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A type of fuzzy ring

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Abstract In this study, by the use of Yuan and Lee's definition of the fuzzy group based on fuzzy binary operation we give a new kind of fuzzy ring. The concept of fuzzy subring, fuzzy ideal and fuzzy ring homomorphism are introduced, and we make a theoretical study their basic properties analogous to those of ordinary rings.

Keywords Fuzzy binary relation · Fuzzy group · Fuzzy ring · Fuzzy subring · Fuzzy ideal

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1 Introduction

In 1971, Rosenfeld [7] introduced fuzzy sets in the realm of group theory and formulated the concept of fuzzy subgroup of a group. Since then many researchers are engaged in extending the concepts of abstract algebra to the broader framework of the fuzzy setting. An increasing number of properties from classical group theory have been generalized, getting a new significance in the new framework. Naturally, there exist a concern in generalizing other types of algebraic structures (e.g. rings, fields, vector spaces) as fuzzy algebraic structures. Consequently, it would be interesting to find modalities to study some

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of these more complex fuzzy algebraic structures by using fuzzy groups. In [5], Liu introduced fuzzy sets in the realm of ring theory. Subsequently, among others, Yue [9], Mukherjee and Sen [6], Dixit et al. [4] studied on fuzzy ring and obtained certain ring theoretic analogues. In the definition of fuzzy subgroups and fuzzy subring they assumed that the subset of group G and ring R are fuzzy and the binary operations on R are nonfuzzy in the classical sense. Another approach is to assume that the set is nonfuzzy or classical and the binary operation is fuzzy in the fuzzy sense. More in line with this latter approach, Demirci [2,3] introduced the concept of smooth group by the use of fuzzy binary operation and concept of fuzzy equality, and Aktaṣ [1] studied its an application. Yuan and Lee [8] introduced a new kind of fuzzy group based on fuzzy binary operation.

In this paper, by the use of Yuan and Lee's fuzzy group, a new kind of fuzzy ring based on binary operation is introduced. The fundamental properties of fuzzy groups fuzzy binary relations are presented in Sect. 2. Concepts of fuzzy ring based on fuzzy relation are introduced in Sect. 3. Fuzzy subrings, fuzzy ideals and factor fuzzy rings are presented in Sect. 4. Finally, in Sect. 5, the concepts of fuzzy ring homomorphisms are introduced and a fundamental homomorphism theorem of fuzzy ring is obtained.

2 Preliminaries

In this section we summarize the preliminary definitions, and results that are required later in this paper. Most contents of this section are contained in [8].

Definition 1 Let G be a nonempty set and R be a fuzzy subset of $G \times G \times G$. R is called a fuzzy binary operation on G if

1. $\forall a, b \in G, \exists c \in G$ such that $R(a, b, c) > \theta$;
2. $\forall a, b, c_1, c_2 \in G, R(a, b, c_1) > \theta$ and $R(a, b, c_2) > \theta$ implies $c_1 = c_2$.

Let R be a fuzzy binary operation on G, then we have a mapping

$$\begin{aligned} R : F(G) \times F(G) &\to F(G) \\ (A, B) &\mapsto R(A, B), \end{aligned}$$

where $F(G) = \{A | A : R \to [0, 1] \text{ is a mapping}\}$ and

$$R(A, B)(c) = \bigvee_{a,b \in G} (A(a)) \wedge B(b) \wedge R(a, b, c)) \tag{1}$$

Let $A = \{a\}$ and $B = \{b\}$, and let $R(A, B)$ be denoted as $a \circ b$, then

$$(a \circ b)(c) = R(a, b, c), \quad \forall c \in G \tag{2}$$

$$((a \circ b) \circ c)(z) = \bigvee_{d \in G} (R(a, b, d) \wedge R(d, c, z)), \tag{3}$$

$$(a \circ (b \circ c)(z) = \bigvee_{d \in G} (R(b, c, d) \wedge R(a, d, z)). \tag{4}$$

Using the notations in Eqs. (2)–(4), we have the following.

Definition 2 Let G be a nonempty set and R be a fuzzy binary operation on G. (G, R) is called a fuzzy group if the following conditions are true:

G1 $\forall a, b, c, z_1, z_2 \in G$, $((a \circ b) \circ c)(z_1) > \theta$ and $(a \circ (b \circ c))(z_2) > \theta$ implies $z_1 = z_2$;
G2 $\exists e_\circ \in G$ such that $(e_\circ \circ a)(a) > \theta$ and $(a \circ e_\circ)(a) > \theta$ for any $a \in G$ ($e_\circ$ is called an identity element of G);
G3 $\forall a \in G, \exists b \in G$ such that $(a \circ b)(e_\circ) > \theta$ and $(b \circ a)(e_\circ) > \theta$ (b is called an inverse element of a and is denoted as a^{-1})

Proposition 3 [8] *H is a fuzzy subgroup of G if and only if*

1. $\forall a, b \in H$, $\forall c \in G$, $(a \circ b)(c) > \theta$ *implies* $c \in H$,
2. $a \in H$ *implies* $a^{-1} \in H$.

Definition 4 Let H be a fuzzy subgroup of G. Let

$$(aH)(z) = \bigvee_{x \in H} R(a, x, z); \quad (Ha)(z) = \bigvee_{x \in H} R(x, a, z) \tag{5}$$

$aH(Ha)$ is called a left (right) coset of H.

Theorem 5 *Let H be a fuzzy subgroup of G, then H is a normal fuzzy subgroup if and only if*

$$\forall a, z \in G, \quad (aH)(z) > \theta \Leftrightarrow (Ha)(z) > \theta.$$

Definition 6 Let (G, R) be a fuzzy group. If

$$(a \circ b)(c) > \theta \Leftrightarrow (b \circ a)(c) > \theta \quad \forall a, b \in G, c \in G$$

then (G, R) is called abelian fuzzy group.

Theorem 7 *Let $[aH] = \{a'H : a'H \sim aH\}$, $\bar{a} = \{a' : a' \in G$ and $a'H \sim aH\}$, $G/H = \{[aH] : a \in G\}$, and*

$$\bar{R} : \frac{G}{H} \times \frac{G}{H} \times \frac{G}{H} \to [0, 1],$$
$$([aH], [bH], [cH]) \to \bar{R}([aH], [bH], [cH]) = \bigvee_{(a', b', c') \in \bar{a} \times \bar{b} \times \bar{c}} R(a', b', c')$$

then $\bar{R}$ is a binary operation on G/H.

Theorem 8 *$(G/H, \bar{R})$ is a fuzzy group.*

3 Fuzzy ring

Let G be a fuzzy binary operation on R. Then we have a mapping

$$\begin{aligned} G : F(R) \times F(R) &\to F(R) \\ (A, B) &\mapsto G(A, B), \end{aligned}$$

where $F(R) = \{A | A : R \to [0, 1] \text{ is a mapping}\}$ and

$$G(A, B)(c) = \bigvee_{a,b \in R} (A(a)) \wedge B(b) \wedge R(a, b, c)) \tag{6}$$

Let $A = \{a\}$ and $B = \{b\}$, and let $G(A, B)$ and $H(A, B)$ be denoted as $a \circ b$ and $a * b$, respectively. Then

$$(a \circ b)(c) = G(a, b, c), \quad \forall c \in R \tag{7}$$
$$(a * b)(c) = H(a, b, c), \quad \forall c \in R \tag{8}$$
$$((a \circ b) \circ c)(z) = \bigvee_{d \in R} (G(a, b, d) \wedge G(d, c, z)), \tag{9}$$
$$(a \circ (b \circ c)(z) = \bigvee_{d \in R} (G(b, c, d) \wedge G(a, d, z)), \tag{10}$$
$$(a * (b \circ c)(z) = \bigvee_{d,e \in R} (G(b, c, d) \wedge H(a, d, z)), \tag{11}$$
$$((a * b) \circ (a * c))(z) = \bigvee_{d \in R} (H(a, b, d) \wedge H(a, c, e) \wedge G(d, e, z)). \tag{12}$$

Using the notations of Eqs. (7)–(12), we can define a kind of fuzzy ring as follows.

Definition 9 Let R be a nonempty set, and G and H be two fuzzy binary operation on R. Then (R, G, H) is called a fuzzy ring if the following conditions hold.

- **R1** (R, G) is an abelian fuzzy group.
- **R2** $\forall a, b, c, z_1, z_2 \in R$, $((a * b) * c)(z_1) > \theta$ and $(a * (b * c))(z_2) > \theta$ implies $z_1 = z_2$
- **R3** $\forall a, b, c, z_1, z_2 \in R$, $((a \circ b) * c)(z_1) > \theta$ and $((a * c) \circ (b * c))(z_2) > \theta$ implies $z_1 = z_2$, and $(a * (b \circ c))(z_1) > \theta$ and $((a * b) \circ (a * c))(z_2) > \theta$ implies $z_1 = z_2$.

If $(a * b)(u) > \theta \Leftrightarrow (b * a)(u) > \theta$ for all $a, b \in R$, then (R, G, H) is said to be a commutative fuzzy ring. If (R, G, H) contains an element e_* such that $(a * e_*)(u) > \theta$ and $(e_* * a)(v) > \theta$ implies $u = v$ for all $a \in R$, then (R, G, H) is said to be a fuzzy ring with identity.

The identity element e_0 is called the zero element of the fuzzy ring.

A fuzzy ring has the following properties.

Theorem 10 *If* (R, G, H) *is a fuzzy ring with zero element* e_0*, then for any* $a, b, \in R$

1. $(a * b)(b) > \theta$ *and* $(a * b)(e_0) > \theta$ *implies* $b = e_0$ *and* $(b * a)(b) > \theta$ *and* $(b * a)(e_0) > \theta$ *implies* $b = e_0$
2. *Let* b^{-1} *be inverse of* b *in* (R, G)*. Then* $(a * b^{-1})(v) > \theta$ *and* $(a * b)(w) > \theta$ *implies* $v = w^{-1}$*, and* $(a^{-1} * b)(s) > \theta$ *and* $(a * b)(t) > \theta$ *implies* $s = t^{-1}$.
3. $(a^{-1} * b^{-1})(u) > \theta$ *and* $(a * b)(v) > \theta$ *implies* $u = v$.

Proof 1. Since e_0 is a zero element of fuzzy ring R,

$$((a * b) \circ e_0)(b) \geq H(a, b, b) \wedge G(b, e_0, b) > \theta$$

and

$$((a \circ e_0) * (b \circ e_0))(e_0) \geq G(a, e_0, a) \wedge G(b, e_0, b) \wedge H(a, b, e_0) > \theta$$
$$((b * a) \circ e_0)(b) \geq H(b, a, b) \wedge G(b, e_0, b) > \theta$$

and

$$((b \circ e_0) * (a \circ e_0))(e_0) \geq G(b, e_0, b) \wedge G(a, e_0, a) \wedge H(b, a, e_0) > \theta.$$

It follows that $b = e_0$ from (R3).

2. Let $c \in R$ such that $G(u, w, c) > \theta$. Then

$$((a * b^{-1}) \circ (a * b))(c) \geq H(a, b^{-1}, v) \wedge H(a, b, w) \wedge G(v, w, c) > \theta$$
$$(a * (b^{-1} \circ b))(e_0) \geq G(b^{-1}, b, e_0) \wedge H(a, e_0, e_0) > \theta.$$

It follows that $c = e_0$ from (R3) and consequently $G(v, w, e_0) > \theta$. Hence we obtain $v = w^{-1}$ from (G3). Similarly, it is shown that $s = t^{-1}$.

3. Let $(a^{-1} * b^{-1})(u) > \theta$. Then $(a^{-1} * b)(u^{-1}) > \theta$ by condition 2, and $H(e_0, b, e_0) > \theta$ by condition 1. If $k \in R$ such that $G(v, u^{-1}, k) > \theta$, then

$$((a * b) \circ (a^{-1} * b))(k) \geq H(a, b, v) \wedge H(a^{-1}, b, u^{-1}) \wedge G(v, u^{-1}, k) > \theta$$
$$((a \circ a^{-1}) * b)(e_0) \geq G(a, a^{-1}, e_0) \wedge H(e_0, b, e_0) > \theta.$$

It follows that $k = e_0$ from (R3), and $G(v, u^{-1}, e_0) > \theta$ and similarly $G(u^{-1}, v, e_0) > \theta$. Consequently $v = u$ from (R3).

□

Definition 11 A nonzero element a in a fuzzy ring (R, G, H) is said to be left (respectively right) zero divisor if there exists a nonzero $b \in R$ such that $(a * b)(e_0) > \theta$ (respectively $(b * a)(e_0) > \theta$). A zero divisor is an element of R which is both a left and right zero divisor.

Proposition 12 *A fuzzy ring (R, G, H) has no zero divisor if and only if for all $a, b, c \in R$ with $a \neq e_0$*

$$(a * b)(u) > \theta \quad \textit{and} \quad (a * c)(u) > \theta \textit{ implies } b = c \tag{13}$$

or

$$(b * a)(u) > \theta \quad \textit{and} \quad (c * a)(u) > \theta \textit{ implies } b = c \tag{14}$$

Proof Suppose that R has no zero divisor. If $(a*c)(u) > \theta$, then $(a*c^{-1})(u^{-1}) > \theta$ from Theorem 10. For all $a, b, c \in R$

$$((a * b) \circ (a * c^{-1}))(e_0) \geq H(a, b, u) \wedge H(a, c^{-1}, u^{-1}) \wedge G(u, u^{-1}, e_0) > \theta$$
$$((a * (b \circ c^{-1}))(e_0) \geq G(b, c^{-1}, k) \wedge H(a, k, e_0) > \theta$$

since R is a fuzzy ring. Thus $H(a, k, e_0) > \theta$ and $k = e_0$ since $a \neq e_0$ and R has no zero divisors. Hence $G(b, c^{-1}, e_0) > \theta$. Therefore $b = c$ since (R, G) is a group. Similarly, it is show that $(b * a)(u) > \theta$ and $(c * a)(u) > \theta$ implies $b = c$.

Conversely, let (13) and (14) be satisfied and $a \neq e_0$. Then, $(a * b)(e_0) > \theta$ and $(a * e_0)(e_0) > \theta$ implies $b = e_0$ from (13). This completes the proof. □

Definition 13 Let (R, G, H) be a fuzzy ring.

1. If $(a*b)(u) > \theta \Leftrightarrow (b*a)(u) > \theta$, then (R, G, H) is said to be a commutative fuzzy ring.
2. If $\exists e_* \in R$ such that $(e_* * a)(a) > \theta$ and $(a * e_*)(a) > \theta$ for every $a \in R$, then (R, G, H) is said to be a fuzzy ring with identity.
3. Let (R, G, H) be a fuzzy ring with identity. If $(a*b)(e_*) > \theta$ and $(b*a)(e_*) > \theta$ $\forall a \in R, \exists b \in R$, then b is said to be an inverse element of a and is denoted by a_*^{-1}.

Theorem 14 *If (R, G, H) is a fuzzy ring with identity, then e_* is unique.*

Proof Let e'_*, e''_* be identity element of (R, G, H). Then $(e'_* * e''_*)(e'_*) > \theta$ and $(e'_* * e''_*)(e''_*) > \theta$, i.e. $H(e'_*, e''_*, e'_*) > \theta$ and $H(e'_*, e''_*, e''_*) > \theta$, it follows that $e'_* = e''_*$. □

4 Fuzzy subrings and fuzzy ideals

Let (R, G, H) be a fuzzy ring and S be a nonempty subset of R. Let $G_S(a, b, c) = G(a, b, c)$ and $H_S(a, b, c) = H(a, b, c)$, $\forall a, b, c \in S$, then we have

$$(a \bigtriangleup b)(c) = G_S(a,b,c) = G(a,b,c) \quad \forall a,b,c \in S \tag{15}$$
$$(a \diamond b)(c) = H_S(a,b,c) = H(a,b,c) \quad \forall a,b,c \in S \tag{16}$$
$$(a \diamond (b \bigtriangleup c))(z) = \bigvee_{x \in S} (G(b,c,x) \wedge H(a,x,z)) \quad \forall z \in S, \ \forall a,b,c \in S \tag{17}$$
$$((a \diamond b) \bigtriangleup (a \diamond c))(z) = \bigvee_{x,y \in S} (H(a,b,x) \wedge H(a,c,y) \wedge G(x,y,z)), \tag{18}$$
$$\forall z \in S, \ \forall a,b,c \in S \tag{19}$$

Then we can give following definitions and results.

Definition 15 Let (R, G, H) be a fuzzy ring and S be a nonempty subset of R.

1. $\forall a, b \in S, \forall c \in R$, $(a \circ b)(c) > \theta$ implies $c \in S$ and $(a * b)(c) > \theta$ implies $c \in S$
2. (S, G_S, H_S) is a fuzzy ring

then, (S, G_S, H_S) is called a fuzzy subring of (R, G, H).

Proposition 16 *Let (R, G, H) be a fuzzy ring and S be a nonempty subset of R. Then (S, G, H) is a fuzzy subring of R if and only if*

1. *$(a \circ b)(c) > \theta$ implies $c \in S$ and $(a * b)(c) > \theta$ implies $c \in S$ for all $a, b \in S$, $c \in G$.*
2. *$a \in S$ implies $a^{-1} \in S$ for all $a \in S$.*

Proof Let (R, G, H) be a fuzzy ring and (S, G, H) be a fuzzy subring of (R, G, H). From Definition 15 conditions (1)and (2) are satisfied.

Conversely, from Proposition 3 (S, G) is a fuzzy subgroup (R, G). Thus (R1) is satisfied. (R2) and (R3) are satisfied for elements of S since they are satisfied every elements of R. □

Proposition 17 *Let (R, G, H) be a fuzzy ring and*

$$C = \{x : x \in R \text{ and } (x * a)(c) > \theta \Leftrightarrow (a * x)(c) > \theta \text{ for any } a, c \in R\}.$$

Then, C is a fuzzy subring of R.

Proof Clearly $e_0 \in C$. Hence $C \neq \phi$

1. $x_1, x_2 \in C$ and $(x_1 \circ x_2)(x) > \theta \Rightarrow x \in C$.
   Let $a, c, d_1, d_2, b_1, b_2 \in R$ such that $H(x, a, c) > \theta$, $H(a, x, d_1) > \theta$, $H(a, x_1, b_1) > \theta$, $H(a, x_2, b_2) > \theta$ and $G(b_1, b_2, d_2) > \theta$. By $G(x_1, x_2, x) > \theta$ and $G(x_2, x_1, x) > \theta$, we have

$$(a * (x_1 \circ x_2))(d_1) \geq G(x_1, x_2, x) \wedge H(a, x, d_1) > \theta$$
$$((a * x_1) \circ (a * x_2))(d_2) \geq H(a, x_1, b_1) > \theta \wedge H(a, x_2, b_2)$$
$$> \theta \wedge G(b_1, b_2, d_2) > \theta$$

Thus $d_1 = d_2$ and $G(b_1, b_2, d_2) > \theta$.
For $x_1, x_2 \in C$, $H(x_2, a, b_2) > \theta$, $H(x_1, a, b_1) > \theta$ and $G(b_2, b_1, d_1) > \theta$, hence

$$((x_2 \circ x_1) * a)(c) \geq G(x_2, x_1, x) \wedge H(x, a, c) > \theta$$

$$((x_2 * a) \circ (x_1 * a))(d_1) \geq H(x_2, a, b_2) \wedge H(x_1, a, b_1) \wedge G(b_2, b_1, d_1) > \theta$$

Thus $c = d_1$ and $H(a, x, c) > \theta$.
Similarly, $H(a, x, c) > \theta$ implies $H(x, a, c) > \theta$. Thus $x \in C$.
And $(x_1 * x_2)(c) > \theta \Rightarrow x \in C$.
Let $a, c, d_1, d_2, b \in R$ such that $H(x, a, c) > \theta, H(a, x, d_1) > \theta, H(a, x_2, b) > \theta$ and $H(b, x_1, d_2) > \theta$. By $H(x_1, x_2, x) > \theta$ and $H(x_2, x_1, x) > \theta$, we have

$$(a * (x_2 * x_1))(d_1) \geq H(x_2, x_1, x) \wedge H(a, x, d_1) > \theta$$

$$(a * x_2) * x_1))(d_2) \geq H(a, x_2, b) \wedge H(b, x_1, d_2) > \theta$$

Thus $d_1 = d_2$ and $H(b, x_1, d_1) > \theta$.
For $x_1, x_2 \in C$, $H(x_2, a, b) > \theta$, $H(x_1, b, d_1) > \theta$, hence

$$((x_1 * x_2) * a)(c) \geq H(x_1, x_2, x) \wedge H(x, a, c) > \theta$$

$$(x_1 * (x_2 * a))(d_1) \geq H(x_2, a, b) \wedge H(x_1, b, d_1) > \theta$$

Thus $c = d_1$ and $H(a, x, c) > \theta$.
Similarly, $H(a, x, c) > \theta$ implies $H(x, a, c) > \theta$, and then $x \in C$.

2. $x \in C \Rightarrow x^{-1} \in C$.
Let $c, b, d \in R$ such that $G(a, x^{-1}, c) > \theta, G(c, x, b) > \theta$, and $G(x^{-1}, a, d) > \theta$. Then

$$((a \circ x^{-1}) \circ x)(b) \geq G(a, x^{-1}, c) \wedge G(c, x, b) > \theta,$$

$$(a \circ (x^{-1} \circ x))(a) \geq G(x^{-1}, x, e_0) \wedge G(a, e_0, a) > \theta.$$

Thus $b = a$ and $G(c, x, a) > \theta$, $G(x, c, a) > \theta$.
Since

$$(x^{-1} \circ (x \circ c))(d) \geq G(x, c, a) \wedge G(x^{-1}, a, d) > \theta,$$

$$((x^{-1} \circ x) \circ c))(c) \geq G(x^{-1}, x, e_0) \wedge G(e_0, c, c) > \theta.$$

we get $c = d$ and $G(x^{-1}, a, c) > \theta$.
Similarly, $x \in C$ and $G(x^{-1}, a, c) > \theta$ implies $G(a, x^{-1}, c) > \theta$. Thus, $x^{-1} \in C$.

Hence, C is a fuzzy subring of G from Proposition 16. □

Definition 18 A nonempty subset $\boldsymbol{I}$ of a fuzzy ring R is called a fuzzy ideal of R the following conditions are satisfied.

1. $\forall x, y \in \boldsymbol{I}$, $(x \circ y)(z) > \theta \Rightarrow z \in \boldsymbol{I}$ for $z \in R$,
2. $\forall x \in \boldsymbol{I}, x^{-1} \in \boldsymbol{I}$,
3. For all $s \in \boldsymbol{I}$, for all $r \in R$ $(r * s)(x) > \theta \Rightarrow x \in \boldsymbol{I}$ and $(s * r)(y) > \theta \Rightarrow y \in \boldsymbol{I}$ $x, y \in R$.

Remark 19 It is clear that, according to Definition 18, a fuzzy ideal of a fuzzy ring R is a fuzzy subring of R.

Theorem 20 *Let $\boldsymbol{I}_i$, $i \in I$, be a fuzzy ideal of R. Then $\bigcap_{i \in I} \boldsymbol{I}_i$ is a fuzzy ideal of R.*

Proof It is trivial. □

Let $\mathbf{I}$ be a fuzzy ideal of fuzzy ring R and let $\Omega = \{a \circ \mathbf{I} : a \in R\}$. We define a relation over Ω

$$a_1 \circ \mathbf{I} \sim a_2 \circ \mathbf{I} \Leftrightarrow \exists u \in \mathbf{I}, \text{ such that } G(a_1^{-1}, a_2, u) > \theta.$$

Then we get the following result.

Theorem 21 $\sim$ *is an equivalent relation over Ω.*

Proof It is proved similarly to the proof of the Theorem 4.1 in [8]. □

Fuzzy ideals play approximately the same role in the theory of fuzzy rings as normal fuzzy subgroups do in the theory of fuzzy groups. For instance, let (R, G, H) be a fuzzy ring and $\mathbf{I}$ be a fuzzy ideal of R. Then, $(\mathbf{I}, G)$ is a fuzzy subgroup of (R, G). Since (R, G) is abelian, $(\mathbf{I}, G)$ is a normal fuzzy subgroup of (R, G) according to Theorem 5. Consequently, from Theorem 8, there is a well-defined quotient fuzzy group $R/\mathbf{I}$ with following operations

$$([a \circ \mathbf{I}] \oplus [b \circ \mathbf{I}])(c \circ \mathbf{I}) = \bar{G}([a \circ \mathbf{I}], [b \circ \mathbf{I}], [c \circ \mathbf{I}]) = \bigvee_{(a', b', c') \in \bar{a} \times \bar{b} \times \bar{c}} G(a', b', c').$$

$$([a \circ \mathbf{I}] \otimes [b \circ \mathbf{I}])(c \circ \mathbf{I}) = \bar{H}([a \circ \mathbf{I}], [b \circ \mathbf{I}], [c \circ \mathbf{I}]) = \bigvee_{(a', b', c') \in \bar{a} \times \bar{b} \times \bar{c}} H(a', b', c').$$

By these operations $R/\mathbf{I}$ can be made into a fuzzy ring.

Let $\bar{G}$ and $\bar{H}$ be two fuzzy binary operation on $R/\mathbf{I}$, so we have

$$\begin{aligned}
(([a\circ\mathbf{I}]\oplus[b\circ\mathbf{I}])\oplus[c\circ\mathbf{I}])([d\circ\mathbf{I}]) &= \bigvee_{x\in R}(\bar{G}([a\circ\mathbf{I}],[b\circ\mathbf{I}],[x\circ\mathbf{I}])\\
&\quad\wedge\bar{G}([x\circ\mathbf{I}],[c\circ\mathbf{I}],[d\circ\mathbf{I}])),\\
([a\circ\mathbf{I}]\oplus([b\circ\mathbf{I}]\oplus[c\circ\mathbf{I}]))([w\circ\mathbf{I}]) &= \bigvee_{x\in R}(\bar{G}([b\circ\mathbf{I}],[c\circ\mathbf{I}],[x\circ\mathbf{I}])\\
&\quad\wedge\bar{G}([a\circ\mathbf{I}],[x\circ\mathbf{I}],[w\circ\mathbf{I}])),\\
([a\circ\mathbf{I}]\otimes([b\circ\mathbf{I}]\oplus[c\circ\mathbf{I}]))([z\circ\mathbf{I}]) &= \bigvee_{d\in R}(\bar{G}([b\circ\mathbf{I}],[c\circ\mathbf{I}],[d\circ\mathbf{I}])\\
&\quad\wedge\bar{H}([a\circ\mathbf{I}],[d\circ\mathbf{I}],[z\circ\mathbf{I}])),\\
(([a\circ\mathbf{I}]\oplus[b\circ\mathbf{I}])\otimes([a\circ\mathbf{I}]\oplus[c\circ\mathbf{I}]))([z\circ\mathbf{I}]) &= \bigvee_{d\in R}(\bar{G}([a\circ\mathbf{I}],[b\circ\mathbf{I}],[d\circ\mathbf{I}])\\
&\quad\wedge\bar{G}([a\circ\mathbf{I}],[c\circ\mathbf{I}],[w\circ\mathbf{I}])\\
&\quad\wedge\bar{H}([d\circ\mathbf{I}],[w\circ\mathbf{I}],[z\circ\mathbf{I}]))
\end{aligned}$$

Theorem 22 *Let (R,G,H) be a fuzzy ring and $\boldsymbol{I}$ be a fuzzy ideal of R. Then the quotient fuzzy group $(R/\boldsymbol{I},\bar{G})$ is a fuzzy ring with*

$$([a\circ\boldsymbol{I}]\otimes[b\circ\boldsymbol{I}])(c\circ\boldsymbol{I}) = \bar{H}([a\circ\boldsymbol{I}],[b\circ\boldsymbol{I}],[c\circ\boldsymbol{I}]) = \bigvee_{(a',b',c')\in\bar{a}\times\bar{b}\times\bar{c}} H(a',b',c').$$

Proof From Theorem 8 $(R/\mathbf{I},\bar{G})$ is a fuzzy group. Let $([a\circ\mathbf{I}]\oplus[b\circ\mathbf{I}])([c\circ\mathbf{I}]) > \theta$. Using (R,G) is an abelian fuzzy group, we get

$$\begin{aligned}
([a\circ\mathbf{I}]\oplus[b\circ\mathbf{I}])([c\circ\mathbf{I}]) &= \bar{G}([a\circ\mathbf{I}],[b\circ\mathbf{I}],[c\circ\mathbf{I}])\\
&= \bigvee_{(a',b',c')\in\bar{a}\times\bar{b}\times\bar{c}} G(a',b',c') > \theta\\
&\Leftrightarrow \bigvee_{(b',a',c')\in\bar{b}\times\bar{a}\times\bar{c}} G(b',a',c') = \bar{G}([b\circ\mathbf{I}],[a\circ\mathbf{I}],[c\circ\mathbf{I}])\\
&= ([b\circ\mathbf{I}]\oplus[a\circ\mathbf{I}])([c\circ\mathbf{I}]) > \theta.
\end{aligned}$$

Thus $(R/\mathbf{I},\bar{G})$ is an abelian fuzzy group.
(R2) can be proved similar to the proof of Theorem 4.3 in [8]. It only remains to check that (R3) is satisfy.

Let $(([a\circ\mathbf{I}]\oplus[b\circ\mathbf{I}])\otimes[c\circ\mathbf{I}])([d\circ\mathbf{I}]) > \theta$ and $(([a\circ\mathbf{I}]\otimes[c\circ\mathbf{I}])\oplus([b\circ\mathbf{I}]\otimes[c\circ\mathbf{I}]))([w\circ\mathbf{I}]) > \theta$. Then, we have $a_1,a_1',b_1,b_1',c_1,c_1',d_1,w_1\in R$ such that $c_1\circ\mathbf{I}\sim c_1'\circ\mathbf{I}\sim c\circ\mathbf{I}$, $a_1\circ\mathbf{I}\sim a_1'\circ\mathbf{I}\sim a\circ\mathbf{I}$, $b_1\circ\mathbf{I}\sim b_1'\circ\mathbf{I}\sim b\circ\mathbf{I}$, $d_1\circ\mathbf{I}\sim d\circ\mathbf{I}$,

$w_1 \circ \mathbf{I} \sim w \circ \mathbf{I}$ and there exist elements $u_1, u_2, u_3 \in \mathbf{I}$, $x_1', x_2', x_3' \in R$ such that

$$G(a_1, b_1, x_1') \wedge H(x_1', c_1, d_1) > \theta$$

$$H(a_1', c_1', x_2') \wedge H(b_1', c_1', x_3') \wedge G(x_2', x_3', w_1) > \theta$$

$$G(a_1', u_1, a_1) > \theta, G(b_1', u_2, b_1) > \theta, G(c_1', u_3, c_1) > \theta$$

Let $z_1 \in R$ such that $G(a_1', b_1', z_1) > \theta$, then by $G(a_1, b_1, x_1') > \theta$, $G(a_1', u_1, a_1) > \theta$, $G(a_1', b_1', z_1) > \theta$, $G(b_1', u_2, b_1) > \theta$, and proof of the Theorem 4.2 in [8], we have $\exists u \in \mathbf{I}$ such that $G(z_1, u, x_1') > \theta$.

Since $\mathbf{I}$ is a fuzzy ideal there exist elements $u_3', u', u_5, u_6, u_7 \in \mathbf{I}$ such that $H(z_1, u_3, u_3') > \theta$, $H(u, c_1', u') > \theta$, $H(u, u_3, u_5) > \theta$, $G(u_3', u', u_6) > \theta$ and $G(u_6, u_5, u_7) > \theta$.

Let $z_2 \in R$ such that $H(z_1, c_1', z_2) > \theta$. By $H(x_1', c_1, d_1) > \theta$, $G(z_1, u, x_1') > \theta$, $G(c_1', u_3, c_1) > \theta$, $H(z_1, c_1', z_2) > \theta$ and proof of the Theorem 4.2 in [8] we have $u_7 \in \mathbf{I}$ $G(z_2, u_7, d_1) > \theta$. Since

$$((a_1' \circ b_1') * c_1')(z_2) \geq (G(a_1', b_1', z_1) \wedge H(z_1, c_1', z_2)) > \theta$$

$$((a_1' * c_1') \circ (b_1' * c_1'))(w_1) \geq H(a_1', c_1', x_2') \wedge H(b_1', c_1', x_3') \wedge G(x_2', x_3', w_1) > \theta$$

so $z_2 = w_1$ and $G(z_2, u_7, d_1) > \theta$. Then $w_1 \circ \mathbf{I} \sim d \circ \mathbf{I}$ and consequently $[w \circ \mathbf{I}] = [d \circ \mathbf{I}]$. Similarly, if $([a \circ \mathbf{I}] \otimes ([b \circ \mathbf{I}] \oplus [c \circ \mathbf{I}]))([d \circ \mathbf{I}]) > \theta$ and $(([a \circ \mathbf{I}] \otimes [b \circ \mathbf{I}]) \oplus ([a \circ \mathbf{I}] \otimes [c \circ \mathbf{I}]))([w \circ \mathbf{I}]) > \theta$, then it can be shown that $[w \circ \mathbf{I}] = [d \circ \mathbf{I}]$. □

Definition 23 $(R/\boldsymbol{I}, \bar{G}, \bar{H})$ is called a factor fuzzy ring of modulo $\mathbf{I}$.

5 Fuzzy ring homomorphisms

Definition 24 Let (R_1, G_1, H_1) and (R_2, G_2, H_2) be two fuzzy rings $f : R_1 \to R_2$ is a mapping. If

1. $G_1(a, b, c) > \theta \Rightarrow G_2(f(a), f(b), f(c)) > \theta$
2. $H_1(a, b, c) > \theta \Rightarrow H_2(f(a), f(b), f(c)) > \theta$

then f is called a fuzzy ring homomorphism. If f is 1-1, it is called a fuzzy ring monomorphism. If f is onto, it is called a fuzzy ring epimorphism. If f is both 1-1 and onto, it is called a fuzzy ring isomorphism.

Proposition 25 *Let $f : (R_1, G_1, H_1) \to (R_2, G_2, H_2)$ be a fuzzy ring homomorphism. Then*

1. $f(e_1) = e_2$,
2. $f(a^{-1}) = (f(a))^{-1}$.

Proof

1. Since f is a fuzzy ring homomorphism, for all $a \in R_1$

$$G_1(a, e_1, a) > \theta \Rightarrow G_2(f(a), f(e_1), f(a)) > \theta$$

and for $f(a) \in R_2$ we have $G_2(f(a), e_2, f(a)) > \theta$. Since

$$\begin{aligned}((f(a)^{-1} \circ f(a)) \circ f(e_1))(e_2) &\geq G(f(a)^{-1}, f(a), e_2)\\ &\quad \wedge G(e_2, f(e_1), f(e_1)) > \theta\\ (f(a)^{-1} \circ (f(a) \circ f(e_1)))(f(e_1)) &\geq G_2(f(a), f(e_1), f(a))\\ &\quad \wedge G_2(f(a)^{-1}, f(a), f(e_1)) > \theta.\end{aligned}$$

We get $f(e_1) = e_2$.

2. Since f is a fuzzy ring homomorphism, for all $a \in R_1$

$$G_1(a, a^{-1}, e_1) > \theta \Rightarrow G_2(f(a), f(a^{-1}), f(e_1)) > \theta.$$

From (1) $f(e_1) = e_2$, then $G_2(f(a), f(a^{-1}), e_2) > \theta$. Thus $f(a^{-1}) = (f(a))^{-1}$.

□

Theorem 26 *Let $f : (R_1, G_1, H_1) \to (R_2, G_2, H_2)$ be a fuzzy ring homomorphism. Then*

1. *Imf is a fuzzy subring of R_2*
2. *$Kerf$ is a fuzzy ideal of R_1.*

Proof

1. Clearly, $f(e_1) = e_2 \in Imf$ and so $Imf \neq \phi$. If $x_1, x_2, x \in R_1$ such that $G_1(x_1, x_2, x) > \theta$, then $G_2(f(x_1), f(x_2), f(x)) > \theta$. Thus $f(x) \in Imf$.
Let $x^{-1} \in R_1$ such that $G_1(x, x^{-1}, e_1) > \theta$, then $G_2(f(x), f(x^{-1}), f(e_1)) > \theta$. From Proposition 25, $G_2(f(x), f(x^{-1}), f(e_1)) = G_2(f(x), f(x)^{-1}, e_2) > \theta$. Thus $f(x)^{-1} \in Imf$.

2. Clearly, $e_1 \in Kerf$ and so $Kerf \neq \phi$. $\forall\, x, y \in Kerf$ such that $G_1(x, y, z) > \theta$. Then $G_2(f(x), f(y), f(z)) > \theta$. Since f is a fuzzy ring homomorphism $G_2(e_2, e_2, f(z)) > \theta$. Thus $f(z) = e_2$. We obtain $z \in Kerf$.
If $x \in Kerf$ such that $G_1(x, x^{-1}, e_1) > \theta$, then $G_2(f(x), f(x^{-1}), f(e_1)) = G_2(e_2, f(x)^{-1}, e_2) > \theta$. Hence $f(x^{-1}) = e_2$ that is $x^{-1} \in Kerf$.
Finally, for all $x \in Kerf$ and for all $r \in R_1$, if $H_1(x, r, u) > \theta$, then $H_2(f(x), f(r), f(u)) > \theta$. Since $f(x) = e_2$, $H_2(e_2, f(r), f(u)) > \theta$. From Theorem 10, $f(u) = e_2$. Similarly, if $H_1(r, x, v) > \theta$, then $H_2(f(r), f(x), f(v)) > \theta$. Since $f(x) = e_2$, $H_2(f(r), e_2, f(v)) > \theta$. From Theorem 10, $f(v) = e_2$. Therefore, $Kerf$ is a fuzzy ideal of R_1. □

Theorem 27 *Let* $f : (R_1, G_1, H_1) \to (R_2, G_2, H_2)$ *be a fuzzy ring epimorphism. Then* R_1/N *is isomorphic to* R_2*, where* $N = Kerf$*.*

Proof From Theorem 5.3 in [8] and its proof, we know that the mapping

$$\psi : R_1/N \to R_2, [r \circ N] \mapsto f(r)$$

is a well defined one-to-one fuzzy group homomorphism. Here it is only remained to check that if $H_1([a \circ N], [b \circ N], [c \circ N]) > \theta$ then $H_2(\psi([a \circ N]), \psi([b \circ N]), \psi([c \circ N])) > \theta$. Let $H_1([a \circ N], [b \circ N], [c \circ N]) > \theta$ then there exist $a_1, b_1, c_1 \in R_1$ and $n_1, n_2, n_3 \in N$ such that $H_1(a, n_1, a_1) > \theta$, $H_1(b, n_2, b_1) > \theta$, $H_1(c, n_3, c_1) > \theta$, $H_1(a_1, b_1, c_1) > \theta$. Let $w \in R_1$ such that $H_1(a, b, w) > \theta$. Similar proof of the Theorem 4.2 in [8] we have $\exists n' \in N$ such that $H_1(w, n', c) > \theta$. Hence $w \circ N \sim c \circ N$ and, consequently, $f(c) = f(w)$. By $H_1(a, b, w) > \theta$, we have $H_2(f(a), f(b), f(w)) > \theta$, and then $H_2(f(a), f(b), f(c)) > \theta$. Hence ψ is a fuzzy isomorphism. □

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