_{J\in fin(I)}\tilde{F}_J$. Using Theorem 2–(4), the mapping f is Rado-compatible with $(\sigma_J)_{J\in fin(I)}$.
$\mathbf{MC} \Rightarrow \mathbf{RL}$: using **MC**, consider an ordinal α and a family $(I_\lambda)_{\lambda\in\alpha}$ of non-empty pairwise disjoint finite sets such that $I = \cup_{\lambda\in\alpha} I_\lambda$. Let $\mathcal{C}$ be the set of elements $F \in fin(I)$ of the form $\cup_{t\in Z} I_t$ where $Z \in fin(\alpha)$: the subset $\mathcal{C}$ of $fin(I)$ is cofinal in the *poset* $(fin(I), \subseteq)$, and $\mathcal{C}$ is well-orderable (because there is a one-to-one mapping from $\mathcal{C}$ into the well-orderable set $fin(\alpha)$). Thus Theorem 2–(5) implies that there exists $f \in \prod_{i\in I} X_i$ which is Rado-compatible with $(\sigma_J)_{J\in fin(I)}$ since $\mathcal{C}$ is cofinal in $(fin(I), \subseteq)$ (see Remark 5).

4 Some equivalent forms of RL

Given set I, consider the following statement:

> $\mathbf{AC}^{fin(I)}$: *"Every infinite family of finite non-empty subsets of I has a non-empty product."*

Every mapping ϕ associating to each non-empty finite subset F of I, an element $\phi(F)$ of F is called a *witness of* $\mathbf{AC}^{fin(I)}$.

Remark 6 Given a set I, if $\mathbf{AC}^{fin(I)}$ holds, then there is a mapping associating to each $F \in fin(I)$ a linear order $\leq_F$ on F (and thus also a linear order on $\{0, 1\}^F$).

We shall now show that the two following consequences of $\mathbf{T}_2$ (or the equivalent statement **BPI**) are equivalent to **RL**:

$\mathbf{RL}_1$: *"For every set I satisfying* $\mathbf{AC}^{fin(I)}$*, the topological product space* $\{0, 1\}^I$ *is compact."*

$\mathbf{RL}_2$: *"Given a boolean algebra* $\mathbb{B}$ *satisfying* $\mathbf{AC}^{fin(\mathbb{B})}$*, every proper filter of* $\mathbb{B}$ *is included in a ultrafilter."*

Proposition 4 (support of an elementary closed subset of a product space) *Let I be an infinite set. For every non-empty elementary closed subset Z of* $\{0, 1\}^I$*, there exists a smallest finite subset S of I, such that Z is of the form* $A \times \{0, 1\}^{I \backslash S}$ *where* $A \subseteq \{0, 1\}^S$*: we call this set S the* support *of Z, and we denote it by* $supp(Z)$.

Proof Let Z be a non-empty elementary closed subset of $\{0, 1\}^I$. For every $f \in \{0, 1\}^I$, every $i_0 \in I$ and every $\varepsilon \in \{0, 1\}$, denote by $f_{i_0 \to \varepsilon}$ the mapping $g : I \to \{0, 1\}$ such that $g(i_0) = \varepsilon$ and $g(i) = f(i)$ for every $i \in I \backslash \{i_0\}$. Let $S := \{i \in I : \exists f \in Z\ (f_{i \to 0} \notin Z \text{ or } f_{i \to 1} \notin Z)\}$. Then S is the smallest subset T of I satisfying $Z = p_T[Z] \times \{0, 1\}^{I \backslash T}$ so S is finite.

Theorem 3 *The following statements are equivalent:* **RL**, $\mathbf{RL}_1$ *and* $\mathbf{RL}_2$.

Proof The equivalence $\mathbf{RL}_0 \Leftrightarrow \mathbf{RL}$ has been proved in Sect. 3.4.
$\mathbf{RL} \Rightarrow \mathbf{RL}_1$. Let I be an infinite set satisfying $\mathbf{AC}^{fin(I)}$: then one can choose some linear order on every $F \in fin(I)$. Let $\mathcal{L}_X$ be the set of elementary closed subsets of $X := \{0, 1\}^I$ (i.e., sets of the form $A \times \prod_{i \in I \backslash F} X_i$ where $F \in fin(I)$ and $A \subseteq \{0, 1\}^F$). Let $\mathcal{F}$ be a filter of the lattice $\mathcal{L}_X$: let us show that $\cap\mathcal{F}$ is non-empty. Using $\mathbf{AC}^{fin(I)}$, for every $F \in fin(I)$, choose some element σ_F in the non-empty subset $\cap_{Z \in \mathcal{F}} p_F[Z]$ of $\{0, 1\}^F$. Using **RL**, consider some $f \in X$ which is Rado-compatible with $(\sigma_F)_{F \in fin(I)}$; then for every $F \in fin(I)$, there exists $G \in fin(I)$ such that $F \subseteq G$ and $f \restriction F = \sigma_G \restriction F$; this implies that $f \restriction F = p_F(\sigma_G) \in \cap_{Z \in \mathcal{F}} p_F[p_G[Z]] = \cap_{Z \in \mathcal{F}} p_F[Z]$, so $f \in \cap\mathcal{F}$.
$\mathbf{RL}_1 \Rightarrow \mathbf{RL}$. Let $(X_i)_{i \in I}$ be an infinite family of finite sets and assume that $(\sigma_F)_{F \in fin(I)}$ is a family such that for every $F \in fin(I)$, $\sigma_F \in \prod_{i \in F} X_i$. Then $(\sigma_F)_{F \in fin(I)}$ is a witness of $\mathbf{AC}^{fin(I)}$, so by $\mathbf{RL}_1$, the (non-empty) space $X := \prod_{i \in I} X_i$ is compact so $\cap_{J \in fin(I)} \tilde{F}_J$ is non-empty—see the notation used in Theorem 2–(3)—; now any element f of $\cap_{J \in fin(I)} \tilde{F}_J$ is Rado-compatible with $(\sigma_J)_{J \in fin(I)}$—-see Theorem 2–(4)—.
$\mathbf{RL}_1 \Rightarrow \mathbf{RL}_2$. Let $\mathbb{B}$ be a boolean algebra satisfying $\mathbf{AC}^{fin(\mathbb{B})}$ and let G be a proper filter of $\mathbb{B}$. For every elements $x, y \in \mathbb{B}$, consider the following closed subsets of $X := \{0, 1\}^{\mathbb{B}}$:

$$F_x := \{h \in X : (x \in \mathcal{G} \Rightarrow h(x) = 1)\}$$
$$G_{x,y} := \{h \in X : (x \wedge y = 0_{\mathbb{B}} \Rightarrow h(x \vee y) = \max(h(x), h(y)))\}$$
$$H_{x,y} := \{h \in X : h(x \wedge y) = \min(h(x), h(y))\}$$

The following set of closed subsets of X satisfies the finite intersection property: $\mathcal{F} := \{F_x : x \in \mathbb{B}\} \cup \{G_{x,y} : x, y \in \mathbb{B}\} \cup \{H_{x,y} : x, y \in \mathbb{B}\}$; using $\mathbf{RL}_1$, X is compact so there exists some element $h \in \cap\mathcal{F}$: such an element h is a boolean morphism $h : \mathbb{B} \to \{0, 1\}$ and $\mathcal{U} := h^{-1}(\{1\})$ is a ultrafilter of $\mathbb{B}$ including $\mathcal{G}$.
$\mathbf{RL}_2 \Rightarrow \mathbf{RL}_1$. Given an infinite set I satisfying $\mathbf{AC}^{fin(I)}$, consider the topological product space $X := \{0, 1\}^I$. Denote by $\mathbb{B}$ the set of elementary closed subsets of X: $\mathbb{B}$ is a boolean sub-algebra of $\mathcal{P}(I) = \{0, 1\}^I$. Using $\mathbf{AC}^{fin(I)}$, $\mathbf{AC}^{fin(\mathbb{B})}$ also holds: indeed, given a non-empty finite subset F of $\mathbb{B}$, then $Z := \cup_{x \in F} supp(x)$ is a finite subset of I; using $\mathbf{AC}^{fin(I)}$, we choose a linear order on $\{0, 1\}^Z$, which implies a linear order on F, which allows to choose an element in F. Assume that $\mathcal{F}$ is a family of elementary closed subsets of X satisfying the finite intersection property; using $\mathbf{RL}_2$, consider a ultrafilter $\mathcal{U}$ of $\mathbb{B}$ including $\mathcal{F}$; then for every $i \in I$, the projection of $\mathcal{U}$ on the ith factor is a singleton $\{\varepsilon_i\}$; and $\varepsilon := (\varepsilon_i)_{i \in I}$ belongs to $\cap\mathcal{U} \subseteq \cap\mathcal{F}$.

Remark 7 The equivalence $\mathbf{RL} \Leftrightarrow \mathbf{RL}_1$ is a slight generalization of Blass's Remark about the equivalence $\mathbf{T}_2 \Leftrightarrow (\mathbf{RL}+\mathbf{AC}^{\mathbf{fin}})$ (see Remark 3).

Consider the following statements:

$\mathbf{UF_{lo}}$: *"Every proper filter on a linearly orderable boolean algebra is included in a ultrafilter."*

$\mathbf{UF_{wo}}$: *"Every filter on a well-orderable set is included in a ultrafilter."*

$\mathbf{UF}_\omega$: *"Every filter on ω is included in a ultrafilter."*

$\mathbf{U}_\omega$: *"There exists a non-trivial ultrafilter on ω."*

Corollary 2 $\mathbf{RL} \Rightarrow \mathbf{UF_{lo}} \Rightarrow \mathbf{UF_{wo}} \Rightarrow \mathbf{UF}_\omega \Rightarrow \mathbf{U}_\omega$.

Proof $\mathbf{RL} \Rightarrow \mathbf{UF_{lo}}$: Given a boolean algebra $\mathbb{B}$ which is linearly orderable, then $\mathbf{AC}^{fin(\mathbb{B})}$ holds thus $\mathbf{RL}_2$ implies that every proper filter of $\mathbb{B}$ is contained in a ultrafilter.
$\mathbf{UF_{lo}} \Rightarrow \mathbf{UF_{wo}}$: Given a well-ordered set $(X, \preceq)$, the set $\mathcal{P}(X) = \{0, 1\}^X$ is linearly orderable (with the lexicographic order) thus $\mathbf{UF_{lo}}$ implies that every filter of the linearly orderable boolean algebra $\mathcal{P}(X)$ is included in a ultrafilter.
The implications $\mathbf{UF_{wo}} \Rightarrow \mathbf{UF}_\omega \Rightarrow \mathbf{U}_\omega$ are trivial.

Question 4 Does $\mathbf{UF_{lo}}$ imply $\mathbf{RL}$? Does $\mathbf{UF_{wo}}$ imply $\mathbf{UF_{lo}}$ or $\mathbf{RL}$?

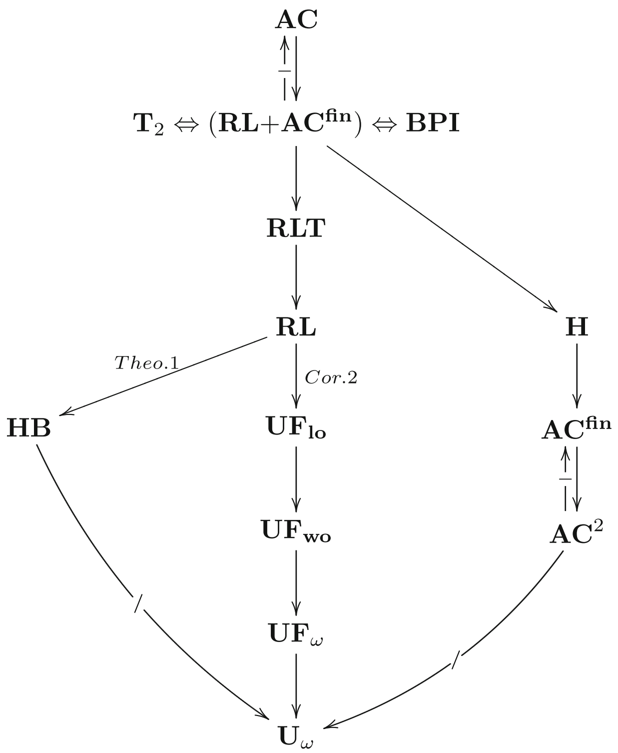
Remark 8 In $\mathbf{ZF}^0$ (set theory with atoms described in [3,5]), $\mathbf{UF_{wo}}$ does not imply $\mathbf{HB}$ (and thus does not imply $\mathbf{RL}$): in a Fraenkel-Mostowski model of $\mathbf{ZF}^0$, for every well-orderable set X, the set $\mathcal{P}(X)$ is also well-orderable thus $\mathbf{UF_{wo}}$ holds, however some of these models do not satisfy the Hahn-Banach axiom (see the model $\mathcal{N}51$ described in [3]).

Remark 9 In **ZF**, **HB** does not imply $\mathbf{U}_\omega$ (see [8]) and thus **HB** does not imply **RL**.

Consider the following statement, which is intermediate between $\mathbf{T}_2$ and $\mathbf{AC^{fin}}$:

H (Hall's marriage theorem, form 107 of [3]): *"Let I be an infinite set of finite sets such that for each finite subset F of I, there is an injective choice function on F. Then there is an injective choice function on I."*

The following implications hold:



The following questions seem to be open:

Question 5 Does **HB**+**H** imply **BPI**? Does **HB**+**H**+$\mathbf{UF_{wo}}$ imply **BPI**?

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