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Mathematical Logic

Comprehension contradicts to the induction within Łukasiewicz predicate logic

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Abstract We introduce the simpler and shorter proof of Hajek's theorem that the mathematical induction on ω implies a contradiction in the set theory with the comprehension principle within Łukasiewicz predicate logic Ł∀ (Hajek Arch Math Logic 44(6):763–782, 2005) by extending the proof in (Yatabe Arch Math Logic, accepted) so as to be effective in any linearly ordered **MV**-algebra.

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1 Introduction

In this paper, we introduce the simpler and shorter proof of Hajek's theorem that the mathematical induction on ω implies a contradiction in the set theory with the comprehension principle within Łukasiewicz predicate logic Ł∀ [2].

A significance of the set theory with the comprehension principle is to allow a *general form of the recursive definition*: For any formula $\varphi(x, \ldots, y)$, the comprehension principle implies

$$(\exists z)(\forall x)[x \in z \equiv \varphi(x, \ldots, z)]$$

within Grišin logic (classical logic minus the contraction rule) [1]. This allows us to represent, for example, the set of natural numbers ω, and any partial recursive function on ω.

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Let **CŁ**$_0$ be a set theory with the comprehension principle within Ł∀, an extension of Grišin logic. **CŁ**$_0$ seems to be enough strong to develop an arithmetic: the general form of recursive definition can be used in place of the mathematical induction to define arithmetic. And it had been expected that the arithmetic is a subset of **PA** in classical logic. However, Petr Hajek showed that the following [2]:

Theorem 1 *The extension* **CL** *of the theory* **CŁ**$_0$ *by the(strong) induction scheme on* ω *is contradictory.*

Hajek's result is very surprising, but his proof is very long. First he developed a crisp arithmetic with the induction scheme in **CL**, and next he constructed a truth predicate in **CŁ**$_0$, and he showed that some logical connectives commute to the truth predicate, however such commutability implies a contradiction within crisp arithmetic.

In [3], we proved the similar result in a simple way that the induction scheme implies a contradiction in the set theory within ∀**Ł** which is weaker than **Ł**∀. In this paper, we extend this proof so as to be effective in Ł∀.

This theorem shows that the general form of recursive definition contradicts to the induction within **Ł**∀ though they are consistent within classical logic. Therefore **CŁ**$_0$ gives a new viewpoint to analyze concepts in arithmetic since it gives a new possibility to give a non-standard arithmetic (an arithmetic developed only by the general form of recursive definition) in a natural way. Since **Ł**∀ is nicely axiomatized, this result might help a study of such recursive definitions.

2 Preliminaries

Our framework in this paper is Łukasiewicz predicate logic **Ł**∀. **Ł**∀ is a fuzzy logic weaker than ∀**Ł**, and is axiomatized in Hilbert style as follows.

Definition 1 The axioms of **Ł**∀ consists of axioms of propositional Łukasiewicz logic **Ł** plus the following two additional rules:

- $\forall x\varphi(x) \to \varphi(t)$,
- $\forall x(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ if x is free in ν.

Ł∀ proves $\neg\exists\neg\varphi \equiv \forall x\varphi$ and $(\nu \to \exists x\varphi) \to \exists x(\nu \to \varphi)$. We note that **Ł**∀ is a predicate logic which is complete for models over **linearly ordered MV-algebras**.

Definition 2 Let **CŁ**$_0$ be a set theory within **Ł**∀, which has a binary predicate $\in$ and terms of the form $\{x : \varphi(x)\}$, and whose axiom scheme is **the comprehension principle**: for any φ not containing u freely, $(\forall u)[u \in \{x : \varphi(x, \dots)\} \equiv \varphi(u, \dots)]$.

We can define Leibniz equality $x = y$ iff $(\forall z)[x \in z \leftrightarrow y \in z]$, the empty set $\emptyset = \{x : x \neq x\}$ in standard way.

As we see, **CŁ**$_0$ proves the general form of the recursive definition [1]. In particular, we can construct a term θ such that $\theta =_{\text{ext}} \{u : \varphi(u, \dots, \theta)\}$ for any formula $\varphi(x, \dots, y)$. By using this, we can prove that the set of natural numbers ω can be defined as follows:

$$(\forall x)x \in \omega \equiv [x = \emptyset \vee (\exists y)[y \in \omega \wedge x = \{y\}]]$$

For simplicity, we write $n + 1$ instead of $\{n\}$ hereafter.

Once Hajek suggested to introduce the induction scheme:

Definition 3 The induction scheme on ω is a scheme of the form: for any formula φ,

$$\varphi(0) \wedge (\forall n \in \omega)[\varphi(n) \equiv \varphi(n+1)] \text{ infer } (\forall x)[x \in \omega \to \varphi(x)]$$

However, Hajek finally proved Theorem 1 in a very complex, long proof.

Let $\forall$**Ł** be Łukasiewicz infinite-valued predicate logic whose algebra of truth functions is the standard **M**V-algebra $[0, 1]_{Ł}$ which is generated by $\langle [0, 1], \Rightarrow, * \rangle$. $\forall$**Ł** is stronger than **Ł**$\forall$, but $\forall$**Ł** is not recursively axiomatizable. And let **H** be the set theory with the comprehension principle within $\forall$**Ł**. In [3], we proved:

Theorem 2 *The extension of* **H** *by the induction scheme on* ω *is contradictory.*

The proof is a very simple, but the proof is only valid for models over Archimedean **M**V-algebras.

3 A short proof of theorem 1

Here, we extend the proof of the theorem 2 of [3]. Let us define

- $\theta = \{\langle n, x\rangle : (n = 0 \wedge x \notin x) \vee (\exists k \in \omega)[n = k + 1 \wedge x \in x \to \langle n, x\rangle \in \theta]\}$,
- $R_\omega = \{x : (\exists n)\langle n, x\rangle \in \theta\}$.

The existence of these sets is guaranteed by the recursion theorem. First we can show that $R_\omega \in R_\omega$, i.e. $(\exists n)\langle n, R_\omega\rangle \in \theta$, is provable in **H**:

$$\dfrac{\dfrac{\dfrac{\dfrac{R_\omega \in R_\omega \equiv (\exists n)\,[\langle n, R_\omega\rangle \in \theta]}{R_\omega \in R_\omega \to (\exists n)\,[\langle n, R_\omega\rangle \in \theta]}}{(\exists n)\,[R_\omega \in R_\omega \to \langle n, R_\omega\rangle \in \theta]}}{(\exists n)\,\langle n+1, R_\omega\rangle \in \theta}}{R_\omega \in R_\omega}$$

Let us assume the induction scheme on ω. We remark that the induction scheme implies the crispness of ω [2]. As we see, $R_\omega \in R_\omega$ is provable, and this means that $\langle 0, R_\omega\rangle \notin \theta$ is provable. For any $n \in \omega$, we can prove $\langle n, R_\omega\rangle \notin \theta \to \langle n+1, R_\omega\rangle \notin \theta$:

$$\dfrac{\dfrac{\dfrac{R_\omega \in R_\omega}{[R_\omega \in R_\omega \to \langle n, R_\omega\rangle \in \theta] \to \langle n, R_\omega\rangle \in \theta}}{\langle n, R_\omega\rangle \notin \theta \to \neg[R_\omega \in R_\omega \to \langle n, R_\omega]}}{\langle n, R_\omega\rangle \notin \theta \to \neg\langle n+1, R_\omega\rangle \in \theta}$$

and $\langle n+1, R_\omega\rangle \notin \theta \to \langle n, R_\omega\rangle \notin \theta$:

$$\dfrac{\dfrac{\dfrac{\langle n+1, R_\omega\rangle \notin \theta}{\neg(R_\omega \in R_\omega \to \langle n, R_\omega\rangle \in \theta)}}{R_\omega \in R_\omega \,\&\, \langle n, R_\omega\rangle \notin \theta}}{\langle n, R_\omega\rangle \notin \theta}$$

Therefore we can conclude $\langle n, R_\omega\rangle \notin \theta \equiv \langle n+1, R_\omega\rangle \notin \theta$ for any $n \in \omega$. The induction scheme proves $(\forall x \in \omega)\langle x, R_\omega\rangle \notin \theta$: This means that $R_\omega \notin R_\omega$, and this contradicts to $R_\omega \in R_\omega$.

We note that, this proof involves that the theory **H** is ω-inconsistent, since $\langle j, R_\omega\rangle \notin \theta$ is provable for any standard natural number j though $(\exists x)\langle x, R_\omega\rangle \in \theta$ is provable. Also we note that, since we use $(\varphi \to \exists x \nu) \to \exists x(\varphi \to \nu)$ and double negation elimination, this proof is not valid in some semantics of **BL**∀. □

4 Conclusion

We introduced the simpler and shorter proof of Hajek's theorem that the mathematical induction on ω implies a contradiction in the set theory with the comprehension principle within **Ł**∀ [2]. We extended the proof of [3] to be effective within **Ł**∀.

This theorem shows that $\mathbf{CŁ}_0$ is ω-inconsistent, and that the recursion contradicts to the induction within **Ł**∀. This means that $\mathbf{CŁ}_0$ gives a new viewpoint to analyze concepts in arithmetic. Since **Ł**∀ is nicely axiomatized, this result might help a proof theoretic study of such recursive definitions.

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