

$\mathbb{L}\Pi$ logic with fixed points

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Abstract We study a system, $\mu\mathbb{L}\Pi$, obtained by an expansion of $\mathbb{L}\Pi$ logic with fixed points connectives. The first main result of the paper is that $\mu\mathbb{L}\Pi$ is standard complete, i.e., complete with regard to the unit interval of real numbers endowed with a suitable structure. We also prove that the class of algebras which forms algebraic semantics for this logic is generated, as a variety, by its linearly ordered members and that they are precisely the interval algebras of real closed fields. This correspondence is extended to a categorical equivalence between the whole category of those algebras and another category naturally arising from real closed fields. Finally, we show that this logic enjoys implicative interpolation.

Keywords $\mathbb{L}\Pi$ logic · Fixed point · Many valued logic

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1 Introduction

Expansions of first order logic (FOL) with fixed point operators have been largely studied, both to better understand inductive properties and to increase the expressiveness of FOL. To our knowledge, so far there is no study in this direction in many valued logic. In this paper we explore this path, in the particular case of $\mathbb{L}\Pi$ logic, showing that it bears to nice properties of the system as well as links with other well established fields of Mathematics.

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The aim is twofolds. On the one hand adding fixed points is stimulating from the algebraic point of view, since it adds new structure to $\mathbb{L}\Pi$ -algebras, leading to structures similar to real closed fields. On the other hand, it is well known that fixed points in logic are strictly related with inductive definitions, whence considering a many-valued system with fixed points explicitly present may give a new insight on inductive definition as well as it may stimulate interesting topics in approximate reasoning.

$\mathbb{L}\Pi$ logic is a combination of two important many-valued logics, Łukasiewicz and Product logic. It was introduced in [10] and deeply explored from the algebraic point of view in [21]. It has been extensively studied, having acquired importance for many reasons: it has been used for formalizing conditional probability [12, 16] and seems to be a good compromise between the foundational formalism of logic and the flexibility of fuzzy logic [26, 27]. Finally, it is the most expressive among the t-norm based logics: it faithfully interprets Łukasiewicz, Product and Gödel Logic. Rational Pavelka Logic (when restricted to finite deductions) is interpretable in $\mathbb{L}\Pi_{\frac{1}{2}}$ logic, which is an expansion of $\mathbb{L}\Pi$ with a constant $\frac{1}{2}$ and the axiom $\frac{1}{2} \leftrightarrow \neg\frac{1}{2}$. Finally, as shown in [18], every logic based on a continuous t-norm with a finite number of idempotents can be defined in $\mathbb{L}\Pi_{\frac{1}{2}}$.

Fixed point theories appear in many different fields of Mathematics, standing at the heart of computer science and being involved in many foundational aspects. The first idea of expanding a logical system by adding fixed points can be found in a proposal by Aho and Ulmann [1]. The first system introduced was the expansion of FOL with minimum fixed points. Several important results were obtained since then, among which some notable links between the expressivity of these logics and the problem $P = NP$ (see, for instance, [9]).

The main obstacle when introducing fixed points in a logical system is to guarantee the existence of a semantical interpretation. Classically this problem is tackled considering formulae as increasing operator over some structure.

Differently from FOL, in the case of many-valued logics based on continuous t -norms there are at least two ways to find a semantics for fixed points operators, grounded on the two known algebraic semantics for those logics. In a classical perspective, one could look at the Kripke semantics of the logic under consideration and define the fixed point of a formula precisely as in modal logic (μ -calculus, see [17]; for details on Kripke semantics for many-valued logic see [4, 24]).

The second way to approach the problem is considering that, in suitable cases, many valued connectives can be considered as functions from $[0, 1]^n$ into $[0, 1]$, that are *continuous*. Therefore Brouwer's theorem ensures a semantical interpretation of the fixed point of a formula.

Theorem 1.1 (Brouwer 1909) *Every continuous function from the closed unit ball D^n to itself has a fixed point.*

Despite the fact that many interesting notions (such as inductive definitions) may appear more easily using the former method, in this article we bias for the latter because of its originality. Still we believe that both approaches deserve consideration and we plan to investigate, at some point, differences and similarities between them.

The paper is organized as follow. In the next section we give a short comparison of the methods used here and the classical approach; in Subsect. 2.1 we introduce basic

definitions useful throughout the paper, furthermore in Subsect. 2.2 the definition of ŁΠ logic and ŁΠ-algebras are given. In Sect. 3 we present μŁΠ logic and its algebraic counterpart, in Subsect. 3.1 algebraic completeness is proved (Theorem 3.10). In Sect. 4 we show that μŁΠ logic enjoys standard completeness (Theorem 4.10), establishing a link between linearly ordered μŁΠ-algebras and real closed fields (Corollary 4.9). Section 5 is devoted to the proof that the class of algebras that form the algebraic semantics for the above logic is categorically equivalent to a class of structures arising as a generalization of real closed fields (see Definition 5.5 and Theorem 5.13). In Sect. 6 we prove that μŁΠ logic enjoys implicative interpolation (Corollary 6.3). We conclude the paper with Sect. 7, in which we outline our future lines of research.

2 Preliminaries

In order to compare the classical approach with the one used in this work, we give a sketch of the ideas used to construct a semantics for FOL with minimum fixed points, similar constructions are used to form different expansions of FOL with fixed points. We suggest to the reader interested in the argument to read [23] for a detailed treatment of the argument and [9] for a perspective on the recent developments.

The various expansions of FOL with fixed point sit between first and second order logic. Given a second order formula $\varphi(R, x)$ such that no second order quantifier appears in φ , x and R are the only free variables and R is the only second order variable in φ , we can associate an operator F , from the subsets of a structure \mathcal{A} to subsets of \mathcal{A} , by the the following definition

$$F(S) = \{a \in A \mid \mathcal{A} \models \varphi([S/R], [a/x])\}$$

where $\varphi[x/y]$ denotes the formula φ in which all free occurrences of y are substituted by x .

An occurrence of a variable x is said to appear *positively* if it is under the scope of an even number of negations. If φ has only positive occurrences of the symbol R then the associated operator is monotone increasing and the set $F = \bigcup_{\alpha} F^{\alpha}$ defined inductively as follows:

$$\begin{aligned} F^0 &= \emptyset \\ F^{\alpha+1} &= \{a \in A \mid \mathcal{A} \models \varphi([F^{\alpha}/R], [a/x])\} \end{aligned}$$

is its (least) fixed point, given by the famous Tarski's theorem.

Theorem 2.1 [28] *Let $\mathcal{L} = \langle L, \leq \rangle$ be any complete lattice. Suppose $f : L \longrightarrow L$ is monotone increasing, i.e., for all x, y in L , $x \leq y$ implies $f(x) \leq f(y)$. Then f has a least fixed point.*

Such fixed point will be the interpretation of the symbol $\mu R, x.\varphi$, more precisely we will have that

$$\mathcal{A} \models \mu R, x.\varphi(t/x) \text{ iff } t^{\mathcal{A}} \in F.$$

In the settings of many-valued logics the domains are still lattice-ordered but the interpretations of formulae are seldom monotone increasing, hence Tarski’s theorem becomes too weak. In this paper we propose a different approach which becomes available when one has to deal with many-valued logics.

The most natural semantics for many-valued logics, and in particular for the logic we study here, is a semantics which interprets formulae as functions from $[0, 1]^n$ to $[0, 1]$. Such semantics of many-valued logics is often called *standard semantics*. Roughly speaking (but see forward for a formal definition) this amount to say that every formula can be seen as a term of a particular algebra based on the real interval $[0, 1]$ and vice-versa; furthermore such a correspondence *commutes with truth*, in the sense that a formula is a tautology in the logic if, and only if, any interpretation of the corresponding term in that particular algebra is equal to 1.

Since in the case of ŁΠ logic, but also for BL, Łukasiewicz logic and other ones, many of those terms are in fact compositions of continuous operations on $[0, 1]$, Brouwer’s theorem ensures that there exists a fixed point for those terms. In our approach we will generalize this introducing explicitly operations which give the minimum fixed point of their associated terms. We will prove this new class of algebras to be exactly the algebraic semantics of the logic introduced and to be generated as a variety by the ŁΠ-algebra $[0, 1]$, endowed with the operations which give the minimum fixed point of their corresponding terms.

2.1 BL, its algebraic semantics and notable extensions

Henceforth we work in the realm of continuous t-norm based logics. To formally specify what this means we need a number of concepts which we will only sketch. The reader not familiar with them may want to check the book [15] for an ample survey on the subject.

Definition 2.2 A continuous **t-norm** is a function $*$ from $[0, 1]^2$ to $[0, 1]$ which is continuous, commutative, associative and non-decreasing, i.e., such that if $x \leq y$ then $x * z \leq y * z$ and finally: $x * 0 = 0$ and $x * 1 = 1$.

The **residuum** \Rightarrow of a t-norm is the unique operation which satisfy the following adjunction:

$$x \leq y \Rightarrow x \text{ if, and only if, } x * y \leq z$$

Example 2.3 The most important continuous t-norms and their residua present in the literature are:

The **Łukasiewicz** t-norm: $x *_L y = \max\{0, x + y - 1\}$ and its residuum: $x \Rightarrow_{*_L} y = \begin{cases} \min\{1, 1 - x + y\} & \text{if } y \leq x \\ 1 & \text{otherwise.} \end{cases}$

The **product** t-norm: $x *_\Pi y = xy$ ¹ and its residuum: $x \Rightarrow_{*_\Pi} y = \begin{cases} \frac{y}{x} & \text{if } y \leq x \\ 1 & \text{otherwise.} \end{cases}$

¹ The ordinary product between real numbers.

The **Gödel** t-norm: $x *_G y = \min\{x, y\}$ and its residuum: $x \Rightarrow_{*G} y = \begin{cases} y & \text{if } y < x \\ 1 & \text{otherwise.} \end{cases}$

Many motivations, which are out of the scope of this article, make reasonable to study logics whose connectives “conjunction” and “implication” can be interpreted as a continuous t-norm and its residuum.

In [15], Hájek introduced BL as a common system to cope with the logics based on continuous t-norms.

Definition 2.4 Basic logic (BL) is a propositional system, with connectives \rightarrow and $\&$ and a constant symbol \perp for falsity, axiomatized by the following formulae:

1. $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \theta) \rightarrow (\varphi \rightarrow \theta))$,
2. $(\varphi \& \psi) \rightarrow \varphi$,
3. $(\varphi \& \psi) \rightarrow (\psi \& \varphi)$,
4. $(\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \& (\varphi \rightarrow \psi))$,
5. $(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\varphi \& \psi) \rightarrow \theta)$,
6. $((\varphi \rightarrow \psi) \rightarrow \theta) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \theta) \rightarrow \theta)$,
7. $\perp \rightarrow \varphi$.

The only rule of the system is *modus ponens*.

In the modern approach, the three most known t-norm based logics can be described as extensions of BL as follows.

Definition 2.5

Lukasiewicz logic is BL plus the axiom: $\varphi = \neg\neg\varphi$, where $\neg\varphi$ stands for $\varphi \rightarrow \perp$;

Gödel logic is BL plus the axiom: $(\varphi \wedge \psi) \rightarrow (\varphi \& \psi)$, where $\varphi \wedge \psi$ stands for $\varphi \& (\varphi \rightarrow \psi)$.

Product logic is BL plus the axiom: $\neg\psi \vee ((\psi \rightarrow \varphi \& \psi) \rightarrow \varphi)$, where $\varphi \vee \psi$ stands for $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$;

We introduce now the concept of **evaluation**, this will allow to see the links between t-norms, logical systems and classes of algebras. In particular it will be possible to give a formal explanation of what we mean by “t-norm based logic”. We only sketch the ideas, the reader interested on the abstract correspondence may consult [3].

Given a propositional system L as above consider a class of algebras \mathbf{A} whose type corresponds to the one of the language of L . Once a bijective correspondence between connectives of the language and operations of the algebras is established one can easily define a function, which sends formulae to terms of the algebras and vice-versa. If, for instance, the logical language has as set of connective the symbols $\&$, \rightarrow , \perp as above and the class of algebra has type $\langle \&, \rightarrow, \perp \rangle$ then such a translation can be recursively defined as:

- $e(\varphi \& \psi) = e(\varphi) \& e(\psi)$,
- $e(\varphi \rightarrow \psi) = e(\varphi) \rightarrow e(\psi)$,
- $e(\perp) = \bar{\perp}$.

Where e sends different propositional variables p_i, p_j in different individual variables x_i, x_j . If we indicate with the symbol \vdash_L the logical consequence in the system L and

with $\models_{\mathbf{A}}$ the equational consequence relation in \mathbf{A} , then we will say that the logic L is **algebraically complete** w.r.t. \mathbf{A} if for any set of formulae $\{\varphi\} \cup \Gamma$:

$$\Gamma \vdash_L \varphi \quad \text{if, and only if,} \quad \Gamma \models_{\mathbf{A}} e(\varphi) = 1.$$

If there exists an algebra \mathcal{S} in \mathbf{A} whose underlying set is the real interval $[0, 1]$ and whose operations $\bar{\&}, \bar{\rightarrow}, \bar{\perp}$ are respectively a t-norm, its residuum and 0, then we will say that L is **standard complete** if

$$\vdash_L \varphi \quad \text{if, and only if,} \quad \models_{\mathcal{S}} e(\varphi) = 1.$$

When this happens we call \mathcal{S} **standard algebra** and we say that L is “t-norm based”. We also say that a system is “the logic of a particular (set of) t-norm”. The following result motivates the choices of names used so far.

Theorem 2.6 [7, 15]

*Lukasiewicz logic is the logic of the Lukasiewicz t-norm $*_L$.
 Product logic is the logic of the product t-norm $*_{\Pi}$.
 Gödel logic is the logic of the Gödel t-norm $*_G$.*

In [14] Hájek conjectured that BL were the logic of all continuous t-norms; this turned out to be true, as shown in [6].

Being so strictly tied to the logic, the algebraic semantics turned out to be a major instrument of study.

Definition 2.7 A **BL-algebra** is an algebra $\mathcal{A} = \langle A, *, \Rightarrow, \wedge, \vee, 0, 1 \rangle$ that satisfies

- $\langle A, \wedge, \vee, 0, 1 \rangle$ is a lattice with greatest and least element,
- $\langle A, *, 1 \rangle$ is a commutative monoid,
- $*$ and \Rightarrow form an adjoint pair, i.e., $z \leq (x \Rightarrow y)$ iff $x * z \leq y$,
- $x \wedge y = x * (x \Rightarrow y)$ (divisibility),
- $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$ (pre-linearity).

Writing $\neg x$ for $x \Rightarrow 0$, an **MV-algebra** is a BL-algebra that satisfies: $\neg\neg x = x$ and a Π -**algebra** is a BL-algebra the satisfies: $\neg x \vee ((x \Rightarrow x * y) \Rightarrow y)$.

Note that if we have a logic L which is standard complete w.r.t. a standard algebra S , L will be algebraically complete w.r.t. the variety generated by S . The following result establishes the missing link among the concepts we have introduced so far.

Theorem 2.8

- [7] *The variety of MV-algebras is generated by the standard algebra $\langle [0,1], *_L, \Rightarrow_{*_L}, 0 \rangle$.*
- [15] *The variety of Π -algebras is generated by the standard algebra $\langle [0,1], *_{\Pi}, \Rightarrow_{*_{\Pi}}, 0 \rangle$.*
- [15] *The variety of Gödel algebras is generated by the standard algebra $\langle [0,1], *_G, \Rightarrow_{*_G}, 0 \rangle$.*

2.2 ŁΠ Logic

ŁΠ Logic was introduced in [10] to deal at the same time with Łukasiewicz and Product logics, its language contains two conjunctions and their respective residua. Its axiomatization was simplified as follows in [8].

Definition 2.9 The language of ŁΠ is built from propositional variables, combined with the following connectives: $\&_L$ and \rightarrow_L , $\&_\Pi$ and \rightarrow_Π and \perp .

ŁΠ denotes the theory whose axioms and rules are the following:

1. All the axioms of Łukasiewicz logic for $\&_L$ and \rightarrow_L
2. All the axioms of Product logic for $\&_\Pi$ and \rightarrow_Π
3. $\varphi \&_\Pi \neg_L (\psi \rightarrow_L \theta) \leftrightarrow_L \neg_L ((\varphi \&_\Pi \psi) \rightarrow_L (\varphi \&_\Pi \theta))$
4. $\Delta(\varphi \rightarrow_L \psi) \rightarrow_L (\varphi \rightarrow_\Pi \psi)$
5. The rules *Modus Ponens* and *necessitation* ($\frac{\varphi}{\Delta(\varphi)}$)

where, following Definition 2.5 we define $\neg_L \varphi = \varphi \rightarrow_L \perp$ and $\neg_\Pi \varphi = \varphi \rightarrow_\Pi \perp$. Furthermore $\Delta \varphi = \neg_\Pi(\neg_L \varphi)$. Lattice operations are also definable as in Definition 2.5. In particular axiom 4 states that the lattice operations defined by $\&_\Pi, \rightarrow_\Pi$ and the ones defined by $\&_L, \rightarrow_L$ are the same.

ŁΠ $\frac{1}{2}$ denotes the logic obtained from ŁΠ by adding a propositional constant $\frac{1}{2}$ together with the axiom $\frac{1}{2} \leftrightarrow_L \neg_L \frac{1}{2}$.

In order to study this logic from an algebraic point of view, the following class of algebras were introduced:

Definition 2.10 A ŁΠ-algebra is a structure $\mathcal{A} = \langle A, *_L, \Rightarrow_L, *_\Pi, \Rightarrow_\Pi, 0, 1 \rangle$ where:

1. $\langle A, *_L, \Rightarrow_L, 0, 1 \rangle$ is a MV-algebra
2. $\langle A, *_\Pi, \Rightarrow_\Pi, 0, 1 \rangle$ is a Π algebra
3. $x *_\Pi (y \ominus z) = (x *_\Pi y) \ominus (x *_\Pi z)$
4. $\Delta(x \Rightarrow_L y) \Rightarrow_L (x \Rightarrow_\Pi y) = 1_A$

where $x \ominus y = \neg_L(x \Rightarrow_L y)$, $x \oplus y = \neg_L x \Rightarrow_L y$ and $\Delta(x)$ is a shorthand for $\neg_\Pi \neg_L x$.

In order to simplify the notation we will often drop the symbol $_L$ both from the connectives and from the operations. Moreover we will use the symbol \cdot or juxtaposition for $*_\Pi$, so $x *_\Pi y = x \cdot y = xy$.

Every linearly ordered ŁΠ-algebra with more than two elements is necessarily infinite and has one element such that $\neg x \Leftrightarrow x$ [21, Lemma 4.3]. So, modulo an expansion of the language, every infinite linearly ordered ŁΠ-algebra is a member of the algebraic semantics of ŁΠ $\frac{1}{2}$ logic, and it is called ŁΠ $\frac{1}{2}$ algebra.

3 μŁΠ Logic

In FOL, and in general in the first approach presented above, fixed points are restricted to formulae having only *positive* occurrences of the relation free variable. Thanks to

our different approach this restriction does not apply here: even the formula $\neg p$ has a fixed point, namely $\frac{1}{2}$.

Nevertheless, in order to meet the requirements in Brouwer’s theorem (Theorem 1.1), we need to restrict only to formulae whose interpretation is continuous. In particular the \rightarrow_{Π} connective may cause problems because its functional interpretation has a discontinuity on the point $(0, 0)$. Of course there are formulae whose interpretation is continuous and present occurrences of \rightarrow_{Π} , but we do not have any syntactical characterisation for all the formulae whose interpretation is continuous. Hence, to be sure of the existence of a semantical counterpart for the fixed point, we shall restrict ourselves only to formulae in which the symbol \rightarrow_{Π} does not occur, we will call those formulae *continuous formulae*.

Notice that another approach could be based on $\mathbb{L}\Pi_q$ -algebras, introduced in [25]. There the discontinuity of \rightarrow_{Π} is overcome by defining a new connective \rightarrow_q which approximates \rightarrow_{Π} as precisely as wanted, still being continuous.

It should also be noted that, in this approach, μ has to “bind” the propositional variable which is under its scope. This is because the formula resulting from an application of μ to a continuous formula it is not necessarily continuous and then it is not suitable for a further application of μ . For this reason, loosely speaking, the same rules of first order logic apply here for substitutions; however, we do not have to worry about this because, as we will see in a moment, there is another suitable way of introducing fixed point in the logic.

To circumvent the problems above we will introduce in the language a new connective $\mu_{\varphi(x, \bar{y})}(\bar{y})$ for any continuous formula φ . This will allow us, when switching to the algebraic semantics of the logic, to consider fixed points as functions on the algebra, rather than operator. If one think of any term $t(x, \bar{y})$ as a generalized connective then it is easy to look also at $\mu x_t(x, \bar{y})(\bar{y})$ as a connective whose arity is the arity of $t(x, \bar{y})$ minus one.

When we write $t(x, \bar{y})$, we mean that the variables x, \bar{y} actually occur in t . We will use this notation for formulae, terms and functions.

It should made precise that $\mu x_t(x, \bar{y})(\bar{y})$ is hence a function which takes a tuple of elements of a structure (the interpretation of \bar{y}) to another element of the structure (the minimum fixed point of the function represented by $t(x, \bar{y})$ under that interpretation). Nevertheless, in order to simplify the exposition we will often refer by an abuse of notation to $\mu x_t(x, \bar{y})(\bar{y})$ as *the fixed point of $t(x, \bar{y})$* .

Definition 3.1 The Fixed Point $\mathbb{L}\Pi$ Logic ($\mu\mathbb{L}\Pi$ logic for short) has the following theory:

The language is an expansion of $\mathbb{L}\Pi$ language by an infinity of new connectives $\mu x_{\varphi(x, \bar{y})}$, where φ is any continuous formula and the arity of $\mu x_{\varphi(x, \bar{y})}$ is the length of \bar{y} .

The axioms are:²

1. All axioms and rules from $\mathbb{L}\Pi$ Logic
2. $\mu x_{\varphi(x, \bar{y})}(\bar{y}) \leftrightarrow_L \varphi(\mu x_{\varphi(x, \bar{y})}(\bar{y}), \bar{y})$

² Since the μ connectives have different arities we will use for them the prefix notation, whereas, to not confuse the reader, we will keep the infix notation for the other connectives.

3. $\Delta(\varphi(z, \bar{y}) \leftrightarrow_L z) \rightarrow_L (\mu x_{\varphi(x, \bar{y})}(\bar{y}) \rightarrow_L z)$
4. $\bigwedge_{i \leq n} \Delta(\psi_i \leftrightarrow_L \xi_i) \rightarrow (\mu x_{\varphi(x, \bar{y})}(\psi_1, \dots, \psi_n) \leftrightarrow_L \mu x_{\varphi(x, \bar{y})}(\xi_1, \dots, \xi_n))$

All axioms have a rather clear meaning: axiom 2 says that $\mu x_{\varphi(x, \bar{y})}(\bar{y})$ is a fixed point of $\varphi(x, \bar{y})$; axiom 3 guarantees that the μ -connectives give the minimum fixed point; finally axiom 4 says that we can substitute equivalent subformulae also under the scope of a μ -connective.

Now we turn to an algebraic approach to this logic, introducing its algebraic counterpart. Following the notation introduced so far we will call **continuous term** a term t in which \Rightarrow_{Π} does not appear; we will call $Cterm$ the set of continuous terms.

Definition 3.2 A $\mu\mathbb{L}\Pi$ algebra is an algebra of type

$$\mathcal{L} = \langle L, *, \Rightarrow, \Rightarrow_{\Pi}, \cdot, \{\mu x_{t(x, \bar{y})}\}_{t(x, \bar{y}) \in Cterm}, 0, 1 \rangle,$$

such that it satisfies the following conditions.

1. $\mathcal{L} = \langle L, *, \Rightarrow, \Rightarrow_{\Pi}, \cdot, 0, 1 \rangle$ is a $\mathbb{L}\Pi$ -algebra
2. $\mu x_{t(x, \bar{y})}(\bar{y}) = t(\mu x_{t(x, \bar{y})}(\bar{y}), \bar{y})$
3. If $t(s, \bar{y}) = s$ then $\mu x_{t(x, \bar{y})}(\bar{y}) \leq s$

Obviously this axiomatization is not finite, still it is worthwhile to note that $\mu\mathbb{L}\Pi$ -algebras form a variety. Indeed from the axiomatization it is evident that $\mu\mathbb{L}\Pi$ -algebras are a quasivariety, but the Δ operator makes possible to define a *discriminator* (see [20], for the definition) and, as proved in [19], any quasivariety with a discriminator is a variety.

We give now some example to clarify the concepts of μ connectives.

Example 3.3 For sake of simplicity let us confine in this example to linearly ordered $\mu\mathbb{L}\Pi$ -algebras (we will see in next section that this is not a very restrictive condition). Consider the term $x * y$, then it is easily seen that $\mu x_{x * y} = 0$ because $0 * a = 0$ for any a . Note that also the term $\mu x_x = 0$, but while the former is a unary term sending any interpretation of y to 0 the latter is an actual constant.

Modulo an expansion of the language $\mu\mathbb{L}\Pi$ -algebras form a subvariety of the variety of $\mathbb{L}\Pi$ algebras. Since the axioms of $\mu\mathbb{L}\Pi$ -algebras force the existence of an element x satisfying $\neg x \Leftrightarrow x$ in every algebra of the variety, they are also, again modulo an expansion of the language, a subvariety of the variety of $\mathbb{L}\Pi_{\frac{1}{2}}$. It is important to keep in mind these relations as in the rest of the paper we will speak freely of $\mathbb{L}\Pi$ and $\mathbb{L}\Pi_{\frac{1}{2}}$ reduct for this class of algebras.

Example 3.4 To construct a concrete example of $\mu\mathbb{L}\Pi$ -algebra we consider the standard $\mathbb{L}\Pi$ -algebra: $\langle [0, 1], *, \Rightarrow, \Rightarrow_{\Pi}, \cdot, 0, 1 \rangle$. To endow it with a family of functions $\{\mu x_{t(x, \bar{y})} \mid t(x, \bar{y}) \in Cterm\}$ we consider the function associating to any tuple $\bar{a} \in [0, 1]^n$, the minimum fixed point of the polynomial $t(x, \bar{a}/\bar{y})$, given by Theorem 1.1. If we call $f_{t(x, \bar{y})}$ such functions, then it is easily see that the algebra

$$\langle [0, 1], *, \Rightarrow, \Rightarrow_{\Pi}, \cdot, \{f_{t(x, \bar{y})}\}_{t(x, \bar{y}) \in Cterm}, 0_L, 1_L \rangle,$$

satisfies the conditions of Definition 3.2

We conclude this section with a proposition which can also be seen as a little exercise to get acquaintance with fixed points.

One may wonder whether the introduction of a *maximum* fixed point would have changed the setting. The following proposition answers negatively to the question.

Proposition 3.5 *Given any continuous formula φ its maximum fixed point is definable in the language of $\mu\text{L}\Pi$.*

Proof Let $\varphi(x, \bar{y})$ any continuous formula with \bar{y} possibly empty. We claim that $m(\bar{y}) = \neg\mu x\psi(x)(\bar{y})$, where $\psi(x, \bar{y}) = \neg(\varphi(\neg x, \bar{y}))$, gives the maximum fixed point of φ . Let us first prove that it is a fixed point of φ .

Fix any tuple \bar{y} so that we can omit it in the following. We have $m = \neg\mu x\psi(x)$ so $\neg m \leftrightarrow \mu x\psi(x)$, i.e., $\neg m$ is a fixed point of $\psi(x)$: $\neg m \leftrightarrow \psi(\neg m)$. Hence $\neg m \leftrightarrow \neg\varphi(\neg\neg m)$ which implies $m \leftrightarrow \varphi(m)$. To show that m is the maximum among fixed points let v be such that $\varphi(v) \leftrightarrow v$, then $\psi(\neg v) \leftrightarrow \neg v$, so $\neg v \rightarrow \mu x\psi(x)$ therefore $v \rightarrow \neg\mu x\psi(x) \leftrightarrow \mu x\neg\varphi(\neg(x)) \leftrightarrow m$.

3.1 Algebraic completeness

The presence of the operator Δ enhances drastically the expressibility of a logic. Whereas in $\text{L}\Pi$ logic Δ is definable due to the presence of both \neg_L and \neg_Π , it can be introduced independently by an axiomatic extension. We present such an extension in the case of MV-algebra for further use.

Definition 3.6 [15] A MV_Δ -algebra is a MV-algebra with an operator Δ that satisfies:

- (1) $\Delta(1) = 1$.
- (2) $\Delta(x \Rightarrow y) \leq \Delta(x) \Rightarrow \Delta(y)$.
- (3) $\Delta(x) \vee \neg\Delta(x) = 1$.
- (4) $\Delta(x) \leq x$.
- (5) $\Delta(\Delta(x)) = \Delta(x)$.
- (6) $\Delta(x \vee y) = \Delta(x) \vee \Delta(y)$.

Notice that, on a linearly ordered structure the behavior of Δ is the following:

$$\Delta(y) = \begin{cases} 1 & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases},$$

so, in a logical perspective, it can be seen as a ‘‘crisp’’ operator stating that a given formula is a theorem.

Example 3.7 Even if pleonastic, we give a further example on the use of fixed point, showing how the delta operator can be defined in a linearly ordered $\text{L}\Pi$ -algebra only using the connectives \neg, \oplus .

Using the fact that, in a linearly ordered structure,

$$\mu x_{x \oplus y}(y) = \begin{cases} 0 & \text{if } y = 0 \\ 1 & \text{otherwise} \end{cases},$$

it is easy to see that $\neg\mu x_{x\oplus\neg y}(y)$ behaves exactly has the delta operator $\Delta(y)$.

Lemma 3.8 *Any $\mu\mathcal{L}\Pi$ -algebra has the same congruences of its underlying MV_Δ -algebra.*

Proof Obviously, every $\mu\mathcal{L}\Pi$ congruence is a MV_Δ congruence. For the other direction note that the property, $x\theta y$ if, and only if, $(x \Leftrightarrow y)\theta 1$, holds for both kinds of congruences. Suppose, for some n and $i \leq n$, x_i, y_i are in a MV_Δ congruence, θ . Then $(x_i \Leftrightarrow y_i)\theta 1$, hence $\Delta(x_i \Leftrightarrow y_i)\theta \Delta(1) = 1$ and since congruences are closed under \wedge we have $\bigwedge_{i \leq n} (\Delta(x_i \Leftrightarrow y_i))\theta 1$. Note now that since $\Delta(z) \leq z$ we also have that

$$\bigwedge_{i \leq n} (\Delta(x_i \Leftrightarrow y_i)) \leq (x_i \Leftrightarrow y_i) = \mu z_{t(z, \bar{w})}(x_1, \dots, x_n) \Leftrightarrow \mu z_{t(z, \bar{w})}(y_1, \dots, y_n),$$

whence:

$$\mu z_{t(z, \bar{w})}(x_1, \dots, x_n) \Leftrightarrow \mu z_{t(z, \bar{w})}(y_1, \dots, y_n)\theta 1$$

that is equivalent to

$$\mu z_{t(z, \bar{w})}(x_1, \dots, x_n)\theta \mu z_{t(z, \bar{w})}(y_1, \dots, y_n).$$

□

Theorem 3.9 *Any $\mu\mathcal{L}\Pi$ -algebra is isomorphic to a subdirect product of linearly ordered $\mu\mathcal{L}\Pi$ -algebras.*

Proof By Birkoff subdirect representation Theorem, every $\mu\mathcal{L}\Pi$ -algebra is isomorphic to a subdirect product of subdirectly irreducible $\mu\mathcal{L}\Pi$ -algebras. Since irreducibility is completely determined by the lattice of congruences, by Lemma 3.8 a $\mu\mathcal{L}\Pi$ -algebra is subdirectly irreducible if, and only if, its underlying MV_Δ -algebra is subdirectly irreducible. But a MV_Δ -algebra which is subdirectly irreducible must be linearly ordered, whence the statement of the theorem. □

Theorem 3.10 *If φ is a formula in the language of $\mu\mathcal{L}\Pi$ logic, then the following are equivalent:*

- (i) φ is provable in $\mu\mathcal{L}\Pi$;
- (ii) For each linearly ordered $\mu\mathcal{L}\Pi$ -algebra \mathcal{A} , $\mathcal{A} \models e(\varphi) = 1$;
- (iii) For each $\mu\mathcal{L}\Pi$ -algebra \mathcal{A} , $\mathcal{A} \models e(\varphi) = 1$;

where e is the canonical evaluation which sends formulae of the $\mu\mathcal{L}\Pi$ logic to their correspondent terms in the language of $\mu\mathcal{L}\Pi$ -algebras.

Proof (i) implies (ii) is easy to prove. For (ii) implies (iii) we argue by contraposition. Suppose that for some $\mu\mathcal{L}\Pi$ -algebra $\mathcal{A} \not\models e(\varphi) = 1$, since validity of $e(\varphi) = 1$ is preserved under subdirect product by Theorem 3.9 some subdirectly irreducible factor \mathcal{A}_i does not satisfies $e(\varphi) = 1$ but \mathcal{A}_i is linearly ordered and the claim follows.

(iii) implies (i) is proved by the usual construction of the Lindenbaum algebra. Indeed, if a formula φ is not provable then $e(\varphi) = 1$ fails in the Lindenbaum algebra. Since the Lindenbaum algebra of $\mu\mathbb{L}\Pi$ logic is a $\mu\mathbb{L}\Pi$ -algebra this implies that there exists at least one $\mu\mathbb{L}\Pi$ -algebra in which $e(\varphi) = 1$ does not hold. \square

4 Standard completeness

In order to prove standard completeness, in this section we will establish a link between linearly ordered $\mu\mathbb{L}\Pi$ -algebras and real closed fields. This result will be generalized in the next section. Nevertheless, considered the importance and the good properties that real closed fields enjoy, this preliminary result has an interest in its own. In Sect. 6, for instance, it will be used to prove the implicative interpolation of $\mu\mathbb{L}\Pi$ logic.

Our construction is based on the results contained in [11], for the reader’s ease we replicate here some definitions.

Definition 4.1 Given a linearly ordered $\mathbb{L}\Pi_{\frac{1}{2}}$ algebra \mathcal{A} , consider the structure $\Phi(\mathcal{A}) = \langle K, +, -, \times, \leq, 0_K, 1_K \rangle$ defined in the following way:

$$K = \{(z, x) \mid z \in Z, x \in A, x \neq 1\}, \quad 0_K = (0, 0), \quad 1_K = (1, 0)$$

$$(n, x) + (m, y) = \begin{cases} (n + m, x \oplus y) & \text{if } x \oplus y < 1 \\ (n + m + 1, x * y) & \text{if } x \oplus y = 1 \end{cases}$$

$$-(n, x) = \begin{cases} (-n, 0) & \text{if } x = 0 \\ (-(n + 1), \neg x) & \text{if } 0 < x < 1 \end{cases}$$

$$(n, x) \leq (m, y) \text{ if } n < m \text{ or } n = m \text{ and } x \leq y$$

$$(n, x) \times (m, y) = (nm, x \cdot y) + m(0, x) + n(0, y)$$

$$\text{Where } m(0, x) = \begin{cases} \underbrace{(0, x) + \dots + (0, x)}_{m\text{-times}} & \text{if } m \geq 0 \\ \underbrace{-(0, x) + \dots + (-(0, x))}_{m\text{-times}} & \text{if } m < 0 \end{cases}$$

$\Phi(\mathcal{A})$ is a linearly ordered, commutative, domain of integrity and it can be extended to a linearly ordered field by taking its field of fractions. The interval algebra of the resulting field is \mathcal{A} .

Vice-versa, given a linearly ordered field it is easy to construct a $\mathbb{L}\Pi$ -algebra: for what said before, such an algebra can be seen as a $\mathbb{L}\Pi_{\frac{1}{2}}$ as soon as it has more than two elements.

Definition 4.2 Given a linearly ordered field \mathcal{K} we define an $\mathbb{L}\Pi$ -algebra \mathcal{A} , called the **$\mathbb{L}\Pi$ -interval algebra** of \mathcal{K} , in the following way

$$\begin{aligned}
 A &= \{x \in K \mid 0 \leq x \leq 1\} \\
 x * y &= \max(0, x + y - 1) \quad x \Rightarrow y = \min\{1, 1 - x + y\} \quad x \cdot y = x \times y \\
 x \Rightarrow_{\Pi} y &= \begin{cases} 1 & \text{if } x \leq y \\ z & \text{otherwise} \end{cases}
 \end{aligned}$$

where z is the only element such that $y = x * z$.

For a complete proof that the previous definitions are correct (i.e., they define respectively a linearly ordered field and a linearly ordered $\mathbb{L}\Pi$ -algebra) see Theorem 7 in [11]:

Theorem 4.3 *Every linearly ordered $\mathbb{L}\Pi_{\frac{1}{2}}$ algebra is isomorphic to the $\mathbb{L}\Pi_{\frac{1}{2}}$ -interval algebra of a linearly ordered field.*

Our aim is now to find a similar result for $\mu\mathbb{L}\Pi$ -algebras. Obviously we need to enrich the structure of linearly ordered fields with something corresponding to fixed points. The natural choice are real closed fields.

Definition 4.4 A **real closed field** is a linearly ordered field such that:

- every positive element is a square: $\forall x \exists y (x = y^2)$
- every polynomial of odd degree has a solution: $\forall a_1, \dots, a_n \exists x_0 (a_0 x_0^n + \dots + a_n x_0 = 0)$

The next lemma shows how the behavior of a polynomial can be locally replicated in the unit interval by another suitable polynomial.

Lemma 4.5 *Given a polynomial $f(x)$ with coefficients in a real closed field \mathcal{F} , there exists another polynomial $g(x)$, with coefficients in the same structure, such that:*

- (i) *the coefficients, the sum of their moduli and the roots of $g(x)$ are all in the unit interval of \mathcal{F} ;*
- (ii) *there exists an $m \in F$ such that $g(t) = 0$ if, and only if, $f(2mt - m) = 0$.*

Proof Let $f(x) = a_n x^n + \dots + a_1 x + a_0$ and call x_1, \dots, x_n its roots. By a Cauchy’s theorem [5] we have that

$$\max_{i \leq n} |x_i| \leq \max_{i \leq n-1} (n \cdot |a_i|)^{\frac{1}{n-i}}.$$

Set $m = \max\{(n \cdot |a_i|) \mid 1 \leq i \leq n\}$. Note that we can assume, without loss of generality, that at least one of the absolute values of the coefficients is greater than 1, otherwise one could pass to the following step, then

$$\max \left\{ (n \cdot a_i)^{\frac{1}{n-i}} \mid i \leq n - 1 \right\} \leq m.$$

Let $g'(x)$ the polynomial obtained by $f(x)$ by the substitution $x \mapsto \frac{x+m}{2m}$. If z is a root of $g'(x)$ then for some solution x_i of $f(x)$:

$$z = \frac{x_i + m}{2m} \leq \frac{m + m}{2m} = m,$$

and

$$z = \frac{x_{\bar{i}} + m}{2m} \geq 0 \quad \text{as } |x_{\bar{i}}| \leq m.$$

Hence all roots of $g'(x)$ are in the unit interval of \mathcal{F} , notice that $x \mapsto \frac{x+m}{2m}$ is a bijection and $g'(t) = 0$ if, and only if, $f(2mt - m) = 0$. So condition (ii) is satisfied. As next step we normalize the coefficients of $g'(x)$ by dividing all of them by $\sum_{i \leq n} |a_i|$. Calling the resulting polynomial $g(x)$, it is readily seen that it meets condition (i), while the transformation from $g'(x)$ to $g(x)$ does not interfere with condition (ii). □

We are ready now to spell out to construct a real closed field starting from a $\mu\mathbb{L}\Pi$ -algebra. By Theorem 4.3 we know that given a $\mathbb{L}\Pi_{\frac{1}{2}}$ -algebra it is possible to build a linearly ordered field. Since any $\mu\mathbb{L}\Pi$ -algebra has a (definable) $\mathbb{L}\Pi_{\frac{1}{2}}$ -reduct, one can, with the same construction, associate to any $\mu\mathbb{L}\Pi$ -algebra a linearly ordered field. Let us call it the field **associated** to a $\mu\mathbb{L}\Pi$ -algebra.

Theorem 4.6 *Any field associated to a $\mu\mathbb{L}\Pi$ -algebra is a real closed field.*

Proof Let \mathcal{K} be the linearly ordered field given by the $\mu\mathbb{L}\Pi$ -algebra \mathcal{A} . We will begin by proving that \mathcal{K} satisfies the second condition of Definition 4.4. To this end we will first restrict to a particular class of polynomials, which we call $[0, 1]$ -polynomials.

An $[0, 1]$ -polynomial is a polynomial of the form $P(x) = a_n x^n + \dots + a_1 x + a_0$ such that the sum of the absolute values of the coefficients as well as all its solutions are contained in the set $\{k \in \mathcal{K} \mid 0_{\mathcal{K}} \leq k \leq 1_{\mathcal{K}}\}$.

Given any $[0, 1]$ -polynomial of odd degree $P(x)$ we will prove that it has a solution in \mathcal{K} by looking at the image under the embedding of interpretation of the fixed point of a particular $\mu\mathbb{L}\Pi$ -term. Indeed notice that if we are able to find a $\mu\mathbb{L}\Pi$ -term, $p'(x)$, such that $p'(x) = 0$ iff $P(x) = 0$, then letting $p(x) = p'(x) \oplus x$ we have that $\mu x.(p(x))$ is the wanted element.

We explain now how to construct such a $p'(x)$. Given a polynomial $P(x)$, as above, we first arrange it in the form $P'(x) = |\sum_{i \in J} a_i x^i - \sum_{i \in K} a_i x^i|$ where all a_i are positive. After such arrangement we can safely replace every occurrence of $+$ with \oplus and the sign minus with $|x - y|$ (defined as $(x \ominus y) \vee (y \ominus x)$). If we call \bar{a}_i the inverse image under the embedding of a_i and we put

$$p'(x) = \left| \bigoplus_{i \in J} \bar{a}_i \odot x^i - \bigoplus_{i \in K} \bar{a}_i \odot x^i \right|.$$

Then we have that $p'(x) = 0$ exactly when $P(x) = 0$, hence is the $\mu\mathbb{L}\Pi$ -term we were looking for.

This proves that all $[0, 1]$ -polynomial of odd degree have a solution in \mathcal{K} . Now, by Lemma 4.5, given any polynomial $P(x)$, of odd degree, with coefficients in \mathcal{K} we can associate to it a $[0, 1]$ -polynomial, which has a solution in \mathcal{K} . Since the solutions of the two polynomials are linked by the correspondence $x \mapsto \frac{x+m}{2m}$, this guarantees that also $P(x)$ has a solution in \mathcal{K} .

The proof that \mathcal{K} satisfy also the first condition of Definition 4.4, i.e that every element of \mathcal{K} is a square of an element in \mathcal{K} , is similar. One has only to notice that in a $\mu\mathbb{L}\Pi$ -algebra every element a can be written as $b \cdot b$ where $b = \mu x |x \cdot x - a|_{\oplus x}$. \square

Corollary 4.7 *Every linearly ordered $\mathbb{L}\Pi^{\frac{1}{2}}$ -algebra can be embedded in at most one linearly ordered $\mu\mathbb{L}\Pi$ -algebra (up to isomorphism).*

Proof Suppose that a linearly ordered $\mathbb{L}\Pi^{\frac{1}{2}}$ -algebra \mathcal{A} can be embedded in two non-isomorphic linearly ordered $\mu\mathbb{L}\Pi$ -algebras \mathcal{B} and \mathcal{C} . Since the fields $\mathcal{F}_{\mathcal{A}}$, $\mathcal{F}_{\mathcal{B}}$ and $\mathcal{F}_{\mathcal{C}}$, associated respectively to \mathcal{A} , \mathcal{B} and \mathcal{C} are definable in the language of $\mathbb{L}\Pi$ -algebras, $\mathcal{F}_{\mathcal{B}}$ and $\mathcal{F}_{\mathcal{C}}$ are not isomorphic and $\mathcal{F}_{\mathcal{A}}$ embeds in both of them. This is a contradiction, for both $\mathcal{F}_{\mathcal{B}}$ and $\mathcal{F}_{\mathcal{C}}$ are real closed fields. \square

The correspondence between linearly ordered $\mu\mathbb{L}\Pi$ -algebras and real closed fields, holds also in the other direction, i.e.

Theorem 4.8 *The $\mathbb{L}\Pi^{\frac{1}{2}}$ interval algebra of every real closed field is a reduct of a $\mu\mathbb{L}\Pi$ -algebra.*

Proof We need to prove that if we take the unitary interval of a real closed field this can be endowed with the structure of a $\mu\mathbb{L}\Pi$ -algebra. Take a real closed field \mathcal{R} and call \mathcal{A} the $\mathbb{L}\Pi^{\frac{1}{2}}$ algebra given by Theorem 4.3. The only thing that has to be checked is that μ can be defined in \mathcal{A} for any term $t(x)$ not containing \Rightarrow_{Π} . To this end note that $t(x)$ can be represented in the \mathcal{R} as

$$\bigvee_{i \in I} \bigwedge_{j \in J} P_{ij}(x),$$

where P_{ij} are polynomial in \mathcal{R} . So our problem reduces to prove that in \mathcal{R} holds:

$$\exists x \left(0 \leq x \leq 1 \text{ and } \left(\bigvee_{i \in I} \bigwedge_{j \in J} P_{ij}(x) \right) = x \right). \tag{1}$$

But this formula is true in the reals (by Theorem 1.1) and every real closed field is elementary equivalent to the reals hence the formula holds also in \mathcal{R} . So $\mu x.t(x)$ is defined as the minimum witness of Eq. (1). \square

Corollary 4.9 *Every linearly ordered $\mu\mathbb{L}\Pi$ -algebra is isomorphic to the interval algebra of some real closed field. Conversely every real closed field is isomorphic to a real closed field associated to a linearly ordered $\mu\mathbb{L}\Pi$ -algebra.*

Proof By construction every $\mu\mathbb{L}\Pi$ -algebra is isomorphic to the $\mu\mathbb{L}\Pi$ -interval algebra of its associated real closed field. In the same way, every real closed field is isomorphic to the real closed field associated to its $\mu\mathbb{L}\Pi$ -interval algebra. \square

Theorem 4.10 *$\mu\mathbb{L}\Pi$ is standard complete, i.e., a formula φ is provable in $\mu\mathbb{L}\Pi$ if, and only if, $e(\varphi) = 1$ is true on the $\mu\mathbb{L}\Pi$ -algebra on $[0, 1]$.*

Proof One direction is obvious. For the other one let us reason by contraposition and suppose that a formula φ is not provable in $\mu\mathbb{L}\Pi$, then by Theorem 3.10 the equation $e(\varphi) = 1$ does not hold in some linearly ordered $\mu\mathbb{L}\Pi$ -algebra \mathcal{A} . Denote by \mathcal{R} the field associated to \mathcal{A} , which is real closed by Theorem 4.6. Call ψ the first order formula, which express the failure of $e(\varphi) = 1$ in the unitary interval of \mathcal{R} . Since \mathcal{R} is a elementary equivalent to the reals, ψ fails in the reals, witnessing the failure of $e(\varphi) = 1$ in its interval algebra. \square

5 Categorical equivalence

In this section we strengthen the result contained in Corollary 4.9. If we want to extend such a representation to any $\mu\mathbb{L}\Pi$ -algebra we have to face the fact that real closed fields are linearly ordered. Hence such structures must be substituted by a more general ones which allow a lattice-order structure. Note that the Naïve idea to use just subdirect product of real closed fields does not suit, for in a subdirect product some solution of some polynomial can be missing.

After recalling some necessary definitions we will cope with the problem described above. We will describe some “characteristic” terms which will serve to axiomatize the structures that we need.

Definition 5.1 (cf [2]) A lattice ordered abelian group, ℓ -**group** for short, is a structure $\mathcal{G} = \langle G, +, -, \vee, \wedge, 0 \rangle$ such that $\langle G, +, -, 0 \rangle$ is an abelian group and $\langle G, \vee, \wedge \rangle$ is a lattice. Furthermore if \leq is the partial order induced by \vee, \wedge then for all $a, b, x \in G$ if $a \leq b$ then $a + x \leq b + x$.

In an ℓ -group \mathcal{G} , an element u is called **strong unit**, if for every $g \in \mathcal{G}$ there is a natural number n such that $g \leq \underbrace{u + \dots + u}_{n \text{ times}}$.

A **lattice ordered ring** is a structure $\mathcal{R} = \langle R, +, -, \times, \vee, \wedge, 0 \rangle$ such that:

- $\langle R, +, -, \times, 0 \rangle$ is a ring,
- $\langle R, +, -, \vee, \wedge, 0 \rangle$ is a ℓ -group and
- for all $a, b, x \in R$ if $a \leq b$ and $x \geq 0$ then $a \times x \leq b \times x$.

An **f-ring** is a lattice ordered ring which is the subdirect product of linearly ordered rings.

In the following a commutative f-ring with strong unit will be called **c-1-f-ring**.

In [21] a class of particular c-1-f-rings was introduced.

Definition 5.2 An **f-semifield** is a c-1-f-ring equipped with an additional operation $^{-1}$ that satisfies:

1. $x^2 \times x^{-1} = x$
2. $0^{-1} = 0$
3. $|x^{-1} - y^{-1}| \leq |x - y| \times (|x^{-1} \times y^{-1}| + |x - y|^{-1} \times (|x^{-1}| + |y^{-1}|))$

Remark 5.3 Note that, the concept of linearly ordered f-semifield and linearly ordered field are equivalent even if they formally differ by the explicit presence of the operator $^{-1}$.

In the same article $\mathbb{L}\Pi_{\frac{1}{2}}$ -algebras were proved to be categorical equivalent to f -semifields. We want to lift this result to $\mu\mathbb{L}\Pi$ -algebras.

In this general setting we will call **f-polynomial** every combination of meets and joins of polynomials. In order to cope with f -polynomial we will need terms which can give us information on the value that they can take. One of such terms, already used in literature is ∇ . It is defined as $\nabla(x) = x \times x^{-1}$, hence, in a linearly ordered f -semifield, it takes value 0, if $x = 0$ and 1 otherwise. For this reason ∇ can be seen as the dual of the Δ of $\mathbb{L}\Pi$ -algebras. Whence it makes sense to define: $\delta(x)$ as $1 - \nabla(1 - x)$.

Finally we define $N(x) = 1 - \nabla(x + |x|)$, its meaning become more clear looking at its value in a linearly ordered f -semifield. Indeed in every linearly ordered f -semifield we have that

$$\delta(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases} \quad N(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

We are now ready to give the definition of the structure that will substitute real closed field in our further results. In order to stress the similarity with real closed fields, we give an equivalent version of Definition 4.4, from which the definition of real closed f -semifield was inspired.

Definition 5.4 A real closed field is a linearly ordered field in which every polynomial which changes its sign in the interval $[a, b]$ has a root in the interval $[a, b]$.

Definition 5.5 A **real closed f-semifield** is a structure:

$$\mathcal{F} = \langle F, +, -, \times, ^{-1}, \{f_n\}_{n \in \mathbb{N}}, \vee, \wedge, 0, 1 \rangle$$

which enjoys the following properties:

- $\mathcal{F} = \langle F, +, -, \times, ^{-1}, \vee, \wedge, 0, 1 \rangle$ is a f -semifield.
- for every f -polynomial $p(x) = \bigvee_{j \in J} \bigwedge_{k \in K} \sum_{i \leq n} a_{ijk} x^i$ and every $a_i, b_i, t \in F$:
 1. $N(p(x)p(y)) \times N(x - y) \leq N(x - f(x, y, p)) \times N(f(x, y, p) - y) \times [1 - \nabla(p(f(x, y, p)))]$
 2. $[1 - \nabla(p(t))] \times N(t - y) \times N(x - t) \leq N[f(x, y, p) - t]$
 3. $\bigvee_{i=0}^n (\nabla(|a_i - b_i|)) \geq \nabla(|f_n(a_n, \dots, a_0) - f_n(b_n, \dots, b_0)|)$
 where $f(x, y, p)$ is a shorthand for $f_n(x, y, a_{000}, \dots, a_{ijk})$ with f_n of the suitable arity.

The reader may check that in the case of a linearly ordered f -semifield the meaning of the axioms is the following. Axiom 1 gives the intended role of the family of functions f_n , namely to force the existence of zeros also in a subdirect product of those structures, for all polynomials which change their sign; axiom 2 guarantees that the solutions given by the f is minimal; axiom 3 ensures that the family of functions is compatible with the congruences of the underlying f -semifield.

Note that a linearly ordered real closed f -semifield only differs from a real closed field by the explicit presence of the family of functions f_i , as proved in the following proposition.

Proposition 5.6 *In a real closed field every f -polynomial which changes its sign in a given interval has a solution in that interval.*

Proof The proof is by induction on the number of polynomials composing the f -polynomial. If this is 1, then the f -polynomial is a polynomial and the statement follows from the axioms of real closed fields.

Suppose now that the f -polynomial, call it $f(x)$, is formed by n polynomials and it is equal to $f'(x) \wedge p(x)$ where $f(x)$ is a f -polynomial formed by $n - 1$ polynomials and $p(x)$ is a polynomial. To prove the theorem let us assume that $f(x)$ changes its sign in an interval $[a, b]$ and suppose, to fix the ideas, that $f(a) > 0$ and $f(b) < 0$.

The proof now splits in several subcases.

If $f(a) = f'(a)$ and $f(b) = f'(b)$ then by induction hypothesis there exists t such that $f'(t) = 0$. If $f(t) = f'(t)$ there is nothing to prove, otherwise, since we are in a linearly ordered structure, we have that $f(t) = p(t) < f'(t) = 0$, so $p(a) \geq 0$ and $p(t) < 0$, which again implies the existence of a $t' \neq t$ such that $p(t') = 0$. If $f(t') \neq p(t')$ then $f'(t') < p(t') = 0$, hence $f'(t') < 0$ and $f'(a) \geq 0$ which in turns implies the existence of a $t'' \neq t'$ (and different from t) for which $f'(t'') = 0$. But this reasoning must stop after a finite number of times, thus leading to a zero for $f(x)$ in the interval.

The case in which $f(a) = p(a)$ and $f(b) = p(b)$ is identical. Also the case where $f(a) = f'(a)$ and $f(b) = p(b)$ or $f(a) = p(a)$ and $f(b) = f'(b)$ are easy to reduce to the same reasoning. Finally for the case in which $f(x) = f'(x) \vee p(x)$ the proof works dually. □

Lemma 5.7 *Every real closed f -semifield has the same congruences of its underlying f -semifield.*

Proof One way is obvious. For the other direction, given a real closed f -semifield \mathcal{F} , we only have to check that the family of functions f_i is compatible with the congruence of the underlying f -semifield \mathcal{F}^- . First of all let us note that given an ideal $J, x \in J$ if, and only if $\nabla(x) \in J$ (because $x = x\nabla(x)$). Suppose now that $a_i \theta b_i$ for any $i \leq n$ then this means that there exists an ideal of \mathcal{F} , call it J , such that $a_i - b_i \in J$, hence $\nabla(|a_i - b_i|) \in J$ so even $\bigvee_{i=0}^n (\nabla(a_i - b_i)) \in J$ and by axiom 3 of Definition 5.5 we have $\nabla|f_n(a_n, \dots, a_0) - f_n(b_n, \dots, b_0)| \in J$, hence $f_n(a_n, \dots, a_0) \theta f_n(b_n, \dots, b_0)$. □

From this and Proposition 5.6 it easily comes the following proposition.

Proposition 5.8 *Every real closed f -semifield has a reduct which is the subdirect product of linearly ordered real closed f -semifields.*

Our next aim is to link $\mu\mathbb{L}\Pi$ -algebras to real closed f -semifields. To this end we need some preliminary facts on how to manipulate f -polynomials on real closed f -semifields.

Lemma 5.9 *Given a f -polynomial $f(x)$ on an f -semifield \mathcal{F} which changes its sign in the interval $[0, 1]$, there exist a term $p(x)$ of the $\mathbb{L}\Pi_{\frac{1}{2}}$ interval algebra of \mathcal{F} and an interpretation i such that $f(t) = 0$ iff $i(p(t)) = 0$.*

Proof Given $f(x) = \bigvee_{j \in J} \bigwedge_{k \in K} \sum_{i \leq n} a_{ijk} x^i$, let $m = \sum_{i \leq n, j \in J, k \in K} a_{ijk}$ and consider the f-polynomial

$$g(x) = \bigvee_{j \in J} \bigwedge_{k \in K} \sum_{i \leq n} \frac{a_{ijk}}{m} x^i = \bigvee_{j \in J} \bigwedge_{k \in K} \sum_{i \leq n} b_{ijk} x^i.$$

Arranging every polynomial in $g(x)$ as

$$\sum_{i \leq n} (b_{ijk} \vee 0) x^i - \sum_{i \leq n} (b_{ijk} \wedge 0) x^i$$

it is easy to realize that $f(t) = 0$ if, and only if, in the interval algebra of \mathcal{F} holds

$$i \left(\bigvee_{j \in J} \bigwedge_{k \in K} \left| \bigoplus_{i \leq n} (y_{ijk} \vee 0) \odot t^i - \bigoplus_{i \leq n} (y_{ijk} \wedge 0) \odot t^i \right| \right) = 0$$

where i is an interpretation which sends each variable y_{ijk} in b_{ijk} . □

Theorem 5.10 *Every $\mu\text{Ł}\Pi$ -algebra is isomorphic to the interval algebra of some closed f-semifield.*

Proof Given a $\mu\text{Ł}\Pi$ -algebra \mathcal{A} consider the f-semifield \mathcal{F} associated to its ŁΠ reduct. We will construct a family of definable f_n in order to make \mathcal{F} a real closed f-semifield.

Given $P(x)$, an f-polynomial and an interval $[a, b]$ in \mathcal{F} consider the bijection $\sigma : [0, 1] \rightarrow [a, b]$ given by $\sigma(x) = (b - a)x + a$, and the new f-polynomial $P'(x) = P(\sigma(x))$, then by the construction of Lemma 5.9 we get a $\text{Ł}\Pi_{\frac{1}{2}}$ term, $p(x)$, associated to $P'(x)$ and an interpretation i . Let $f(a, b, P) = \sigma^{-1}(i(\mu x.[(p(x) \oplus x)]))$.

To show that indeed those f satisfy the axioms in Definition 5.5, let $P_i(x)$ be the i th projection of $P(x)$ in the subdirect representation of \mathcal{F} . In the factors of the representation, both parts of the inequality of axioms 1 are evaluated either to 0 or to 1. Let $a, b \in A_i$, if $N(P_i(a)P_i(b)) \times N(a - b) = 0$ then there is nothing to check. Otherwise, supposing $a \leq b$, we have to show that $a \leq f(a, b, P_i) \leq b$ and $P_i(f(a, b, P_i)) = 0$. The first two inequalities hold, because $a \leq \sigma(x) \leq b$. For the last one, note that by Lemma 5.9 $P'(i(\mu x.[p(x) \oplus x])) = 0$ hence $P(\sigma^{-1}(i(\mu x.[(p(x) \oplus x)]))) = 0$.

Axiom 2 is satisfied because, by axiom 2 of Definition 3.2, $\mu x.[p(x) \oplus x]$ is the smallest element for which $p(x) = 0$ and σ preserves the order.

Axiom 3 directly follows from the fact that the f_n are definable. □

The established link holds also in the other direction

Theorem 5.11 *Every real closed f-semifield contains a $\mu\text{Ł}\Pi$ -interval algebra.*

Proof If \mathcal{F} is a real closed f-semifield then it has a ŁΠ-interval algebra, \mathcal{A} . To prove that \mathcal{A} can be endowed with the structure of a $\mu\text{Ł}\Pi$ -algebra we need to find a fixed point for every term $t(x)$ in \mathcal{A} that is \Rightarrow_{Π} -free. Let $T(x)$ the corresponding f-polynomial in \mathcal{F} . If $T(0) = 0$, then set $\mu x.t(x) = 0$, otherwise consider $T'(x) = T(x) - x$. Note that $T'(0) = T(0) - 0 > 0$ and $T'(1) = T(1) - 1 \leq 0$ hence $T'(0)T'(1) \leq 0$ so, letting $t = f(0, 1, T')$, we have $T'(t) = 0$ which means $T(t) = t$. □

Definition 5.12 Let $\bar{\Phi}$ the functor from the category of $\mu\mathbb{L}\Pi$ -algebras to the one of real closed f-semifields. $\bar{\Phi}$ assigns to each $\mu\mathbb{L}\Pi$ -algebra, the real closed f-semifield constructed as in Theorem 5.10.

Let $\bar{\Psi}$ the functor from the category of real closed f-semifields to the one of $\mu\mathbb{L}\Pi$ -algebras. $\bar{\Psi}$ assigns to every real closed f-semifield, the $\mu\mathbb{L}\Pi$ -algebra of Theorem 5.11.

To define $\bar{\Phi}$ and $\bar{\Psi}$ on morphisms just note that a function is a morphism of $\mu\mathbb{L}\Pi$ -algebras if, and only if, it is a morphism of $\mathbb{L}\Pi$ -algebras. Then $\bar{\Phi}$ and $\bar{\Psi}$ act as Φ and Ψ the functors defined as in [21].

Theorem 5.13 *The category of $\mu\mathbb{L}\Pi$ -algebras and the category of real closed f-semifields are equivalent.*

Proof We have to prove that there are natural isomorphisms,

$$\eta : \bar{\Phi}\bar{\Psi} \cong id_{\mathbb{F}} \text{ and } \beta : \bar{\Psi}\bar{\Phi} \cong id_{\mathbb{L}\mathbb{P}}.$$

In other words we have to show that for any pair \mathcal{A}, \mathcal{B} of $\mu\mathbb{L}\Pi$ -algebras, and any $f : \mathcal{A} \rightarrow \mathcal{B}$ a morphism, and for every pair \mathcal{F}, \mathcal{G} of real closed f-semifields with a morphism $g : \mathcal{F} \rightarrow \mathcal{G}$ there exist two pairs of morphisms, $\beta_{\mathcal{A}}, \beta_{\mathcal{B}}$ and $\eta_{\mathcal{F}}, \eta_{\mathcal{G}}$ such that the following diagrams commute:

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\
 \beta_{\mathcal{A}} \downarrow & & \downarrow \beta_{\mathcal{B}} \\
 \bar{\Psi}\bar{\Phi}(\mathcal{A}) & \xrightarrow{\bar{\Psi}\bar{\Phi}(f)} & \bar{\Psi}\bar{\Phi}(\mathcal{B})
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{F} & \xrightarrow{f} & \mathcal{G} \\
 \eta_{\mathcal{F}} \downarrow & & \downarrow \eta_{\mathcal{G}} \\
 \bar{\Phi}\bar{\Psi}(\mathcal{F}) & \xrightarrow{\bar{\Phi}\bar{\Psi}(g)} & \bar{\Phi}\bar{\Psi}(\mathcal{G})
 \end{array}$$

Considering the $\mathbb{L}\Pi^{\frac{1}{2}}$ reducts, $\beta_{\mathcal{A}}$ and $\beta_{\mathcal{B}}$ exist and they are also morphisms of $\mu\mathbb{L}\Pi$ -algebras. Moreover \mathcal{A} and $\bar{\Psi}\bar{\Phi}(\mathcal{A})$, as well as \mathcal{B} and $\bar{\Psi}\bar{\Phi}(\mathcal{B})$ are isomorphic as $\mathbb{L}\Pi^{\frac{1}{2}}$ -algebras by construction and since all the constructions involved in $\bar{\Psi}$ and $\bar{\Phi}$ are definable, they are isomorphic as $\mu\mathbb{L}\Pi$ -algebras. The case of real closed f-semifields is similar. □

6 Amalgamation

We conclude this work with an easy application of the ideas contained in [22]. In this article the relation between amalgamation and interpolation are pointed out and explored in the most important cases. A key remark is that, due to the absence of a deduction theorem for most of these logic, deductive interpolation and implicative interpolation are not equivalent in this framework.

Given a class of structures \mathbf{K} , let \mathbf{K}_{lin} be the linearly ordered members of \mathbf{K} . In [22], Lemma 3.3 and 3.4 the following result is proved.

Lemma 6.1 *Let \mathbf{K} a quasi-variety of BL-algebras possibly with additional operators such that \mathbf{K}_{lin} has the amalgamation property. Then \mathbf{K} has the amalgamation property.*

Theorem 6.2 *Linearly ordered $\mu\mathbb{L}\Pi$ -algebras enjoy amalgamation.*

Proof For sake of simplicity we show only that for every \mathcal{A}, \mathcal{B} and \mathcal{C} linearly ordered $\mu\mathbb{L}\Pi$ -algebras such that $\mathcal{A} = \mathcal{B} \cap \mathcal{C}$, there exist \mathcal{D} and embeddings h and k of \mathcal{B} and \mathcal{C} respectively into \mathcal{D} such that the restriction of h and k to \mathcal{A} coincide. This will readily implies the theorem. Let \mathcal{I}, \mathcal{J} and \mathcal{K} be the real closed fields built respectively from \mathcal{A}, \mathcal{B} and \mathcal{C} , we can safely suppose $\mathcal{I} = \mathcal{J} \cap \mathcal{K}$. By the amalgamation property for real closed fields we know that there is a real closed field \mathcal{L} and embeddings h and k from \mathcal{J}, \mathcal{K} into \mathcal{L} such that h and k coincide on \mathcal{I} , but then h and k coincide also on \mathcal{A} . Hence the interval algebra of \mathcal{L} plus the restrictions of h and k to \mathcal{J} and \mathcal{K} is the amalgam we were looking for. \square

Since in commutative residuated lattices the amalgamation property is equivalent to the interpolation for the correspondent logic (see [13]), this proves:

Corollary 6.3 *$\mu\mathbb{L}\Pi$ logic enjoys deductive interpolation.*

7 Conclusions and open problems

We expanded $\mathbb{L}\Pi$ logic with new connectives which allow to define fixed points of a subset of formulae in the language of $\mathbb{L}\Pi$. In Theorem 3.10 we proved that the algebraic structures defined in Definition 3.2 are precisely the algebraic semantics of this logic. This result helped us to prove Theorem 4.10 which states that $\mu\mathbb{L}\Pi$ logic enjoys standard completeness.

The proof of Theorem 4.10 sheds light on the tight relation between the algebraic semantics of $\mu\mathbb{L}\Pi$ logic and real closed fields. This correspondence is explored and generalized in Sect. 5 where, in Theorem 5.13, we proved that the class of algebras that form the algebraic semantics for the above logic is categorically equivalent to the class of real closed f-semifields (see Definition 5.5).

Real closed f-semifields can be seen as the meeting point of real closed fields and f-rings. Indeed, to generalize the correspondence between linearly ordered $\mathbb{L}\Pi$ -algebras and real closed fields, one needs new structures with the properties of both classes above. On one hand the result established in Proposition 3.9 tells that the structures suitable for a generalization of the above correspondence need all to be subdirect product of linearly ordered structures; on the other hand the presence of fixed point functions in $\mu\mathbb{L}\Pi$ -algebras forces the existence of solutions for a suitable class of polynomials which must be kept when taking subdirect product. The methods we used to guarantee that both properties are met at the same time is to include in the language of real closed fields, explicit functions which give the solutions of their associated polynomial. The relation between real closed fields and real closed f-semifields can be found through Propositions 5.6 and 5.8.

The last result of the paper uses, as a simple application, the correspondence between linearly ordered $\mathbb{L}\Pi$ -algebras and real closed fields, to show that $\mu\mathbb{L}\Pi$ logic enjoys implicative interpolation (Corollary 6.3).

We wish to stress once more that our approach to fixed points in logic is radically different from the classical one, and that this is possible just because we deal with many-valued logics. Working with a new *modus operandi*, during our research we found many interesting questions which we did not have the time to answer. We propose some of them here both to present our future lines of research and with the ambition to stimulate the interest of other researchers.

We are exploring at the moment the results of adding a (min or max) fixed point operator to BL, or Łukasiewicz logics. Note that the only extension of BL logic having a continuous residuum and standard completeness is Łukasiewicz logic. Whereas in Łukasiewicz logic all connectives have a continuous interpretation, in the case of BL the fixed points which are guaranteed to exist are for formulae in which only the symbols $*$, \vee , \wedge occur.

We had to menage an annoying connective here (namely \rightarrow_{Π}) would be possible to develop the same work in PMV-algebras? Note that in Łukasiewicz logic Δ is already definable as $\mu x \neg(\neg y \& x)$, so μ PMV and μ Ł Π should have interesting relations.

μ Ł Π logic is decidable, due to the elimination of quantifiers in real closed fields, but which is its complexity? (The problem is not trivial since a sharp bound on the complexity of Ł Π is unknown)

Does μ Ł Π Logic enjoys deductive interpolation or, putting in a semantical perspective, does the class of μ Ł Π -algebras enjoys the strong amalgamation property? Characterize the free algebra of μ Ł Π (i.e., find an equivalent of McNaughton's theorem for μ Ł Π).

Is it possible to lift all the machinery to first order many-valued logic?

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