Arch. Math. Logic (2007) 46:281–287
DOI 10.1007/s00153-007-0043-5

Mathematical Logic

Distinguishing non-standard natural numbers in a set theory within Łukasiewicz logic

Shunsuke Yatabe

Received: 26 February 2006 / Revised: 25 September 2006 /
Published online: 21 February 2007

Abstract In **H**, a set theory with the comprehension principle within Łukasiewicz infinite-valued predicate logic, we prove that a statement which can be interpreted as "there is an infinite descending sequence of initial segments of ω" is truth value 1 in any model of **H**, and we prove an analogy of Hájek's theorem with a very simple procedure.

Keywords Set theory · Arithmetic · Łukasiewicz logic · The comprehension principle · Non-standard natural numbers

Mathematics Subject Classification (2000) 03E72

1 Introduction

In this paper, we investigate an arithmetic defined in **H**, a set theory with the comprehension principle within **ŁQ**, Łukasiewicz infinite-valued predicate logic. We base on a result in [6]: the theory **H** is ω-inconsistent. Informally speaking, this means that a statement which can be interpreted as "ω must contain a non-standard natural number" is valid in **H**, i.e. it has the truth value 1 in every model of **H**. Its proof is by constructing a formula $\varphi(x)$ such that, if n is a standard natural number then the truth value of $\varphi(n)$ is 0, and that of $(\exists n)\varphi(n)$ is 1 in any model of **H**. This suggests that we can distinguish standard and non-standard natural numbers to some extent in **H**. Using this, we prove two corollaries of this theorem.

S. Yatabe (✉)
Department of Computer and Systems Engineering, Kobe University, Rokko-dai chou, Nada-ku, Kobe 657-8501, Japan
e-mail: yatabe@kurt.scitec.kobe-u.ac.jp

First we can prove an "*overspill*" result: if $\tau(x)$ defines a set $\{x : \tau(x)\}$ of elements of ω in each model of **H** and $\tau(n)$ is valid in **H** for infinitely many standard numbers n, then $(\exists x \in \omega)(\tau(x) \,\&\, \varphi'(x))$ is valid in **H** for the formula φ' which is very similar to φ. Moreover, iterating this argument, we can show that a statement which can be interpreted as "there is an infinite descending sequence of initial segments of ω" is valid in **H**.

Unlike **PA**, we can distinguish some non-standard natural numbers in **H**. This proves a difference: the induction scheme on ω implies a contradiction in **H**. We prove this in the very simple way as the second corollary. We note that this is an analogy of the theorem proved by Hájek [3].

The first corollary suggests that an arithmetic in **H** is somehow similar to one in "non-standard models" of **PA**. However the second shows that they are very different in spots. We should study further how different they are.

2 Preliminaries

We work within **ŁQ**, Łukasiewicz infinite-valued predicate logic with its standard semantics. It is known that **ŁQ** is not recursively axiomatizable. So, for simplicity, we introduce **ŁQ** by defining its models. As for syntactic characterization of **H**, see [7].

Given $\mathbf{M} = \langle M, (r_P)_{P\,\mathrm{predicate}}, (m_c)_{c\,\mathrm{constant}} \rangle$ where $M \neq \emptyset$, $m_c \in M$, $r_P : M^n \to [0,1]$ (if P is n-ary relation) and a valuation v i.e. v : (object variables) $\to M$, let $\|\varphi\|_{M,v}$ be the truth value of φ in M, v iff

- $\|P(x, \ldots, c \ldots)\|_{M,v} = r_P(v(x), \ldots, m_c, \ldots)$,
- $\|A \to B\|_{M,v} = \|A\|_{M,v} \Longrightarrow \|B\|_{M,v}$ and $\|A\&B\|_{M,v} = \|A\|_{M,v} * \|B\|_{M,v}$,
- $\|(\forall x)\varphi\|_{M,v} = \inf\{\|\varphi\|_{M,v'} : v'(y) = v(y)$ for all variable y except possibly $x\}$,

where $x \Longrightarrow y = \min(1, 1 - x + y)$ and $x * y = \max(0, x + y - 1)$. The rest connectives are defined by using $\to$ and constant $\bar{0}$ (for example $\|\neg A\|_{M,v} = \|A \to \bar{0}\|_{M,v} = \|A\|_{M,v} \Longrightarrow 0$ and $\|A \wedge B\|_{M,v} = \|\neg(A \to \neg(A \to B))\|_{M,v}$).

Set $\|\varphi\|_{\mathbf{M}} = \inf\{\|\varphi\|_{M,v} : v$ is a valuation on $\mathbf{M}\}$.

Definition 1 For T a theory within **ŁQ**, $\mathbf{M} = \langle M, (r_P)_{P\,\mathrm{predicate}}, (m_c)_{c\,\mathrm{constant}} \rangle$ is a model of T (or a natural Tarskian semantics for T) if $\|\varphi\|_{\mathbf{M}} = 1$ for any axiom $\varphi \in T$.

We call φ is valid in T when φ has truth value 1 in any model of T.

We note that, Hájek defined the Łukasiewicz logic **Ł**∀ in more general semantics [2]. **Ł**∀ considers models over all linearly ordered **MV**-algebras and it is recursively axiomatizable. The above natural Tarskian semantics is called a $([0,1], *, \Longrightarrow, 0, 1)$-structure **M** where $([0,1], *, \Longrightarrow, 0, 1)$ forms the standard **MV**-algebra: The strength of **Ł**∀ is between Grišin logic and **ŁQ**.

Definition 2 (Set theory **H**) Let **H** be the set theory within **ŁQ**

- which has a binary predicate $\in$, and terms of the form $\{x : \varphi(x)\}$,

– whose only axiom scheme is the comprehension principle: For any φ not containing u freely,

$$(\forall u)[u \in \{x : \varphi(x, \ldots)\} \equiv \varphi(u, \ldots)]$$

This theory is consistent (see [7]). Next we develop an arithmetic within **H** in the track of Hájek's paper [3].

Definition 3 $\emptyset$ is the term defined as $\{x : x \neq x\}$.

Theorem 1 (Recursion theorem) *For any formula* $\varphi(x, \ldots, y)$*,*

$$\mathbf{H} \vdash (\exists z)(\forall x)[x \in z \equiv \varphi(x, \ldots, z)]$$

In particular, there is a term θ *such that* $\theta =_{\text{ext}} \{u : \varphi(u, \ldots, \theta)\}$.

We note that $X =_{\text{ext}} Y$ iff $(\forall x)[x \in X \equiv x \in Y]$. For the proof, see [1]. Using this, we can define a term which represents a set of all natural numbers as follows:

Definition 4 ω is a term such that by

$$\omega =_{\text{ext}} \{z : z = \emptyset \vee (\exists y)[y \in \omega \wedge z = \{y\}]\}$$

We define natural number $n > 0$ to be $\emptyset$ within n iterate of $\{\}$. For simplicity, we write $n + 1$ instead of $\{n\}$ hereafter. We note that $\|n \in \omega\|_{\mathbf{M}} = 1$ for any standard natural number n. Next we summarize a theorem in [6].

Theorem 2 *The theory* **H** *is* ω*-inconsistent.*

Informally speaking, a statement which can be interpreted as "there is a non-standard natural number" is valid in **H**.

Proof First let us define a term θ by

$$\begin{aligned}\langle n, x\rangle \in \theta \iff & [n = \emptyset \wedge x \notin x] \\ & \vee[(\exists k \in \omega)\,[n = \{k\} \wedge (x \in x \to \langle k, x\rangle \in \theta)]]\end{aligned}$$

Claim For any standard natural number n, $\|\langle n, x\rangle \in \theta\|_{\mathbf{M}} = \min\{(n+1)(1 - \|x \in x\|_{\mathbf{M}}), 1\}$ in any model **M** of **H**.

Next we produce the Russell-like set $R_\omega = \{x : (\exists n \in \omega)\langle n, x\rangle \in \theta\}$.

Lemma 1 $\|R_\omega \in R_\omega\|_{\mathbf{M}} = 1$.

Proof Let $p = \|R_\omega \in R_\omega\|_{\mathbf{M}} \in [0, 1]$. Then we have

$$\begin{aligned}\|R_\omega \in R_\omega\|_{\mathbf{M}} &= \sup\{\|n \in \omega \wedge \langle n, R_\omega\rangle \in \theta\|_{\mathbf{M}} : n \in \mathbf{M}\} \\ &\geq \min\{\sup\{(n+1)(1-p) : n \in \mathbb{N}\}, 1\}\end{aligned}$$

Assume $p \neq 1$, then there must be some $j \in \mathbb{N}$ such that $j \cdot (1 - p) \geq 1$, so p must be 1, a contradiction. □

We note that $\|\langle n, R_\omega\rangle \in \theta\|_{\mathbf{M}} = 0$ for any standard natural number n. So the statement "$\langle n, R_\omega\rangle \in \theta$" can be interpreted as "n is a non-standard natural number", and "$R_\omega \in R_\omega$" means that $(\exists n \in \omega)\langle n, R_\omega\rangle \in \theta$, so it can be interpreted as "there is a non-standard natural number", and it has truth value 1 in any model of **H**. □

This theorem is an analogy of Greg Restall's result [5]: He proved ω-inconsistency of some system of arithmetic with addition, multiplication and truth predicate within Łukasiewicz infinite-valued predicate logic by using a diagonalization argument.

The above proof is a generalization of the derivation of Moh's paradox [4]: the comprehension principle implies a contradiction within $\mathbf{Ł}_m$, Łukasiewicz m-valued propositional logic, for any finite natural number $m > 2$. Its derivation is that the truth value of the sentence "$R_m \in R_m$" can't be decided within $\mathbf{Ł}_m$, where

$$R_m = \{x : \underbrace{x \in x \rightarrow (x \in x \rightarrow (x \in x \cdots (x \in x \rightarrow}_{m-2 \text{ times}} x \notin x) \cdots))\}$$

The proof of Theorem 2 itself doesn't work within **Ł**∀ which admits models over non-archimedean linearly ordered **MV**-algebras. However, since Moh's argument works to show that the comprehension principle implies a contradiction within any finite **BCK**-logic (this is a logic whose model is a finite **BCK** algebra), Yuichi Komori suggested that an analogy of Theorem 2 could be proved probably within **BCK** logic.

3 Overspill

In the proof of Theorem 2, we show $\|\langle n, R_\omega\rangle \in \theta\|_{\mathbf{M}} = 0$ for any standard natural number n. Next question is whether the converse, $\|\langle n, R_\omega\rangle \in \theta\|_{\mathbf{M}} = 0$ implies n is a standard natural number, holds or not. If this holds, then $\omega' = \{n \in \omega : \langle n, R_\omega\rangle \notin \theta\}$ is similar to the set of standard natural numbers. However, it is easy to see that the statement which can be interpreted as "ω' must contain non-standard numbers" is valid in **H**: By defining $R_{\omega'} = \{x : (\exists n \in \omega')\langle n, x\rangle \in \theta\}$, we can prove this in a very similar way to Theorem 2. Moreover, we can prove the overspill-like phenomenon as mentioned in the introduction. Assume $\tau(x)$ defines a set $\{x : \tau(x)\}$ of elements of ω in each model of **H** and $\tau(n)$ is valid in **H** for infinitely many standard natural numbers. Let us define a set R'_ω as

$$R'_\omega = \{x : (\exists n \in \omega)\, [\langle n, x\rangle \in \theta \,\&\, \tau(n)]\}$$

then just the same argument shows $\|R'_\omega \in R'_\omega\| = 1$. Let us define $\varphi'(n)$ by $\langle n, R'_\omega\rangle \in \theta$. We can see that $(\exists x \in \omega)(\tau(x) \,\&\, \varphi'(x))$ is valid in **H**.

Next we investigate the order-type of ω. By iterating the overspill-like argument infinitely-many times, we can easily prove an analogy of Cantor's theorem which says that the order-type of the domain of any countable non-standard model of **PA** is isomorphic to $\mathbb{N} + \mathbb{Z} \times \mathbb{Q}$. Let us formalize this within **H**, and construct a concrete term which corresponds to them.

First let us present our argument implicitly. We define subsets of ω inductively as follows:

- $\omega^{(0)} = \omega, R_{\omega^{(0)}} = R_\omega$
- $\dots$
- $\omega^{(n+1)} = \{k \in \omega^{(n)} : \langle k, R_{\omega^{(n)}} \rangle \notin \theta\}, R_{\omega^{(n+1)}} = \{x : (\exists j \in \omega^{(n+1)}) \langle j, x \rangle \in \theta\}$
- $\dots$

As above, we can prove a negative answer.

Lemma 2 *For any model* **M** *of* **H** *and any standard natural number n, a statement which can be interpreted as "there is a non-standard natural number in* $\omega^{(n)}$*" has truth value* 1 *in* **M**.

Proof Here we only prove the case $n = 1$. Let **M** be any model of **H**. Let us consider the truth value of $R_{\omega^{(1)}} \in R_{\omega^{(1)}}$: We can see, for any standard natural number n, $\|\langle n, R_{\omega^{(1)}} \rangle \in \theta\|_{\mathbf{M}} = 0$, and $\|R_{\omega^{(1)}} \in R_{\omega^{(1)}}\|_{\mathbf{M}} = 1$. This means, the statement which can be interpreted as "there is a non-standard natural number in $\omega^{(1)}$" has truth value 1 in **M**. □

This proof shows the followings: for any natural number m, k, j,

- $\|m \in \omega^{(k)}\|_{\mathbf{M}} \geq \|m \in \omega^{(k+1)}\|_{\mathbf{M}}$ holds (in this sense $\omega^{(n+1)}$ is an initial segment of ω),
- if $\|R_{\omega^{(k)}} \in R_{\omega^{(k)}}\| = 1$, then $\|j \in \omega^{(k+1)}\|_{\mathbf{M}} = \|m \in \omega^{(k+1)}\|_{\mathbf{M}} = \|m + 1 \in \omega^{(k+1)}\|_{\mathbf{M}}$ where $m = j + 1$.

In particular, for any standard natural number n,

- $\|k \in \omega^{(n)}\|_{\mathbf{M}} = 1$ holds for any standard natural number k,
- for any non-standard natural number d, if $\|\langle d, R_{\omega(n)} \rangle \in \theta\|_{\mathbf{M}} = 1$ holds, then $\|d \in \omega(n+1)\|_{\mathbf{M}} = 0$ holds: In this sense $\omega(n+1)$ is a cutoff of some non-standard natural numbers in $\omega(n)$,
- $\|R_{\omega^{(n)}} \in R_{\omega^{(n)}}\| = 1$ holds: this sentence can be interpreted as "$\omega^{(n)}$ must contain some non-standard natural numbers", and furthermore this means "there is a descending sequence $\omega(0) \supseteq \cdots \supseteq \omega(n+1)$" by construction.

So we can take an infinite "downward" sequence $\langle \omega^{(n)} \rangle$ of initial segments of ω such that $\omega^{(n+1)}$ is an initial segment of $\omega^{(n)}$.

Next we formalize this argument: Let us construct a formula which corresponds to "the existence of an infinite downward sequence".

Theorem 3 *The statement which can be interpreted as "there is a infinite descending sequence of initial segments of* ω*" is valid in* **H**.

Proof Let us formalize the argument in the proof of Lemma 2. First we define the term Ω such that

– $\langle 0, \langle n, x\rangle\rangle \in \Omega$ iff $x \in \omega^{(n)}$,
– $\langle 1, \langle n, x\rangle\rangle \in \Omega$ iff $x \in R_{\omega^{(n)}}$ (so $R_{\omega^{(n)}} = \{y : (\exists j)\langle 0, \langle n, j\rangle\rangle \in \Omega \wedge \langle j, y\rangle \in \theta\}$),

for any standard natural number n. In concrete terms,

$$\begin{aligned}\langle i, \langle n, x\rangle\rangle \in \Omega \Longleftrightarrow [i = 0 \wedge [(n = \emptyset \wedge x \in \omega) \\ \vee(\exists\, k \in \omega)(n = \{k\} \wedge \langle 0, \langle k, x\rangle\rangle \in \Omega \\ \wedge\langle x,\, \{y : (\exists j)\langle 0, \langle n, j\rangle\rangle \in \Omega \wedge \langle j, y\rangle \in \theta\}\rangle \notin \theta)]] \\ \vee[i = 1 \wedge [(\exists l)(\langle 0, \langle n, l\rangle\rangle \in \Omega \wedge \langle l, x\rangle \in \theta)]]\end{aligned}$$

Theorem 1 guarantees the existence of Ω: This means the term $R_{\omega^{(n)}}$ is definable for every natural number n.

We have seen that $\|R_{\omega(n)} \in R_{\omega(n)}\| = 1$ for every standard natural number n. Then, by overspill, we can show the following:

$$\|(\exists x)\varphi'(x)\,\&\,(R_{\omega(x)} \in R_{\omega(x)})\|_{\mathbf{M}} = 1$$

As we see, this sentence can be interpreted as "there is a infinite descending sequence of initial segments of ω". □

4 An analogy of Hájek's theorem

In **PA**, it is well-known that we can never distinguish standard and non-standard natural numbers. It is because of the induction scheme on ω. On the other hand, $\|\langle d, R_\omega\rangle \in \theta\|_{\mathbf{M}} > 0$ implies that d is a non-standard natural number in **H**. This gives a way of distinguishing them, and this causes a big difference between an arithmetic in **H** and **PA**.

Definition 5 The induction scheme on ω is a scheme of the form: for any formula φ,

$$\varphi(0) \wedge (\forall n \in \omega)[\varphi(n) \equiv \varphi(n+1)] \text{ infer } (\forall x)[x \in \omega \rightarrow \varphi(x)]$$

Theorem 4 *The induction scheme on* ω *implies a contradiction in* **H**.
This means that the induction scheme is valid in no model of **H**.

Proof Let us assume the induction scheme on ω. It is easy to see that $\|\langle 0, R_\omega\rangle \notin \theta\| = \|R_\omega \in R_\omega\| = 1$. And

$$\begin{aligned}\|\langle n+1, R_\omega\rangle \notin \theta\| &= 1 - (\min\{1 - \|R_\omega \in R_\omega\| + \|\langle n, R_\omega\rangle \in \theta\|, 1\}) \\ &= \|\langle n, R_\omega\rangle \notin \theta\|\end{aligned}$$

So the induction scheme proves that $(\forall x)[\langle x, R_\omega\rangle \notin \theta]$, but this contradicts to Theorem 2. □

Last we note about the definability of arithmetical functions in **H**. For example, is *addition* $+$ definable in **H**? The graph $Plus(x, y, z)$ of addition $(x + y = z)$ itself is definable in **H** by using Theorem 1. However, Theorem 1 is not strong enough to show the following:

– $Plus$ is a crisp relation, i.e. $\|Plus(x, y, z)\| = 0$ or 1 for any x, y, z,
– we can define a function $plus : \omega \times \omega \to \omega$ (where $plus(x, y)$ means $x + y$),
– the totality of the function *plus* (if we can define it),
– $(\omega, \leq)$ becomes a linear ordering where $x \leq y$ iff $(\exists z)[x + z = y]$ for any $x, y \in \omega$.

We probably need the stronger axioms or rules.

Hájek [3] investigated how much we can develop an arithmetic in more general semantics. He works in $\mathbf{CŁ}_0$, a set theory with the comprehension principle within **Ł**∀, and he showed that, for example, ω becomes a crisp set and *plus* can be defined as a total crisp function using the induction scheme. So $(\omega, \leq)$ becomes a linear ordering. However, he eventually proved that the induction scheme on ω implies a contradiction. His original proof was very complex: he constructed a truth predicate which commutes with connectives by using the induction scheme, and such a predicate implies a contradiction.

Theorem 4 is an analogy of Hájek's theorem, and this shows that we can't assume any Hájek's style arithmetic with the induction scheme in **H**. And we do not know that an arithmetic with the axiom that ω is crisp is consistent. Hájek himself raised a question as follows:

> can we add consistently to the theory axioms guaranteeing the existence of the crisp structure ω of natural numbers and further add all the axioms of Peano arithmetic for ω?

Acknowledgements The author would like to thank Petr Hájek for his advises.

References

1. Cantini, A.: The undecidability of Grišin's set theory. Stud. Log. **74**, 345–368 (2003)
2. Hájek, P.: Metamathematics of Fuzzy Logic. Kluwer, Dordrecht (2001)
3. Hájek, P.: On arithmetic in the Cantor-Łukasiewicz fuzzy set theory. Arch. Math. Log. **44**, 763–82 (2005)
4. Moh S.-K.: Logical paradoxes for many-valued systems. J. Symbolic Log. **19**, 37–40 (1954)
5. Restall, G.: Arithmetic and truth in Łukasiewicz's infinitely valued logic. Log. Anal. **36**, 25–38 (1993)
6. Yatabe, S.: A note on Hajek, Paris and Shepherdson's theorem. Log. J. IGPL **13**, 261–266 (2005)
7. White, R.B.: The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz. J. Philos. Log. **8**, 509–534 (1979)