Arch. Math. Logic (2012) 51:695–707
DOI 10.1007/s00153-012-0293-8

Mathematical Logic

Glivenko theorems and negative translations in substructural predicate logics

Hadi Farahani · Hiroakira Ono

Received: 30 July 2011 / Accepted: 25 June 2012 / Published online: 5 July 2012

Abstract Along the same line as that in Ono (Ann Pure Appl Logic 161:246–250, 2009), a proof-theoretic approach to Glivenko theorems is developed here for substructural predicate logics relative not only to classical predicate logic but also to arbitrary involutive substructural predicate logics over intuitionistic linear predicate logic without exponentials $\mathbf{QFL_e}$. It is shown that there exists the weakest logic over $\mathbf{QFL_e}$ among substructural predicate logics for which the Glivenko theorem holds. Negative translations of substructural predicate logics are studied by using the same approach. First, a negative translation, called *extended Kuroda translation* is introduced. Then a translation result of an arbitrary involutive substructural predicate logics over $\mathbf{QFL_e}$ is shown, and the existence of the weakest logic is proved among such logics for which the extended Kuroda translation works. They are obtained by a slight modification of the proof of the Glivenko theorem. Relations of our extended Kuroda translation with other standard negative translations will be discussed. Lastly, algebraic aspects of these results will be mentioned briefly. In this way, a clear and comprehensive understanding of Glivenko theorems and negative translations will be obtained from a substructural viewpoint.

Authors would like to thank anonymous referees for their useful comments. Also the first author would like to thank Prof. E. Turunen for accepting him at TUT-Finland in 2010 and for introducing him to problems on Glivenko theorem.

H. Farahani
Department of Mathematics, Shahid Beheshti University, G.C., Evin, Tehran, Iran
e-mail: hadi.g.farahani@gmail.com

H. Ono (✉)
Japan Advanced Institute of Science and Technology, Ishikawa, Japan
e-mail: ono@jaist.ac.jp

Keywords Glivenko's theorem · Negative translations · Double negation shift · Substructural predicate logics · Proof-theoretic methods

Mathematics Subject Classification 03B47 · 03F03

1 Introduction

In [10], the second author has developed a proof-theoretic approach to Glivenko theorems for substructural propositional logics and gave alternative proofs of results in [6]. We will discuss in this paper Glivenko theorems and negative translations in substructural predicate logics along the same line as [10].

Results on Glivenko theorems for substructural propositional logics obtained in [6] and [10] can be naturally extended to those for substructural predicate logics. It is shown that the following *double negation shift* scheme (DNS) plays an essential role: $\forall x \neg\neg\varphi(x) \to \neg\neg\forall x \varphi(x)$. Among others, the Glivenko theorem relative to classical predicate logic is shown for a logic $\mathbf{G}^\sharp(\mathbf{QCl})$, which has (DNS) as an axiom scheme and which is an extension of intuitionistic linear predicate logic without exponentials $\mathbf{QFL_e}$. Moreover, this is the weakest logic over $\mathbf{QFL_e}$ among substructural predicate logics for which the Glivenko theorem holds. A classical result on the Glivenko theorem for predicate logics over intuitionistic logic relative to classical predicate logic by Umezawa [15] and Gabbay [3] follows from this.

Using the same proof-theoretic approach, we study negative translations of substructural predicate logics in the second part. After introducing a particular negative translation, called *extended Kuroda translation*, we show that a formula is provable in classical predicate logic iff its translation is provable in an extension $\mathbf{N}^\sharp(\mathbf{QCl})$ of $\mathbf{QFL_e}$. The logic $\mathbf{N}^\sharp(\mathbf{QCl})$ is properly weaker than $\mathbf{G}^\sharp(\mathbf{QCl})$, and moreover $\mathbf{N}^\sharp(\mathbf{QCl})$ is the weakest logic among such logics for which the extended Kuroda translation works. It is verified easily that some other standard negative translations including Kolmogorov translation and Gödel-Gentzen translation are equivalent to extended Kuroda translation over $\mathbf{QFL_e}$. Therefore we can give a unified view of Glivenko theorems and negative translations through a substructural perspective. Throughout the present paper, we assume familiarities with the arguments and techniques in [10].

To begin with, we give a brief historical sketch of Glivenko theorems for predicate logics over intuitionistic logic relative to classical predicate logic. We say the Glivenko theorem holds for a logic $\mathbf{L}$ relative to another logic $\mathbf{K}$, when

for any formula α, α is provable in $\mathbf{K}$ iff $\neg\neg\alpha$ is provable in $\mathbf{L}$.

In his book [8], Kleene discussed the Glivenko theorem for intuitionistic logic relative to classical logic. While the Glivenko theorem holds between them for the propositional logics, Kleene pointed out that the theorem does not hold for the predicate case, as the following formula

(K) $\neg\neg\forall x(\varphi(x) \vee \neg\varphi(x))$

is not provable in intuitionistic predicate logic. The axiom scheme (K) is shown to be equivalent over intuitionistic predicate logic to the following (see e.g. [14]), which is sometimes called the *Kuroda's formula* (see [9]), or the *double negation shift scheme* (DNS):

$$\text{(DNS)} \quad \forall x \neg\neg\varphi(x) \to \neg\neg\forall x \varphi(x).$$

In the same book, Kleene pointed out the following fact.

Proposition 1 *For all formulas α not containing any universal quantifier, α is provable in classical predicate logic iff $\neg\neg\alpha$ is provable in intuitionistic predicate logic.*

Later, Umezawa [14] and Gabbay [3] showed that in fact the Glivenko theorem holds in the following form. See also [4].

Proposition 2 *For all formulas α, α is provable in classical predicate logic* **QCl** *iff $\neg\neg\alpha$ is provable in the predicate logic obtained from intuitionistic predicate logic* **QInt** *by adding the axiom scheme (DNS).*

Suppose that the Glivenko theorem holds for an intermediate predicate logic **L** (i.e. a predicate logic between intuitionistic logic and classical logic) relative to classical predicate logic. Then, clearly the scheme (K) must be provable in **L** since $\forall x(\varphi(x) \vee \neg\varphi(x))$ is provable in classical logic. Therefore, Proposition 2 implies that the logic **QInt** with (DNS) is the weakest intermediate logic for which the Glivenko theorem holds relative to classical predicate logic. Our first step in generalizing the above proposition is to see what will happen if we consider also substructural predicate logics.

2 Glivenko theorems relative to classical logic

For this purpose, we will introduce here key results on Glivenko theorems for substructural propositional logics shown in [10]. Let $\mathbf{FL_e}$ be the sequent calculus obtained from the sequent calculus **LJ** for intuitionistic propositional logic by deleting both weakening rules and contraction rule, and then adding rules for the fusion (or multiplicative conjunction) $\cdot$. Define $\mathbf{FL_e}^\dagger$ is the sequent calculus obtained from $\mathbf{FL_e}$ by adding the following (A1), (A2), (AC) and (AW) as axiom schemes (i.e., initial sequents).

$$\begin{array}{ll} \text{(A1)} & \neg(\alpha \to \beta) \Rightarrow \neg(\neg\neg\alpha \to \neg\neg\beta), \\ \text{(A2)} & \neg(\alpha \wedge \beta) \Rightarrow \neg(\neg\neg\alpha \wedge \neg\neg\beta), \\ \text{(AC)} & \neg(\alpha \cdot \alpha) \Rightarrow \neg\alpha, \\ \text{(AW)} & \neg\beta \Rightarrow \neg(\alpha \cdot \beta). \end{array}$$

Similarly to [10], we use the word "logics" in an ambiguous way, sometimes denoting calculi and sometimes sets of formulas, since only the set of formulas provable in a given calculus does matter to Glivenko's theorem. Now, we can show the following (see Theorems 3 and 5 in [10]).

Proposition 3 *The Glivenko theorem holds for $\mathbf{FL_e}^\dagger$ relative to classical propositional logic* **Cl**. *In fact, $\mathbf{FL_e}^\dagger$ is the weakest substructural propositional logic over $\mathbf{FL_e}$ for which the Glivenko theorem holds relative to* **Cl**.

In Theorem 1 below, we show an extension of Proposition 3. In the following, the logic obtained from a given logic **L** by adding axiom schemes $\varphi_1, \ldots, \varphi_m$ is denoted by $\mathbf{L}[\varphi_1, \ldots, \varphi_m]$. As an axiom scheme, we may take either a formula scheme (like $\alpha \to \beta$) or an equivalent sequent scheme (like $\alpha \Rightarrow \beta$) as long as it does not cause any confusion.

Now define $\mathbf{QFL_e}$ to be the minimum predicate extension of $\mathbf{FL_e}$. Also define $\mathbf{QFL_e}^\dagger$ and $\mathbf{QFL_e}^\ddagger$ as follows.

$$\mathbf{QFL_e}^\dagger = \mathbf{QFL_e}[(A1), (A2), (AC), (AW)],$$
$$\mathbf{QFL_e}^\ddagger = \mathbf{QFL_e}[(A1), (A2), (AC), (AW), (DNS)].$$

To be precise, we give here the sequent calculus $\mathbf{QFL_e}$ explicitly. It consists of three initial sequents below and rules in the following.

$$\alpha \Rightarrow \alpha \qquad \Rightarrow 1 \qquad 0 \Rightarrow$$

It has weakening rules for constants 1 and 0, exchange rule and cut;

$$\frac{\Gamma \Rightarrow \gamma}{1, \Gamma \Rightarrow \gamma}(1 \text{ weakening}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}(0 \text{ weakening})$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma}(\text{exchange}) \qquad \frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma}(\text{cut})$$

and also rules for logical connectives and quantifiers;

$$\frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha \vee \beta, \Gamma \Rightarrow \gamma}(\text{left}\vee)$$

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}(\text{right} \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}(\text{right} \vee 2)$$

$$\frac{\Gamma \Rightarrow \alpha \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}(\text{right}\wedge)$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma}(\text{left} \wedge 1) \qquad \frac{\beta, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma}(\text{left} \wedge 2)$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \gamma}{\alpha \cdot \beta, \Gamma \Rightarrow \gamma}(\text{left}\cdot) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta}(\text{right}\cdot)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha \to \beta, \Gamma, \Delta \Rightarrow \gamma}(\text{left} \to) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}(\text{right} \to)$$

$$\frac{\varphi(z), \Gamma \Rightarrow \Delta}{\exists x \varphi(x), \Gamma \Rightarrow \Delta}(\text{left}\exists) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)}(\text{right}\exists)$$

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta}(\text{left}\forall) \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(z)}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)}(\text{right}\forall)$$

In (right $\exists$) and (left $\forall$), t is an arbitrary term. Also in (left $\exists$) and (right $\forall$), z is any variable which has no free occurrences in the lower sequent (*eigenvariable condition*)

(For more information on sequent calculi, see e.g. [13]). The constant 0 is used to define the negation $\neg\alpha$ of a formula α by $\alpha \to 0$. The following are derived rules for $\neg$.

$$\frac{\Gamma \Rightarrow \alpha}{\neg\alpha, \Gamma \Rightarrow}\text{(left}\neg\text{)} \qquad \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\alpha}\text{(right}\neg\text{)}$$

The following Lemma is an extension of Proposition 2 of [10] to predicate case, which is essential to the proof of Proposition 3.

Lemma 1 *For any sequent* $\Gamma \Rightarrow \Delta$, *if it is provable in the sequent calculus* **LK** *for classical predicate logic then the sequent* $\neg\Delta, \Gamma \Rightarrow$ *is provable in* $\mathbf{QFL_e}^{\ddagger}$.

Proof This is shown by using induction on the height of the proof of $\Gamma \Rightarrow \Delta$. In addition to the proof of Proposition 2 in [10], we need to consider cases where the last inference I of the proof $\Gamma \Rightarrow \Delta$ is one of rules for quantifiers. Clearly our Lemma holds when it is either of left rules for quantifiers. Now consider the case where I is (right $\forall$):

$$\frac{\Gamma \Rightarrow \Delta, \varphi(z)}{\Gamma \Rightarrow \Delta, \forall x\varphi(x)}$$

By the hypothesis of induction, $\neg\varphi(z), \neg\Delta, \Gamma \Rightarrow$ is provable in $\mathbf{QFL_e}^{\ddagger}$. Using (right $\neg$) and then (right $\forall$), $\neg\Delta, \Gamma \Rightarrow \forall x\neg\neg\varphi(x)$ is provable. Since (DNS) is an axiom scheme of $\mathbf{QFL_e}^{\ddagger}$, we can show that $\neg\Delta, \Gamma \Rightarrow \neg\neg\forall x\varphi(x)$ and hence $\neg\forall x\varphi(x), \neg\Delta, \Gamma \Rightarrow$ are also provable in $\mathbf{QFL_e}^{\ddagger}$. Thus our Lemma holds for this case. The case where I is (right $\exists$) can be treated in the same way. Note that the sequent $\exists x\neg\neg\varphi(x) \Rightarrow \neg\neg\exists x\varphi(x)$ is provable in $\mathbf{QFL_e}$. □

Theorem 1 (*1*) *The Glivenko theorem holds for* $\mathbf{QFL_e}^{\ddagger}$ *relative to the classical predicate logic* **QCl**. *In fact,* $\mathbf{QFL_e}^{\ddagger}$ *is the weakest substructural predicate logic over* $\mathbf{QFL_e}$ *for which the Glivenko theorem holds relative to* **QCl**.

(*2*) *For all formulas* α *not containing any universal quantifier,* α *is provable in classical predicate logic* **QCl** *iff* $\neg\neg\alpha$ *is provable in* $\mathbf{QFL_e}^{\dagger}$.

Proof The first statement of our Theorem follows immediately from Lemma 1. To show that $\mathbf{QFL_e}^{\ddagger}$ is the weakest among logics over $\mathbf{QFL_e}$ for which the Glivenko theorem holds relative to **QCl**, it is enough to show that (DNS) must hold always in such logics (see also the proof of Theorem5 in [10]). So suppose that the Glivenko theorem holds for **L** relative to **QCl** where **L** is a predicate logic over $\mathbf{QFL_e}$. As $\forall x\neg\neg\varphi(x) \to \forall x\varphi(x)$ is provable in **QCl**, $\neg\neg(\forall x\neg\neg\varphi(x) \to \forall x\varphi(x))$ is provable in **L**. Since in general $\neg\neg(\alpha \to \beta) \Rightarrow \alpha \to \neg\neg\beta$ is provable in $\mathbf{QFL_e}$ and hence in **L**, (DNS) is provable in **L**. The second part of our Theorem is shown in almost the same way as the first part, simply by neglecting the case analysis for universal quantifiers. □

We have not mentioned the logical connective fusion $\cdot$ explicitly in the above. But the presence of fusion does not cause any problem, since $\neg(\alpha \cdot \beta) \Rightarrow \neg(\neg\neg\alpha \cdot \neg\neg\beta)$

is provable in $\mathbf{QFL_e}$, as shown in [10]. We note also that $\alpha \cdot \beta$ is equivalent to $\alpha \wedge \beta$ for all α and β in every logic over $\mathbf{QFL_{eci}}$.

As the scheme (A2) is equivalent to the scheme: $\neg\neg\alpha \wedge \neg\neg\beta \Rightarrow \neg\neg(\alpha \wedge \beta)$, the scheme (DNS) can be regarded as an infinite analogue of (A2). In defining the logic $\mathbf{QFL_e}^{\ddagger}$ we took (DNS) instead of (K). The scheme (K) is provable in $\mathbf{QFL_e}^{\ddagger}$. Conversely, (DNS) follows from (K) in $\mathbf{QFL_{ew}}$, i.e. $\mathbf{QFL_e}$ with weakening rules, but it is unlikely that (DNS) is derived from (K) in $\mathbf{QFL_e}^{\dagger}$ without using weakening rules (see discussions after Theorem 3).

Recall that (AC) and (AW) are restricted forms of contraction rule and left-weakening rule, respectively. Also, (A1) (and (A2)) follows from weakening rules and (AC) (and left-weakening rule and (AC), respectively) over $\mathbf{QFL_e}$, as shown in [10]. So as an immediate consequence of Theorem 1, we can see how the conditions will change when we consider logics with weakening rules or with contraction rule. As typical examples, let us consider $\mathbf{QFL_{ew}}$ and $\mathbf{QFL_{eci}}$ which are minimum predicate extensions of $\mathbf{FL_{ew}}$ and $\mathbf{FL_{eci}}$, respectively. The logic $\mathbf{QFL_{eci}}$ is known as *minimal predicate logic*. This logic has both contraction rule and left-weakening rule. Hence (A2), (AC) and (AW) are derived in it. In this way, we can show the following.

Corollary 1 *The logic* $\mathbf{QFL_{ew}}[(AC), (DNS)]$ *is the weakest substructural predicate logic over* $\mathbf{QFL_{ew}}$*, and the logic* $\mathbf{QFL_{eci}}[(A1), (DNS)]$ *is the weakest substructural predicate logic over* $\mathbf{QFL_{eci}}$*, respectively, for which the Glivenko theorem holds relative to the logic* **QCl***.*

In [7], a natural deduction system $\mathbf{MQC}^+$ was introduced and its computational aspect was discussed. It is an extension of minimal predicate logic with (DNS), for which the Glivenko theorem relative to **QCl** was shown. The above Corollary implies that (A1) is provable in $\mathbf{MQC}^+$.

3 Glivenko theorems for substructural predicate logics

We show next Glivenko theorems for substructural predicate logics over $\mathbf{QFL_e}$ relative to an arbitrary *involutive* substructural predicate logic **K** over $\mathbf{QFL_e}$, by modifying the proof of Theorem 1. We discuss these Glivenko theorems briefly since proofs go almost in the same way as those in [10].

First, we introduce the deducibility in logics over $\mathbf{QFL_e}$. For an arbitrary substructural predicate logic **K** over $\mathbf{QFL_e}$, define the deducibility relation $\vdash_{\mathbf{K}}$ as follows. For any set of *closed* formulas Π and any formula φ, $\Pi \vdash_{\mathbf{K}} \varphi$ holds if and only if the sequent $\Rightarrow \varphi$ can be provable in the system obtained from **K** by adding $\Rightarrow \delta$ as new initial sequent for each $\delta \in \Pi$. Then the following local deduction theorem for $\mathbf{QFL_e}$ is shown essentially in the same way as that for $\mathbf{FL_e}$. For a proof of the local deduction theorem for $\mathbf{FL_e}$, see e.g. in §2.4 of [5]. We note here that since all formulas in Π are closed, they will not disturb eigenvariable conditions in an application of either (left $\exists$) or (right $\forall$). Thus the following holds. Here, $(\alpha \wedge 1)^m$ denotes the formula obtained by multiplying $\alpha \wedge 1$ by fusion for m times.

Theorem 2 *For every substructural predicate logic* **K***,* $\Pi, \alpha \vdash_{\mathbf{K}} \varphi$ *if and only if* $\Pi \vdash_{\mathbf{K}} (\alpha \wedge 1)^m \rightarrow \varphi$ *for some* m*.*

Following [6], we introduce three kinds of involutiveness and three kinds of Glivenko theorems. A logic **K** is *involutive* whenever $\neg\neg\alpha \rightarrow \alpha$ is provable in it for every formula α. It is *weakly involutive* whenever $\neg\neg\alpha \vdash_{\mathbf{K}} \alpha$ holds for every formula α. It is *Glivenko involutive* when for every formula α if $\neg\neg\alpha$ is provable in **K** then so is α. Clearly, involutiveness implies weak involutiveness, which implies Glivenko involutiveness.

As for Glivenko theorems, the first one is the Glivenko theorem which we have discussed already. We say that the *deductive Glivenko theorem* holds for a logic **L** relative to another logic **K** if

> for any set Π of closed formulas and any formula α, $\Pi \vdash_{\mathbf{K}} \alpha$ if and only if $\Pi \vdash_{\mathbf{L}} \neg\neg\alpha$.

We say that the *equational Glivenko theorem* holds for **L** relative to **K** if

> for all formulas α and β, $\alpha \leftrightarrow \beta$ is provable in **K** if and only if $\neg\neg\alpha \leftrightarrow \neg\neg\beta$ is provable in **L**.

It can be shown that this condition is equivalent to the following:

> for all formulas α and β, $\alpha \rightarrow \beta$ is provable in **K** if and only if $\alpha \rightarrow \neg\neg\beta$ is provable in **L**.

We can show that if the equational Glivenko theorem holds for **L** relative to **K** then the deductive Glivenko theorem holds for **L** relative to **K**, by using the local deduction theorem (Theorem 2). Obviously the deductive Glivenko theorem implies the Glivenko theorem.

It was shown in [10] that for a given (Glivenko) involutive substructural propositional logic **K** there exists the weakest logic **G**(**K**) among substructural propositional logics over $\mathbf{FL_e}$ for which the Glivenko theorem holds relative to **K**. This result can be extended to the predicate case.

For any logic **K** over $\mathbf{QFL_e}$, let $\mathbf{G}^{\sharp}(\mathbf{K})$ to be the logic obtained from $\mathbf{QFL_e}$ by adding axiom schemes $\{\neg\neg\beta : \beta$ is a closed formula which is provable in $\mathbf{K}\}$, in addition to axiom schemes (A1), (A2) and (DNS). For instance, $\mathbf{G}^{\sharp}(\mathbf{QCl})$ is equal to $\mathbf{QFL_e}^{\ddagger}$. Clearly, $\mathbf{G}^{\sharp}(\mathbf{K})$ is finitely axiomatizable when **K** is so. Now theorems in §3 of [10] can be extended as follows.

Theorem 3 (*1*) *Suppose that* **K** *is a Glivenko involutive substructural predicate logic over* $\mathbf{QFL_e}$. *Then, for each substructural predicate logic* **L** *over* $\mathbf{QFL_e}$, *the Glivenko theorem holds for* **L** *relative to* **K** *if and only if* **L** *is an extension of* $\mathbf{G}^{\sharp}(\mathbf{K})$ *and is included by* **K**.

(*2*) *The same statement on the deductive Glivenko theorem (the equational Glivenko theorem) holds for a weakly involutive (involutive, respectively) logic* **K**.

Obviously, the logic **QCl** is involutive. Thus it follows from Theorem 3 that Theorem 1 can be extended to the equational Glivenko theorem for $\mathbf{QFL_e}^{\ddagger}$ relative to **QCl**. In fact, this is obtained immediately from Lemma 1.

Now consider the involutive logic $\mathbf{InQFL_e}$ obtained from $\mathbf{QFL_e}$ by adding the axiom of involutiveness $\neg\neg\alpha \rightarrow \alpha$, which is known as linear predicate logic without exponentials. Then, the above Theorem 3 implies that the equational Glivenko theorem relative to $\mathbf{InQFL_e}$ holds for $\mathbf{G}^\sharp(\mathbf{InQFL_e})$, i.e. $\mathbf{QFL_e}$[(A1), (A2), (DNS)].

We show here that (DNS) can not be replaced by the scheme (K) in the definition of $\mathbf{G}^\sharp(\mathbf{InQFL_e})$. Suppose that (K) is provable in $\mathbf{G}^\sharp(\mathbf{InQFL_e})$. Then $\forall x(\varphi(x) \vee \neg\varphi(x))$ must be provable in $\mathbf{InQFL_e}$, since the Glivenko theorem holds for $\mathbf{G}^\sharp(\mathbf{InQFL_e})$ relative to $\mathbf{InQFL_e}$. On the other hand, it can be shown that $\mathbf{QFL_e}$ with $\forall x(\varphi(x) \vee \neg\varphi(x))$ is equal to classical logic (see e.g. §5 of [11]). But this is a contradiction.

In [14], Umezawa gave several schemes including (DNS) which are equivalent to the scheme (K) in **QInt**. We note here that among others the following schemes are equivalent to (DNS) even in $\mathbf{QFL_e}$.

(1) $\neg\forall x\varphi(x) \to \neg\neg\exists x\neg\varphi(x)$,
(2) $\neg\neg(\forall x\neg\neg\varphi(x) \to \neg\neg\forall x\varphi(x))$,
(3) $\neg\neg(\neg\forall x\varphi(x) \to \neg\neg\exists x\neg\varphi(x))$.

4 Substructural aspect of negative translations

We have developed proof-theoretic analysis of Glivenko theorems so far by focusing on roles of structural rules. Along the same line, we discuss *negative translations* among substructural predicate logics. Various negative translations of **QCl** into **QInt** have been introduced and studied (see e.g. [2]). A common feature among them is to attach double negations to some or all subfomulas of a given formula. Then it is shown that a given formula is provable in **QCl** if and only if its translation is provable in **QInt**.

Here we consider negative translations not only of classical logic **QCl** into **QInt**, but also of arbitrary involutive substructural predicate logics into another substructural predicate logic over $\mathbf{QFL_e}$ which lacks the involutiveness. We introduce first a particular negative translation, which we call *extended Kuroda translation*, and show its close connection with Glivenko theorems. Then, we show that some of standard negative translations will be reduced to this translation.

Let us begin with negative translations of **QCl** into **QInt** (see e.g. [2] for various translations). Among others, we first focus on a translation $\circ$, called *Kuroda translation*, which is defined as follows. (In the rest of our paper, we assume always that $(\neg\alpha)^\tau = \neg(\alpha^\tau)$ for any negative translation τ).

$$
\begin{array}{ll}
p^\circ = p, \quad \text{for atomic } p & (\alpha \wedge \beta)^\circ = \alpha^\circ \wedge \beta^\circ \\
(\alpha \vee \beta)^\circ = \alpha^\circ \vee \beta^\circ & (\alpha \to \beta)^\circ = \alpha^\circ \to \beta^\circ \\
(\forall x\alpha)^\circ = \forall x\neg\neg\alpha^\circ & (\exists x\alpha)^\circ = \exists x\alpha^\circ
\end{array}
$$

Recall that our Theorem 1 says that the Glivenko theorem holds for $\mathbf{QFL_e}^\ddagger$ relative to **QCl**, where $\mathbf{QFL_e}^\ddagger = \mathbf{QFL_e}$[(A1), (A2), (AC), (AW), (DNS)]. Since all (A1), (A2), (AC) and (AW) but (DNS) are provable in **QInt**, Proposition 2 follows from this theorem if we restrict our attention only to logics over **QInt**.

The reason why we need (DNS) in this Proposition 2 comes from the induction step for the case of (right $\forall$), as noted in the proof of Lemma 1. To resolve this difficulty and to get a translation of **QCl** into **QInt**, Kuroda introduced $\circ$ by defining $(\forall x\alpha)^\circ = \forall x\neg\neg\alpha^\circ$, and proved the following in his paper [9].

Proposition 4 *For all formulas α, α is provable in* **QCl** *iff* $\neg\neg\alpha^\circ$ *is provable in the predicate logic* **QInt**.

Proposition 4 can be obtained as a corollary of the following stronger result. We note that all of (A1), (A2), (AC) and (AW) are provable in **QInt**.

Theorem 4 *For all formulas α, α is provable in* **QCl** *iff* $\neg\neg\alpha^\circ$ *is provable in the logic* $\mathbf{QFL_e}^\dagger$, *where* $\mathbf{QFL_e}^\dagger = \mathbf{QFL_e}[(A1), (A2), (AC), (AW)]$.

Theorem 4 can be derived from the following Lemma 2 (cf. Lemma 1). Here, Γ° denotes the sequence of formulas $(\gamma_1)^\circ, \ldots, (\gamma_m)^\circ$ when Γ is $\gamma_1, \ldots, \gamma_m$, and similarly $\neg\Delta^\circ$ denotes the sequence of formulas $\neg(\delta_1)^\circ, \ldots, \neg(\delta_m)^\circ$ when Δ is $\delta_1, \ldots, \delta_m$.

Lemma 2 *For any sequent $\Gamma \Rightarrow \Delta$, if it is provable in* **LK** *then the sequent* $\neg\Delta^\circ, \Gamma^\circ \Rightarrow$ *is provable in* $\mathbf{QFL_e}^\dagger$.

Proof To see this, it is enough to check this for the induction step of the case of (right $\forall$). It goes similarly to the proof of Lemma 1. By the hypothesis of induction, $\neg\varphi(z)^\circ, \neg\Delta^\circ, \Gamma^\circ \Rightarrow$ is provable in $\mathbf{QFL_e}^\dagger$. Using (right $\neg$) and then (right $\forall$), $\neg\Delta^\circ, \Gamma^\circ \Rightarrow \forall x\neg\neg\varphi(x)^\circ$ is provable. Thus, $\neg(\forall x\varphi(x))^\circ, \neg\Delta^\circ, \Gamma^\circ \Rightarrow$ is provable in $\mathbf{QFL_e}^\dagger$.

Before extending Kuroda translation, we give the following preliminary lemma which is sometimes used in the rest of this section. It is obvious that the converse of each of the following sequents is also provable in $\mathbf{QFL_e}$.

Lemma 3 *The following sequents are provable in* $\mathbf{QFL_e}$.

(*1*) $\neg\neg(\neg\neg\alpha \wedge \neg\neg\beta) \Rightarrow \neg\neg\alpha \wedge \neg\neg\beta$,
(*2*) $\neg\neg(\neg\neg\alpha \vee \neg\neg\beta) \Rightarrow \neg\neg(\alpha \vee \beta)$,
(*3*) $\neg\neg(\neg\neg\alpha \cdot \neg\neg\beta) \Rightarrow \neg\neg(\alpha \cdot \beta)$,
(*4*) $\neg\neg(\neg\neg\alpha \to \neg\neg\beta) \Rightarrow \neg\neg\alpha \to \neg\neg\beta$,
(*5*) $\alpha \to \neg\neg\beta \Rightarrow \neg\neg\alpha \to \neg\neg\beta$,
(*6*) $\neg\neg\forall x\neg\neg\alpha \Rightarrow \forall x\neg\neg\alpha$,
(*7*) $\neg\neg\exists x\neg\neg\alpha \Rightarrow \neg\neg\exists x\alpha$.

The essential role of the Kuroda translation is to *trivialize* the scheme (DNS). Similarly, by modifying the translation of formulas of the form $\alpha \to \beta$ and $\alpha \wedge \beta$, we can also trivialize both schemes (A1) and (A2). This idea leads us to another translation $\bullet$, called *extended Kuroda translation*:

$$
\begin{array}{ll}
p^\bullet = p, \quad \text{for atomic } p & \\
(\alpha \wedge \beta)^\bullet = \neg\neg\alpha^\bullet \wedge \neg\neg\beta^\bullet & (\alpha \vee \beta)^\bullet = \alpha^\bullet \vee \beta^\bullet \\
(\alpha \to \beta)^\bullet = \alpha^\bullet \to \neg\neg\beta^\bullet & (\alpha \cdot \beta)^\bullet = \alpha^\bullet \cdot \beta^\bullet \\
(\forall x\alpha)^\bullet = \forall x\neg\neg\alpha^\bullet & (\exists x\alpha)^\bullet = \exists x\alpha^\bullet
\end{array}
$$

It can be easily verified that $\neg\neg\alpha^\circ$ is equivalent to $\neg\neg\alpha^\bullet$ in **QInt** for any formula α without fusion, and that $\neg\neg\alpha^\bullet$ is equivalent to α in every involutive logic.

In the case of the extended Kuroda translation double negations are put exactly at places in subformulas so that consideration to (A1), (A2) and (DNS) is not necessary any more. Thus, we can show the following Theorem 5, which strengthens Theorem 4.

Theorem 5 (*1*) *For all formulas α, α is provable in* **QCl** *iff* $\neg\neg\alpha^\bullet$ *is provable in the predicate logic* $\mathbf{QFL_e}[(AC), (AW)]$.

(*2*) *Let* **L** *be a predicate logic over* $\mathbf{QFL_e}$ *which is included by* **QCl**. *Moreover suppose that for all formulas α, α is provable in* **QCl** *iff* $\neg\neg\alpha^\bullet$ *is provable in* **L**. *Then* **L** *is an extension of the logic* $\mathbf{QFL_e}[(AC), (AW)]$.

(*3*) *For all formulas α, α is provable in* **QCl** *iff* $\neg\neg\alpha^\bullet$ *is provable in minimal predicate logic* $\mathbf{QFL_{eci}}$.

(*4*) *For all formulas α, α is provable in* $\mathbf{InQFL_e}$ *iff* $\neg\neg\alpha^\bullet$ *is provable in the predicate logic* $\mathbf{QFL_e}$.

Proof The statement (1) is shown similarly to Theorem 4. The statement (3) is a corollary of (2). To show (2), let us suppose that a logic **L** satisfies the above condition. Since the instance of the axiom for weakening $q \to (p \to q)$ is provable in **QCl** where p and q are atomic formulas, $\neg\neg(q \to (p \to q))^\bullet$ must be provable in **L**. But this formula is equal to $\neg\neg(q \to \neg\neg(p \to \neg\neg q))$. By using Lemma 3 (3), (4) and their converses, the subformula $\neg\neg(p \to \neg\neg q)$ is shown to be equivalent to $p \to \neg\neg q$. Thus, $\neg\neg(q \to \neg\neg(p \to \neg\neg q))$ is equivalent to $\neg\neg(q \to (p \to \neg\neg q))$ and hence to $\neg\neg((p\cdot q) \to \neg\neg q)$. From this it follows that the formula $\neg q \to \neg(p\cdot q)$ is provable in **L**. Since every logic is closed under substitution, each formula of the form $\neg\beta \to \neg(\alpha \cdot \beta)$ is provable in **L**. Similarly, (AC) is shown to be provable in **L**. The statement (4) can be shown similarly to (1). Since $\mathbf{InQFL_e}$ contains neither contraction rule nor weakening rules, both (AC) and (AW) are dispensable. □

As Lemma 2 suggests, every statement of Theorem 5 still holds if we replace every occurrence of α by $\beta \to \alpha$ and every occurrence of $\neg\neg\alpha^\bullet$ by $\beta^\bullet \to \neg\neg\alpha^\bullet$ where α and β are arbitrary formulas. We notice also that we can keep the translation of formulas of the form $\alpha \wedge \beta$ unchanged in Kuroda negative translation only to get the statement (2) of the above theorem (cf. Corollary 1). The result is mentioned in the last paragraph of §6.3 of [2]. Also, compare $\bullet$ with a translation M' in §6.1 of [2].

We will show next a connection of extended Kuroda negative translation with other standard negative translations like Kolmogorov translation κ and Gödel-Gentzen translation γ. They are defined as follows. (See also [6] for discussions on a generalization of Kolmogorov translation in the substructural setting).

$$
\begin{array}{ll}
p^\kappa = \neg\neg p, \quad \text{for atomic } p & \\
(\alpha \wedge \beta)^\kappa = \neg\neg(\alpha^\kappa \wedge \beta^\kappa) & (\alpha \vee \beta)^\kappa = \neg\neg(\alpha^\kappa \vee \beta^\kappa) \\
(\alpha \to \beta)^\kappa = \neg\neg(\alpha^\kappa \to \beta^\kappa) & (\alpha \cdot \beta)^\kappa = \neg\neg(\alpha^\kappa \cdot \beta^\kappa) \\
(\forall x\alpha)^\kappa = \neg\neg(\forall x\alpha^\kappa) & (\exists x\alpha)^\kappa = \neg\neg(\exists x\alpha^\kappa)
\end{array}
$$

$$
\begin{array}{ll}
p^\gamma = \neg\neg p, \quad \text{for atomic } p & \\
(\alpha \wedge \beta)^\gamma = \alpha^\gamma \wedge \beta^\gamma & (\alpha \vee \beta)^\gamma = \neg\neg(\alpha^\gamma \vee \beta^\gamma) \\
(\alpha \to \beta)^\gamma = \alpha^\gamma \to \beta^\gamma & (\alpha \cdot \beta)^\gamma = \neg\neg(\alpha^\kappa \cdot \beta^\gamma) \\
(\forall x\alpha)^\gamma = \forall x\alpha^\gamma & (\exists x\alpha)^\gamma = \neg\neg(\exists x\alpha^\gamma)
\end{array}
$$

The first statement (1) in the next lemma is a folklore, but an important point is the fact that the equivalence holds in $\mathbf{QFL_e}$. Both statements are proved by using the induction on the length of the formula α.

Lemma 4 (*1*) *For any formula* α*,* α^{κ} *is equivalent to* α^{γ} *in* $\mathbf{QFL_e}$.
(*2*) *For any formula* α*,* $\neg\neg\alpha^{\bullet}$ *is equivalent to* α^{κ} *in* $\mathbf{QFL_e}$.

We can extend our idea to Krivine-type negative translation (see e.g. [12] and [1]). Let us define a translation $\star$ as follows.

$$
\begin{array}{ll}
p_\star = \neg p, \quad \text{for atomic } p & \\
(\alpha \wedge \beta)_\star = \alpha_\star \vee \beta_\star & (\alpha \vee \beta)_\star = \neg\neg\alpha_\star \wedge \neg\neg\beta_\star \\
(\alpha \to \beta)_\star = \neg\alpha_\star \cdot \beta_\star & (\alpha \cdot \beta)_\star = \neg\alpha_\star \to \neg\neg\beta_\star \\
(\forall x\alpha)_\star = \exists x\alpha_\star & (\exists x\alpha)_\star = \forall x\neg\neg\alpha_\star
\end{array}
$$

Lemma 5 *For any formula* α*,* α^{γ} *is equivalent to* $\neg\alpha_\star$ *in* $\mathbf{QFL_e}$.

Now we can conclude that Theorem 5 holds also when we replace $\neg\neg\alpha^{\bullet}$ by any of α^{κ}, α^{γ} and $\neg\alpha_\star$ of α. Thus, we have the following.

Corollary 2 *Let* α^{τ} *be any one of Kolmogorov translation* α^{κ}*, Gödel-Gentzen translation* α^{γ} *and Krivine-type negative translation* $\neg\alpha_\star$ *for any formula* α*. Then the following* (*1*) *and* (*2*) *are mutually equivalent for each extension* $\mathbf{L}$ *of the logic* $\mathbf{QFL_e}$ *which is included by* $\mathbf{QCl}$*:*

(*1*) *For all formulas* α*,* α *is provable in* $\mathbf{QCl}$ *iff* α^{τ} *is provable in* $\mathbf{L}$.
(*2*) $\mathbf{L}$ *is an extension of the logic* $\mathbf{QFL_e}[(AC), (AW)]$.

Similarly to Theorem 3 in the previous section, we can state our result in a general form. Take any negative translation τ in Corollary 2. For any logic $\mathbf{K}$ over $\mathbf{QFL_e}$, let $\mathbf{N}^{\sharp}(\mathbf{K})$ to be the logic obtained from $\mathbf{QFL_e}$ by adding axiom schemes $\{\beta^{\tau} : \beta$ is a closed formula which is provable in $\mathbf{K}\}$. (Note that the set of these axiom schemes does not depend on the choice of τ). We can show that $\mathbf{N}^{\sharp}(\mathbf{K})$ is properly weaker than $\mathbf{G}^{\sharp}(\mathbf{K})$ for every consistent involutive substructural predicate logic $\mathbf{K}$ over $\mathbf{QFL_e}$, and that $\mathbf{N}^{\sharp}(\mathbf{K})$ is finitely axiomatizable when $\mathbf{K}$ is so. Also it can be shown that $\mathbf{N}^{\sharp}(\mathbf{QCl})$ and $\mathbf{N}^{\sharp}(\mathbf{InQFL_e})$ are equal to $\mathbf{QFL_e}$[(AC), (AW)] and $\mathbf{QFL_e}$, respectively. (See also §7 of [6] for Kolmogorov translation among substructural propositional logics).

Corollary 3 *Let* $\mathbf{K}$ *be an involutive substructural predicate logic over* $\mathbf{QFL_e}$ *and* $\mathbf{L}$ *be a substructural predicate logic over* $\mathbf{QFL_e}$ *which is included by* $\mathbf{K}$*. Moreover suppose that for each formula* α*,* α *is provable in* $\mathbf{K}$ *iff* α^{τ} *is provable in* $\mathbf{L}$*. Then* $\mathbf{L}$ *is an extension of* $\mathbf{N}^{\sharp}(\mathbf{K})$*. Hence,* $\mathbf{N}^{\sharp}(\mathbf{K})$ *is the weakest logic among such logics.*

5 Algebraic view of negative translations

Lastly, we will mention briefly algebraic meaning of these negative translations. Only for the simplicity's sake, we restrict our attention only to the propositional part. The following remarks are essentially described in §7 of [6] but in a slightly different way. We assume some familiarity with algebraic notions related to substructural logics, e.g. in [6] or [5]. An algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \rightarrow, 1, 0\rangle$ is a an FL_e-*algebra* if $\langle A, \vee, \wedge, \cdot, \rightarrow, 1\rangle$ is a commutative residuated lattice and 0 is a fixed element in A. FL_e-algebras are algebraic structures for the propositional logic $\mathbf{FL_e}$, and algebraic operations $\vee, \wedge, \cdot$ and $\rightarrow$ will give an interpretation of the corresponding logical connectives. (By abuse of symbols, we use the same symbols for algebraic operations and for logical connectives). We define the algebraic operation $\neg$ by $\neg x = x \rightarrow 0$, which gives an interpretation of the negation.

For a given FL_e-algebra $\mathbf{A}$, an operation c on A is a *nucleus* iff it is a closure operator satisfying that $c(x) \cdot c(y) \leq c(x \cdot y)$. Let A_c be the set of all c-*closed* elements of A, i.e. elements x such that $c(x) = x$ holds. For c-closed x and y, either of $x \vee y$ and $x \cdot y$ is not always c-closed, while both $x \wedge y$ and $x \rightarrow y$ are c-closed. We define operations $\vee_c$ and $\cdot_c$ on A by $x \vee_c y = c(x \vee y)$ and $x \cdot_c y = c(x \cdot y)$, respectively. Define an algebra $\mathbf{A}_c = \langle A_c, \vee_c, \wedge, \circ_c, \rightarrow, c(1), c(0)\rangle$. Then $\mathbf{A}_c$ is an FL_e-algebra in which $c(z) = z$ holds always. The algebra $\mathbf{A}_c$ is called the c-*retraction* of $\mathbf{A}$.

Suppose that an operation c is defined by a unary FL_e-term for which all the conditions of nuclei and the condition that $c(0) = 0$ hold. We define α^c for each term α as follows. (Compare the definition with the definition of Gödel-Gentzen translation. We identify formulas with algebraic terms in the following).

$$p^c = c(p), \quad \text{for atomic } p$$
$$(\alpha \wedge \beta)^c = \alpha^c \wedge \beta^c \qquad (\alpha \vee \beta)^c = c(\alpha^c \vee \beta^c)$$
$$(\alpha \rightarrow \beta)^c = \alpha^c \rightarrow \beta^c \qquad (\alpha \cdot \beta)^c = c(\alpha^c \cdot \beta^c)$$

By using induction of the length of α, we can show that for any term α, α^c is valid in an FL_e-algebra $\mathbf{A}$ if and only if α is valid in its c-retraction $\mathbf{A}_c$.

Let us define $g(x) = \neg\neg x$ for all x in $\mathbf{A}$. Then the double negation g is shown to be a nucleus satisfying $g(0) = 0$. The g-retraction $\mathbf{A}_g$ is an FL_e-algebra satisfying $\neg\neg z = z$. Moreover as a particular case of the above result, we have that for any term α, α is valid in the g-retraction $\mathbf{A}_g$ if and only if α^g is valid in an FL_e-algebra $\mathbf{A}$. From this it follows that $\mathbf{A}_g$ is a Boolean algebra if and only if $\mathbf{A}$ satisfies both $\neg(x \cdot x) \leq \neg x$ and $\neg y \leq \neg(x \cdot y)$. Using this, we can derive that for any term α, α is valid in every Boolean algebra if and only if α^g $(= \alpha^\gamma)$ is valid in every FL_e-algebra satisfying $\neg(x \cdot x) \leq \neg x$ and $\neg y \leq \neg(x \cdot y)$. Note that these two inequalities are algebraic counterparts of (AC) and (AW), respectively. Thus they are the algebraic contents of (the propositional part of) the result in Corollary 2 and the statements just below it. In fact, our arguments mentioned above give an algebraic proof of these results. See also related arguments in §5 and §7 of [6].

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