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Homogeneous iteration and measure one covering relative to HOD

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Abstract Relative to a hyperstrong cardinal, it is consistent that measure one covering fails relative to HOD. In fact it is consistent that there is a superstrong cardinal and for every regular cardinal κ, κ^+ is greater than κ^+ of HOD. The proof uses a very general lemma showing that homogeneity is preserved through certain reverse Easton iterations.

Keywords HOD · Homogeneity · Measure one covering

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1 Introduction

Assuming that there is no inner model with a Woodin cardinal, Steel constructs in [8] a certain inner model K^c, from which the "true" core model K for a Woodin cardinal is obtained. An important lemma in the derivation of K from K^c is the following.

Lemma 1 (Measure one covering relative to K^c, see [8]) *Assume there is no inner model with a Woodin cardinal. If κ is a measurable cardinal with a normal measure μ then $\{\alpha < \kappa \mid \alpha^+ = \alpha^+ \text{ of } K^c\}$ has μ-measure one.*

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A natural question is whether such a result will hold in the context of stronger large cardinal properties, such as superstrength or supercompactness. The inner model K^c is contained in HOD, the universe of hereditarily ordinal definable sets, and therefore it is natural to ask:

Question Can measure one covering fail relative to HOD? I.e., is it consistent that for some measurable cardinal κ with normal measure μ, the set $\{\alpha < \kappa \mid \alpha^+ = \alpha^+$ of HOD$\}$ has μ-measure zero?

Theorem 1 *Relative to a hyperstrong cardinal, it is consistent that measure one covering fails relative to HOD. In fact it is consistent that there is a superstrong cardinal and for every regular cardinal κ, κ^+ is greater than κ^+ of HOD.*

Definition 1 κ is *superstrong* iff κ is the critical point of a $j : V \to M$ with $H(j(\kappa)) \in M$. κ is *hyperstrong* iff κ is the critical point of a $j : V \to M$ with $H(j(\kappa)^+) \in M$.

In the hierarchy of consistency strengths we have: Measurable < Strong < Woodin < Superstrong < κ^+ supercompact < Hyperstrong < κ^{++} supercompact.

Along the way to obtaining Theorem 1, we prove two very general lemmas showing that different weak forms of homogeneity are preserved in reverse Easton iterations (see Lemmas 5 and 6). At the end we mention a further application of homogeneity-preservation in reverse Easton iterations to the study of morasses (due to Brooke-Taylor and Friedman).

2 Proof of Theorem 1

This section will culminate in the proof of Theorem 1. Starting with a hyperstrong cardinal κ (or just a cardinal κ which is κ^+-supercompact), we show in Lemma 2 that a certain class-length iterated forcing P, a reverse Easton iteration of collapses, keeps at least a superstrong cardinal in the extension universe. Next, we note in Lemma 3 that any open-dense homogeneous forcing (see Definition 2) preserves HOD. Finally, we prove in Lemma 5 that P is open-dense homogeneous, in fact weakly homogeneous (see Definition 2). Theorem 1 then follows.

Lemma 2 *Suppose κ is hyperstrong and GCH holds. Let P be the class-length reverse Easton iteration where at each stage α which is regular after the first α iteration stages, α^+ is collapsed to α by the collapse $Coll(\alpha, \alpha^+)$ (and the iteration is trivial at stages α not of this form). Then κ remains superstrong after forcing with P.*

Proof This is a standard master condition argument (see [2]) combined with the hyperultrapower methods of [3] and [4]. For cofinally many successor cardinals γ, P can be written as $P({<}\gamma) * P({\geq}\gamma)$ where $P({<}\gamma)$ has size less than γ and $P({\geq}\gamma)$ is γ-closed. It follows that P preserves ZFC.

Let $j : V \to M$ witness the hyperstrength of κ. We may assume that every element of M is of the form $j(f)(a)$ where $f : H(\kappa^+) \to V$ and a belongs to $H(j(\kappa)^+)$. Let G be P-generic over V. To show that κ is superstrong in $V[G]$ it suffices to show that there is a P^M-generic G^M over M such that $j[G] \subseteq G^M$, G^M is definable in $(V[G], G)$ and such that the $H(j(\kappa))$ of $V[G]$ is in $M[G^M]$. Here, P^M denotes M's version of P.

Choose $G^M_{j(\kappa)}$ to be $G_{j(\kappa)}$, the generic for the first $j(\kappa)$ stages of the iteration. At stage $j(\kappa)$ in the iteration P^M we must choose $G^M(j(\kappa))$ to contain the pointwise image under (the lifting to $V[G_\kappa]$ of) j of $G(\kappa)$. But $j[G(\kappa)]$ is a compatible set of conditions in $P^M(j(\kappa))$ which belongs to $M[G_{j(\kappa)}]$ (as $j \restriction H(\kappa^+)$ belongs to M) and has size κ^+ of V, less than $j(\kappa)$. As the forcing $P^M(j(\kappa))$ is $j(\kappa)$-directed closed, there is a single condition p in $P^M(j(\kappa))$ which is stronger than all conditions in $j[G(\kappa)]$ (a "master" condition). By the homogeneity of the forcing $P^M(j(\kappa)) = \mathrm{Coll}(j(\kappa), j(\kappa)^+)$, in $V[G_{j(\kappa)}][G(j(\kappa))]$ we can choose a generic $G^M(j(\kappa))$ for $P^M(j(\kappa))$ which contains the condition p. Thus we have $j[G_{\kappa+1}] \subseteq G^M_{j(\kappa)+1}$, providing a lifting of j to $V[G_{\kappa+1}]$, which we continue to write as j.

The forcing $P(>\kappa)$ is κ^+-closed in $V[G_{\kappa+1}]$ and κ^+ of V is collapsed in $V[G_{\kappa+1}]$. Now we claim that if $D \in M[G^M_{j(\kappa)+1}]$ is a set-sized maximal antichain in the forcing $P^M(>j(\kappa))$ then D is met by an element of $j[G(>\kappa)]$. Indeed, D can be written as $j(f)(a)^{G^M_{j(\kappa)+1}}$ where $f : H(\kappa^+ \text{ of } V) \to V$, a belongs to $H(j(\kappa)^+)$ and $H(\kappa^+)$ of V has cardinality κ in $V[G_{\kappa+1}]$. By the κ^+ closure of the forcing $P(>\kappa)$, there is a condition $\bar{p} \in G(>\kappa)$ which meets each maximal antichain $\bar{D}$ on $P(>\kappa)$ of the form $f(\bar{a})^{G_{\kappa+1}}$, $\bar{a} \in H(\kappa^+)$ of V; but then $j(\bar{p}) = p \in j[G(>\kappa)]$ meets each maximal antichain D' on $P^M(>j(\kappa))$ of the form $j(f)(a')^{H_{j(\kappa)+1}}$, $a' \in H(j(\kappa^+))$ of $M = H(j(\kappa^+))$. In particular, p meets the original $D = j(f)(a)^{G^M_{j(\kappa)+1}}$. A similar argument works for definable dense classes D and not just set-sized maximal antichains.

Thus we can take $G^M(>j(\kappa))$ to be the class of conditions extended by some condition in $j[G(>\kappa)]$. The resulting $G^M = G^M_{\kappa+1} * G^M(>\kappa)$ is the desired P^M generic containing the pointwise image of G under j. □

Thus if G is P-generic over V and V has a hyperstrong, then in $V[G]$ we have that the V-successor of every regular cardinal is collapsed and there is a superstrong cardinal. It remains only to show that HOD of $V[G]$ is contained in V.

Remark 1 In fact, in Lemma 2, we could have started with a cardinal κ which is just κ^+-supercompact. After the forcing, κ will remain the limit of superstrong cardinals (although κ itself may not be superstrong).

We now define three versions of weak homogeneity.

Definition 2 Let P be a set partial ordering.

1. [5,6] P is *weakly homogeneous* iff for any two conditions p, q in P there is an automorphism π of P such that $\pi(p), q$ are compatible.
2. P is *open-dense homogeneous* iff for any two conditions p, q in P, there is an open dense set $D \subseteq P$, an isomorphism $\pi : D \to D$, and a p' in D such that $p' \leq p$ and $\pi(p'), q$ are compatible.
3. For $p \in P$, let $\mathrm{Cone}(p)$ denote $\{r \in P : r \leq p\}$, the cone of conditions in P below p. P is *cone homogeneous* iff for any two conditions $p, q \in P$, there exist $p' \leq p$, $q' \leq q$, and an isomorphism $\pi : \mathrm{Cone}(p') \to \mathrm{Cone}(q')$.

Remark 2 Weakly homogeneous is called *almost homogeneous* in [7].

Fact 1 *Let P be a set partial ordering.*

1. *If P is weakly homogeneous, then P is open-dense homogeneous.*
2. *If P is open-dense homogeneous, then P is cone homogeneous.*
3. *Suppose for each $p \in P$, there are two incompatible elements of P below p (P being separative and atomless is sufficient). Then cone homogeneity of P implies open-dense homogeneity of P.*

Proof 1. is immediate from the definitions.

Suppose P is a set partial ordering which is open-dense homogeneous. Let $p, q \in P$. There exist an open dense set $D \subseteq P$, $p' \leq p$, $q' \leq q$ in D, and an isomorphism $\pi : D \to D$ such that $\pi(p') = q'$. Claim: $\pi \restriction \text{Cone}(p') : \text{Cone}(p') \to \text{Cone}(q')$ is a cone isomorphism. Certainly π is 1–1 and order preserving. $\pi \restriction \text{Cone}(p')$ is onto $\text{Cone}(q')$, since given $r \leq q'$, $\pi^{-1}(r) \leq \pi^{-1}(q') = p'$, and $\pi(\pi^{-1}(r)) = r$. Therefore, $\pi \restriction \text{Cone}(p')$ is a cone isomorphism between $\text{Cone}(p')$ and $\text{Cone}(q')$.

Now suppose P is a cone homogeneous set partial ordering such that for each $p \in P$, there are two incompatible elements of P below p. Let $p, q \in P$. We will find an open dense set D, an isomorphism π of D, and $p_0 \leq p$, $q_0 \leq q$ in D such that $\pi(p_0) = q_0$.

First take $p' \leq p$ and $q' \leq q$ such that p' and q' are incompatible. The existence of such p' and q' follows easily from our assumption that below any element of P there are two incompatible elements. By cone homogeneity, there exist $p_0 \leq p'$ and $q_0 \leq q'$ and an isomorphism $\sigma_0 : \text{Cone}(p_0) \to \text{Cone}(q_0)$. Since p_0 and q_0 are incompatible, we obtain for free an isomorphism $\pi_0 : \text{Cone}(p_0) \cup \text{Cone}(q_0) \to \text{Cone}(p_0) \cup \text{Cone}(q_0)$ by defining $\pi_0 \restriction \text{Cone}(p_0) = \sigma$ and $\pi_0 \restriction \text{Cone}(q_0) = \sigma^{-1}$.

Let $X_0 = \{p_0, q_0\}$. By induction, for each $\alpha < |P|^+$, build an antichain X_α and π_α an isomorphism of $D_\alpha := \{s \in P : \exists r \in X_\alpha (s \leq r)\}$ such that for all $\beta < \alpha$, $X_\beta \subseteq X_\alpha$ and $\pi_\alpha \restriction D_\beta = \pi_\beta$ (actually, by enumerating P at the beginning, one can do this construction in $\leq |P|$ steps). Let $\alpha < |P|^+$. If X_α is a maximal antichain in P, then π_α is the desired automorphism witnessing open-dense homogeneity of P. If X_α is not a maximal antichain in P, then there exist $p'_{\alpha+1}, q'_{\alpha+1}$ incompatible with each other and with every element in X_α, since for each $p \in P$ there are two incompatible elements of P below p. There exist $p_{\alpha+1} \leq p'_{\alpha+1}$ and $q_{\alpha+1} \leq q'_{\alpha+1}$ and a cone isomorphism $\sigma_{\alpha+1} : \text{Cone}(p_{\alpha+1}) \to \text{Cone}(q_{\alpha+1})$. Let $X_{\alpha+1} = X_\alpha \cup \{p_{\alpha+1}, q_{\alpha+1}\}$ and $D_{\alpha+1} = \{s \in P : \exists r \in X_{\alpha+1} (s \leq r)\}$. Let $\pi_{\alpha+1}$ be $\pi_\alpha \cup \sigma_{\alpha+1} \cup \sigma_{\alpha+1}^{-1}$. Then $\pi_{\alpha+1}$ is an isomorphism on $D_{\alpha+1}$ as $\pi_\alpha, \sigma_{\alpha+1}$ and $\sigma_{\alpha+1}^{-1}$ are defined on incompatible sets of conditions. At limit ordinals α, let $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$, $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$ and $\pi_\alpha = \bigcup_{\beta<\alpha} \pi_\beta$.

At some ordinal $\alpha < |P|^+$, X_α will be a maximal antichain of P. Let $D = D_\alpha$ and $\pi = \pi_\alpha$. Then D is an open dense subset of P, $p_0, q_0 \in D$, π is an isomorphism of D, and $\pi(p_0) = q_0$. Hence, P is open-dense homogeneous. □

Lemma 3 *Suppose that P is a cone homogeneous set forcing, P belongs to HOD and G is P-generic over V. Then HOD of $V[G]$ is contained in HOD of V.*

Proof The following argument can be found in [5]. It suffices to show that if a is a set of ordinals in $V[G]$ which is definable in $V[G]$ with ordinal parameters then a belongs to V. Write $x = \{\alpha \mid V[G] \vDash \varphi(\alpha, \beta)\}$. Then $\alpha \in x$ iff $p \Vdash \varphi(\alpha, \beta)$ for some $p \in G$. We claim that if $p \Vdash \varphi(\alpha, \beta)$ then in fact $q \Vdash \varphi(\alpha, \beta)$ for all $q \in P$. Indeed, suppose $p \Vdash \varphi(\alpha, \beta)$ and $q \in P$. By cone homogeneity there are $p' \leq p$ and $q' \leq q$ and an isomorphism $\pi : \mathrm{Cone}(p') \to \mathrm{Cone}(q')$. Since $p' \Vdash \varphi(\alpha, \beta)$, also $q' \Vdash \varphi(\alpha, \beta)$. Hence, there is a dense set of elements which force $\varphi(\alpha, \beta)$. So we have $x = \{\alpha \mid p \Vdash \varphi(\alpha, \beta) \text{ for some } p \in P\}$, and since P and its forcing relation are ordinal-definable in V, it follows that x belongs to HOD of V. □

In the rest of the paper, we will show that forcings like that used in Lemma 2 are weakly homogeneous. We first consider iterations of length equal to some ordinal.

Definition 3 Let $\lambda \in \mathrm{Ord}$. We say that $\langle P_\alpha \mid \alpha \leq \lambda\rangle$ is an *iteration* with support $\mathscr{S} = \langle \mathscr{S}_\alpha \mid \alpha \leq \lambda\rangle$ if

1. For each $\alpha \leq \lambda$, $\mathscr{S}_\alpha \subseteq \mathscr{P}(\alpha)$;
2. For each $\alpha \leq \lambda$, $x \subseteq y$ and $y \in \mathscr{S}_\alpha \to x \in \mathscr{S}_\alpha$;
3. For all $\beta < \alpha \leq \lambda$, $\mathscr{S}_\beta \subseteq \mathscr{S}_\alpha$;
4. If $x, y \in \mathscr{S}_\alpha$, then $x \cup y \in \mathscr{S}_\alpha$.
5. $P_{\alpha+1}$ is the preorder $P_\alpha * \dot{Q}_\alpha$, where $P_\alpha \Vdash \dot{Q}_\alpha$ is a set partial ordering;
6. For limit α, P_α is the preorder consisting of all α-sequences p such that for $\beta < \alpha$, $p \restriction \beta \Vdash_{P_\beta} p(\beta) \in \dot{Q}_\beta$, and $\mathrm{supp}(p) \in \mathscr{S}_\alpha$.

For $q, p \in P_\lambda$, $q \leq p$ iff for each $\alpha < \lambda$, $q \restriction \alpha \Vdash_\alpha q(\alpha) \leq p(\alpha)$.

Let $\langle P_\alpha \mid \alpha \leq \lambda\rangle$ be an iteration with support $\mathscr{S}$. For $\beta < \alpha \leq \lambda$ and $a_\alpha \in P_\alpha$, $a_\alpha \restriction \beta$ denotes the restriction of a_α to P_β. For $a_\alpha, c_\alpha \in P_\alpha$, define $a_\alpha \sim_\alpha c_\alpha$ iff $a_\alpha \leq c_\alpha$ and $c_\alpha \leq a_\alpha$. $P_\alpha/_{\sim_\alpha}$ is a partial ordering.

Definition 4 A total function $\pi_\alpha : P_\alpha \to P_\alpha$ is a *pre-automorphism* of a preorder P_α iff π_α satisfies the following:

1. For all $a_\alpha, c_\alpha \in P_\alpha$, $c_\alpha \leq a_\alpha \leftrightarrow \pi_\alpha(c_\alpha) \leq \pi_\alpha(a_\alpha)$.
2. For each $c_\alpha \in P_\alpha$, there is an $a_\alpha \in P_\alpha$ such that $\pi_\alpha(a_\alpha) \sim_\alpha c_\alpha$.

If π_α is a pre-automorphism of P_α, then $\tilde{\pi}_\alpha$ is an automorphism of $P_\alpha/_{\sim_\alpha}$, where $\tilde{\pi}_\alpha([a_\alpha]_{\sim_\alpha}) = [\pi_\alpha(a_\alpha)]_{\sim_\alpha}$. For P_α-names σ, $\pi_\alpha(\sigma)$ is inductively defined to be $\{\langle \pi_\alpha(\tau), \pi_\alpha(p)\rangle : \langle \tau, p\rangle \in \sigma\}$.

Lemma 4 (Weak Homogeneity Preservation Lemma) *Suppose that* $\langle P_\alpha \mid \alpha \leq \lambda\rangle$ *($\lambda \in Ord$) is an iteration with support $\mathscr{S}$, where $P_0 = \{\mathbb{1}_0\}$ and for each α, $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, where P_α forces $\dot{Q}_\alpha$ to be a weakly homogeneous partial ordering. Also suppose that for each α and each pre-automorphism π_α of P_α, $P_\alpha \Vdash \pi_\alpha(\dot{Q}_\alpha) = \dot{Q}_\alpha$. Then $P_\lambda/_{\sim_\lambda}$ is weakly homogeneous.*

Proof Fix $p, r \in P_\lambda$. By induction, we obtain for each $\alpha \leq \lambda$ a pre-automorphism π_α of P_α and a $t_\alpha \in P_\alpha$ such that

1. For all $\beta \leq \alpha$ and all $a_\beta \in P_\beta$, $\mathrm{supp}(\pi_\beta(a_\beta)) = \mathrm{supp}(a_\beta)$;

2. For all $\gamma < \beta \leq \alpha$ and all $a_\beta \in P_\beta$, $\pi_\gamma(a_\beta \restriction \gamma) = \pi_\beta(a_\beta) \restriction \gamma$;
3. For all $\gamma < \beta \leq \alpha$, $t_\gamma = t_\beta \restriction \gamma$;
4. For all $\beta \leq \alpha$, if both $p \restriction \beta \Vdash p(\beta) = \mathbb{1}_\beta$ and $r \restriction \beta \Vdash r(\beta) = \mathbb{1}_\beta$, then $t(\beta)$ is a P_β-name for the trivial condition $\mathbb{1}_\beta$ in $\dot{Q}_\beta$;
5. For all $\beta \leq \alpha$, $t_\beta \leq \pi_\beta(p \restriction \beta), r \restriction \beta$.

$P_0 = \{\mathbb{1}_0\}$. Let π_0 denote the automorphism of P_0, and let $t_0 = \mathbb{1}_0$. (1)–(5) are trivially satisfied.

Let $0 < \alpha \leq \lambda$, and assume for all $\beta < \alpha$, π_β is a pre-automorphism of P_β and π_β, t_β satisfy (1)–(5).

Case 1 $\alpha = \beta + 1$. Let $p \restriction \alpha$ be $\langle p_\beta, \dot{q}_\beta \rangle$, where p_β is $p \restriction \beta$, and let $r \restriction \alpha$ be $\langle r_\beta, \dot{s}_\beta \rangle$, where r_β is $r \restriction \beta$. π_β is a pre-automorphism of P_β and $p_\beta \Vdash \dot{q}_\beta \in \dot{Q}_\beta$, so $\pi_\beta(p_\beta) \Vdash \pi_\beta(\dot{q}_\beta) \in \pi_\beta(\dot{Q}_\beta)$. By hypothesis, $P_\beta \Vdash \pi_\beta(\dot{Q}_\beta) = \dot{Q}_\beta$, so $\pi_\beta(p_\beta) \Vdash \pi_\beta(\dot{q}_\beta) \in \dot{Q}_\beta$. Hence $t_\beta \Vdash \pi_\beta(\dot{q}_\beta), \dot{s}_\beta \in \dot{Q}_\beta$. Since $P_\beta \Vdash \dot{Q}_\beta$ is weakly homogeneous, there is a P_β-name $\dot{\sigma}_\beta$ such that $P_\beta \Vdash \dot{\sigma}_\beta$ is an automorphism of $\dot{Q}_\beta$, and $t_\beta \Vdash \dot{\sigma}_\beta(\pi_\beta(\dot{q}_\beta))$ is compatible with $\dot{s}_\beta$. Let $\dot{u}_\beta$ be a P_β-name for an element of $\dot{Q}_\beta$ such that $t_\beta \Vdash \dot{u}_\beta \leq \dot{\sigma}_\beta(\pi_\beta(\dot{q}_\beta)), \dot{s}_\beta$. If both $p_\beta \Vdash \dot{q}_\beta = \mathbb{1}_\beta$ and $r_\beta \Vdash \dot{s}_\beta = \mathbb{1}_\beta$, then $t_\beta \Vdash \dot{\sigma}_\beta(\pi_\beta(\dot{q}_\beta)) = \dot{s}_\beta = \mathbb{1}_\beta$. In this case, let $\dot{u}_\beta$ be a P_β-name for the trivial condition $\mathbb{1}_\beta \in \dot{Q}_\beta$. Let $t_\alpha = \langle t_\beta, \dot{u}_\beta \rangle$. Given $\langle a_\beta, \dot{b}_\beta \rangle \in P_\beta * \dot{Q}_\beta$, define $\pi_\alpha(\langle a_\beta, \dot{b}_\beta \rangle) = \langle \pi_\beta(a_\beta), \dot{\sigma}_\beta(\pi_\beta(\dot{b}_\beta)) \rangle$. π_α is a total function on P_α, and $t_\alpha \leq \pi_\alpha(p \restriction \alpha), r \restriction \alpha$. π_α and t_α satisfy (1)–(5).

Case 2 α is a limit ordinal. Define π_α on P_α by $\pi_\alpha(a_\alpha) = \langle \pi_0(a_0), \dot{\sigma}_0(\pi_0(\dot{b}_0)), \dot{\sigma}_1(\pi_1(\dot{b}_1)), \ldots \rangle$, for each $a_\alpha = \langle a_0, \dot{b}_0, \dot{b}_1, \ldots \rangle \in P_\alpha$. Note that for each $\beta < \alpha$, $\pi_\beta(a_\alpha \restriction \beta) = \pi_\alpha(a_\alpha) \restriction \beta$. Let $t_\alpha = \bigcup_{\beta < \alpha} t_\beta$. By (4) of the induction hypothesis and Definition 3 (3), $t_\alpha \in P_\alpha$, since $\mathrm{supp}(t_\alpha) = \mathrm{supp}(p \restriction \alpha) \cup \mathrm{supp}(r \restriction \alpha)$. For all $\beta < \alpha$, $t_\alpha \restriction \beta = t_\beta \leq \pi_\alpha(p \restriction \alpha) \restriction \beta, r \restriction \beta$. Therefore, $t_\alpha \leq \pi_\alpha(p \restriction \alpha), r \restriction \alpha$. Hence, (1)–(5) hold.

Now we show that π_α is a pre-automorphism.

Claim 1 For all $a_\alpha, c_\alpha \in P_\alpha$, $c_\alpha \leq a_\alpha$ iff $\pi_\alpha(c_\alpha) \leq \pi_\alpha(a_\alpha)$.

Proof Suppose $\alpha = \beta + 1$. Let $\langle c_\beta, \dot{d}_\beta \rangle$ denote c_α and $\langle a_\beta, \dot{b}_\beta \rangle$ denote a_α. Suppose $c_\alpha \leq a_\alpha$. Then $c_\beta \leq a_\beta$ and $c_\beta \Vdash \dot{d}_\beta \leq \dot{b}_\beta$. It follows that $\pi_\beta(c_\beta) \leq \pi_\beta(a_\beta)$ and $\pi_\beta(c_\beta) \Vdash \pi_\beta(\dot{d}_\beta) \leq \pi_\beta(\dot{b}_\beta)$, since π_β is a pre-automorphism of P_β. $\pi_\beta(c_\beta) \Vdash \dot{\sigma}_\beta(\pi_\beta(\dot{d}_\beta)) \leq \dot{\sigma}_\beta(\pi_\beta(\dot{b}_\beta))$, since $P_\beta \Vdash \dot{\sigma}_\beta$ is an automorphism of $\dot{Q}_\beta$. Therefore, $\langle \pi_\beta(c_\beta), \dot{\sigma}_\beta(\pi_\beta(\dot{d}_\beta)) \rangle \leq \langle \pi_\beta(a_\beta), \dot{\sigma}_\beta(\pi_\beta(\dot{b}_\beta)) \rangle$.

Now assume that $c_\alpha \not\leq a_\alpha$. Then either $c_\beta \not\leq a_\beta$, or else ($c_\beta \leq a_\beta$ and $c_\beta \not\Vdash \dot{d}_\beta \leq \dot{b}_\beta$). If $c_\beta \not\leq a_\beta$, then $\pi_\beta(c_\beta) \not\leq \pi_\beta(a_\beta)$, since π_β is a pre-automorphism. Otherwise, there is some $e_\beta \leq c_\beta$ such that $e_\beta \Vdash \dot{d}_\beta \not\leq \dot{b}_\beta$. Since π_β is a pre-automorphism of P_β, $\pi_\beta(e_\beta) \Vdash \pi_\beta(\dot{d}_\beta) \not\leq \pi_\beta(\dot{b}_\beta)$. Since $P_\beta \Vdash \dot{\sigma}_\beta$ is an automorphism of $\dot{Q}_\beta$, it follows that $\pi_\beta(e_\beta) \Vdash \dot{\sigma}_\beta(\pi_\beta(\dot{d}_\beta)) \not\leq \dot{\sigma}_\beta(\pi_\beta(\dot{b}_\beta))$. Therefore, $\pi_\beta(c_\beta) \not\Vdash \dot{\sigma}_\beta(\pi_\beta(\dot{d}_\beta)) \leq \dot{\sigma}_\beta(\pi_\beta(\dot{b}_\beta))$. In both cases, $\langle \pi_\beta(c_\beta), \dot{\sigma}_\beta(\pi_\beta(d_\beta)) \rangle \not\leq \langle \pi(a_\beta), \dot{\sigma}_\beta(\pi_\beta(\dot{b}_\beta)) \rangle$.

Now suppose α is a limit ordinal. $c_\alpha \leq a_\alpha \leftrightarrow$ for all $\beta < \alpha$, $c_\alpha \restriction \beta \leq a_\alpha \restriction \beta \leftrightarrow$ for all $\beta < \alpha$, $\pi_\beta(c_\alpha \restriction \beta) \leq \pi_\beta(a_\alpha \restriction \beta) \leftrightarrow$ for all $\beta < \alpha$, $\pi_\alpha(c_\alpha) \restriction \beta \leq \pi_\alpha(a_\alpha) \restriction \beta$ $\leftrightarrow \pi_\alpha(c_\alpha) \leq \pi_\alpha(a_\alpha)$. □

Claim 2 For each $a_\alpha \in P_\alpha$, there is a $c_\alpha \in P_\alpha$ such that $a_\alpha \sim_\alpha \pi_\alpha(c_\alpha)$.

Proof Let $a_\alpha \in P_\alpha$. We construct such a $c_\alpha \in P_\alpha$ by induction on $\beta \leq \alpha$. Let $c_0 = a_\alpha(0) = \mathbb{1}_0$. Suppose $\beta < \alpha$ and $c_\beta \in P_\beta$ satisfies $\pi_\beta(c_\beta) \sim_\beta a_\alpha \upharpoonright \beta$. $\pi_{\beta+1}$ is a pre-automorphism of $P_{\beta+1}$, so there exists an $e_{\beta+1} = \langle e_\beta, \dot{f}_\beta \rangle \in P_{\beta+1}$ such that $\pi_{\beta+1}(e_{\beta+1}) \sim_{\beta+1} a_\alpha \upharpoonright (\beta+1)$. $\pi_{\beta+1}(e_{\beta+1}) = \langle \pi_\beta(e_\beta), \dot{\sigma}_\beta(\pi_\beta(\dot{f}_\beta))\rangle$. So, $\pi_\beta(e_\beta) \sim_\beta a_\alpha \upharpoonright \beta \sim_\beta \pi_\beta(c_\beta)$. Hence, $e_\beta \sim_\beta c_\beta$, since π_β is a pre-automorphism. Let $c_{\beta+1} = \langle c_\beta, \dot{f}_\beta \rangle$. Then $\pi_{\beta+1}(c_{\beta+1}) \sim_{\beta+1} a_\alpha \upharpoonright (\beta+1)$, and $c_{\beta+1} \upharpoonright \beta = c_\beta$.

If $\beta \leq \alpha$ is a limit ordinal, let $c_\beta = \bigcup_{\gamma<\beta} c_\gamma$. Since π_β is support-preserving, $\text{supp}(c_\beta) = \text{supp}(\pi_\beta(c_\beta)) = \text{supp}(a_\beta)$; so $c_\beta \in P_\beta$. For all $\gamma < \beta$, $\pi_\beta(c_\beta) \upharpoonright \gamma = \pi_\gamma(c_\gamma) \sim_\gamma a_\gamma$. This implies that $\pi_\beta(c_\beta) \sim_\beta a_\beta$, since $\pi_\beta(c_\beta) = \bigcup_{\gamma<\beta} \pi_\gamma(c_\gamma)$. □

Hence, π_α is a pre-automorphism of P_α. □

The previous lemma also applies to iterations of length Ord.

Definition 5 We say that $\langle P_\alpha \mid \alpha \leq \text{Ord}\rangle$ is an *iteration* with supports $\mathscr{S}_\alpha$, $\alpha \in \text{Ord}$, if

1. For each $\alpha \in \text{Ord}$, $\mathscr{S}_\alpha \subseteq \mathscr{P}(\alpha)$;
2. For all $\beta < \alpha \in \text{Ord}$, $\mathscr{S}_\beta \subseteq \mathscr{S}_\alpha$;
3. If $x, y \in \mathscr{S}_\alpha$, then $x \cup y \in \mathscr{S}_\alpha$;
4. $P_{\alpha+1}$ is the preorder $P_\alpha * \dot{Q}_\alpha$, where $P_\alpha \Vdash \dot{Q}_\alpha$ is a set partial ordering;
5. For limit α, P_α is the preorder consisting of all α-sequences p such that for $\beta < \alpha$, $p \upharpoonright \beta \Vdash_{P_\beta} p(\beta) \in \dot{Q}_\beta$, and $\text{supp}(p) \in \mathscr{S}_\alpha$;
6. $P_{\text{Ord}} = \bigcup_{\alpha \in \text{Ord}} P_\alpha$ ordered by $q \leq p$ iff $\text{supp}(q) \supseteq \text{supp}(p)$ and for each $\alpha \in \text{supp}(p)$, $q \upharpoonright \alpha \Vdash q(\alpha) \leq p(\alpha)$.

$p \sim_{\text{Ord}} q$ iff $p \sim_\alpha q$ for sufficiently large α. We say that $\langle P_\alpha : \alpha \leq \text{Ord}\rangle$ is *weakly or open-dense homogeneous* iff for each ordinal λ, $\langle P_\alpha : \alpha \leq \lambda\rangle$ is weakly or open-dense homogeneous, respectively.

Lemma 5 (Weak Homogeneity Preservation Lemma for Ordinal Length Iteration) *Suppose that $\langle P_\alpha \mid \alpha \leq Ord\rangle$ is an iteration with supports $\mathscr{S}_\alpha$, $\alpha \in Ord$, where $P_0 = \{\mathbb{1}_0\}$ and for each α, $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, where P_α forces $\dot{Q}_\alpha$ to be a weakly homogeneous set partial ordering. Also suppose that for each α and each pre-automorphism π_α of P_α, $P_\alpha \Vdash \pi_\alpha(\dot{Q}_\alpha) = \dot{Q}_\alpha$. Then $P = P_{\text{Ord}}/_{\sim_{\text{Ord}}}$ is weakly homogeneous.*

The above proof also works for iterations of open-dense homogeneous partial orderings.

Lemma 6 (Open Dense Homogeneity Preservation Lemma) *Suppose that $\langle P_\alpha \mid \alpha \leq \lambda\rangle$, where $\lambda \leq Ord$, is an iteration with supports $\mathscr{S}_\alpha$, $\alpha \in \lambda$, where $P_0 = \{\mathbb{1}_0\}$ and for each α, $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$, where P_α forces $\dot{Q}_\alpha$ to be an open-dense homogeneous set partial ordering. Also suppose that for each α, each open dense $D_\alpha \subseteq P_\alpha$, and each pre-isomorphism π_α of D_α, $P_\alpha \Vdash \pi_\alpha(\dot{Q}_\alpha(D_\alpha)) = \dot{Q}_\alpha(D_\alpha)$, where $\dot{Q}_\alpha(D_\alpha)$ denotes the collection of D_α-names for elements in $\dot{Q}_\alpha$. Then $P = P_\lambda/_{\sim_\lambda}$ is open-dense homogeneous.*

Finally, we prove the main theorem.

Proof of Theorem 1.3 Suppose κ is hyperstrong and $V \models$ GCH. Let P denote the forcing used in Lemma 2. By Lemma 5, the partial ordering P is weakly homogeneous. Let G be P-generic over V. Then replacement holds in $V[G]$ with G as a predicate. It follows that any element of HOD of $V[G]$ belongs to HOD of $V[G_\lambda]$ for some $\lambda <$ Ord. By Lemma 3, HOD of $V[G_\lambda]$ is contained in HOD of V. By Lemma 2, κ is still superstrong in $V[G]$. □

In a forthcoming paper, Brooke-Taylor and Friedman make essential use of Lemma 6 to prove the following theorem for preserving large cardinals while adding morasses.

Theorem 2 [1] *Let V be a model of ZFC + GCH. Then there is a class generic extension $V[G]$ of V such that cardinals are preserved, n-superstrong cardinals are preserved for all n in $\omega + 1$, hyperstrong cardinals are preserved, and in $V[G]$ there is a gap-1 morass at every regular cardinal.*

We conclude this paper with the following open problem.

Open Problem Is it consistent that $(\kappa^+)^{\mathrm{HOD}}$ be less than κ^+ for all infinite cardinals κ?

Cummings and Woodin (unpublished) have established the consistency of $(\kappa^+)^{\mathrm{HOD}} < \kappa^+$ for a closed unbounded class of cardinals κ.

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