Arch. Math. Logic (2012) 51:475–501
DOI 10.1007/s00153-012-0277-8

Mathematical Logic

How to assign ordinal numbers to combinatory terms with polymorphic types

William R. Stirton

Received: 1 November 2010 / Accepted: 6 March 2012 / Published online: 15 April 2012
© Springer-Verlag 2012

Abstract The article investigates a system of polymorphically typed combinatory logic which is equivalent to Gödel's **T**. A notion of (strong) reduction is defined over terms of this system and it is proved that the class of well-formed terms is closed under both bracket abstraction and reduction. The main new result is that the number of contractions needed to reduce a term to normal form is computed by an ε_0-recursive function. The ordinal assignments used to obtain this result are also used to prove that the system under consideration is indeed equivalent to Gödel's **T**. It is hoped that the methods used here can be extended so as to obtain similar results for stronger systems of polymorphically typed combinatory terms. An interesting corollary of such results is that they yield ordinally informative proofs of normalizability for sub-systems of second-order intuitionist logic, in natural deduction style.

Keywords Proof theory · Combinatory logic · Primitive recursive functionals · Gödel's **T** · Ordinal analysis · Second-order logic

Mathematics Subject Classification 03Fxx

Traditionally the language used for talking about the primitive recursive functionals (henceforth: prf's) has been one constructed from the language of the simply typed λ-calculus (or, alternatively, combinatory logic) by adding numerals and a method of forming a new term from two given terms that is designed to represent the definition of a new functional, by primitive recursion of finite type, from two functionals already defined. When rules of inference for reasoning in such a language are provided, which

W. R. Stirton (✉)
Department of Philosophy, University of Edinburgh, 3 Charles Street, Edinburgh EH8 9AD, UK
e-mail: william_stirton@yahoo.co.uk

are designed to be faithful to the intended interpretation, the result is one of the theories that go by the name of "Gödel's **T**".

In recent years, starting with [1] and [12], there has been some interest in using a different kind of system for reasoning about the prf's. While the terms of these systems are all typed (or typable) versions of terms used in the untyped λ-calculus (combinatory logic), the types in question are not the familiar simple types but polymorphic types of a relatively simple kind. In both these respects, the systems exemplified by [1] and [12] differ from **T**: the types of terms in **T** are all simple types, while on the other hand **T** contains terms (with associated reduction rules) which are not typed (or typable) versions of terms used in the untyped λ-calculus (combinatory logic).

The present paper is a contribution to the study of the kind of systems described in [1] and [12]. The reader may consult the introductory discussion in [12] for further information on the purpose for which these systems were constructed. At present, whereas **T** has been investigated very thoroughly indeed, much less is known about the systems exemplified by [1] and [12]. In particular, fruitful connections between terms and ordinal numbers have not yet been established.[1]

In Sect. 1 of this paper, a system of polymorphically typed combinatory logic will be described. This is very much stronger than **T**, but by placing restriction on the formation of terms we arrive at a special class of terms called ***BI**-terms*, which are just strong enough to represent the prf's (Sect. 3). In Sect. 5, a function is defined which takes each **BI**-term to an ordinal number below ε_0. This function is based on ideas first presented in [9] and [14] and, as there, it is demonstrated that contracting redexes within a **BI**-term lowers the assigned ordinal. Consequences of this are deduced in Sect. 6, one being that every functional defined by a **BI**-term is a prf. Thus the calculus of **BI**-terms is, in a sense that can be made precise, equivalent both to **T** and to the systems described in [1] and [12]. Furthermore, using the main result of Sect. 5, it is possible to define (by ε_0-recursion) a function which predicts how many reduction steps are needed to normalize a **BI**-term. Nothing like this is proved in [1] or [12] and the use of ordinal-theoretic methods is presumably essential to this last result.[2]

A fundamental choice facing researchers in this area is whether to work with combinatory terms or λ-terms. Combinatory terms are studied here because they are much more amenable to the Howard–Schütte [9, 14] style of ordinal assignments used in Sect. 5. The principal ordinal assigned to a combinatory term, as in [14, pp. 105–107], is computed from *sequences* of ordinals assigned to its immediate subterms. However when λ-terms are treated, as in [9], it is necessary to assign to each term not a sequence of *ordinals* but a sequence of *functions over the ordinals* and the calculations required become much more complicated in all respects, as can be seen from comparing [9] with [14, pp. 105–112]. The advantages of the Howard–Schütte style of proof, compared with others in the literature, are discussed in Sect. 7.

[1] Coquand [5] contains a reference to a proof, by I. Takeuti, of normalizability of a system of typed λ-calculus apparently of similar strength to the system of **BI**-terms. It makes use of an ordinal-theoretic cut-elimination proof by G. Takeuti [16]. The reference is to a book called *Proof Theory and Reverse Mathematics*, but I cannot find any other references to this book on the internet (maybe it is in Japanese). The goals of this paper are in any case different from the goal of proving normalizability. See Sect. 7 for general observations on method.

[2] I am indebted to Dr. Altenkirch for discussions on this matter.

In standard works on combinatory logic, two concepts of reduction are principally considered, called *strong* and *weak* reduction respectively.[3] Here, for reasons explained following Proposition 4.8, it is necessary to deal with strong reduction, which is defined by Definitions 2.4 and 2.10 below. As explained in Sects. 3 and 8, the ordinal analysis of strong reduction is of especial interest because of its implications for the proof theory of second-order logic.

1 Polymorphically typed combinatory terms

Definition 1.1 defines the *polymorphic types*, henceforth simply *types*. Letters like $A, B, C, D, E, A_0, A_1, \ldots$ range over types. There are two kinds of atomic type: *free type-variables* and *bound type-variables*; and denumerably many of each. $\alpha, \beta, \gamma, \delta, \alpha_0, \alpha_1, \ldots$ range over the former and $\phi, \psi, \chi, \phi_0, \phi_1, \ldots$ over the latter. “Type”, “semi-type” and “VarF” are defined simultaneously.

Definition 1.1 (i) Every type-variable, free or bound, is a semi-type. $\mathrm{VarF}(\alpha) = \{\alpha\}$. $\mathrm{VarF}(\phi) = \{\phi\}$.

(ii) If A and B are semi-types, so is $A \to B$. $\mathrm{VarF}(A \to B) = \mathrm{VarF}(A) \cup \mathrm{VarF}(B)$.

(iii) If A is a semi-type and $\phi \in \mathrm{VarF}(A)$, $\forall\phi(A)$ is a semi-type. $\mathrm{VarF}(\forall\phi(A)) = \mathrm{VarF}(A) - \{\phi\}$.

(iv) $\mathrm{VarF}(A)$ shall be read as “the set of (free or bound) variables occurring free in A”.

(v) A type is a semi-type in which no bound variable occurs free.

Definition 1.2 (*Substitution*) Types can be thought of as formulae of second-order propositional logic (cf. [20, p. 345f]). Substitution of the semi-type C for the free variable α within the type A can therefore be defined in the usual way and shall be denoted with “$A({}^{\alpha}{}_{C})$”.

As it will generally be clear when two different substitutions are substitutions for the same variable, the notation can be simplified. For example, if $A(\alpha)$ and $A(C)$ are used in the same context, this means that, for some $\beta \in \mathrm{VarF}(A)$, $A(\alpha)$ is $A({}^{\beta}{}_{\alpha})$ while $A(C)$ is $A({}^{\beta}{}_{C})$.

Definition 1.3 (*t-variables*) A second kind of variable are the *term-variables*; henceforth *t-variables*. For every type A there are denumerably many *t-variables of type A*, over which $x^A, y^A, z^A, w^A, x_0^A, x_1^A, \ldots$ will range.

Definition 1.4 (**CL**-*terms*) A **CL**-term is built up from t-variables and certain constants (see below). $M, N, P, Q, M_0, M_1, \ldots, X_0, X_1, \ldots, Y$ shall range over **CL**-terms in general and $a, b, c, \ldots$ over atomic **CL**-terms. ‘$\equiv$’ shall denote identity between syntactic entities (terms and their types).

The following clauses (i)–(viii) define simultaneously both the concept of a **CL**-term and a function denoted by *type* which takes any **CL**-term M to a type, called the “type of M”.

[3] For a more expansive discussion of weak and strong reduction, their respective properties and various ways of defining strong reduction, see [6,7].

(i) Any t-variable x^A is a **CL**-term. $type\left(x^A\right) \equiv A$.
(ii) For any types A, B, C, $\mathbf{S}_{ABC}$ is a **CL**-term. $type(\mathbf{S}_{ABC}) \equiv (A \to B \to C) \to (A \to B) \to A \to C$.
(iii) For any types A, B, $\mathbf{K}_{AB}$ is a **CL**-term. $type(\mathbf{K}_{AB}) \equiv A \to B \to A$.
(iv) For any type A, $\mathbf{I}_A$ is a **CL**-term. $type(\mathbf{I}_A) \equiv A \to A$.
(v) For any types $A(\alpha)$ and C, and any variable ϕ not occurring in $A(\alpha)$, $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}$ is a **CL**-term. $type(\mathbf{U}^{\forall\phi A(\phi)\to A(C)}) \equiv \forall\phi A(\phi) \to A(C)$.
(vi) For any type $F(\alpha)$ and variable ϕ which does not occur in $F(\alpha)$, $\Pi_{F(\alpha),\alpha,\phi}$ is a **CL**-term. $type(\Pi_{F(\alpha),\alpha,\phi}) \equiv F(\alpha) \to \forall\phi F(\phi)$.
(vii) For every variable α, the class Cr_α of α-*critical* terms is defined inductively by four clauses:
  (1) $\Pi_{F(\alpha),\alpha,\phi} \in \mathrm{Cr}_\alpha$, for every $F(\alpha)$ and ϕ;
  (2) If M is an α-critical term built up by clauses (1)–(3) only, if $M \equiv \mathbf{K}_{(B\to C)A}(M_0)$ for some M_0 so that $type(M) \equiv A \to B \to C$, then $\mathbf{S}_{ABC}M \in \mathrm{Cr}_\alpha$, provided $\alpha \notin \mathrm{VarF}(A)$;
  (3) If M is an α-critical term built up by clauses (1)–(3) only, if $type(M) \equiv B$ and the last clause applied is (1) or (2), then $\mathbf{K}_{BA}M \in \mathrm{Cr}_\alpha$, provided $\alpha \notin \mathrm{VarF}(A)$;
  (4) If $\mathbf{K}_{AB}M \in \mathrm{Cr}_\alpha$ and $\alpha \notin \mathrm{VarF}\,(type\,(x))$ for any t-variable x in N, $\mathbf{K}_{AB}MN \in \mathrm{Cr}_\alpha$.
(viii) For any **CL**-terms M and N and any types A and B, if $type(M) = A \to B$ and if $type(N) = A$, then (MN) is a **CL**-term provided the following conditions are satisfied, for every α:
  (a) If $N \in \mathrm{Cr}_\alpha$, $MN \in \mathrm{Cr}_\alpha$ too, in virtue of clause (vii) (2)–(3).
  (b) If $M \in \mathrm{Cr}_\alpha$, then $\alpha \notin \mathrm{VarF}(type(x))$ for any t-variable x which occurs in N.
  (c) If M is α-critical and not identical to $\mathbf{K}_{AB}X$ for any X, $\alpha \notin \mathrm{VarF}(type(MN))$.
  When these conditions hold, $type(MN) \equiv B$. As usual, outermost brackets are omitted.
(ix) X shall be *independent* if $X \notin \mathrm{Cr}_\alpha$ for every α and, if X has an α-critical subterm, $\alpha \notin \mathrm{VarF}(type(y))$, for every variable y in X, and $\alpha \notin \mathrm{VarF}(type(X))$.

Remark regarding the conditions in clause (viii), see the discussion following Proposition 2.11. Letters ranging over constants, i.e. "**S**", "**K**", "**I**", "**U**", "Π", will often be written without type super- or subscripts. An expression like "$\mathbf{S}(\mathbf{K}(\mathbf{KI}))(\mathbf{KU}^{\forall\psi F(\psi)\to F(D)})$" should then be understood as ranging over all possible **CL**-terms of the shape indicated.

Corollaries *(i) for every* **CL***-term M, type(M) is uniquely determined. This is proved by induction on the construction of terms; (ii) every term M which is not α-critical for any α can be transformed into an independent term merely by changing some variables which do not occur in type(M). Hence we can assume without loss of generality that every* **CL***-term is either independent or α-critical for some α; (iii) if M is independent and N is an α-critical subterm of M, N is either a subterm of another α-critical subterm of M or else N occurs in the context NP, for some P.*

Definition 1.5 (i) **CL**-terms of the kind introduced by Definition 1.4 (v) resp. (vi) will be called **U**-terms (mnemonically, the letter "U" stands for "universal instantiation") resp. Π-terms.
(ii) $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}$ shall be called *simple* just in case the semi-type $A(\phi)$ contains no quantifiers.
(iii) A ***BI**-term* is a **CL**-term within which any **U**-term occurring is simple.
(iv) Type-substitution shall also be thought of as an operation performed upon terms as well as types and similar notation used. Thus "$M(^{\alpha}_{C})$" shall denote the result of substituting C for the variable α throughout all atomic subterms of M.
(v) $[N/x]M$ shall denote the result of substituting N for the t-variable x within M.

Corollary $type(M(^{\alpha}{}_{C})) \equiv (type(M))(^{\alpha}{}_{C})$.

2 Reduction of terms

Definition 2.1 (*The mapping* $M \mapsto [x]M$) This is defined using the algorithm (abcf) of [6, p. 190]; cf. also [7, p. 43].

Definition 2.2 (i) The *level* of a semi-type A, $l(A)$, is defined by the clauses:

$$l(A) = 0 \text{ if } A \text{ does not have } \to \text{ as its main connective.}$$
$$l(B \to C) = \max\{l(B) + 1, l(C)\}.$$

(ii) The level of a **CL**-term M, $l(M)$, is $l(type(M))$.
(iii) A *supplementation* of a term M shall be any term of the shape $Mx_1x_2\ldots x_n$ $(n > 0)$ having level 0, where $x_1, x_2, \ldots, x_n$ are t-variables not occurring in M.
(iv) A subterm N of M occurs *in head position* within M (or *is a head* of$M$) iff either $N \equiv M$ or, for some terms $P_1, P_2, \ldots P_n$, $M \equiv NP_1P_2\ldots P_n$.
(v) The class Ir of (*strongly*) *irreducible terms* is defined inductively.
  (a) If M is a t-variable, $M \in$ Ir.
  (b) If $M \in$ Ir, $[x]M \in$ Ir, provided either $[x]M$ is a **U**-term or $[x]M$ is not obtainable from M by the algorithm (c) (Definition 2.1) alone.
  (c) If ΠQ is a **CL**-term and $Q \in$ Ir, then $\Pi Q \in$ Ir.
  (d) If PQ is a **CL**-term,$P \in$ Ir, $Q \in$ Ir and P has a variable or a **U**-term in head position (but is not itself a **U**-term), then $PQ \in$ Ir.
  (e) If $\mathbf{U}Q$ is a **CL**-term,$Q \in$ Ir and Q does not have a Π-term in head position, $\mathbf{U}Q \in$ Ir.

Definition 2.3 The notions of a *J-term* and of a *λ-bound t-variable* are defined simultaneously.

(i) Every **CL**-term is a J-term in which no t-variable is λ-bound.
(ii) If M is an independent J-term in which x^A is not λ-bound, $\lambda x^A(M)$ is a J-term in which the λ-bound t-variables are x^A plus those that are in λ-bound in M. $type(\lambda x^A(M)) \equiv A \to type(M)$.

(iii) If M and N are J-terms related as in Definition 1.4 (viii), then (MN) is a J-term in which the only variables λ-bound are those λ-bound in either M or N.
(iv) The *body* of a J-term is the result of stripping away any λ-binding variables at the beginning.
(v) A *reducible* J-term is any J-term which is not an irreducible **CL**-term (Definition 2.2 (v)).
(vi) A subterm N of a J-term M is *in head position* within M iff N is in head position within the body of M.
(vii) A J_0-*term* is a J-term which either (a) is a **CL**-term or (b) is a proper J-term whose body is a **CL**-term of level 0.

Definition 2.4 (*Redexes and corresponding contracta*) Certain J-terms are identified as (strong) redexes[4] and a contractum associated with each one.

(i) A *type I redex* or *weak redex* is a J-term of the shape $\mathbf{S}_{ABC}MNP$, $\mathbf{K}_{AB}MN$, $\mathbf{I}_A M$, $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}M)$ provided (in the last case) $M \in$ Ir. These four kinds of redex have associated *contracta*, viz.: $MP(NP)$, M, M, $M(^{\alpha}{}_{C})$.
(ii) An *open term* is a J-term which does not have a weak redex in head position, while some supplementation of it does, e.g. $\mathbf{S}_{ABC}MN$.
(iii) If a J-term M satisfies the following conditions: (a) $type(M) \equiv A_1 \to A_2 \to \ldots \to A_n \to B$ for some $A_1, A_2, \ldots A_n$ $(n > 0)$ and some B where $l(B) = 0$; (b) M is a reducible open term, then M is a *type II redex*. Its contractum is $\lambda x_1^{A_1} x_2^{A_2} \ldots x_n^{A_n}(Mx_1^{A_1}x_2^{A_2}\ldots x_n^{A_n})$, $x_1^{A_1}, x_2^{A2}, \ldots x_n^{A_n}$ being the first n t-variables in some enumeration of the t-variables of types $A_1, A_2, \ldots A_n$ that are neither free nor λ-bound in M.
(iv) A *type III redex* is any J-term of the shape $\lambda x_1 x_2 \ldots x_n(N)$ $(n > 0)$ where $N \in$ Ir. The contractum of $\lambda x_1 x_2 \ldots x_n(N)$ is $[x_1, x_2, \ldots x_n]N$.

Corollaries *A redex has the same type as its contractum. No redex has more than one contractum.*

Definition 2.5 For every J-term X, there is an atom a and J-terms $U_1, U_2, \ldots U_n (0 \leq n)$ so that $X \equiv \lambda x_1 x_2 \ldots x_m a U_1 U_2 \ldots U_n (0 \leq m)$.

(i) Within X, the subterms $a, U_1, U_2, \ldots U_n$ shall be *components of rank 0*.
(ii) Let $\lambda y_1 y_2 \ldots y_q b V_1 V_2 \ldots V_p$ be a component of rank k within X. Then $b, V_1, V_2, \ldots V_p$ shall be *components* of X *of rank* k+1.
(iii) A *component* of X is a subterm of X which is a component of rank k, for some k.
(iv) Within X resp. $bV_1V_2 \ldots V_p$, a resp. b shall be the *head atom*.

Corollary *Not all subterms are components, e.g. xy is a subterm, but not a component, of (xy)z.*

Definition 2.6 For any property π defined over J-terms, the *leftmost subterm* of $\lambda x_1 x_2 \ldots x_m a U_1 U_2 \ldots U_n$ *which has* π is picked out as follows.

4 For an explanation of why strong reduction can be defined like this, see [6, p. 222f].

(i) If $\lambda x_1 x_2 \ldots x_m a U_1 U_2 \ldots U_n$ itself has π, then $\lambda x_1 x_2 \ldots x_m a U_1 U_2 \ldots U_n$ is the leftmost subterm of itself having π.

(ii) If $\lambda x_1 x_2 \ldots x_m a U_1 U_2 \ldots U_n$ does not itself have π but one of the heads $a, aU_1, aU_1U_2, \ldots$ has π, then the smallest head to have π is also the leftmost subterm of $\lambda x_1 x_2 \ldots x_m a U_1 U_2 \ldots U_n$ that has π.

(iii) Otherwise, let i $(1 \leq i \leq n)$ be the smallest number such that, for some N, N is the leftmost subterm of U_i that has π. Then N is the leftmost subterm of $\lambda x_1 x_2 \ldots x_m a U_1 U_2 \ldots U_n$ that has π.

Definition 2.7 (i) $\boldsymbol{f}_1$ shall be a function that counts the number of sequences of contiguous λ-binding variables $\lambda x_1 x_2 \ldots . x_m$ within a J-term.

(ii) For any J-term M which contains a type III redex, $\boldsymbol{f}_2(M)$ shall be the J-term which results from replacing the leftmost type III redex within M with its contractum.

(iii) For any J-term M containing only type III redexes, $\boldsymbol{f}_3(M)$ shall be that J-term, free of type III redexes, to which M is reduced by repeatedly replacing the leftmost redex with its contractum.

Remark The existence of simple (and certainly primitive recursive) functions having the properties of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ should be obvious. $\boldsymbol{f}_3$ is then defined from $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ by primitive recursion. Contracting a type III redex within M reduces $\boldsymbol{f}_1(M)$ by 1; hence it is obvious that at most $\boldsymbol{f}_1(M)$ contractions of type III redexes are needed to take M to a new term free of type III redexes.

Definition 2.8 A subterm N of a J-term M is called an *active redex within M* iff one of the following holds.

(i) There are redexes in M which are of type I or II and, of these, N is the leftmost.

(ii) No redexes in M are of type I/II, while there is at least one subterm $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi} Q)$ such that $\boldsymbol{f}_3(Q) \in \mathrm{Ir}$; and N is the leftmost type III redex in the leftmost subterm of that kind.

(iii) No redex in M is active in the sense of (i) or (ii); and N is the leftmost type III redex in M.

Corollary *No J-term contains more than one active redex.*

Proposition 2.9 *The following conditions are all equivalent:*

(i) $M \in$ *Ir.*

(ii) *M contains no active redex.*

(iii) *M contains no redexes.*

Proof That (i) implies (ii) and (iii) is easily seen by induction on the construction of Ir (Definition 2.2 (v)). (iii) implies (ii) because an active redex is a redex. Proving that (ii) implies (i) is harder, but follows from the fact that every J-term M has one of the following shapes: (a) M has a combinator in head position and is reducible; (b) M has a combinator in head position and is irreducible; (c) M has a variable in head position; (d) M has a Π-term in head position; (e) a subterm of the shape $\mathbf{U}^{\forall\phi A(\phi)\to A(C)} X_1$ is in head position, $X_1 \in \mathrm{Ir}$ and X_1 has a Π-term in head position; (f) like (e), $X_1 \in \mathrm{Ir}$, but X_1 does not have a Π-term in head position; (g) like (e) and (f) but $X_1 \notin$ Ir. These seven cases are treated separately assuming by way of I.H. that the proposition holds for all proper subterms of M. In case (a) M either has a weak redex in head position or is open and, being reducible by stipulation, constitutes a type II redex. □

Remark This strategy of proving a theorem by enumerating the seven possible shapes a J-term can have will be used in the following without explicit advertisement, eg in proving Proposition 4.4.

Definition 2.10 (i) $P \rhd_1 Q$ shall mean that Q is the result of contracting the active redex in P.

(ii) A *reduction* (*sequence*) is a sequence $P_0, P_1, P_2, \ldots$ of J-terms in which P_0 is an independent **CL**-term and, for every i such that P_i is nonterminal, $P_i \rhd_1 P_{i+1}$.

(iii) A *complete reduction* is a reduction which either continues *ad infinitum* or else has a last element M which is irreducible; and no element of the reduction other than M is irreducible.

(iv) $P \rhd Q$ holds iff there is a reduction sequence with P and Q as its first and last elements.

(v) If a complete reduction starting with P has a last element, we call it the *normal form* of P.

Remark Since reductions are determinate, a term can have at most one normal form. The strategy of reducing a term by replacing, at every stage, the *active* redex with its contractum was chosen for reasons that will be explained following the proof of Theorem 2 in Sect. 6.[5]

Proposition 2.11 *(i) For any independent* **CL**-*term M which contains no reducible α-critical subterms and any t-variable x, $[x]M$ is a* **CL**-*term with these same properties.*

(ii) If $P \rhd_1 Q$ and P is an independent **CL**-*term, then so is Q.*

Proof (i) is proved by induction on the construction of M. It is obvious when $M \equiv x$ or when M does not contain x. If $M \equiv Nx$ when N does not contain x, $[x]\,M \equiv N$ is independent, as otherwise Nx would violate either condition (b) or condition (c) of Definition 1.4 (viii), in view of the fact that α occurs within the type of any α-critical term.

Suppose now $M \equiv M_0M_1$ and none of the foregoing holds. M_1 cannot be α-critical, as M would not then be independent, in view of Definition 1.4 (viii) (a). When M_0 is independent, $[x]M_0$ and $[x]M_1$ are independent by I.H. and therefore so is $[x]\,M \equiv \mathsf{S}\,([x]\,M_0)\,([x]\,M_1)$. If M_0 is α-critical, for some α, $[x]M_0 \equiv \mathsf{K}M_0$ for some M_0 because, since M_0 is a well-formed α-critical term which contains no reducible α-critical subterms, it can contain no variables. But then $[x]M_0 \equiv \mathsf{K}M_0$ is α-critical too by 1.4 (vii) (3); and the combination $\mathsf{S}([x]M_0)$ satisfies both 1.4 (vii) (2) and 1.4 (viii) (a), because $\alpha \notin \mathrm{VarF}(type(x))$. Condition 1.4 (viii) (b) is satisfied as well because M_0 is α-critical and M_0M_1 must satisfy 1.4 (viii) (b). Let $type([x]M_0) \equiv A \to B \to C$.

[5] The reduction sequences defined here, generated by contracting the *active* redex at every stage, are akin to the *normal reductions* defined in [6, p. 226f].

We have established $\alpha \notin \mathrm{VarF}(A)$; we also have $\alpha \notin \mathrm{VarF}(C)$ because $M_0 M_1$ satisfies 1.4 (viii) (c). Because $type(\mathbf{S}([x]M_0)([x]M_1)) \equiv A \to C$, the last term satisfies condition 1.4 (viii) (c) too.

(ii) is obvious when the redex contracted is of type II and follows from (i) when it is of type III. Let the active redex in P be $\mathbf{S}_{ABC}LMN$ resp. $\mathbf{K}_{AB}LM$ resp. $\mathbf{U}^{\forall\phi A(\phi)\to A(\mathrm{C})}(\Pi_{A(\alpha),\alpha,\phi}M)$. There is no difficulty when none of the components L, M, N is α-critical while, by 1.4 (vii)–(viii), at most $\mathbf{S}_{ABC}L$, $\mathbf{K}_{AB}LM$ and L can be α-critical while $\Pi_{A(\alpha),\alpha,\phi}$ necessarily is. Only the case of the redex $\mathbf{S}_{ABC}LMN$ raises any difficulty. If $\mathbf{S}_{ABC}L$ is α-critical, L must, by 1.4 (vii) (2), be both irreducible and identical to $\mathbf{K}Y$ for some Y. $\mathbf{K}YN$ is therefore also α-critical by 1.4 (vii) (4). On the other hand $\alpha \notin \mathrm{VarF}(type(\mathbf{S}_{ABC}LMN))$ from which follows that the contractum $LN(MN)$ satisfies condition 1.4 (viii) (c). Satisfaction of 1.4 (viii) (b) follows in every case from the fact that P is an independent term. $LN(MN)$ is therefore well-formed. That independence is preserved by reduction is obvious. ◻

Discussion: Proposition 2.11 (i) shows that, despite the restrictions in Definition 1.4 (viii) on the formation of **CL**-terms, at least the class of independent **CL**-terms which contain no reducible α-critical subterms is combinatorially complete, while every other independent **CL**-term can be easily reduced to such a term by contracting all α-critical subterms of the shape $\mathbf{K}MN$. Part (ii) shows that the same class is closed under reduction; so we can freely reduce independent terms and apply the operation $M \mapsto [x]\,M$ without worrying about these restrictions. Completeness of another sort is established by Proposition 3.2 below.

3 Second-order implicational logic: BI-terms

The formulae of second-order intuitionist implicational logic, $\to\forall^2$**Nip**^{2} [20, p. 345f.] are just the polymorphic types. $\to\forall^2$**Nip**^{2} can be formalized by (among other possibilities) the four natural deduction rules ∀I, ∀E, →I, →E [20, p. 345f.].

Definition 3.1 (i) For any set Γ of types and any type A, $\Gamma \Rightarrow A$ shall be a *sequent*.

(ii) $M \vdash \Gamma \Rightarrow A$ (read as "M proves $\Gamma \Rightarrow A$") shall hold just in case M is independent (in the sense of Definition 1.4 (ix)), $type(M) \equiv A$ and Γ includes all types of t-variables occurring in M.

(iii) A sequent is ***CL**-derivable* (resp. ***BI**-derivable*) just in case some **CL**-term (**BI**-term) proves it.

(iv) ***NBI***$^{\to i}$ shall be the result of restricting $\to\forall^2$**Nip**^{2} so as to permit ∀E only when the premiss-formula contains only one quantifier.[6]

[6] Takeuti [16] investigated a second-order *sequent* calculus characterized by a similar restriction on the second order ∀ (left) rule, viz.: when an *implicit* inference is made in accordance with this rule, the main formula may contain only one second-order quantifier. *LBI* is the name Arai gave to this fragment of second-order logic [2]. The nomenclature "**BI**-term" is derived from this.

Proposition 3.2 *(i) Every sequent provable in* $\rightarrow\forall^2$**Nip**^{2} *is* **CL**-*derivable.*
(ii) Every sequent provable in **NBI**$^{\rightarrow i}$ *is* **BI**-*derivable.*

Proof of (i): For any formula A, $x^A \vdash A \Rightarrow A$. Closure under ∀I and ∀E is easily shown using Π-terms and **U**-terms. If $M \vdash \Gamma \Rightarrow B$, then $\left[x^A\right] M \vdash \Gamma - \{A\} \Rightarrow A \rightarrow B$. That $[x^A]M$ is a **CL**-term was established by Proposition 2.11 (i). The class of **CL**-derivable sequents is also closed under →E, because the result of applying one *independent* **CL**-term to another is always itself an independent **CL**-term, by Definition 1.4 (viii)–(ix). □

The implications of 3.2 (ii), when combined with Theorem 2 below, are discussed in Sect. 8.

As anticipated in the introduction, the main goal of this paper is to show that there is an ε_0-recursive function which gives a bound to the lengths of reduction sequences commencing with a **BI**-term. Restriction to **BI**-terms is motivated by the facts that the full class of **CL**-terms is far too complicated to be amenable to ordinal-theoretic treatment at the present time, while the **BI**-terms are both more tractable and quite interesting, as the following definition and theorem show.

Definition 3.3 (i) For any natural number n, the term $\Pi_{(\alpha\rightarrow\alpha)\rightarrow\alpha\rightarrow\alpha,\alpha,\phi}$ $\{[y^{\alpha\rightarrow\alpha}, x^{\alpha}]M\}$, where M is $y^{\alpha\rightarrow\alpha}(y^{\alpha\rightarrow\alpha}\ldots(y^{\alpha\rightarrow\alpha}x^{\alpha})\ldots)$ with n occurrences of $y^{\alpha\rightarrow\alpha}$, shall be called the nth *Church-Girard numeral*, henceforth $[\![n]\!]$.

(ii) The condition for an m-place function F that takes m-tuples of natural numbers to natural numbers to be *combinatorially defined* by a **CL**-term M, relative to the Church-Girard numerals, shall be defined in the standard way. To wit, M defines F when, for every m-tuple $\langle n_1, \ldots n_m\rangle$ of natural numbers, $F(n_1, \ldots n_m) = n_{m+1}$ just when $M[\![n_1]\!][\![n_2]\!]\ldots[\![n_m]\!] \rhd [\![n_{m+1}]\!]$.

(iii) Clearly, every Church-Girard numeral has type $\forall\phi((\phi \rightarrow \phi) \rightarrow \phi \rightarrow \phi)$. Let this be abbreviated to I.

(iv) An *I-type* shall be a type built up from I by the operation $A, B \mapsto A \rightarrow B$.

Theorem 1 *Every prf is combinatorially defined by a* **BI**-*term, relative to Church-Girard numerals.*

Sketch of proof: The basic idea is both simple and well-known. Let A be an I-type; then the term $\mathbf{U}^{I\rightarrow(A\rightarrow A)\rightarrow A\rightarrow A}$ functions as an *iterator of type* A in the sense defined in [14, pp. 98–100]. That is, for any natural number n and any terms M, N of types $A \rightarrow A$ resp. A, $\mathbf{U}^{I\rightarrow(A\rightarrow A)\rightarrow A\rightarrow A}[\![n]\!]MN \rhd M(M\ldots(MN)\ldots)$ (with n occurrences of M). $\mathbf{U}^{I\rightarrow(A\rightarrow A)\rightarrow A\rightarrow A}$ is moreover simple because I contains only a single quantifier. □

While it is not the case that every term of a reduction commencing with a **BI**-term is itself a **BI**-term, such reductions nonetheless have certain pleasant properties that makes their ordinal analysis far simpler than that of arbitrary reductions. To uncover these properties is the goal of Sect. 4.

4 Behaviour of U-terms in reductions: marked U-terms

Most of the results of this section concern arbitrary J-terms; but the final goal (Proposition 4.2) is to establish a property enjoyed by just those J-terms that have a **BI**-term in head position.

Definition 4.1 (i) The notion of an *occurrence* of a J-term M within a J-term N will be assumed as familiar. If desired, a precise definition could be given, e.g. it is a triple $\langle M, n, N\rangle$ where n is a natural number that indexes a node within the construction tree for N (cf. [7, p. 16]).

(ii) Let $P \rhd Q$ and let # be an occurrence within P of a subterm N. A *footprint* of # in Q shall be an occurrence of N within Q, the concept of a footprint being defined like the standard concept of a residual [7, p. 29], but more general, in that N does not need to be a redex.

Heuristic discussion: the purpose of this section can be explained as follows. Let $N_0, N_1, \ldots$ be a reduction. Then occurrences of **U**-terms in N_0 divide into two classes:

(A) {#: # is an occurrence of a **U**-term in N_0 and a footprint of # occurs in N_i, for some i, as the head of a weak redex}

(B) {#: # is an occurrence of a **U**-term in N_0 and no footprint of # occurs in N_i, for any i as the head of a weak redex}

Ordinal numbers will be assigned to **BI**-terms in the next section; these will be either finite or infinite. A head of N_0 shall have a finite ordinal assigned to it only if it contains no occurrences of **U**-terms belonging to class (A). Moreover the proof presented in Sect. 5 below requires that all irreducible **BI**-terms shall have finite ordinals. This means, since it cannot be expected that an irreducible **BI**-term will contain no **U**-terms at all, it is necessary to prove:

Proposition 4.2 *Let* $\mathbf{U}^{\forall\phi A(\phi)\rightarrow A(C)}\left(\Pi_{A(\alpha),\alpha,\phi} M(^{\beta}{}_{\alpha})\right)$ *be a* **BI**-*term with* $M \in Ir$ *and let* N_0*, the first term in a reduction, be* $\mathbf{U}^{\forall\phi A(\phi)\rightarrow A(C)}(\Pi_{A(\alpha),\alpha,\phi} M(^{\beta}{}_{\alpha}))L_1L_2 \ldots L_r$*, for some* $L_1, L_2, \ldots L_r$ $(0 < r)$*. Then, in the terminology of the last paragraph, all occurrences of* **U***-terms in* $M(^{\beta}{}_{\alpha})$ *belong to class (B).*

Historically, the distinction between two kinds of occurrences of **U**-terms corresponds at least remotely to the distinction drawn in [16] between *implicit* and *explicit* inferences which introduce second-order ∀ (left) in a sequent derivation. A sequent derivation has a transfinite ordinal number assigned to it if and only if it contains implicit such inferences. A redex of the shape $\mathbf{U}^{\forall\phi A(\phi)\rightarrow A(C)} P$ within a **CL**-term has a function somewhat analogous to that of a cut, where the cut formula has second-order ∀ as its main connective, within a second-order sequent derivation.

The simplest known proof of Proposition 4.2, presented here, makes use (in Proposition 4.4) of the fact that every reduction sequence terminates.[7] This can be proved using the Tait–Girard computability predicates (see [3] and the appendix below). Because of this, every J-term can be thought of as the conclusion of what will be called an

[7] Proofs of Proposition 4.2 that do not use this premiss are indeed known, but are more complicated.

expansion tree, this having the property that if a redex Y is contracted in the reduction sequence beginning with X, then Y occurs *in head position* at some point in the expansion tree which generates X. If Proposition 4.2 were false, it would be possible that a footprint of a **U**-term in $M(^{\beta}{}_{\alpha})$ should occur higher up in the tree as the head of a redex. But it can be shown that this is impossible.

Definition 4.3 An *expansion tree for M* shall be a tree made up of J-terms with M at the bottom. Every initial term in the tree is irreducible. Every other term is derived from the term(s) immediately above it by one of the following rules of inference.

$$\text{(I)}\quad \frac{\lambda x_1 x_2 \ldots x_m\,(X Y_1 Y_2 \ldots Y_n)}{\lambda x_1 x_2 \ldots x_m (X' Y_1 Y_2 \ldots Y_n)}$$
where $0 \leq n, 0 \leq m$, X' is a weak redex and $X' \rhd_1 X$

$$\text{(II)}\quad \frac{X}{\lambda x_1 x_2 \ldots x_m Y}$$
where $X \equiv [x_1, x_2, \ldots, x_m]\, Y$, $Y \in \text{Ir}$ and $l\,(Y) = 0\,(0 < m)$

$$\text{(III)}\quad \frac{\lambda x_1 x_2 \ldots x_n\,(X x_1 x_2 \ldots x_n)}{X}$$
X is an open **CL**-term, $l\,(X x_1 x_2 \ldots x_n) = 0$, $X \notin \text{Ir}\ \ (0 < n)$

$$\text{(IV)}\quad \frac{\lambda x_1 \ldots x_m (\mathbf{U}^{\forall\psi F(\psi)\rightarrow F(D)}(\Pi Q) Y_1 \ldots Y_n)\ \ P}{\lambda x_1 \ldots x_m (\mathbf{U}^{\forall\psi F(\psi)\rightarrow F(D)} P Y_1 \ldots Y_n)}$$
provided $Q \in \text{Ir}$, $P \notin \text{Ir}$ and $P \rhd \Pi Q$

$$\text{(V)}\quad \frac{X_1, X_2, \ldots, X_n}{\lambda x_1 x_2 \ldots x_m\,(y X_1 X_2 \ldots X_n)}$$
$(0 < n, 0 \leq m)$

$$\text{(VI)}\quad \frac{X_1, X_2, \ldots, X_n}{\lambda x_1 \ldots x_m (\mathbf{U}^{\forall\phi A(\phi)\rightarrow A(C)} X_1 \ldots X_n)}$$
where $0 < n, 0 \leq m$ and for no V does $X_1 \rhd \Pi V$ hold

$$\text{(VII)}\quad \frac{X}{\lambda x_1 x_2 \ldots x_m (\Pi X)}$$
$(0 \leq m)$

To ensure each term has a unique tree, it is necessary to stipulate that the premisses of rules (V)–(VII) may not all be irreducible. An instance of rule (I) shall be called an inference by **S**-expansion, **K**-expansion, **I**-expansion or **U**-expansion, depending on what atomic term stands at the head of X'.

Inspection of the rules shows that, in an instance of (V)–(VII), there corresponds to every premiss X_i a component X_i in the conclusion. In any instance of (I)–(IV) there are, corresponding to most (though not all) components in any premiss, necessarily identical components in the conclusion, eg Y_i $(1 \leq i \leq n)$ in (I) and (IV). These occurrences in the conclusion will now be called *immediate descendants* of those in the premiss and the latter *immediate ancestors* of the former. The more general concepts *descendant* and *ancestor* are defined in the obvious way.

Proposition 4.4 *Every J_0-term X is the conclusion of exactly one expansion tree; and it is composed of J_0-terms.*

Proof uniqueness is easy (cf. Proposition 2.9). For existence, we use the fact that X has a normal form and therefore the reduction starting with X has a finite length, say n. The proposition is proved by induction on n and, within that, on the construction of X. If $n = 1$, X is irreducible and belongs to a singleton expansion tree. If X has a redex in head position, the proposition follows immediately from the main induction hypothesis. Otherwise, X must have a variable, a Π-term or a $\mathbf{U}$-term in head position and all the rank 0 components of X will be **CL**-terms, from which follows, by the induction hypotheses, that they will be conclusions of expansion trees. From these components (other than the head atom), X itself can be derived by one of (IV)–(VII). □

Remark Since our main interest is in expansion trees terminating with a **BI**-term, and since Proposition 4.4 shows that every subtree of such a tree will terminate with a J_0-term, it will be necessary from now on to consider only expansion trees terminating with a J_0-term.

Proposition 4.5 *Let M be a J_0-term, N a redex that is contracted in the reduction of M and N' the contractum of N. Then, in the expansion tree τ which generates M, there is a term Y having N in head position while the term immediately above Y has N' in head position.*

Proof By induction on the height of τ. As in 2.9, cases are distinguished regarding the possible shape of M. In cases (a) and (e), M is either itself a redex or has a weak redex M_0 in head position, M resp. M_0 is the active redex and M is a conclusion of (I), (II) or (III). Moreover, if L is the term immediately above M, $L$is either the contractum of M or has the contractum of M_0 in head position. So if N is M or M_0, the proposition holds. If not, N is contracted in the reduction of L and the I.H. applies. In cases (c), (d), (f) and (g), M is a conclusion of rule (V), (VII), (VI) or (IV) respectively. Then N is contracted in reducing one of the terms immediately above M in τ and the I.H. applies. □

Definition 4.6 An atom c shall be said to *protect* an occurrence \$ of a $\mathbf{U}$-term within a J-term N iff one of the conditions (i)–(iv) holds:

(i) c is a variable or $\mathbf{U}$-term, \$ occurs in P_j for some $j \leq n$ and $cP_1P_2...P_n$ is a subterm of N. Moreover, if c is a $\mathbf{U}$-term, $cP_1 \in \mathrm{Ir}$.
(ii) $\$(cP_1P_2\ldots P_n)$ is a subterm of N and c is either a variable or a $\mathbf{U}$-term such that $cP_1 \in \mathrm{Ir}$.
(iii) For some term Q, c protects \$ in Q and $N \equiv [x]Q$ for some $x \not\equiv c$.
(iv) For some term Q, c protects \$ in $Q({}^{\beta}{}_{C})$ and $N \equiv \mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}\, Q({}^{\beta}{}_{\alpha}))$.

(v) If c protects \$ in N, then it does so in virtue of one of the conditions (i)–(iv).

Remark Especially in Proposition 4.8, *irreducible* terms N within which \$ is protected will be of special interest and when considering such terms it is useful to think of them as built up by the five clauses of Definition 2.2 (v), adapted to ensure that at least one of (i)–(iii) above also holds.

Definition 4.7 For any **CL**-term M, we shall say that:

(i) M is *essentially quantified* if there is no term M_0 such that *type*(M_0) is quantifier-free and M is obtained from M_0 by type-substitution.
(ii) M is *essentially quantified outside its final subtype* iff there is no term M_0 of type $C_0 \to C_1 \to \cdots C_n \to D$ such that $C_0, C_1, \ldots C_n$ are quantifier-free, $l(D) = 0$ and M is derived from M_0 by type-substitution.
(iii) M is *essentially quantified in its antecedent* iff there is no term M_0 of type $C \to D$ such that C is quantifier-free and M is derived from M_0 by type-substitution.

Proposition 4.8 *(i) If M is an irreducible term in which there is an unprotected occurrence \$ of some* **U***-term, then M is essentially quantified outside its final subtype.*
(ii) If x is the only atom in N that protects \$, then $[x]N$ is essentially quantified in its antecedent.

Proof (i) and (ii) are proved simultaneously by induction on the build-up of M (Definition 2.2 (v)). When the operation $X \mapsto [x]X$ (Definition 2.1) is used in constructing M, cases are distinguished according to whether or not an atom other than x protects \$ in M. Let $M \equiv [x]N$. If there is an atom in N other than x which protects \$, the same atom also protects \$ in M. If there is no such atom, cases are distinguished according to the shape of N. N cannot be atomic, so suppose $N \equiv N_0 N_1$. There are three possibilities: (A) \$ occurs in N_i $(i = 0, 1)$ and is protected by x within N_i. Then (ii) holds for $[x]N_i$ by the induction hypothesis and hence for $[x]N$. (B) $N_0 \equiv xP_0P_1 \ldots P_j$ while \$ is in N_1. Then $M \equiv \mathbf{S}[x](xP_0P_1 \ldots P_j)[x]N_1$ and, by I.H. of (i), $type([x]N_1)$ is essentially quantified irrespective of whether x occurs in N_1 or not. Inspection of N shows that, if *type*$(x) \equiv A_1 \to A_2 \to \cdots \to A_n \to B$, where $l(B) = 0$, then $type(N_1) \equiv A_i$ for some i. So $[x]N_1$ is essentially quantified in its antecedent, even if x does not protect \$ in N_1, and the proposition follows from the fact that the antecedents of $type(M)$ and of $type([x]N_1)$ coincide. (C) $N_0 \equiv$ \$ while$N_1 \equiv xP_0P_1 \ldots P_j$. The proposition now follows from the fact that \$ is essentially quantified in its antecedent. □

Remark This proposition would fail if the notion of reduction under consideration were *weak* reduction. A *weakly* irreducible term can contain an unprotected **U**-term without being essentially quantified, e.g. $\mathbf{S}(\mathbf{K}(\mathbf{KI}))(\mathbf{KU}^{\forall\psi F(\psi)\to F(D)})$. But this is not strongly irreducible: it reduces to $\mathbf{I}_{F(D)}$.

Definition 4.9 In the rest of this section, we shall be concerned with an arbitrary expansion tree τ within which $\lambda x_1 x_2 \ldots x_n(\mathbf{U}^{\forall\psi F(\psi)\to F(D)}(\Pi_{F(\delta),\delta,\psi}Q)$

$X_1X_2\dots X_q$) occurs. Let # be the indicated occurrence of $\mathbf{U}^{\forall\psi F(\psi)\to F(D)}$. Let $\lambda y_1y_2\dots y_m\left(aN_1N_2\dots N_p\right)$ be an arbitrary term on the path leading to the end of τ from $\lambda x_1x_2\dots x_n(\mathbf{U}^{\forall\psi F(\psi)\to F(D)}(\Pi_{F(\delta),\delta,\psi}Q)X_1X_2\dots X_q)$. More specifically, we are concerned with the case where an irreducible term M is in head position within $aN_1N_2\dots N_p$ and ♮ is an ocurrence within M of some $\mathbf{U}$-term. We investigate the conditions that must hold for ♮ to be a descendant of #.

If the component $\Pi_{F(\delta),\delta,\psi}Q$ within $\lambda x_1x_2\dots x_n(\mathbf{U}^{\forall\psi F(\psi)\to F(D)}(\Pi_{F(\delta),\delta,\psi}Q)$ $X_1X_2\dots X_q$) is going to be replaced using rule (IV), this needs to be done before any other rules are applied. It will be assumed that this is done and $\Pi_{F(\delta),\delta,\psi}Q$ is replaced with P. Propositions 4.10 and 4.11 are proved by induction on the number of inferences between $\lambda x_1x_2\dots x_n(\mathbf{U}^{\forall\psi F(\psi)\to F(D)}PX_1X_2\dots X_q)$ and $\lambda y_1y_2\dots y_m(aN_1N_2\dots N_p)$ in τ. Only those inferences are discussed in detail where it is not obvious that the property in question is transmitted from premiss to conclusion.

Proposition 4.10 *If $\lambda y_1y_2\dots y_m(aN_1N_2\dots N_p)$ is derived from $\lambda x_1x_2\dots x_n$ $(\mathbf{U}^{\forall\psi F(\psi)\to F(D)}PX_1X_2\dots X_q)$ using expansion inferences only, if L is a head of $aN_1N_2\dots N_p$, and ♮ is descended from #, ♮ is not protected in L.*

Proof If ♮ is the head atom in within $\lambda x_1x_2\dots x_n(\mathbf{U}^{\forall\psi F(\psi)\to F(D)}PX_1X_2\dots X_q)$, ♮ could only be protected if it were so in the sense of Definition 4.6 (ii); but it is not. For an atom capable of protecting ♮ cannot occur in head position in P; if it did, P would not reduce to $\Pi_{F(\delta),\delta,\psi}Q$. We assume now that other inferences come between $\lambda y_1y_2\dots y_m\left(aN_1N_2\dots N_p\right)$ and $\lambda x_1x_2\dots x_n(\mathbf{U}^{\forall\psi F(\psi)\to F(D)}PX_1X_2\dots X_q)$ in τ; and that $\lambda y_1y_2\dots y_m\left(aN_1N_2\dots N_p\right)$ is the *first* term in this branch (working downwards) to have a head within which a descendant of # is protected.

♮ is not protected in L in virtue of clause (i) or (ii) of Definition 4.6, because inspection of the expansion rules shows that no conclusion can contain a subterm of the required shape unless that subterm also occurs in the premiss of the rule. If ♮ is protected in virtue of clause (iv), then it is already protected in the premiss of the rule (in consequence of clause (v)), contradicting our assumption. So it must be protected in virtue of clause (iii). Moreover, the head atom in L must be $\mathbf{S}$, because the assumption would also be contradicted if it were $\mathbf{K}$ or $\mathbf{I}$.

Then $L\equiv\mathbf{S}N_1N_2$ and the term immediately above $\lambda y_1y_2\dots y_m(aN_1N_2\dots N_p)$ has $N_1N_3(N_2N_3)$ in head position. Again by clause (v) of Definition 4.6, if some atom c protects ♮ in $\mathbf{S}N_1N_2$, c already protects ♮ in $N_1x(N_2x)$, for any x not in $\mathbf{S}N_1N_2$; and $c\not\equiv x$. On the other hand, we have assumed that c does not protect ♮ in $N_1N_3(N_2N_3)$. If c protects ♮ in N_1 or N_2 within $N_1x(N_2x)$ in virtue of 4.6 (i) or (ii), then it also does so in $N_1N_3(N_2N_3)$, contradicting our assumption. Otherwise $N_1x(N_2x)$ is itself a term formed by applying the operation $X\mapsto[y]X$ to another term Z within which c protects ♮. This means $N_1x(N_2x)\equiv\mathbf{S}x(N_2x)$ and $Z\equiv xy(N_2xy)$. But now, if c protects ♮ in Z, it protects it in N_2 in virtue of 4.6 (i) or (ii); and we have just seen that this contradicts our assumption. □

Proposition 4.11 *If any of rules (IV)–(VII) is used in the derivation of $\lambda y_1y_2\dots y_m(aN_1N_2\dots N_p)$ and ♮ is protected in the irreducible head M (see Definition 4.9), then either ♮ is not a descendant of # or M is essentially quantified outside its final subtype.*

Proof If $\lambda y_1 y_2 \ldots y_m(aN_1N_2 \ldots N_p)$ is a conclusion of one of rules (IV)–(VII), no descendant of # can occur in any irreducible head of $aN_1N_2 \ldots N_p$, because only one component N_i will contain a descendant of #; but that component will be reducible, as $\lambda x_1 x_2 \ldots x_n(\mathbf{U}^{\forall\psi F(\psi)\to F(D)} P X_1 X_2 \ldots X_q)$ is, and hence will not be part of M.

The remaining possibility is that $\lambda y_1 y_2 \ldots y_m(aN_1N_2 \ldots N_p)$ is derived by expansion inferences from a term, say $\lambda y_1 y_2 \ldots y_m(bP_1P_2 \ldots P_r)$, which is itself derived by one of rules (IV)–(VII). As above, only one component P_i will contain a descendant of #; we need to consider what happens to P_i in the course of the subsequent expansion inferences. In fact a component of a premiss of an expansion inference will also be a component of the conclusion, *unless* the inference is **S**-expansion in which case a component $cU_1U_2 \ldots U_{j+1}$ may occur in the premiss while only $cU_1U_2 \ldots U_j$ occurs in the conclusion. This process will be called "stripping away the last component of P_i". As above, P_i must be reducible; but it may happen that $cU_1U_2 \ldots U_{j+1}$ is reducible while $cU_1U_2 \ldots U_j$ is not. Let P' be the largest irreducible head of P_i; so that ♮ must be in P' and P_i has the shape $P'Q_0Q_1 \ldots Q_n$.

Case (1): ♮ is unprotected within P'. When one of the components $Q_s (0 \le s \le n)$ is stripped away, this happens through an inference which leads from $XQ_s(P'Q_0Q_1 \ldots Q_s)$ to $\mathbf{S}X(P'Q_0Q_1 \ldots Q_{s-1})Q_s$. Within this last term, at most the head $\mathbf{S}X(P'Q_0Q_1 \ldots Q_{s-1})$ can be irreducible and its type is $type\,(Q_s) \to type(XQ_s(P'Q_0Q_1 \ldots Q_s))$. By proposition 4.8 (i), P' is essentially quantified outside of its last subtype; hence, for at least one s, $\mathbf{S}X(P'Q_0Q_1 \ldots Q_{s-1})$ is essentially quantified in its antecedent.

Case (2): ♮ is protected in P'. This case is handled just like case (1), except that this time the proposition that P' is essentially quantified outside of its last subtype is derived not from 4.8 (i) but from the induction hypothesis mentioned in 4.9.

Proposition 4.12 *If M is a head of $\lambda y_1 y_2 \ldots y_m(aN_1N_2 \ldots N_p)$, if $M \in Ir$ and M is not essentially quantified, then M contains no descendants of #.*

Proof follows from Propositions 4.8 (i), 4.10 and 4.11.

Proof of Proposition 4.2 The first step in the reduction of N_0 takes it to $M({}^{\beta}C)L_1 L_2 \ldots L_r$. $type(M({}^{\beta}\alpha))$ must be quantifier-free because $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}$ is simple, hence $M({}^{\beta}C)$ is not essentially quantified. By Definition 2.4 (i), $M({}^{\beta}\alpha)$ is also irreducible; hence so is $M({}^{\beta}C)$ and it therefore satisfies the conditions mentioned in the antecedent of 4.12. Proposition 4.2 now follows from Propositions 4.5 and 4.12. □

Definition 4.13 A *marked reduction* is a (finite or infinite) sequence of pairs $\langle P_0, S_0\rangle, \langle P_1, S_1\rangle, \ldots$ where $P_0, P_1, \ldots$ constitute a reduction and, for each i, S_i is a set of occurrences of **U**-terms in P_i, called *marked occurrences*. The following conditions are moreover satisfied.

(i) For any i, if the active redex in P_i is $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)$ for some Q, then all occurrences of **U**-terms in Q are also elements of S_i.

(ii) For any i, if $\langle P_{i+1}, S_{i+1}\rangle$ is an element of the sequence, if # is an occurrence of a **U**-term in P_i and # belongs to S_i, then any footprint of # in P_{i+1} is an element of S_{i+1}.

(iii) An occurrence of a $\mathbf{U}$-term in P_i belongs to S_i only if it does so in virtue of (i) or (ii) above.
(iv) All $\mathbf{U}$-terms occurring in P_0 are either simple (Definition 1.5 (ii)) or elements of S_0. Note that this condition is always satisfied when P_0 is a **BI**-term.

Proposition 4.14 *If $\langle P_i, S_i\rangle$ is an element of a marked reduction, then:*

(a) Every occurrence in P_i of a $\mathbf{U}$-term not in S_i is simple.
(b) If # in P_i is an element of S_i, then either (α) # occurs inside Q in the active redex of P_i, which moreover has the shape $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)$; or ($\beta$) for some M and some $j < i$, $\mathbf{U}^{\forall\phi F(\phi)\to F(U)}(\Pi_{F(\alpha),\alpha,\phi}M)$ is the active redex in P_j and # is a footprint of a $\mathbf{U}$-term inside M.
(c) For every occurrence # of a $\mathbf{U}$-term in P_i in the context $\#(\Pi_{A(\alpha),\alpha,\phi}Q)$, $\# \notin S_i$.

Proof (a) and (b) are proved simultaneously by induction on i; (c) is a consequence of (b) and Proposition 4.2. Basis: (a) holds by Definition 4.13 (iv). As for (b), alternative (α) holds thanks to Definition 4.13 (i). Induction step: follows from Proposition 4.2 and Definition 4.13. □

Remark Proposition 4.14 vindicates the claim in Sect. 3, that **BI**-terms are very much more amenable to ordinal analysis than **CL**-terms in general. The essential point is that, in a marked reduction, only a *simple* $\mathbf{U}$-term can find itself at the head of a weak redex.

5 Assignment of ordinal numbers to terms in a marked reduction

In the following definition + denotes the natural sum of ordinal numbers and $\times$ in the context $3^\sigma \times \tau$ the natural product as defined in [14, p. 106]. The properties of + and $\times$ that are needed for the following proofs are listed in [9, p. 448] and [14].

Definition 5.1 Let a marked reduction $\langle P_0, S_0\rangle, \langle P_1, S_1\rangle, \ldots$ be given. For every natural number i and every subterm M of any term P_j occurring in the reduction, $[M]_i$ shall be an ordinal number below ε_0:

(i) If M is a t-variable or a Π-term, then, for every i, $[M]_i = 0$.
(ii) $[\mathbf{S}_{ABC}]_i = 1$ if $i \leq l(\mathbf{S}_{ABC})$; otherwise $[\mathbf{S}_{ABC}]_i = 0$.
(iii) $[\mathbf{K}_{AB}]_i = 1$ if $i \leq l(\mathbf{K}_{AB})$; otherwise $[\mathbf{K}_{AB}]_i = 0$.
(iv) $[\mathbf{I}_A]_i = 1$ if $i \leq l(\mathbf{I}_A)$; otherwise $[\mathbf{I}_A]_i = 0$.
(v) $[\mathbf{U}^{\forall\phi A(\phi)\to A(C)}]_i = \omega$ if $i = l(A(C))+1$ and $\mathbf{U}^{\forall\phi A(\phi)\to A(C)} \notin S_j$, i.e. is unmarked.
(vi) $[\mathbf{U}^{\forall\phi A(\phi)\to A(C)}]_i = 1$ if $i \leq l(A(C))$ and $\mathbf{U}^{\forall\phi A(\phi)\to A(C)} \notin S_j$, i.e. is unmarked.
(vii) $[\mathbf{U}^{\forall\phi A(\phi)\to A(C)}]_i = 0$ if $i > l(A(C))+1$ or $\mathbf{U}^{\forall\phi A(\phi)\to A(C)} \in S_j$, i.e. is marked.
(viii) If $M \equiv PQ$ and if either $i > l(Q)$ or Q is a variable, then $[M]_i = [P]_i$.
(ix) If $M \equiv PQ$, $i \leq l(Q)$ and Q is not a variable, then $[M]_i = 3^{[M]_{i+1}} \times ([P]_i + [Q]_i)$.
(x) If $M \equiv \lambda x P$, $[M]_i = [P]_i$ for every i.

Corollary *By clauses (viii) and (x),* $[M]_i = [\lambda x_1 x_2 \ldots x_n\, (M x_1 x_2 \ldots x_n)]_i$ *for every* M *and* i*. The specification on variables in (viii) admittedly complicates the proof of Proposition 5.3 below, but is necessary in order to prevent contraction of a type II redex within a term* M *from yielding a new term* N *such that* $[N]_i > [M]_i$ *for some or all values of* i*.*

In Propositions 5.2, 5.3 and 5.5, the ordinals $[M]_i$ and $[N]_i$ are compared where M is a weak redex, N its contractum and i is a natural number not exceeding the level of the head atom in M. It is assumed, to begin with, that the marked terms in N are just the descendants of those in M.

Proposition 5.2 *If* $0 \le i \le l(A)$, $[M]_i < [\boldsymbol{I}_A M]_i$. *If* $0 \le i \le l(A)$, $[M]_i < [\boldsymbol{K}_{AB} M N]_i$.

Proposition 5.3 *If* $0 \le i \le l(M)$, $[MP(NP)]_i < [\mathbf{S}_{ABC} MNP]_i$.

Proof It is assumed to begin with that none of M, N, P is a variable: the various cases in which this assumption does not hold will be dealt with subsequently.

Values of i are divided into three ranges, yielding a case distinction: (I) $l(N) < i \le l(M)$; (II) $l(P) < i \le l(N)$; (III) $0 \le i \le l(P)$. Proposition 5.3 is proved by induction on $l(M) - i$; that is, in the induction step we assume the proposition holds when i is replaced with $i + 1$.

(I) In this case, $[MP(NP)]_i = [M]_i$, while $[\mathbf{S}_{ABC} MNP]_i = [\mathbf{S}_{ABC} M]_i$.

For (II) and (III), the stronger proposition $[MP\,(NP)]_i + [MP]_i + [NP]_i + 2 \le [\mathbf{S}_{ABC} MNP]_i$ is proved.

(II) *First subcase:* $l(NP) < i$ Then

$$
\begin{aligned}
[MP(NP)]_i + [MP]_i + [NP]_i + 2 &= [M]_i + [M]_i + [N]_i + 2 \text{ by clause (viii)},\\
&\qquad \text{because } l\,(NP) < i\\
&< 3 \times ([M]_i + [N]_i + 1)\\
&< 3^{[\mathbf{S}MNP]_{i+1}} \times ([\mathbf{S}_{ABC} M]_i + [N]_i)\\
&= [\mathbf{S}_{ABC} MN]_i \text{ (using } [\mathbf{S}_{ABC} MNP]_{i+1}\\
&\qquad = [\mathbf{S}_{ABC} MN]_{i+1})\\
&= [\mathbf{S}_{ABC} MNP]_i \text{ by clause (viii)},\\
&\qquad \text{because } l\,(P) < i
\end{aligned}
$$

Second subcase: $i \le l(NP)$ Then

$$
\begin{aligned}
[MP\,(NP)]_i + [MP]_i + [NP]_i + 2 &= 3^{[MP(NP)]_{i+1}} \times ([M]_i + [N]_i) + [M]_i + [N]_i + 2 \\
&\quad \text{(by Definition 5.1 clauses (viii) , (ix))} \\
&< 3^{[MP(NP)]_{i+1}+1} \times ([M]_i + [N]_i + 1) \\
&\le 3^{[\mathbf{S}MNP]_{i+1}} \times ([M]_i + [N]_i + 1) \\
&\quad \text{(using either the I.H.or the result of case (I))} \\
&\le 3^{[\mathbf{S}MNP]_{i+1}} \times ([\mathbf{S}_{ABC}M]_i + [N]_i) \\
&= [\mathbf{S}_{ABC}MN]_i \text{ (using } [\mathbf{S}_{ABC}MNP]_{i+1} \\
&\qquad = [\mathbf{S}_{ABC}MN]_{i+1}) \\
&= [\mathbf{S}_{ABC}MNP]_i \text{ by clause (viii), because } l(P) < i
\end{aligned}
$$

(III)
$$
\begin{aligned}
[MP\,(NP)]_i + [MP]_i + [NP]_i + 2 &= 3^{[MP(NP)]_{i+1}} \times ([MP]_i + [NP]_i) \\
&\quad + [MP]_i + [NP]_i + 2 \\
&< 3^{[MP(NP)]_{i+1}+1} \times ([MP]_i + [NP]_i + 1) \\
&= 3^{[MP(NP)]_{i+1}+1} \times \{(3^{[MP]_{i+1}} \times ([M]_i + [P]_i)) \\
&\quad + (3^{[NP]_{i+1}} \times ([N]_i + [P]_i)) + 1\} \\
&< 3^{[MP(NP)]_{i+1}+1+[MP]_{i+1}+[NP]_{i+1}} \\
&\quad \times \{[M]_i + [P]_i + [N]_i + [P]_i + 1\} \\
&\le 3^{[MP(NP)]_{i+1}+[MP]_{i+1}+[NP]_{i+1}+2} \\
&\quad \times \{[M]_i + [N]_i + [P]_i + 1\} \\
&\le 3^{[\mathbf{S}MNP]_{i+1}} \times ([M]_i + [N]_i \\
&\quad + [P]_i + 1) \text{ by I.H.} \\
&< 3^{[\mathbf{S}MNP]_{i+1}} \times \{(3^{[\mathbf{S}MN]_{i+1}} \\
&\quad \times ([\mathbf{S}M]_i + [N]_i)) + [P]_i\} \\
&= 3^{[\mathbf{S}MNP]_{i+1}} \times \{[\mathbf{S}_{ABC}MN]_i + [P]_i\} \\
&= [\mathbf{S}_{ABC}MNP]_i
\end{aligned}
$$

Cases where one or more of M, N, P is a variable can generally be dealt with by simple modifications of the foregoing. For example, if only P is a variable, the argument given for case (II) works also for case (III). When only N is a variable, the argument dealing with the first subcase of (II) is modified so as to end like this:

$$
\begin{aligned}
3 \times ([M]_i + [N]_i + 1) &\le 3^{[\mathbf{S}M]_{i+1}} \times ([M]_i + 1) \text{ because } [N]_i = 0 \\
&= 3^{[\mathbf{S}M]_{i+1}} \times ([\mathbf{S}_{ABC}]_i + [M]_i) \text{ by Definition 5.1 clause (ii)} \\
&= [\mathbf{S}_{ABC}M]_i \text{ by clause (ix)} \\
&= [\mathbf{S}_{ABC}MNP]_i \text{ by two applications of clause (viii)}
\end{aligned}
$$

and similar adjustments are made in the second subcase and part (III).

The most difficult case is where M and N are variables, but not P. This time we prove

$$[MP\,(NP)]_i + [MP]_i + 1 \leq [\mathbf{S}_{ABC}MNP]_i \tag{*}$$

When $l(P) < i$ and M and N are variables, (*) is easily proved, as $[\mathbf{S}_{ABC}MNP]_i = [\mathbf{S}_{ABC}]_i = 1$, while $[MP(NP)]_i = [MP]_i = 0$. For part (III), a function $n, i \mapsto [P]_i^n$ is defined by primitive recursion: $[P]_i^0 = 0$, $[P]_i^{n+1} = 3^{[P]_{i+1}^n} \times [P]_i$. By induction on $l(P)+1–i$, we easily prove $[MP]_i = [NP]_i = [P]_i^{l(\mathrm{P})+1-\mathrm{i}}$. Hence (*) follows from $[MP\,(NP)]_i + [P]_i^{l(\mathrm{P})+1-\mathrm{i}} + 1 \leq [\mathbf{S}_{ABC}MNP]_i$. Part (III) is dealt with by induction on $l(P) + 1 - i$. The case $i = l(P)+1$ has already been treated. Now:

$$\begin{aligned}
[MP(NP)]_i + [P]_i^{l(\mathrm{P})+1-\mathrm{i}} + 1 &= 3^{[MP(NP)]_{i+1}} \times ([MP]_i + [NP]_i) \\
&\quad + [P]_i^{l(\mathrm{P})+1-\mathrm{i}} + 1 \\
&= 3^{[MP(NP)]_{i+1}} \times \left([P]_i^{\mathrm{l(P)+1-i}} + [P]_i^{\mathrm{l(P)+1-i}}\right) \\
&\quad + [P]_i^{l(\mathrm{P})+1-\mathrm{i}} + 1 \text{(see above)} \\
&\leq (3^{[MP(NP)]_{i+1}+1} \times [P]_i^{\mathrm{l(P)+1-i}}) + 1 \\
&= (3^{[MP(NP)]_{i+1}+1} \times 3^{[P]_{\mathrm{i+1}}^{\mathrm{l(P)-i}}} \times [P]_i) + 1 \\
&= (3^{[MP(NP)]_{i+1}+1+\{[P]_{\mathrm{i+1}}^{l\mathrm{(P)+1-(i+1)}}\}} \times [P]_i) + 1 \\
&\quad \text{(because } l(P) - i = l\,(P) + 1 - (i+1)) \\
&\leq (3^{[\mathbf{S}MNP]_{i+1}} \times [P]_i) + 1 \text{ by the induction hypothesis} \\
&< 3^{[\mathbf{S}MNP]_{i+1}} \times ([P]_i + 1) \text{ because} [\mathbf{S}MNP]_{i+1} \geq 1 \\
&= 3^{[\mathbf{S}MNP]_{i+1}} \times ([\mathbf{S}_{ABC}]_i + [P]_i) \\
&= 3^{[\mathbf{S}MNP]_{i+1}} \times ([\mathbf{S}_{ABC}MN]_i + [P]_i) \\
&\quad \text{because} M \text{and} N \text{are variables} \\
&= [\mathbf{S}_{ABC}MNP]_i
\end{aligned}$$

□

Definition 5.4 Let N be the active redex in $[N/x]M$. In consequence of Definitions 2.4 and 2.8, if N is a type II redex with contractum $\lambda x_1 \ldots x_n(Nx_1 \ldots x_n)$, then some head of $\lambda x_1 x_2 \ldots x_n\,(Nx_1x_2 \ldots x_n)$, say $Nx_1x_2 \ldots x_j\,(j \leq n)$, is a weak redex and is, moreover, the active redex in $[\lambda x_1x_2 \ldots x_n(Nx_1x_2 \ldots x_n)/x]M$. If N is a type I redex, let its contractum be N'. We define a primitive recursive function $\boldsymbol{f}_4$ by the clauses:

(i) If N is a type I redex, $\boldsymbol{f}_4([N/x]M)$ shall be $([N'/x]M)$.
(ii) If N is a type II redex, $\boldsymbol{f}_4([N/x]M)$ shall be the term which results from contracting the weak redex $Nx_1x_2 \ldots x_j$ within $[\lambda x_1x_2 \ldots x_n(Nx_1x_2 \ldots x_n)/x]M$.
(iii) If N is a type III redex within a subterm $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)$, such that $\boldsymbol{f}_3(Q) \in \mathrm{Ir}$, $\boldsymbol{f}_4(M)$ shall be the term which results from M by replacing $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)$ with $(\boldsymbol{f}_3(Q))({}^{\alpha}{}_C)$.

Proposition 5.5 *Let* $\langle P_0, S_0\rangle, \langle P_1, S_1\rangle, \ldots$. *be as in Definition* 5.1. *If* P_{j+1} *is obtained from* P_j *by replacing the active redex* $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)$ *with* $Q({}^{\alpha}{}_C)$ *and* $0 \leq i \leq l(A(C))$, $[Q({}^{\alpha}{}_C)]_i < [\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)]_i$.

Proof By Definition 4.13 (i) all occurrences of **U**-terms in Q are marked, so that $[Q]_i$ for every i is calculated without using clause (v) of Definition 5.1 and is therefore finite. The same goes for $[Q(^{\alpha}C)]_i$ by Definition 4.13 (ii) (although $[Q(^{\alpha}C)]_i$ will in general be larger than $[Q]_i$, this does not affect the main point). By contrast, the **U**-term in head position in $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)$ is unmarked by Proposition 4.14 (c), so, by clause (v) of Definition 5.1, $[\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)]_{l(A(C))} = 3^{\omega}\times\big(1+[Q]_{l(A(C))}\big)\geq\omega$. By induction on $l(A(C))-i$, $[\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)]_i \geq \omega$ for $0\leq i\leq l(A(C))$. □

Proposition 5.6 *Let $\langle P_n, S_n\rangle$ be a pair in a marked reduction and let the active redex in P_n be of type I or II. Then $[f_4(P_n)]_0 < [P_n]_0$ and $[f_4(P_n)]_i \leq [P_n]_i$ for every $i \leq l(P_n)$.*

Proof If the active redex in P_n is of type II, then $f_4(P_n)$ is obtained from P_n by two successive contractions, of which the first (by Definition 5.1 and corollary) yields a term to which the same ordinals are assigned as to P_n itself. Everything therefore turns on what happens when the active redex is of type I and this is what we now investigate.

P_n must be (a) itself a weak redex, or (b) have the shape $\lambda x X$, where X contains a weak redex, or have the shape (c) XY where exactly one of X, Y contains the redex that is contracted.

Case (a) is essentially dealt with by Propositions 5.2, 5.3 and 5.5. Sub-cases have admittedly to be distinguished according to whether the marked terms in $f_4(P_n)$ are just the footprints of the marked terms in P_n or whether additional **U**-terms become marked as a result of the contraction. But calculation of $[f_4(P_n)]_0$ on the assumption that some occurrence # of a **U**-term in $f_4(P_n)$ is marked can only yield a smaller value than the same calculation on the assumption that # is unmarked, by Definition 5.1 especially clauses (v)–(vii).

Cases (b) and (c) are proved by induction on the construction of P_n. The basis of the induction is just case (a). The treatment of (b) follows trivially from the induction hypothesis, using Definition 5.1 (x). For (c), let $X'Y'$ be the result of contracting the active redex in XY. $[XY]_i =_{\text{df.}} [X]_i$ (if Y is a variable or $l(Y) < i$) or $3^{[XY]_{i+1}} \times ([X]_i + [Y]_i)$ (otherwise).

If Y is a variable, the redex contracted must be in X and both $[X'Y']_i \leq [XY]_i$ and $[X'Y']_0 < [XY]_0$ follow from the induction hypothesis.

If $[XY]_i =_{\text{df.}} [X]_i$ because $l(Y) < i$, and if $Y \equiv Y'$, $[X'Y']_i = [X']_i \leq [X]_i$ (by I.H.) = $[XY]_i$. If $X \equiv X'$, clearly $[X']_i \leq [X]_i$.

Finally, if $[XY]_i = 3^{[XY]_{i+1}} \times ([X]_i + [Y]_i)$, we use a subsidiary induction on $l(Y) - i$. The basis follows from the result of the last paragraph together with the hypothesis of the main induction. When we reach $i = l(Y)$, $[XY]_0 = 3^{[XY]_1} \times ([X]_0 + [Y]_0)$. By the hypothesis of the main induction, $([X']_0 + [Y']_0) < ([X]_0 + [Y]_0)$. By the hypothesis of the subsidiary induction, $3^{[X'Y']_1} \leq 3^{[XY]_1}$. So $3^{[X'Y']_1} \times ([X']_0 + [Y']_0) < 3^{[XY]_1} \times ([X]_0 + [Y]_0)$.

Proposition 5.7 *If P_k occurs in a reduction and the active redex in P_k is a type III redex within a subterm $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)$, where $f_3(Q) \in Ir$, at most $f_1(Q)$ + 1 further reduction steps yield $f_4(P_k)$.*

Proof After P_k, at most $f_1(Q)$ further steps in the reduction will yield the normal form of Q. The $(k+f_1(Q))$th term in the reduction will contain $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}\boldsymbol{f}_{\mathbf{3}}(Q))$ and, because $\boldsymbol{f}_{\mathbf{3}}(Q) \in \text{Ir}$, $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}\boldsymbol{f}_{\mathbf{3}}(Q))$ is a type I redex and is moreover the active redex in $P_{k+f_1(Q)}$, by Definition 2.8 (i). Contracting this redex itself yields $f_4(P_k)$, by Definition 5.4 (iii). □

Proposition 5.8 *Let P_k be as in Proposition 5.7 and $\langle P_k, S_k\rangle$ a pair in a marked reduction. Then, for* $0 \leq i \leq l(A(C))$, $[f_4(P_k)]_0 < [P_k]_0$.

Proof By Definition 5.4 (ii), P_k must contain a subterm $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)$ with the property $[\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)]_{l(A(C))} = 3^{\omega} \times \left(1 + [Q]_{l(A(C))}\right) \geq \omega$, just as in Proposition 5.6. However $[(\boldsymbol{f}_{\mathbf{3}}(Q))\,({}^{\alpha}C)]_{l(A(C))}$ is finite by 4.14 (c) and 5.1. Therefore, when $0 \leq i \leq l(A(C))$, $[Q({}^{\alpha}C)]_i < [\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)]_i$, just as in Proposition 5.5. $[f_4(P_k)]_0 < [P_k]_0$ now follows as in 5.6. □

6 The functions Nf and Count

The goal of Definition 6.1 is to define by ε_0-recursion two two-place functions **Nf** and **Count** so that, for any pair $\langle M, S\rangle$ that can occur in a marked reduction: (I) $M \rhd \mathbf{Nf}(M, [M]_0)$; (II) $\mathbf{Nf}(M, [M]_0) \in \text{Ir}$; (III) $\mathbf{Count}\,(M, [M]_0)$ is a natural number not less than the number of terms in the reduction leading from M to $\mathbf{Nf}(M, [M]_0)$.

Definition 6.1 Here σ is an arbitrary ordinal number below ε_0.

$$\mathbf{Nf}(M,\sigma) =_{\text{df.}} M \quad \text{when } \sigma < [M]_0 \tag{1}$$
$$\mathbf{Count}(M,\sigma) =_{\text{df.}} 0 \quad \text{when } \sigma < [M]_0 \tag{2}$$
$$\mathbf{Nf}(M,\sigma) =_{\text{df.}} \mathbf{Nf}(M,[M]_0) \quad \text{when } [M]_0 < \sigma \tag{3}$$
$$\mathbf{Count}(M,\sigma) =_{\text{df.}} \mathbf{Count}(M,[M]_0) \quad \text{when } [M]_0 < \sigma \tag{4}$$

Thanks to clauses (1)–(4), the problem is reduced to that of defining **Nf** and **Count** for the case where the arguments are M and $[M]_0$, for some M. We put:

$$\mathbf{Nf}(M,[M]_0) =_{\text{df.}} f_3(M) \quad \text{when } [M]_0 = 0 \tag{5}$$
$$\mathbf{Count}\,(M,[M]_0) =_{\text{df.}} f_1(M) \quad \text{when } [M]_0 = 0 \tag{6}$$

$\mathbf{Nf}(M, [M]_0)$ and $\mathbf{Count}\,(M, [M]_0)$ when $[M]_0 \neq 0$ are differently defined according to which clause of Definition 2.8 picks out the active redex in M. If it is a type I or II redex, then:

$$\mathbf{Nf}(M,[M]_0) =_{\text{df.}} \mathbf{Nf}\left(f_4(M), \left[f_4(M)\right]_0\right) \quad \text{when } 0 < [M]_0 \tag{7}$$
$$\mathbf{Count}\,(M,[M]_0) =_{\text{df.}} \mathbf{Count}\left(f_4(M), \left[f_4(M)\right]_0\right) + 2 \tag{8}$$

If the active redex is a type III redex within a term $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)$, where $f_3(Q) \in \text{Ir}$, then:

$$\mathbf{Count}\,(M,[M]_0) =_{\text{df.}} \mathbf{Count}\left(f_4(M), \left[f_4(M)\right]_0\right) + f_1(Q) + 1 \tag{9}$$

while $\mathbf{Nf}(M, [M]_0)$ is again defined by (7) above. If neither of the foregoing:

$$\mathbf{Nf}\,(M, [M]_0) =_{\mathrm{df}} \boldsymbol{f}_3(M) \tag{10}$$

$$\mathbf{Count}\,(M, [M]_0) =_{\mathrm{df}} \boldsymbol{f}_1(M) \tag{11}$$

Theorem 2 *The definitions of* **Nf** *and* **Count** *are definitions by* ε_0*-recursion; and the functions defined do indeed have the properties mentioned at the beginning of Sect. 6.*

Proof In clauses (1), (2), (5), (6), (10), (11) **Nf** and **Count** are defined explicitly in terms of functions already defined. The correctness of the remaining clauses follows from the facts that $\boldsymbol{f}_4(M)$ can be primitive recursively computed from M, while $[\boldsymbol{f}_4(M)]_0 < [M]_0$ by 5.6 and 5.8 [8]

That the functions have the properties advertised is proved by transfinite induction. It can sometimes happen that, when a term contains only type III redexes, contractions of these redexes lead to the creation of new type I redexes. This is the case dealt with in Definitions 2.8 (ii), 5.4 (iii), Propositions 5.7–5.8 and clause (9) above. However this only happens when some of the type III redexes occur inside subterms of the shape $\mathbf{U}^{\forall\phi A(\phi)\to A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)$. When none of them do, the term can be reduced to normal form simply by repeatedly contracting the leftmost type III redex; and this is the case dealt with in clauses (5), (6), (10) and (11).

That **Nf**, in the remaining cases, has the properties advertised follows from the facts that $M \rhd \boldsymbol{f}_4(M)$ (by Definition 2.8; see also Definition 5.4) and the hypothesis of the transfinite induction, which enables us to infer, from $[\boldsymbol{f}_4(M)]_0 < [M]_0$, that **Nf** $(\boldsymbol{f}_4(M), [\boldsymbol{f}_4(M)]_0)$ is the normal form of $\boldsymbol{f}_4(M)$. That **Count** has the properties advertised follows from the hypothesis of the transfinite induction together with the fact that, in clauses (8) and (9), no more than 2 resp. $\boldsymbol{f}_1(Q) + 1$ contractions are needed to get from M to $\boldsymbol{f}_4(M)$. This last claim follows from Definition 5.4 and Proposition 5.7. □

Discussion: it should be clear by now why we work with a reduction strategy of contracting the *active* redex at every stage (Definition 2.10 (i)–(ii)). Superficially, there might seem to be a difficulty caused by the fact that, when N is obtained from M by contracting a type III redex, $[N]_0$ may be *larger* than $[M]_0$. Even if the redex contracted is type II, we have not $[N]_0 < [M]_0$ but only $[N]_0 = [M]_0$ (by the corollary to Definition 5.1). The solution to this difficulty is to observe that if we choose our reduction strategy carefully and continue the reduction steps until the next step at which a weak redex is contracted, the result of this contraction being $\boldsymbol{f}_4(M)$, we have $[\boldsymbol{f}_4(M)]_0 < [M]_0$, even if $[N]_0 > [M]_0$ for some terms N coming between M and $\boldsymbol{f}_4(M)$ in the reduction.

Theorem 3 *(converse to Theorem 1)*
If F is an m-place function that takes m-tuples of natural numbers to natural numbers and is defined by a **BI***-term, then F is definable by* $<\varepsilon_0$*-recursion; and is therefore a primitive recursive functional.*

[8] For schemes of ordinal recursion, see [13, chapter 3], or [19]. Only *unnested* ordinal recursion is needed here.

Sketch of proof: Theorem 3 follows from Theorem 2 because, given a **BI**-term M that defines F, it is always possible to define, using M and **Nf**, a $<\varepsilon_0$-recursive function that is extensionally equivalent to F. Detailed instructions for drawing up a definition have been given in several places, e.g. [11]. Finally, every $<\varepsilon_0$-recursive function is (when considered extensionally) also a prf [19].

7 Discussion of the methods used

Theorem 2 is the main concrete innovation of the present paper, as analogues of Theorems 1 and 3 were proved in [1] and [12]. [1] and [12], however, make no use of ordinal assignments, so the use of ordinal numbers in order to prove Theorem 3 also constitutes a methodological innovation. As Leivant has argued [12], given any natural class of typed combinatory or lambda terms, one can ask which subclass of the recursive functions is combinatorially defined (λ-defined) by terms in this class, relative to the Church-Girard numerals. There is a rich field of open problems here. However the characterization of subclasses of the recursive functions very often makes use of transfinite ordinal numbers [13], so it is unlikely that Leivant's programme could be carried very much further without use of ordinal-theoretic methods.

Definition 1.4 shows how *type*(M), for any **CL**-term M, can be deduced from the types of its atomic subterms. Formal rules for carrying out such a deduction can be given and the result is what is called a *system of type-assignment* [3, p. 148]. A **CL**-term M can be very naturally [15, chapter 5] regarded as encoding a deduction of $M : A$ (read as "M has type A") in *axiomatic* style, for some type A such that $type(M) \equiv A$.

Many proofs of normalizability for terms of Gödel's **T** and typed terms generally work by considering *sequent* style rules for the derivation of $M : type(M)$ and proving (using cut-elimination methods developed by Gentzen, Takeuti or others) that, if $M : type(M)$ is derivable by such rules, there is a term M' to which M reduces and such that $M' : type(M)$ is derivable without cuts. On the other hand, the rules can be so set up that $M' : type(M)$ is derivable without cuts only if M' is in normal form; therefore we get a proof of normalizability via cut-elimination.

This method of proving normalizability was initiated in [6] and important applications of it to Gödel's **T** are found in [4,8,10]. But sequent calculi have the disadvantage that a typed λ-term M will not in general encode a *unique* sequent derivation of M : *type*(M). E.g., given the information that a derivation of M : *type*(M), for some M, ends with a cut, one cannot yet deduce what the premisses of the cut were. If a term is wanted which encodes a sequent derivation of $M : type(M)$, it must therefore contain more information than M itself, thus multiplying definitions.

A more serious disadvantage of sequent theories of type-assignment is that, with them, one has to work rather hard in order to define a function which, like **Count** in the present paper, tells how many contractions are needed to reduce a term to normal form.[9] If eliminability of cuts is proved by transfinite induction, it is easy enough to

[9] Beckmann and Weiermann [4] have adapted a proof of normalizability of terms of **T**, via cut-elimination [10], so as to establish the existence of a function, operating on terms of **T**, that is analogous to **Count**. However the adaptation is not particularly straightforward: [4] appears rather more complicated than [10].

define a function which tells how many reduction steps on *derivations* are needed to reduce a derivation to a cut-free equivalent; but nothing follows obviously about the *terms* whose types are determined by the derivations, as there is no obvious correspondence between reduction of terms and reduction of derivations.

In the present paper, by contrast, precisely because a **CL**-term M can be so naturally read as encoding a determinate deduction of $M : type(M)$, no distinction needs to be drawn between reduction-steps on **CL**-terms and reduction-steps on the deductions that they encode. For this reason, Theorem 2 followed from the results of Sect. 5 without much further work.

8 Applications to the proof theory of second-order logic

The five clauses in the definition of Ir (Definition 2.2(v)) above were expressly designed to ensure that a proof in tree form of $M \in \text{Ir}$, using those five clauses, could automatically be transformed into a *normal* derivation in $\rightarrow\forall^2$**Nip**^{2} of any sequent proved by M. The transformation takes place simply by writing, in place of the terms in the tree, sequents which they prove and noting that clause (d), for example, is equivalent to: the result of applying $\rightarrow$E to normal derivations of $\Gamma \Rightarrow A \rightarrow B$ and $\Delta \Rightarrow A$ is a normal derivation of $\Gamma, \Delta \Rightarrow B$, provided that the derivation of $\Gamma \Rightarrow A \rightarrow B$ does not end with $\rightarrow$I. Proposition 3.2 (ii) and Theorem 2 together therefore entail: there is an ε_0-recursive function defined with the help of **BI**-terms and **Nf**, which, for any proof π in **NBI**$^{\rightarrow i}$ of $\Gamma \Rightarrow A$, takes π to a *normal* proof of the same sequent. This result is not substantially new: analogues for the *classical sequent* calculus were established long ago [2,16] but it should be methodologically interesting for researchers in proof theory to see that the achievement of ordinally informative normalization proofs is not tied to the use of sequent calculi.

To a great extent, the interest of the methods used here will turn on whether they can be extended so as to yield similar results for classes of **CL**-terms (and the corresponding subsystems of second-order logic) which are wider than the class of **BI**-terms (and stronger than the subsystem **NBI**$^{\rightarrow i}$). It is doubtful if Proposition 4.2 can then be used, because of the essential restriction to **BI**-terms there, and it will be necessary to consider redexes $\mathbf{U}^{\forall\phi A(\phi)\rightarrow A(C)}(\Pi_{A(\alpha),\alpha,\phi}Q)$ where $[Q]_i$ will be infinite for many values of i. Thus any analogue of Proposition 5.5, if provable at all, cannot be proved by the simple argument used here and must presumably require ordinal numbers very much larger than ε_0. However it is encouraging to note that Takeuti, in his investigation of second-order sequent calculi, was faced with a similar problem and solved it for a variety of such calculi ([17,18]; cf. also [2]). Whether similar results can be achieved with the methods used here and whether these methods might have any advantage in terms of simplicity, brevity or perspicuity are very exciting questions.[10]

A proof of normalizability via cut-elimination for **BI**-terms is indeed known (unpublished; see also footnote 1 above) but it is far from clear that a function like **Count** could be extracted from it.

[10] The author would like to express his heartfelt thanks to two anonymous referees, whose comments on an earlier version of this paper have stimulated him to improve it (or so he hopes) considerably.

Appendix: Sketch of a proof that every J-term has a normal form

This appendix gives hints on how to adapt Barendregt's [3] proof for polymorphic λ-terms. It is convenient to use his terminology while changing the definitions just slightly. Propositions A2 (i) and (ii) correspond to his Lemma 4.3.8 and Proposition 4.3.10. *Strong* normalizability does not here come into question as we are concerned only with the deterministic reductions of Definition 2.10.

Definition A1 (i) A set X of J-terms shall be *saturated* iff every term in X has a normal form (nf) and:
- (a) X is closed under rules (I)–(III) of Sect. 4;
- (b) A conclusion of (IV) belongs to X if the left premiss does and the right premiss has an nf;
- (c) A conclusion of rules (V)–(VII) belongs to X so long as each premiss has an nf.

(ii) ξ ranges over *assignments*, sc. of a saturated set of terms of type α to each type-variable α.

(iii) With every assignment ξ and every type A is associated a class $[\![A]\!]_\xi$ of terms:
- (a) $[\![\alpha]\!]_\xi =_{\mathrm{df.}} \xi(\alpha)$.
- (b) $[\![A \rightarrow B]\!]_\xi =_{\mathrm{df.}} \{M : type(M) \equiv A \rightarrow B$ and, for every $N \in [\![A]\!]_\xi$, $MN \in [\![B]\!]_\xi\}$.
- (c) $[\![\forall\phi F(\phi)]\!]_\xi =_{\mathrm{df}} \{M : M \in [\![F(\alpha)]\!]_{\xi'}$ for every ξ' taking α to a saturated set of terms of type α and otherwise agreeing with $\xi\}$.

Proposition A2 *(i) For every valuation ξ and every type A, $[\![A]\!]_\xi$ is saturated.*
(ii) If M is an independent J-term, $M \in [\![type(M)]\!]_\xi$ for every ξ.

Remark the restriction to *independent* terms is crucial. $\Pi_{F(\alpha),\alpha,\phi} \in [\![F(\alpha) \rightarrow \forall\phi F(\phi)]\!]_\xi$ is not true for any ξ. But it is easy to see that if $M \in [\![F(\alpha)]\!]_\xi$ for every ξ, then $\Pi_{F(\alpha),\alpha,\phi} M \in [\![\forall\phi F(\phi)]\!]_\xi$ for every ξ. Every other term with an α-critical term in head position is formed either by abstraction, or by **K**-expansion, or both, from a term of the same kind.

References

1. Altenkirch, T., T. Coquand: A finitary subsystem of the polymorphic lambda calculus. In: S. Abramsky (ed.), Typed Lambda Calculi and Applications (Lecture Notes in Computer Science 2044) Springer, pp. 22–28 (2001)
2. Arai, T.: An introduction to finitary analyses of proof figures. In: Cooper, S.B., Truss, J.K. (eds.) Sets and Proofs, pp. 1–26. Cambridge University Press, Cambridge (1999)
3. Barendregt, H.P. et al.: Lambda calculi with types. In: Abramsky, S. (ed.) Handbook of Logic in Computer Science, vol 2, pp. 117–309. Oxford University Press, Oxford (1992)
4. Beckmann, A., Weiermann, A.: Analysing Gödel's **T** by means of expanded head reduction trees. Math. Logic Q. **46**, 517–536 (2000)
5. Coquand T.: Completeness theorems and λ-Calculus. In: P. Urzyczyn (ed.) Typed Lambda Calculi and Applications (Lecture Notes in Computer Science 3461) Springer, pp. 1–9 (2005)
6. Curry, H.B., Feys, R.: Combinatory Logic, vol I. North-Holland, Amsterdam (1958)
7. Curry, H.B., Hindley, J.R., Seldin, J.P.: Combinatory Logic, vol II. North-Holland, Amsterdam (1972)
8. Hinata, S.: Calculability of primitive recursive functionals of finite type. Sci. Rep. Tokyo Kyoiku Daigaku, Section A **9**, 218–235 (1967)

9. Howard, W.A.: Assignment of ordinals to terms for primitive recursive functionals of finite type. In: Kino, A., Myhill, J., Vesley, R.E. (eds.) Intuitionism and Proof Theory, pp. 443–458. North-Holland, Amsterdam (1970)
10. Howard, W.A.: Ordinal analysis of terms of finite type. J. Symbol. Logic **45**, 493–504 (1980)
11. Howard W.A.: Computability of ordinal recursion of type level two. In: F. Richman (ed.), Constructive Mathematics (Lecture Notes in Mathematics 873) Springer, Berlin, pp. 87–104, (1981)
12. Leivant, D.: Peano's lambda calculus. In: Anderson, C.A., Zeleny, M. (eds.) Logic, Meaning and Computation, pp. 313–330. Kluwer, Dordrecht (2001)
13. Rose, H.E.: Subrecursion: Functions and Hierarchies. Clarendon Press, Oxford (1984)
14. Schütte K.: Proof Theory, translated by J. N. Crossley Springer, Berlin (1977)
15. Sørensen, M.H.B., Urzyczyn, P.: Lectures on the Curry–Howard Isomorphism. Elsevier, Amsterdam (2006)
16. Takeuti, G.: On the fundamental conjecture of GLC I. J. Math. Soc. Jpn. **7**, 249–275 (1955)
17. Takeuti, G.: On the fundamental conjecture of GLC V. J. Math. Soc. Jpn. **10**, 121–134 (1958)
18. Takeuti, G.: Proof Theory. 2nd edn. North-Holland, Amsterdam (1987)
19. Terlouw, J.: On definition trees of ordinal recursive functions. J. Symbol. Logic **47**, 395–403 (1982)
20. Troelstra, A.S., Schwichtenberg, H.: Basic Proof Theory. 2nd edn. Cambridge University Press, Cambridge (2000)