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Strong reducibility of partial numberings

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Abstract. A strong reducibility relation between partial numberings is introduced which is such that the reduction function transfers *exactly* the numbers which are indices under the numbering to be reduced into corresponding indices of the other numbering. The degrees of partial numberings of a given set with respect to this relation form an upper semilattice.

In addition, Ershov's completion construction for total numberings is extended to the partial case: every partially numbered set can be embedded in a set which results from the given set by adding one point and which is enumerated by a total and complete numbering. As is shown, the degrees of complete numberings of the extended set also form an upper semilattice. Moreover, both semilattices are isomorphic.

This is not so in the case of the usual, weaker reducibility relation for partial numberings which allows the reduction function to transfer arbitrary numbers into indices.

1. Introduction

Numberings have turned out to be an important tool for lifting computability notions to abstract structures. In the development of numbering theory mostly total numberings have been considered. (For an overview of at least an important part of this development see [3–7].) This can be done as long as purely algebraic structures are considered. Canonical numberings of topological spaces, however, are only partial maps, in general. Moreover, as has been shown by the author [9], they are necessarily so. They are total, only in case that the space has sufficiently many finite points, i.e., points with a finitely based neighbourhood filter.

Total numberings are usually compared by the reducibility preorder. The collection of the induced equivalence classes, called degrees, is known to be an upper semilattice. It is easy to lift the reducibility relation to the case of partial numberings. As has been shown in a recent paper [2], these numberings and their degrees behave very differently from the total case. Their collection is now a distributive lattice. Moreover, a computable function can reduce several such numberings to

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one numbering. As a consequence of this, degrees are uncountable in the partial case.

In the generalised reducibility preorder the reduction function is a partial computable map that is at least defined for all indices of the numbering ν to be reduced and maps them to fitting indices of the second numbering κ. But it may also map numbers which do not appear among the indices of ν to indices of κ. Thus, if we know that some function value is an index of some point under numbering κ, we cannot conclude that there is a corresponding argument which is an index of the point with respect to numbering ν.

In this note we strengthen the reducibility preorder for partial numberings so that one *can* reason this way. Note that the strengthened reducibility has already been considered in [3, 9]. With respect to this preorder partial numberings behave as in the total case: the collection of degrees is an upper semilattice and the numberings given by the infimum operation defined for the weaker reducibility relation in [2], are no longer infima with respect to the stronger relation.

Ershov [3] demonstrated that every totally numbered set S can be embedded in a set $\widehat{S}$ enumerated by a complete numbering. To this end a special element is added to the given set S. In this note we will see that the construction works also in the case of partial numberings. By this way, a total complete numbering of the extension $\widehat{S}$ of S is assigned to each numbering of S. As will be shown, the degrees of complete numberings of $\widehat{S}$ also form an upper semilattice. Note that in general it is not a sub-semilattice of the upper semilattice of the degrees of all total numberings of $\widehat{S}$: the operations of taking least upper bounds do not necessarily coincide. Moreover, we will prove that the upper semilattice of degrees of partial numberings of S with respect to the strengthened reducibility relation is isomorphic to the upper semilattice of degrees of complete numberings of $\widehat{S}$, thus showing that with respect to this reducibility partial numberings behave like the total ones.

The paper is organised as follows. Section 2 contains basic definitions and results. In Section 3, Ershov's completion construction is extended to partial numberings and in Section 4 some properties of the operator which maps a partial numbering to its completion are derived. Section 5 discusses the relationship between the upper semilattice of the degrees of partial numberings of a set S and the upper semilattices of the degrees of total and/or complete numberings of the extended set $\widehat{S}$. Final remarks are made in Section 6.

2. Basic definitions and results

In what follows, let $\langle\ ,\ \rangle\colon \omega^2 \to \omega$ be a recursive pairing function with corresponding projections π_1 and π_2 such that $\pi_i(\langle a_1, a_2\rangle) = a_i$. Furthermore, let $P^{(n)}$ $(R^{(n)})$ denote the set of all n-ary partial (total) computable functions and let φ be a Gödel numbering of $P^{(1)}$. We let $\varphi(a)\downarrow \in C$ mean that the computation of $\varphi(a)$ stops with value in C. Set $u(\langle m, n\rangle) = \varphi_m(n)$ and for $a \in \omega$, let $u'(a)$ be the first $c \in \omega$ found in some dovetailing computation of u with $u(c) = a$. Then u' is computable and a right inverse of u. Since u is surjective, u' is even total.

Definition 1. *Let S be a countable set. A* (partial) numbering ν *of S is a surjective partial map* $\nu\colon \omega \rightharpoonup S$ *with domain* $\mathrm{dom}(\nu)$.

The value of ν at $n \in \operatorname{dom}(\nu)$ is denoted, interchangeably, by ν_n and $\nu(n)$. For a given $s \in S$, any $n \in \operatorname{dom}(\nu)$ with $\nu(n) = s$ is called *index* of s. In case that $\operatorname{dom}(\nu) = \omega$ we say that ν is a *total numbering*. The set of all partial numberings of the set S is denoted by $\mathrm{Num}_p(S)$ and $\mathrm{Num}(S)$ stands for the set of all total numberings of S.

Among the total numberings of set S those being complete will be of special importance for us.

Definition 2. *Let S be a countable set. A total numbering ν of S is* complete*, if there is some element $\perp$ in S so that for every $p \in P^{(1)}$ there is some $g \in R^{(1)}$ with*

$$\nu(g(a)) = \begin{cases} \nu(p(a)) & \text{if } a \in \operatorname{dom}(p), \\ \perp & \text{otherwise.} \end{cases}$$

The distinguished element $\perp$ is called *special* element. Let $\mathrm{CNum}_\perp(S)$ be the set of all complete numberings of S that have $\perp$ as special element.

Numbered sets form a category. Morphisms are the effective maps, where for two numbered sets (S, ν) and (S', ν') a map $F\colon S \to S'$ is called *effective*, if there is a function $f \in P^{(1)}$ such that $f(a)\downarrow \in \operatorname{dom}(\nu')$ and $F(\nu_a) = \nu'_{f(a)}$, for all $a \in \operatorname{dom}(\nu)$. One says in this case that f *tracks* F.

Definition 3. *Let $\nu, \kappa \in \mathrm{Num}_p(S)$.*

1. *$\nu \leq_p \kappa$, read ν is* partially reducible *to κ, if there is some* witness *function $f \in P^{(1)}$ such that* $\operatorname{dom}(\nu) \subseteq \operatorname{dom}(f)$, $f(\operatorname{dom}(\nu)) \subseteq \operatorname{dom}(\kappa)$, *and* $\nu(a) = \kappa(f(a))$, *for all $a \in \operatorname{dom}(\nu)$.*
2. *$\nu \leq_s \kappa$, read ν is* strongly reducible *to κ, if $\nu \leq_p \kappa$ via $f \in P^{(1)}$ so that* $\operatorname{dom}(\nu) = f^{-1}(\operatorname{dom}(\kappa))$.
3. *$\nu \equiv_s \kappa$, read ν is* strongly equivalent *to κ, if $\nu \leq_s \kappa$ and $\kappa \leq_s \nu$. Similarly for* partial equivalence $\equiv_p$.

If the numberings ν and κ are total we speak of *reducibility* of ν to κ and denote it by $\nu \leq \kappa$. In addition, we write $\nu \equiv \kappa$ if both $\nu \leq \kappa$ and $\kappa \leq \nu$.

In the case of total numberings ν and κ we have that $\nu \leq \kappa$ via $f \in R^{(1)}$ just if $\nu = \kappa \circ f$. As follows from the definition, if ν and κ are partial numberings and ν is partially reducible to κ via $f \in P^{(1)}$, then we only have that $\nu(a) = \kappa(f(a))$, for all $a \in \operatorname{dom}(\nu)$, whereas ν is strongly reducible to κ via $f \in P^{(1)}$ exactly if $\nu = \kappa \circ f$, where, if read pointwise, this equality means that either both sides are defined and equal, or both sides are undefined. It follows that $\nu \leq_s \kappa$ via $f \in P^{(1)}$ if and only if for every $s \in S$ and all $i \in \omega$,

$$i \in \nu^{-1}(\{s\}) \Leftrightarrow f(i)\downarrow \in \kappa^{-1}(\{s\}), \tag{1}$$

and that $\nu \leq_p \kappa$ via $f \in P^{(1)}$ if and only if for every $s \in S$ and all $i \in \omega$,

$$i \in \nu^{-1}(\{s\}) \Rightarrow f(i)\downarrow \in \kappa^{-1}(\{s\}). \tag{2}$$

This shows that strong reducibility extends Ershov's notion of pm-reducibility for sets and families of sets [3] to partial numberings. Moreover, we see that in the case that $\nu \leq_p \kappa$ it is only required that the witness function f behaves correctly when transforming indices i of elements $s \in S$ with respect to ν into indices $f(i)$ of s with respect to κ. We do not demand that if $f(i)$ in an index of some s with respect to κ, then i must be an index of s with respect to ν. Though in some cases we need be able to reason in this way.

In the theory of effective topological spaces [9, 10], e.g., one usually works with numberings ν for which there is a function $\mathrm{pt} \in P^{(1)}$ such that if m is an index of a certain normed enumeration of a base of basic open sets of the neighbourhood filter of some point s, then $\mathrm{pt}(m)\downarrow \in \mathrm{dom}(\nu)$ and $s = \nu(\mathrm{pt}(m))$. This property is invariant under partial equivalence. For some results, however, we had to use numberings ν with the additional property that if $\mathrm{pt}(m)\downarrow \in \mathrm{dom}(\nu)$, then, conversely, m is also an index of an enumeration of a base of basic open sets of the neighbourhood filter of $\nu(\mathrm{pt}(m))$. This property is only invariant under strong equivalence and it was the search for the appropriate invariance notion in this case, which led us to consider strong reducibility and strong equivalence in [9].

As a consequence of the weaker condition (2), one function may reduce many numberings to the same given numbering. If e.g. α is a numbering of S then any numbering β of S the graph of which is included in the graph of α is reduced to α by the identity function on ω. The behaviour of partial numberings with respect to partial reducibility has been studied in [2]. Here, we will investigate the stronger reducibility notion.

Both, $\leq_s$ and $\leq$, respectively, are reflexive and transitive relations on $\mathrm{Num}_p(S)$ and $\mathrm{Num}(S)$. Therefore we can introduce degrees of numberings as follows:

$$\begin{aligned} \deg_s(\nu) &= \{\kappa \in \mathrm{Num}_p(S) \mid \nu \equiv_s \kappa\} \quad (\nu \in \mathrm{Num}_p(S)), \\ \deg(\nu) &= \{\kappa \in \mathrm{Num}(S) \mid \nu \equiv \kappa\} \quad (\nu \in \mathrm{Num}(S)). \end{aligned}$$

As usual the reducibilities $\leq_s$ and $\leq$, respectively, induce partial orderings on the sets of degrees which we also denote by $\leq_s$ and $\leq$. Thus, we have the following partial orders:

$$\begin{aligned} \mathcal{L}_s(S) &= (\{\deg_s(\nu) \mid \nu \in \mathrm{Num}_p(S)\}, \leq_s), \\ \mathcal{L}(S) &= (\{\deg(\nu) \mid \nu \in \mathrm{Num}(S)\}, \leq), \\ \mathcal{C}_\perp(S) &= (\{\deg(\nu) \mid \nu \in \mathrm{CNum}_\perp(S)\}, \leq). \end{aligned}$$

The first two structures are upper semilattices in which the supremum of the degrees of ν and κ is induced by the *join* $\nu \oplus \kappa$ defined as follows: for $a \in \omega$

$$(\nu \oplus \kappa)(2a) = \begin{cases} \nu(a) & \text{if } a \in \mathrm{dom}(\nu), \\ \text{undefined} & \text{otherwise,} \end{cases}$$

$$(\nu \oplus \kappa)(2a+1) = \begin{cases} \kappa(a), & \text{if } a \in \mathrm{dom}(\kappa), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The join $\nu \oplus \kappa$ of two complete numberings ν and κ needs not be complete again. Nevertheless, we will see that also $\mathcal{C}_\perp(S)$ is an upper semilattice.

In the remainder of this section we recall some notions and facts on partially ordered sets and monotone maps which we will need later (cf. [1, 8]).

Definition 4. *Let P and Q be partially ordered sets. A pair $s\colon P \to Q$, $r\colon Q \to P$ of monotone maps is called a* monotone section retraction pair *if $r \circ s$ is the identity on P. In this situation P is said to be a* monotone retract *of Q.*

One sees immediately that in a section retraction pair the retraction is surjective and the section is injective.

Definition 5. *Let P and Q be partially ordered sets and $l\colon P \to Q$ and $u\colon Q \to P$ be maps. One says that (l, u) is an* adjunction *between P and Q if for all $x \in P$ and $y \in Q$,*

$$x \le u(y) \Leftrightarrow l(x) \le y.$$

The maps l and u, respectively, are called lower *and* upper adjoint.

Lower adjoints preserve existing suprema and upper adjoints preserve existing infima.

Definition 6. *Let P be a partially ordered set.*

1. A projection *is an idempotent, monotone self-map $p\colon P \to P$.*
2. A closure operator *is a projection c on P with $x \le c(x)$, for all $x \in P$.*

Note that the image of a closure operator is closed under the formation of existing infima. Moreover, its co-restriction $c^\circ\colon P \to c(P)$ preserves arbitrary suprema, i.e., $\sup_{c(P)} X = c(\sup_P X)$, for all $X \subseteq c(P)$.

Lemma 1. *Let P and Q be partially ordered sets and $(s\colon P \to Q, r\colon Q \to P)$ be a section retraction pair. If $s \circ r$ is a closure operator on Q, then (r, s) is an adjunction between Q and P.*

3. Completing partial numberings

In this section we transfer Ershov's completion construction [3] to partial numberings. The result will be weaker as in the case of total numberings. Let (S, ν) be a partially numbered set and for $s \in S$, $A_s = \nu^{-1}(\{s\})$. Set $\widehat{A}_s = u^{-1}(A_s)$, for $s \in S$, and for $a \in \omega$ define $\widehat{\nu}(a) = \{\, s \in S \mid a \in \widehat{A}_s \,\}$. Then

$$\widehat{\nu}(a) = \begin{cases} \{\nu(u(a))\} & \text{if } a \in u^{-1}(\mathrm{dom}(\nu)), \\ \emptyset & \text{otherwise.} \end{cases}$$

Set $\widehat{S} = \{\, \{s\} \mid s \in S \,\} \cup \{\emptyset\}$. Then $(\widehat{S}, \widehat{\nu})$ is a totally numbered set. Moreover, S is embedded in $\widehat{S}$ by the effective map $\iota\colon S \to \widehat{S}$ with $\iota(s) = \{s\}$. In order to see that ι is effective let $a \in \mathrm{dom}(\nu)$. Then $u'(a) \in u^{-1}(\mathrm{dom}(\nu))$ and hence

$$\widehat{\nu}(u'(a)) = \iota(\nu(u(u'(a)))) = \iota(\nu(a)).$$

Theorem 1. *For any partial numbering ν of S, $\widehat{\nu}$ is a complete numbering of $\widehat{S}$ with special element $\emptyset$.*

Proof. Let $p \in P^{(1)}$ and m be a Gödel number of $u \circ p$. Define $f \in R^{(1)}$ by $f(a) = \langle m, a \rangle$. Then $u \circ f = u \circ p$ and we have for $a \in \mathrm{dom}(p)$ with $p(a) \in u^{-1}(\mathrm{dom}(\nu))$ that

$$\widehat{\nu}(f(a)) = \{\nu(u(f(a)))\} = \{\nu(u(p(a)))\} = \widehat{\nu}(p(a)).$$

If $a \in \mathrm{dom}(p)$, but $p(a) \notin u^{-1}(\mathrm{dom}(\nu))$, it follows that

$$\widehat{\nu}(f(a)) = \{\, s \in S \mid f(a) \in u^{-1}(\nu^{-1}(\{s\}))\,\} = \emptyset = \widehat{\nu}(p(a)).$$

Similarly, we obtain that $\widehat{\nu}(f(a)) = \emptyset$, if $a \notin \mathrm{dom}(p)$.

4. Some properties

Let S be a countable set. In this section we study the completion operation introduced in the preceding section and its relationship to the operation which maps every numbering in $\mathrm{Num}(\widehat{S})$ onto a numbering in $\mathrm{Num}_p(S)$ by co-restricting it to $\widehat{S} \setminus \{\emptyset\}$.

Lemma 2. *The map* $\widehat{\cdot}\colon \mathrm{Num}_p(S) \to \mathrm{Num}(\widehat{S})$ *is injective.*

Proof. Let $\nu, \kappa \in \mathrm{Num}_p(S)$ with $\widehat{\nu} = \widehat{\kappa}$. Then we have for $a \in \omega$ that

$$\begin{aligned} a \notin \mathrm{dom}(\nu) &\Leftrightarrow u'(a) \notin u^{-1}(\mathrm{dom}(\nu)) \Leftrightarrow \widehat{\nu}(u'(a)) = \emptyset \\ &\Leftrightarrow \widehat{\kappa}(u'(a)) = \emptyset \Leftrightarrow a \notin \mathrm{dom}(\kappa). \end{aligned}$$

Thus $\mathrm{dom}(\nu) = \mathrm{dom}(\kappa)$. Similarly, it follows that $\nu(a) = \kappa(a)$, for $a \in \mathrm{dom}(\nu)$.

Lemma 3. *The map* $\widehat{\cdot}$ *is monotone.*

Proof. Let $\nu, \kappa \in \mathrm{Num}_p(S)$ so that $\nu \leq_s \kappa$ with witness function $f \in P^{(1)}$. Moreover, let $g \in R^{(1)}$ with $\varphi_{g(m)}(n) = f(\varphi_m(n))$. Set $h(\langle m, n \rangle) = \langle g(m), n \rangle$. We will show that $\widehat{\nu} = \widehat{\kappa} \circ h$.

By definition, $f \circ u = u \circ h$. Hence,

$$u^{-1}(\mathrm{dom}(\nu)) = u^{-1}(f^{-1}(\mathrm{dom}(\kappa)) = h^{-1}(u^{-1}(\mathrm{dom}(\kappa)).$$

Therefore, we have for $a \in u^{-1}(\mathrm{dom}(\nu))$ that

$$\widehat{\nu}(a) = \{\nu(u(a))\} = \{\kappa(f(u(a)))\} = \{\kappa(u(h(a)))\} = \widehat{\kappa}(h(a)).$$

If $a \notin u^{-1}(\mathrm{dom}(\nu))$, it follows that $h(a) \notin u^{-1}(\mathrm{dom}(\kappa))$. Thus, $\widehat{\nu}(a) = \emptyset = \widehat{\kappa}(h(a))$.

Now, for $\rho \in \mathrm{Num}(\widehat{S})$, let $\overline{\rho} \in \mathrm{Num}_p(S)$ be the numbering with $\overline{\rho}(a) = \iota^{-1}(\rho(a))$, if $\rho(a) \neq \emptyset$, and $\overline{\rho}(a)$ being undefined, otherwise.

Lemma 4. *The map* $\overline{\cdot}\colon \operatorname{Num}(\widehat{S}) \to \operatorname{Num}_p(S)$ *is monotone.*

Proof. Let $\rho, \gamma \in \operatorname{Num}(\widehat{S})$ so that $\rho \leq \gamma$ with witness function $f \in R^{(1)}$. If $a \in \operatorname{dom}(\overline{\rho})$, then $\gamma(f(a)) = \rho(a) \neq \emptyset$. Hence $f(a) \in \operatorname{dom}(\overline{\gamma})$ and therefore

$$\{\overline{\rho}(a)\} = \rho(a) = \gamma(f(a)) = \{\overline{\gamma}(f(a))\},$$

i.e., $\overline{\rho}(a) = \overline{\gamma}(f(a))$.

It remains to show that $\operatorname{dom}(\overline{\rho}) = f^{-1}(\operatorname{dom}(\overline{\gamma}))$. We have already seen that $\operatorname{dom}(\overline{\rho}) \subseteq f^{-1}(\operatorname{dom}(\overline{\gamma}))$. The converse inclusion follows analogously.

Lemma 5. *For* $\nu \in \operatorname{Num}_p(S)$, $\nu \equiv_s \overline{\widehat{\nu}}$.

Proof. By definition we have that $\operatorname{dom}(\overline{\widehat{\nu}}) = u^{-1}(\operatorname{dom}(\nu))$. Moreover,

$$\{\overline{\widehat{\nu}}(a)\} = \widehat{\nu}(a) = \{\nu(u(a))\},$$

for $a \in \operatorname{dom}(\overline{\widehat{\nu}})$. Thus $\overline{\widehat{\nu}} \leq_s \nu$.

Conversely, we obtain for $a \in \operatorname{dom}(\nu)$ that

$$\{\nu(a)\} = \{\nu(u(u'(a)))\} = \widehat{\nu}(u'(a)) = \{\overline{\widehat{\nu}}(u'(a))\}.$$

Since, in addition,

$$\operatorname{dom}(\nu) = u'^{-1}(u^{-1}(\operatorname{dom}(\nu)) = u'^{-1}(\operatorname{dom}(\overline{\widehat{\nu}})),$$

it also follows that $\nu \leq_s \overline{\widehat{\nu}}$.

Lemma 6. *Let* $\rho \in \operatorname{Num}(\widehat{S})$. *Then the following two statements hold:*
1. $\rho \leq \widehat{\overline{\rho}}$.
2. If $\rho \in \operatorname{CNum}_{\emptyset}(\widehat{S})$ *then* $\rho \equiv \widehat{\overline{\rho}}$.

Proof. Let $a \in \omega$ with $\rho(a) \neq \emptyset$. Then $a \in \operatorname{dom}(\overline{\rho})$ and hence $u'(a) \in u^{-1}(\operatorname{dom}(\overline{\rho}))$. Thus,

$$\rho(a) = \{\overline{\rho}(a)\} = \{\overline{\rho}(u(u'(a)))\} = \widehat{\overline{\rho}}(u'(a)).$$

If $\rho(a) = \emptyset$, then $u'(a) \notin u^{-1}(\operatorname{dom}(\overline{\rho}))$ and therefore $\rho(a) = \emptyset = \widehat{\overline{\rho}}(u'(a))$. This shows that $\rho \leq \widehat{\overline{\rho}}$.

Now, assume that ρ is complete with special element $\emptyset$. In this case there is some function $g \in R^{(1)}$ so that

$$\rho(g(a)) = \begin{cases} \rho(u(a)) & \text{if } a \in \operatorname{dom}(u), \\ \emptyset & \text{otherwise.} \end{cases}$$

If $a \in u^{-1}(\operatorname{dom}(\overline{\rho}))$, it follows that

$$\widehat{\overline{\rho}}(a) = \{\overline{\rho}(u(a))\} = \rho(u(a)) = \rho(g(a)).$$

On the other hand, if $a \in \operatorname{dom}(u)$, but $u(a) \notin \operatorname{dom}(\overline{\rho})$, we obtain that

$$\widehat{\overline{\rho}}(a) = \emptyset = \rho(u(a)) = \rho(g(a)).$$

If, finally, $a \notin \operatorname{dom}(u)$, then $a \notin u^{-1}(\operatorname{dom}(\overline{\rho}))$ as well. Hence, $\widehat{\overline{\rho}}(a) = \emptyset = \rho(g(a))$ again. Thus, $\widehat{\overline{\rho}} \leq \rho$.

5. Main results

Let $H\colon \mathcal{L}_s(S) \to \mathcal{L}(\widehat{S})$ and $B\colon \mathcal{L}(\widehat{S}) \to \mathcal{L}_s(S)$ be defined by

$$H(\deg_s(\nu)) = \deg(\widehat{\nu}) \qquad \text{and} \qquad B(\deg(\rho)) = \deg_s(\overline{\rho}),$$

for $\nu \in \mathrm{Num}_p(S)$ and $\rho \in \mathrm{Num}(\widehat{S})$. By the results of the last section both maps are well defined and monotone. Moreover, as a consequence of Lemma 5, (H, B) is a monotone section retraction pair. Hence, H is injective and B is surjective. In addition, we obtain the following result which shows that with respect to strong reducibility partial numberings behave as total ones.

Theorem 2. *$\mathcal{L}_s(S)$ is a monotone retract of $\mathcal{L}(\widehat{S})$.*

With Lemma 6(2) one readily verifies that $\mathcal{C}_\emptyset(\widehat{S})$ is the range of the embedding H. Let $C = H \circ B$. Then we obtain with Lemma 6(1) that $C\colon \mathcal{L}(\widehat{S}) \to \mathcal{L}(\widehat{S})$ is a closure operator. The next result is therefore a consequence of Lemma 1.

Proposition 1. *(B, H) is an adjunction between $\mathcal{L}(\widehat{S})$ and $\mathcal{L}_s(S)$.*

It follows that H preserves existing infima and B preserves suprema.

Corollary 1. *Let $\rho, \sigma \in \mathrm{Num}(\widehat{S})$. Then $\overline{\rho \oplus \sigma} \equiv_s \overline{\rho} \oplus \overline{\sigma}$.*

Since C is a closure operator, its co-restriction $C^\circ\colon \mathcal{L}(\widehat{S}) \to \mathcal{C}_\emptyset(\widehat{S})$ preserves arbitrary suprema.

Proposition 2. *$\mathcal{C}_\emptyset(\widehat{S})$ is an upper semilattice with*

$$\sup_{\mathcal{C}_\emptyset(\widehat{S})}\{\deg(\rho), \deg(\sigma)\} = \deg(\widehat{\overline{\rho} \oplus \overline{\sigma}}),$$

for $\rho, \sigma \in \mathrm{CNum}_\emptyset(\widehat{S})$

Because of Lemma 5 we obtain for $\nu, \kappa \in \mathrm{Num}_p(S)$ that

$$H(\sup_{\mathcal{L}_s(S)}\{\deg_s(\nu), \deg_s(\kappa)\}) = H(\deg_s(\nu \oplus \kappa)) = \sup_{\mathcal{C}_\emptyset(\widehat{S})}\{H(\nu), H(\kappa)\}.$$

Theorem 3. *The two upper semilattices $\mathcal{L}_s(S)$ and $\mathcal{C}_\emptyset(\widehat{S})$ are isomorphic.*

6. Final remarks

A strong reducibility relation between partial numberings was introduced in this note that requires the reduction function to transfer exactly the numbers which are indices under the numbering to be reduced to corresponding indices of the other numbering. In the case of the weaker reducibility relation studied in [2] the reduction function is allowed to map arbitrary numbers onto indices. As a result, a reduction function may reduce several partial numberings to one numbering. Thus, the degrees of partial numberings with respect to the weaker relation are uncountable. Moreover, they form a distributive lattice.

It was shown here that this is not the case if the degrees are formed with respect to the strong reducibility relation. To this end, Ershov's completion construction was extended to the partial case: modulo a canonical embedding every partial numbering of a given set can be extended to a total and complete numbering of a larger set which results from the given set by adding an extra point. The degrees of complete numberings of the extended set form an upper semilattice. As the operation of taking suprema is different from the usual operation of taking suprema for degrees, this upper semilattice in general is not a sub-semilattice of the upper semilattice of the degrees of all total numberings of the extended set.

The partial order of strong degrees of partial numberings of a given set was proved to be a monotone retract of the partial order of the degrees of all total numberings of the extended set. Moreover, the upper semilattice of the strong degrees was shown to be isomorphic to the upper semilattice of the degrees of complete numberings of the extended set. Both results confirm that with respect to the strong reducibility relation partial numberings behave as in the case of total numberings.

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