Arch. Math. Logic 43, 1009–1039 (2004)
Digital Object Identifier (DOI):
10.1007/s00153-004-0240-4

Mathematical Logic

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Non-dual fuzzy connections

Received: 20 August 2003 / Accepted: 2 December 2003 /
Published online: 14 June 2004 –

Abstract. The lack of double negation and de Morgan properties makes fuzzy logic unsymmetrical. This is the reason why fuzzy versions of notions like closure operator or Galois connection deserve attention for both antiotone and isotone cases, these two cases not being dual. This paper offers them attention, comming to the following conclusions:

– some kind of hardly describable "local preduality" still makes possible important parallel results;
– interesting new concepts besides antitone and isotone ones (like, for instance, *conjugated pair*), that were classically reducible to the first, gain independency in fuzzy setting.

1. Introduction

By revisiting the concept of set [26], fuzzy logic has to take a new look at some derived (at least in their initial, "concrete" forms) concepts too, like closure operator or Galois connection. Antitone fuzzy forms of these were introduced and studied in [1], [2], [5], [6], [9]. This paper argues that, within fuzzy framework, the isotone forms are also worth studying, since the duality that made possible transporting results between antitone and isotone in classical setting does no longer exist here. Let us consider, for exemplification, the following crisp notions:

Let X and Y be two sets. A pair $(\uparrow, \downarrow)$ of functions, $\uparrow: \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$, $\downarrow: \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ is said to be:

(1) an **antitone Galois connection** between X and Y if

$$A \subseteq B^{\downarrow} \text{ iff } B \subseteq A^{\uparrow} \ .$$

(2) an **isotone Galois connection** between X and Y if

$$A \subseteq B^{\downarrow} \text{ iff } A^{\uparrow} \subseteq B \ .$$

(3) a **conjugated pair** between X and Y if

$$A \cap B^{\downarrow} = \emptyset \text{ iff } B \cap A^{\uparrow} = \emptyset \ .$$

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Key words or phrases: Duality – Isotone structure – Fuzzy set theory – Galois connection, conjugated pair – Closure operator

The above three types of structure appear canonically from binary relations between X and Y. Let R be such a relation. Define:

-(1) $A^{\uparrow_R} = \{y \in Y \ / \ \forall x \in X, xRy\}$;
$B^{\downarrow_R} = \{x \in X \ / \ \forall y \in B, xRy\}$.
-(2) $A^{\Uparrow_R} = \{y \in Y \ / \ \exists x \in X, xRy\}$;
$B^{\Downarrow_R} = \{x \in X \ / \ \forall y, xRy \Longrightarrow y \in B\}$.
-(3) $A^{\boldsymbol{\Uparrow}_R} = \{y \in Y \ / \ \exists x \in X, xRy\}$;
$B^{\boldsymbol{\Downarrow}_R} = \{x \in X \ / \ \exists y \in B, xRy\}$.

What does it mean that two of the above three notions are dual? Duality is not a very well defined concept here, since we deal not with abstract, but concrete partial orders (inclusions). Still, duality is self-evident. Take, for example, antitone Galois connections and conjugated pairs. Any Galois connection $(\uparrow, \downarrow)$ produces, using the complement operation $\bar{\ }$, a conjugated pair $(\bar{\uparrow}, \bar{\downarrow})$, and vice versa. The two correspondences are inverse to each other. Moreover, if a Galois connection is given by a relation R, then its corresponding conjugated pair is given by $\bar{R}$, and vice versa. And same goes for antitone versus isotone Galois connections. These facts make that many relevant statement regarding one of the concepts to be immediately dualizable to the other two. This is also the situation of opening-closure operators or systems. In addition, duality is present even in statements mixing these concepts:

- composing the components of an antitone Galois connections produces two closure operators $\Longrightarrow$ composing the components of an isotone Galois connection produces an opening and a closure operator;
- any antitone Galois connection is induced by a relation $\Longrightarrow$ any conjugated pair is induced by a relation;
- opening operators are in bijection to opening systems $\Longrightarrow$ closure operators are in bijection to closure systems.

But, of course, there are some limits - we do not have, for instance, a notion similar to the one of closure or opening operator that comes by composition from conjugated pairs. This means that duality and its application is, regarding these set-theoretical notions, junctural: although many relevant statements are easy to dualize, we do not have a uniform and consistent way of transporting results or concepts like at a real duality. Still, if one takes the trouble to formalize all the above notions inside a common language (and then isolate between sentences some that are dualizable), one can notice that the real cause of duality is usually the fact that $(\mathcal{P}(X), \subseteq)$ is isomorphic, as a complete lattice, to $(\mathcal{P}(X), \supseteq)$, the complement operation being the isomorphism. This is not however the case in fuzzy setting, where "the set of parts of X" becomes L^X, with L being the structure of truth values. If L is not a Boolean algebra (and it is in fuzzy logic's nature not to be), there is still possible to define the above structures, but not as dual structures; although sometimes, similar results can be proved and even a common approach is possible. Some of these common approches and similar results are studied in this paper, which is structured as follows.

Section 2 prepares the field for discussion.

In Section 3, starting from the consideration of three (fuzzy versions of) basic connections between sets (isotone and antitone Galois connection, and conjugated pair), we introduce an abstract notion of connection and provide some results about all the three discussed examples.

In Section 4, fuzzy opening operators are introduced and studied, using similar techniques to the ones about fuzzy closure operators from [5] and [6].

Section 5 discusses the operators induced by fuzzy isotone Galois connections; two characterizations are provided (one of them similar to that from [5]).

Operators comming from fuzzy conjugated pairs are the issue of Section 6. These operators, constructed inductively, reveal the important role played by transitive closures of relations when conjugated pairs are considered.

The hierarchy induced by a conjugated pair constitutes the main subject of Section 7. A characterization theorem (in the spirit of those of antitone or isotone concept lattices) is proved under certain additional (quite natural) hypotheses.

Some concluding remarks end the paper.

2. Preliminaries

Given a lattice L, we denote by L^d its dual.

Definition 1. *A residuated lattice is a structure* $(L, \vee, \wedge, *, \to, 0, 1)$ *such that the following are true:*

$(GR1)$ $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice;*
$(GR2)$ $(L, *, 1)$ *is a monoid;*
$(GR3)$ *(residuation)*

For all $a, b, c \in L$, $a \leq b \to c$ *iff* $a * b \leq c$.

Lemma 1 ([18]). *The following hold in any residuated lattice (whenever we use family suprema or infima, we assume that they exist; the* **negation** $^{-}$ *is defined by* $\bar{a} = a \to 0$*):*

(1) $a \to 1 = 1$; $1 \to a = a$;
(2) $a \leq (a \to b) \to b$;
(3) $(a \to b) * a \leq b$;
(4) $\left(\bigvee_{i\in I} a_i\right) * a = \bigvee_{i\in I}(a_i * a)$ *and* $\left(\bigwedge_{i\in I} a_i\right) * a \leq \bigwedge_{i\in I}(a_i * a)$;
(5) $\left(\bigvee_{i\in I} a_i\right) \to a = \bigwedge_{i\in I}(a_i \to a)$;
(6) $a \to \left(\bigwedge_{i\in I} a_i\right) = \bigwedge_{i\in I}(a \to a_i)$;
(7) $\bigwedge_{i\in I}(a_i \to b_i) \leq \left(\bigwedge_{i\in I} a_i\right) \to \left(\bigwedge_{i\in I} b_i\right)$;

(8) $\bigwedge_{i\in I}(a_i \to b_i) \le \left(\bigvee_{i\in I} a_i\right) \to \left(\bigvee_{i\in I} b_i\right)$;
(9) $a \to b \le (a * c) \to (b * c)$;
(10) $(a \to b) * (b \to c) \le (a \to c)$;
(11) $a \to b \le (b \to c) \to (a \to c)$;
(12) $a \to b \le (c \to a) \to (c \to b)$;
(13) $b \le a \to (a * b)$;
(14) $a \to (b \to c) = b \to (a \to c)$;
(15) $a \le b$ *iff* $a \to b = 1$;
(16) $\to$ *is antitone in the first and isotone in the second argument;*
(17) $*$ *is isotone in both arguments;*
(18) $a * b \le a$ *and* $a * b \le b$;
(19) $0 * a = a * 0 = 0$;
(20) $a \le \bar{\bar{a}}$;
(21) $a \to b \le \bar{b} \to \bar{a}$;
(22) $a \to \bar{b} = b \to \bar{a}$;
(23) *The following axioms are equivalent:*
- $a = \bar{\bar{a}}$,
- $a \ge \bar{\bar{a}}$,
- $\bar{a} \to \bar{b} = b \to a$,
- $\bar{a} \to \bar{b} \le b \to a$,
- $\bar{a} \to b = \bar{b} \to a$,
- $\bar{a} \to b \le \bar{b} \to a$;

(24) $(a * b) \to c = a \to (b \to c)$.

When L satisfies one of the equivalent axioms from Lemma 1.(23), it is said to have the **double negation property**. Define $/ : L \times L \longrightarrow L$ by $a/b = b \to a$.

For the whole paper, we fix a complete residuated lattice L.

We shall deal with fixed non-empty sets like X or Y (called **universes**). The complete residuated lattice L that stands for the truth degree structure. Elements from L^X and L^Y shall be called L**-sets** or **fuzzy sets**, while elements from $L^{X\times Y}$ L**-relations** or **fuzzy relations**.

Notice that, if we naturally write $A \subseteq B$ whenever $A(x) \le B(x)$ for all $x \in X$, $(L^X, \subseteq)$ is a complete lattice in which suprema and infima are the fuzzy unions and intersections (denoted $\bigcup$ and $\bigcap$), that is the suprema and infima taken pointwise.

Define $S, T : L^X \times L^X \longrightarrow L$ by

$$S(A, B) = \bigwedge_{x\in X} A(x) \to B(x) \ ,$$

$$T(A, B) = \bigvee_{x\in X} A(x) * B(x) \ ,$$

for all $A, B \in L^X$. Notice that S and T are actually fuzzy relations on L^X. While S is the famous **subsethood degree**, $T(A, B)$ expresses the degree to which intersection of A with B is non-empty. The following proposition lists some immediate properties of S and T (among which, that S is a fuzzy order in the sense of [3], [9]).

Proposition 1. *For all* $A, B, C \in L^X$,

(1) $S(A, A) = 1$;
(2) $T(A, 0) = T(0, A) = 0$ (*where the first two* 0*-s are the constant functions*);
(3) $S(A, B) * S(B, C) \leq S(A, C)$;
(4) $S(A, B) = 1$ *iff* $A \subseteq B$;
(5) $S(A, B) = S(B, A) = 1$ *implies* $A = B$;
(6) $T(A, B * C) = T(A * B, C)$.

3. Fuzzy connections

In this section, we are going to treat three important types of connections between fuzzy sets, by means of a general notion of connection associated to some binary operators, the **(α, β)-connection**. We shall prove that, in some cases, (α, β)-connections come from binary fuzzy relations.

Definition 2. *Let* X, Y *be two universes. A pair of functions* $(\uparrow, \downarrow)$, $\uparrow: L^X \longrightarrow L^Y$, $\downarrow: L^Y \longrightarrow L^X$ *is called*

– **L-antitone Galois connection** *between* X *and* Y *if*

$$S(A, B^{\downarrow}) = S(B, A^{\uparrow})$$

– **L-isotone Galois connection** *between* X *and* Y *if*

$$S(A, B^{\downarrow}) = S(A^{\uparrow}, B)$$

– **L-conjugated pair** *between* X *and* Y *if*

$$T(A, B^{\downarrow}) = T(B, A^{\uparrow})$$

for all $A \in L^X, B \in L^Y$.

Since L is understood, we can call them **fuzzy antitone (isotone) Galois connection** *or* **fuzzy conjugated pair**, *that we abreviate by* "FAC", "FIC", *and* "FCP" *respectively.*

There are canonical ways to obtain such couples of functions, using fuzzy relations $R \in L^{X\times Y}$. To such a relation, associate a pair $(\uparrow_R, \downarrow_R)$ by:

$$(1)\ A^{\uparrow_R}(y) = \bigwedge_{x'\in X} A(x') \to R(x', y)\ , \quad B^{\downarrow_R}(x) = \bigwedge_{y'\in Y} B(y') \to R(x, y')\ ,$$

$$(2)\ A^{\uparrow_R}(y) = \bigvee_{x'\in X} A(x') * R(x', y)\ , \quad B^{\downarrow_R}(x) = \bigwedge_{y'\in Y} R(x, y') \to B(y')\ ,$$

$$(3)\ A^{\uparrow_R}(y) = \bigvee_{x'\in X} A(x') * R(x', y)\ , \quad B^{\downarrow_R}(x) = \bigvee_{y'\in Y} B(y') * R(x, y')\ ,$$

for all $A \in L^X, B \in L^Y, x \in X, y \in Y$. One can immediately check, using the residuated lattice properties, the following:

Proposition 2. *The pair of functions defined by*

-(1) *is a FAC,*
-(2) *is a FIC,*
-(3) *is a FCP.*

Fags were introduced and studied in [1] and [2]. There, among others, it is proved that all FACs are induced by fuzzy relations, and moreover, that $R \mapsto (\uparrow_R, \downarrow_R)$ is a bijection. What about FICs and FCPs? Is there any (duality) relationship with FACs that can transport the above result? In classical setting (taking L to be the Boolean algebra $\{0, 1\}$), because L^X (which is $\mathcal{P}(X)$) is isomorphic, as a lattice, to $(L^d)^X$, there is indeed a perfect duality between both the notions FAC-FIC and the constructions (1)–(2) - this is why one does not need to study (crisp) isotone Galois connetcions, all the results comming by duality from the antitones. Also, crisp conjugate pairs stay in duality with both isotone and antitone Galois connections (see the introduction) - thus, we can actually speak about a “triality” in the crisp case.[1] However, because of the lack of symmetry of residuated lattices, we do not have this nice situation in fuzzy case. All we can recover from it is that isotones induce antitones:

Proposition 3.

(1) *Every FIC* $C = (\uparrow, \downarrow)$ *gives a FAC* $C' = (\Uparrow, \Downarrow)$ *defined by*;

$$A^{\Uparrow} = \overline{A^{\uparrow}}, \quad B^{\Downarrow} = \bar{B}^{\downarrow} .$$

(2) *If* C *is induced by some* L*-relation* R, *then* C' *is induced by* $\bar{R}$.
(3) *The mapping* $C \mapsto C'$ *is not necessarily injective or surjective.*
(4) *If* L *has the double negation property, then the mapping* $C \mapsto C'$ *is bijective, its inverse being* $(\Uparrow, \Downarrow) \mapsto (\uparrow, \downarrow)$, *where* $A^{\uparrow} = \overline{A^{\Uparrow}}$ *and* $B^{\downarrow} = \bar{B}^{\Downarrow}$ *for all* $A \in L^X$ *and* $B \in L^Y$.

Proof.

(1): Let $A \in L^X$ and $B \in L^Y$. We need to show $S(A, B^{\Downarrow}) = S(B, A^{\Uparrow})$. But this means $S(A, \bar{B}^{\downarrow}) = S(B, \overline{A^{\uparrow}})$, that is, buy the isotone Galois connection property of $(\uparrow, \downarrow)$, $S(A^{\uparrow}, \bar{B}) = S(B, \overline{A^{\uparrow}})$. The last is true by Lemma 1.(22).
(2): Suppose $C = (\uparrow, \downarrow)$ is induced by R. Then

$$A^{\Uparrow}(y) = \overline{A^{\uparrow}(x)} = \overline{\bigvee_{x \in X} A(x) * R(x, y)} = \bigwedge_{x \in X} \overline{A(x) * R(x, y)}$$
$$= \bigwedge_{x \in X} A(x) \to \bar{R}(x, y)$$

and

$$B^{\Downarrow}(x) = \bar{B}^{\downarrow}(x) = \bigwedge_{y \in Y} R(x, y) \to \overline{B(y)} = \bigwedge_{y \in Y} B(y) \to \bar{R}(x, y) ,$$

for all x, y, A, B (we applied Lemma 1.(5,22,24)). Hence $(\Uparrow, \Downarrow)$ is induced by $\bar{R}$.

[1] It is in fact a “quadrality”, if we define a notion of connection that is symmetric to the one of isotone Galois connection by $S(B^{\downarrow}, A) = S(B, A^{\uparrow})$ - this is not however interesting to consider separately, since it is just an isotone Galois connection between Y and X.

(3): We consider the structure of truth values $L = \{0, 1/3, 2/3, 1\}$ having the usual number order and the operations $*$ and $\to$ given in the following tables (the values of the first argument are displayed vertically and the values of the second argument horizontally):

$*$	**0**	**1/3**	**2/3**	**1**
0	0	0	0	0
1/3	0	0	0	1/3
2/3	0	0	0	2/3
1	0	1/3	2/3	1

$\to$	**0**	**1/3**	**2/3**	**1**
0	1	1	1	1
1/3	2/3	1	1	1
2/3	2/3	2/3	1	1
1	0	1/3	2/3	1

It is easy to verify that the above structure is a residuated lattice. The negation is given below:

x	**0**	**1/3**	**2/3**	**1**
$\bar{x}$	1	2/3	2/3	0

Let $X = \{x\}$ and $Y = \{y\}$, two singleton universes. We identify L^X, L^Y and $L^{X\times Y}$ with L. Consider the following functions from L to L (i.e. from L^X to L^Y or from L^Y to L^X):

x	**0**	**1/3**	**2/3**	**1**
$x^{\Uparrow}$	1	1	2/3	1/3
$x^{\Downarrow}$	1	1	2/3	1/3
$x^{\Uparrow'}$	1	1	1	2/3
$x^{\Downarrow'}$	1	1	1	2/3
$x^{\uparrow_1}$	0	0	0	1/3
$x^{\downarrow_1}$	2/3	1	1	1
$x^{\uparrow_2}$	0	0	0	2/3
$x^{\downarrow_2}$	2/3	2/3	1	1

One can easily check the following facts:

- $(\Uparrow, \Downarrow)$ and $(\Uparrow', \Downarrow')$ are L-antitone Galois connections;
- $(\uparrow_1, \downarrow_1)$ and $(\uparrow_2, \downarrow_2)$ are linear L-isotone Galois connections;
- there is no L-isotone Galois connection C such that $C \mapsto (\Uparrow, \Downarrow)$ (beacuse $1^{\Uparrow} = 1/3$, which is not the negation of any element);
- $(\uparrow_1, \downarrow_1) \mapsto (\Uparrow', \Downarrow')$ and $(\uparrow_2, \downarrow_2) \mapsto (\Uparrow', \Downarrow')$.

These facts prove (3).

(4): If the mapping from antitones to isotones is well defined, it is clearly the inverse of the mapping from point (1). Thus, all we need to prove is that, if $(\Uparrow, \Downarrow)$ is FAC, then $(\uparrow, \downarrow)$ is a FIC. Consider for this $A \in L^X$ and $B \in L^Y$. The required property is $S(A^{\uparrow}, B) = S(A, B^{\downarrow})$. This means $S(\overline{A^{\Uparrow}}, B) = S(A, \bar{B}^{\Downarrow})$, that is $S(\overline{A^{\Uparrow}}, B) = S(\bar{B}, A^{\Uparrow})$, which is true by the double negation property (see Lemma 1.(23)). □

It very much seems that no other natural connection between these notions can be found, unless we take the very harsh Boolean algebra assumptions. In fact, we can conjecture the following non-mathematical statement:

Consider any two of the three notions, $n, m \in \{\text{FAC,FIC,FCP}\}$, $n \neq m$. Then, the following statements are equivalent for a residuated lattice L:

(1) For each non-empty sets X, Y, there exist two *natural* bijections

$$\phi : L^{X\times Y} \longrightarrow L^{X\times Y} ,$$
$$\chi : \{\text{the } n\text{-s between } X \text{ and } Y\} \longrightarrow \{\text{the } m\text{-s between } X \text{ and } Y\} ,$$

such that, if $(\uparrow, \downarrow)$ is an n given by R, then $\phi(\uparrow, \downarrow)$ is an m given by $\chi(R)$.

(2) L is a Boolean algebra.

If one notices that the only mappings ϕ, χ that have any chance of being *natural* are those that make use of the negation in L, one tends to agree with the above. In fact, we can make this statement a mathematical one, if we translate "natural" by "polynomially defined using the residuated lattice operators", but we do not enter such details - all that we want to point out is that, inside this fuzzy context, and inside any fuzzy context really,[2] the discussed three notions are far from being dual.

We are now going to answer the question of relationship between these structures and fuzzy relations, in a common, but artificial framework, which puts together, in spite of their lack of duality, the three cases.

Consider five complete lattices L', L_1, L_2, L_3, L_4, L_5 and two operations

$$\alpha : L_1 \times L_2 \longrightarrow L' , \quad \beta : L_3 \times L_4 \longrightarrow L' .$$

The infima in the lattices L_i shall have i as an upper index. The same thing happens to intersections of families of elements from L_i^X or L_i^Y, which are pointwise infima. The "zero" and "one" elements from L_i shall be denoted by 0_i and 1_i. All the entities connected to L' (infima, "zero", "one" etc.) shall be denoted without any additional index ($\bigvee$, 0, 1 etc.). About α and β, we assume that, for each b_1, b_2, b_3, b_4 from L_1, L_2, L_3, L_4 respectively,

- the applications $b_2 \mapsto \alpha(\_, b_2)$ and $b_4 \mapsto \beta(\_, b_4)$ are injective;
- α, β commute, on each argument, with arbitrary infima (in particular, if one of the arguments of β or α is 1_i, with $i \in 1, 2, 3, 4$, then the result is 1).

For each $A_i \in L_i^X$, $B_i \in L_{i+2}^Y$, with $i \in \{1, 2\}$, define

$$S_\alpha(A_1, A_2) = \bigwedge_{x\in X} \alpha(A_1(x), A_2(x)) , \quad S_\beta(B_1, B_2) = \bigwedge_{y\in Y} \beta(B_1(y), B_2(y)) .$$

In any of the lattices, say L', and with respect to any set, say X, define, for each $b \in L'$ and $x \in X$, **the hypersingleton** $\{b|x\}$ to be the fuzzy subset of X given by $\{b|x\}(x') = b$ if $x = x'$ and 1 otherwise.

Definition 3. *An* (α, β)**-connection**[3] *between two non-empty sets* X, Y *is a pair* $(\uparrow, \downarrow)$ *of functions* $\uparrow: L_1^X \longrightarrow L_4^Y$, $\downarrow: L_3^Y \longrightarrow L_2^X$ *such that* $S_\alpha(A, B^\downarrow) = S_\beta(B, A^\uparrow)$ *for all* $A \in L_1^X$, $B \in L_3^Y$.

[2] "Fuzzy" is usually related to the unit interval $([0, 1], \leq)$, which cannot be organized as a Boolean algebra.

[3] Notice that the notion of (α, β)-connection uses some complete lattices, without any given implication (residua) - the "connection" condition is expressed only in terms of the operators α, β and the lattice infima.

(α, β)-connections include FACs, FICs and FCPs as particular cases, as follows (suppose, as usual, that their complete residuated lattice is L - denote by D the complete lattice underlying L):

- FAC: $L' = L_2 = L_4 = D$, $L_3 = L_1 = D^d$, $\alpha = \beta = \rightarrow$ - use Lemma 1.(1) to get the required injectivity and Lemma 1.(5,6) for commutation with infima.
- FIC: $L_1 = L_4 = D, L_2 = L_3 = D^d, \alpha = /, \beta = \rightarrow$ - for injectivity of $a \mapsto \_ \rightarrow a$, use again Lemma 1.(1); as for $a \mapsto \_/a$, notice that $a \mapsto \_/a = a' \mapsto \_/a'$ implies that fort each x, $a \leq x$ iff $a' \leq x$, hence $a = a'$; Lemma 1.(5,6) can be used to establish commutation with infima.
- FCP: $L' = L_1 = L_2 = L_3 = L_4 = D^d$, $\alpha = \beta = *$ - injectivity follows from 1 being identity element for $*$; commutation with infima is assured by Lemma 1.(4).

We now come to explore the structure of (α, β)-connections, reaching in particular common properties of the three types of connections.

Lemma 2. *Every (α, β)-connection $(\uparrow, \downarrow)$ between X and Y is uniquely determined by each one of its pair members, $\uparrow$ or $\downarrow$.*

Proof. Let us prove, for instance, that the pair is determined by its first member. For this, assume that $(\uparrow, \downarrow)$ and $(\uparrow, \downarrow')$ are (α, β)-connections - we need to show that $\downarrow = \downarrow'$. Let $B \in L_3^Y$. We have that

$$S_\alpha(A, B^{\downarrow}) = S_\beta(B, A^{\uparrow}) = S_\alpha(A, B^{\downarrow'}) ,$$

for each $A \in L_1^X$. Let $x \in X$ and let $b_1 \in L_1$. We put $A = \{b_1|x\}$, and obtain $S_\alpha(\{b_1|x\}, B^{\downarrow}) = S_\alpha(\{b_1|x\}, B^{\downarrow'})$, that is

$$\alpha(b_1, B^{\downarrow}(y)) = \alpha(b_1, B^{\downarrow'}(y)) ,$$

and this happens for all $b_1 \in L_1$. Applying the injectivity of $b_2 \mapsto \alpha(\_, b_2)$, we get $B^{\downarrow}(y) = B^{\downarrow'}(y)$. □

Lemma 3. *For all families $(A_i)_{i\in I} \subseteq L_1^X$, $(B_i)_{i\in I} \subseteq L_3^Y$,*

$$\left(\bigcap_{i\in I}^{3} B_i\right)^{\downarrow} = \bigcap_{i\in I}^{2} B_i^{\downarrow} , \quad \left(\bigcap_{i\in I}^{1} A_i\right)^{\uparrow} = \bigcap_{i\in I}^{4} A_i^{\uparrow} .$$

Proof. We only prove the first equality. For this, let $A \in L_1^X$. Aplying infima properties, definition of connection and commutation with infima, we get

$$\begin{aligned} S_\alpha(A, \left(\bigcap_{i\in I}^{3} B_i\right)^{\downarrow}) &= S_\beta\left(\bigcap_{i\in I}^{3} B_i, A^{\uparrow}\right) = \bigwedge_{y\in Y} \beta\left(\bigwedge_{i\in I}^{3} B_i(y), A^{\uparrow}(y)\right) \\ &= \bigwedge_{y\in Y}\bigwedge_{i\in I} \beta(B_i(y), A^{\uparrow}(y)) = \bigwedge_{i\in I}\bigwedge_{y\in Y} \beta(B_i(y), A^{\uparrow}(y)) \\ &= \bigwedge_{i\in I} S_\beta(B_i, A^{\uparrow}) = \bigwedge_{i\in I} S_\alpha(A, B_i^{\downarrow}) \end{aligned}$$

$$= \bigwedge_{i\in I}\bigwedge_{x\in X}\alpha(A(x), B_i^{\downarrow}(x)) = \bigwedge_{x\in X}\bigwedge_{i\in I}\alpha(A(x), B_i^{\downarrow}(x))$$

$$= \bigwedge_{x\in X}\alpha\left(A(x), \overset{2}{\bigwedge_{i\in I}} B_i^{\downarrow}(x)\right) = S_\alpha\left(A, \overset{2}{\bigcap_{i\in I}} B_i^{\downarrow}\right) .$$

Let $x \in X$ and let $A = \{b_1|x\}$, where $b_1 \in L_1$. The above proved equality becomes

$$\alpha(b_1, \left(\overset{3}{\bigcap_{i\in I}} B_i\right)^{\downarrow}) = \alpha\left(b_1, \overset{2}{\bigcap_{i\in I}} B_i^{\downarrow}\right), \quad \forall b_1 \in L_1 .$$

This implies, via the injectivity from the connection definition, that

$$\left(\overset{3}{\bigcap_{i\in I}} B_i\right)^{\downarrow} = \overset{2}{\bigcap_{i\in I}} B_i^{\downarrow} . \qquad \square$$

Corollary 1. *Let* $(\uparrow, \downarrow)$ *and* $(\uparrow', \downarrow')$ *be two* (α, β)*-connections between* X *and* Y*. Then, in order that* $(\uparrow, \downarrow) = (\uparrow', \downarrow')$*, it suffices that* $\uparrow$ *and* $\uparrow'$*, or* $\downarrow$ *and* $\downarrow'$*, coincide on hypersingletons.*

Proof. Assume, for instance, that $\uparrow$ and $\uparrow'$ coincide on hypersingletons. Knowing that each $A \in L_1^X$ can be written as $\bigcap^1_{x\in X}\{A(x)|x\}$, then applying Lemma 3, we obtain $\uparrow=\uparrow'$. It now suffices to apply Lemma 2 in order to get the desired result. $\square$

Even at this very general level, we can speak, in certain conditions, about connections given by fuzzy relations. Let us assume that the parameterized (with parameters b_1, b_3) equation

$$\alpha(b_1, z_2) = \beta(b_3, z_4)$$

has a solution that can be functionally expressed with values in a complete lattice L_5. Namely, we assume that there exist $\alpha' : L_3 \times L_5 \longrightarrow L_2, \beta' : L_1 \times L_5 \longrightarrow L_4$, such that, for all b_1, b_3, b_5 from L_1, L_3, L_5,

$$\alpha(b_1, \alpha'(b_3, b_5)) = \beta(b_3, \beta'(b_1, b_5)) .$$

Now, let $R \subseteq L_5^{X\times Y}$ be a fuzzy relation. Define $\uparrow_R, \downarrow_R$ by

$$A^{\uparrow_R}(y) = \overset{4}{\bigwedge_{x'\in X}} \beta'(A(x'), R(x', y)) = S_{\beta'}(A, R(\_, y)) ,$$

$$B^{\downarrow_R}(x) = \overset{2}{\bigwedge_{y'\in Y}} \alpha'(B(y'), R(x, y')) = S_{\alpha'}(B, R(x, \_)) ,$$

for all $A \in L_1^X, B \in L_3^Y, x \in X, y \in Y$.

It is easy to see that, particularizing this for our main three cases, we obtain the standard ways to construct them from binary fuzzy relations (see Proposition 2), if we instanciate L_5, α', β' as follows:

– FAC: $L_5 = D$, $\alpha' = \beta' = \rightarrow$ - the requested equality becomes

$$b_1 \rightarrow (b_3 \rightarrow b_5) = b_3 \rightarrow (b_1 \rightarrow b_5) \ ,$$

which is true according to Lemma 1.(14).

– FIC: $L_5 = D^d$, $\alpha' = *$, $\beta' = /$ - the equality becomes

$$b_1/(b_3 * b_5) = b_3 \rightarrow (b_1/b_5) \ ,$$

which means

$$(b_3 * b_5) \rightarrow b_1 = b_3 \rightarrow (b_5 \rightarrow b_1) \ ,$$

true according to Lemma 1.(24).

– FCP: $L_5 = D^d$, $\alpha' = \beta' = *$ - the equality becomes

$$b_1 * (b_3 * b_5) = b_3 * (b_1 * b_5) \ ,$$

true by associativity and commutativity.

Proposition 4. *$(\uparrow_R, \downarrow_R)$ is an (α, β)-connection between X and Y.*

Proof. Let $A \in L_1^X$ and $B \in L_3^Y$. We only use the assumed equality with $\alpha, \beta, \alpha', \beta'$ and commutation with infima:

$$\begin{aligned} S_\alpha(A, B^{\downarrow_R}) &= \bigwedge_{x\in X} \alpha(A(x), B^{\downarrow_R}(x)) \\ &= \bigwedge_{x\in X} \alpha(A(x), \bigwedge_{y\in Y}^{2} \alpha'(B(y), R(x,y))) \\ &= \bigwedge_{x\in X} \bigwedge_{y\in Y} \beta(B(y), \beta'(A(x), R(x,y))) \\ &= \bigwedge_{y\in Y} \bigwedge_{x\in X} \beta(B(y), \beta'(A(x), R(x,y))) \\ &= \bigwedge_{y\in Y} \beta(B(y), \bigwedge_{x\in X}^{4} \beta'(A(x), R(x,y))) \\ &= \bigwedge_{y\in Y} \beta(B(y), A^{\uparrow_R}(y)) = S_\beta(B, A^{\uparrow_R}) \ . \end{aligned}$$

□

We have thus provided a sufficient criterion for the solvability of a more complex system of equations with $(\uparrow, \downarrow)$ as unknown part,

$$S_\alpha(A, B^{\downarrow}) = S_\beta(A^{\uparrow}, B) \ , \quad (A, B) \in L_1^X \times L_2^Y \ ,$$

by reducing it to a simpler functional equation:

$$\alpha(x, \alpha'(y, z)) = \beta(y, \beta'(x, z)),$$

having α' and β' as unknown part. In case this functional equation has a solution, we obtain a whole family of solutions for the initial system.

Of course, at this stage, it is hard to say whether each solution of the system can be obtained in the above way. However, if we "strategically" equalize some of the lattices and operations, we come to a positive answer, that will be relevant for the three cases of connections we care about.

Proposition 5. *Assume that $L_5 = L_2$, $L_4 = L'$, $\beta' = \alpha$ and 0_3 is neutral for β (i.e. $\beta(0_3, b)) = b$ for all $b \in L'$). Then each (α, β)-connection between X and Y is given by a fuzzy relation. Moreover, (α, β)-connections between X and Y and fuzzy relations from $L_2^{X\times Y}$ are in bijection.*

Proof. Let us prove that the application $R \mapsto (\uparrow_R, \downarrow_R)$ is a bijection. Let $(\uparrow, \downarrow)$ be a (α, β)-connection. We show that there exists a unique $R \in L_2^{X\times Y}$ such that $(\uparrow, \downarrow) = (\uparrow_R, \downarrow_R)$.

Unicity: Let $x \in X$, $y \in Y$. A presumptive R as above would satisfy

$$\begin{aligned}\{b_1|x\}^{\uparrow}(y)) = \{b_1|x\}^{\uparrow_R}(y) &= \bigwedge_{x'\in X}^{4} \beta'(\{b_1|x\}(x'), R(x', y)) \\ &= \bigwedge_{x'\in X} \alpha(\{b_1|x\}(x'), R(x', y)) \\ &= \alpha(b_1, R(x, y)) \wedge \bigwedge_{x'\neq x} \alpha(1, R(x', y)) = \alpha(b_1, R(x, y)) ,\end{aligned}$$

for each $b_1 \in L_1$. So, because of the injectivity of $b_2 \mapsto \alpha(\_, b_2)$, $R(x, y)$ is uniquely determined; and this happens for each x, y.

Existence: Define, for each $x \in X$, $y \in Y$, $R(x, y) = \{0_3|y\}^{\downarrow}(x)$. In order to prove that $(\uparrow, \downarrow) = (\uparrow_R, \downarrow_R)$, it would be sufficient, according to Corollary 1, to show that $\uparrow$ and $\uparrow_R$ coincide on hypersingletons; that is, for all $x \in X$, $y \in Y$, $b_1 \in L_1$,

$$\{b_1|x\}^{\uparrow}(y) = \{b_1|x\}^{\uparrow_R}(y) ,$$

which means

$$\{b_1|x\}^{\uparrow}(y) = \alpha(b_1, R(x, y)) .$$

To see that the last is true, apply the connection property for $A = \{0_3|y\}$ and $B = \{b_1|x\}$, to get

$$\beta(0_3, \{b_1|x\}^{\uparrow}(y)) = \alpha(b_1, \{1_3|y\}^{\downarrow}(x)) ,$$

that is

$$\{b_1|x\}^{\uparrow}(y) = \alpha(b_1, R(x, y)) .$$

□

All the identifications requested in the above propositions $L_5 = L_2$, $L_4 = L'$ and $\beta' = \alpha$, hold in the three cases; also, 0_3 being neutral for β means, in L, $1 \to x = x$ or $1 * x = x$, which is true. We get:

Corollary 2. *Each FAC (FIC, FCP) between X and Y is induced by a fuzzy relation.*

Thus indeed, all the three types of structures are given by fuzzy relations, and this is proved by a common reasoning, keeping within itself bits and pieces of duality, remains from the classical case.

4. Fuzzy opening operators and systems

Fuzzy closure operators were introduced in [5]. In classical set theory, the dual notion is the one of opening operator. However, fuzzy opening operators are not perfectly dual to fuzzy closure operators. But they are "dual enough" to parallel some important properties, like the bijecive correspondence to appropriate systems of fuzzy sets. The results from this section (as well as Proposition 11 from Section 5) have form and use techniques very similar to the ones from [5] (but they are not, as far as we see, deducible from those[4]). Unless otherwise stated, we work with a fixed universe X.

Definition 4. *A* **fuzzy opening operator** *on X is a mapping $\circ : L^X \longrightarrow L^X$ such that, for all $A, B \in L^X$,*

- *$A^\circ \subseteq A$;*
- *$A^{\circ\circ} = A^\circ$;*
- *$S(A, B) \leq S(A^\circ, B^\circ)$.*

Definition 5. *A* **fuzzy opening system** *on X is a set $\mathcal{O} \subseteq L^X$ such that, for each $A \in L^X$,*

$$\bigcup_{B \in \mathcal{O}} S(B, A) * B \in \mathcal{O} .$$

The condition from Definition 5 is a (more or less) fuzzy version of the statement that "for each set A, the union of all B from $\mathcal{O}$ that are included in A is in $\mathcal{O}$"; of course, as for all fuzzy translations of a crisp sentence, more choices could be made - here, one chosed that "B is included in A" is expressed *along* the union.

Proposition 6. *Let $\mathcal{O} = (A_i)_{i \in I} \subseteq L^X$. Then $\mathcal{O}$ is a fuzzy opening system iff, for each family $(a_i)_{i \in I} \subseteq L^X$, $\bigcup_{i \in I} a_i * A_i \in \mathcal{O}$.*

Proof. For proving the "if" statement, let $A \in L^X$. Take $a_i = S(A_i, A)$, for all $i \in I$.

Conversely, let $(a_i)_{i \in I} \subseteq L$ and denote $A = \bigcup_{i \in I} a_i * A_i$. For proving $A \in \mathcal{O}$, it would be enough that $\bigcup_{i \in I} S(A_i, A) * A_i = A$. The $\subseteq$ part is obvious. The other inclusion means

$$\bigcup_{i \in I} a_i * A_i \subseteq \bigcup_{i \in I} S(A_i, A) * A_i \ .$$

It suffices to show that, for all $i \in I$,

$$a_i \leq S(A_i, A) = \bigwedge_{x \in X} A_i(x) \rightarrow A(x) \ ,$$

that is,

[4] Although finding a common framework for proving these and perhaps others, together with their antitone forms, in the spirit of Section 3, would be interesting.

$$\forall x \in X, \quad a_i \leq A_i(x) \to A(x) \ ,$$
$$\forall x \in X, \quad a_i * A_i(x) \leq A(x) \ ,$$

which is true by the choice of A. □

Corollary 3. *Let $\mathcal{O} \subseteq L^X$. Then $\mathcal{O}$ is a fuzzy opening operator iff $\mathcal{O}$ is closed to arbitrary unions and, for all $a \in L$, $A \in \mathcal{O}$, $a * A \in \mathcal{O}$.*

Proof. Suppose $\mathcal{O} = (A_i)_{i \in I}$. For the "if" statement, let $(a_i)_{i \in I}$. Since, for each $i \in I$, $a_i * A_i \in \mathcal{O}$ and $\mathcal{O}$ is closed under unions, we get $\bigcup_{i \in I} a_i * A_i \in \mathcal{O}$. Thus, by the last proposition, $\mathcal{O}$ is a fuzzy opening system. The "only if" statement also follows from Proposition 6. In order to prove closure under union of the family $(A_j)_{j \in J}$, with $J \subseteq I$, we chose a_i to be 1 if $i \in J$ and 0 otherwise. To prove that $a * A_j \in \mathcal{O}$ for some $j \in I$, we cose a_j to be a and the other a_i-s to be 0. □

Proposition 7. *Let $\mathcal{O} \subseteq L^X$ be a fuzzy opening system. Then, for all $A \in L^X$,*

$$\bigcup_{B \in \mathcal{O}} S(B, A) * B = \bigcup_{B \in \mathcal{O}, B \subseteq A} B \ ,$$

Proof.

$$\bigcup_{B \in \mathcal{O}, B \subseteq A} B \subseteq \bigcup_{B \in \mathcal{O}, S(B,A)=1} B \subseteq \bigcup_{B \in \mathcal{O}} S(B, A) * B \ .$$

Conversely, $\bigcup_{B \in \mathcal{O}} S(B, A) * B$ is in $\mathcal{O}$ and is included in A, hence it is also included in $\bigcup_{B \in \mathcal{O}, B \subseteq A} B$. □

Proposition 8. *There exists a bijective correspondence between fuzzy opening systems and fuzzy opening operators, given by:*

– *$\mathcal{O} \mapsto \circ_{\mathcal{O}}$, where, for all $A \in L^X$, $A^{\circ_{\mathcal{O}}} = \bigcup_{B \subseteq A, B \in \mathcal{O}} B$;*
– *$\circ \mapsto \mathcal{O}_\circ = \{A \in L^X \ / \ A = A^\circ\}$.*

Proof. (a) Let us show that $\mathcal{O}_\circ$ is a fuzzy opening operator. Obviously, $A^{\circ_{\mathcal{O}}} \subseteq A$. In addition, $A^{\circ_{\mathcal{O}}} \in \mathcal{O}$, hence $A^{\circ_{\mathcal{O}} \circ_{\mathcal{O}}} = A^{\circ_{\mathcal{O}}}$. Now, using Proposition 7 and Lemma 1.(8,9),

$$\begin{aligned} S(A^{\circ_{\mathcal{O}}}, A'^{\circ_{\mathcal{O}}}) &= \bigwedge_{x \in X} \left(\bigcup_{B \subseteq A, B \in \mathcal{O}} B \right)(x) \to \left(\bigcup_{B \subseteq A', B \in \mathcal{O}} B \right)(x) \\ &= \bigwedge_{x \in X} \left(\bigvee_{B \in \mathcal{O}} S(B, A) * B(x) \right) \to \left(\bigvee_{B \in \mathcal{O}} S(B, A') * B(x) \right) \\ &\geq \bigwedge_{x \in X} \bigwedge_{B \in \mathcal{O}} (S(B, A) * B(x)) \to (S(B, A') * B(x)) \\ &\geq \bigwedge_{x \in X} A(x) \to A'(x) = S(A, A') \ . \end{aligned}$$

(b) We prove that $\mathcal{O}_\circ$ is a fuzzy opening system. Let $A \in L^X$. It is sufficient to show

$$\bigcup_{B \in \mathcal{O}_\circ} S(B, A) * B = A^\circ .$$

Because $S(A^\circ, A) = 1$, we have

$$A^\circ = S(A^\circ, A) * A^\circ \subseteq \bigcup_{B \in \mathcal{O}_\circ} S(B, A) * B .$$

On the other hand, for all $x \in X$,

$$A^\circ(x) \geq \bigvee_{B \in \mathcal{O}_\circ} S(B, A) * B(x)$$

means that, for all $B \in \mathcal{O}_\circ$, $S(B, A) * B(x) \leq A^\circ(x)$, that is $S(B, A) \leq B(x) \to A^\circ(x)$. But $B = B^\circ$, so the last is true, because

$$S(B, A) \leq S(B^\circ, A^\circ) = S(B, A^\circ) \leq B(x) \to A^\circ(x) . \qquad \square$$

Let us now look at the relation between fuzzy opening and closure operators. Like in the case of fuzzy Galois connections, this relation turns up to be "one way". First, recall from [5] the definitions of fuzzy closure operator and system.

Definition 6. *A* **fuzzy closure operator** *on X is a mapping $\bullet : L^X \longrightarrow L^X$ such that, for all $A, B \in L^X$,*

- *$A \subseteq A^\bullet$;*
- *$A^{\bullet\bullet} = A^\bullet$;*
- *$S(A, B) \leq S(A^\bullet, B^\bullet)$.*

Definition 7. *A* **fuzzy closure system** *on X is a set $\mathcal{C} \subseteq L^X$ such that, for each $A \in L^X$,*

$$\bigcap_{B \in \mathcal{C}} S(A, B) \to A \in \mathcal{C} .$$

There exists a one-to-one correspondence between fuzzy closure operators and systems [5], by which each operator $\bullet$ is taken into the set of its closed fuzzy sets A (i.e. such that $A = A^\bullet$).

Proposition 9.

1. *Every fuzzy opening operator $\circ$ on X naturally gives a fuzzy closure operator on X $\bullet$ defined by*[5]

$$A^\bullet = \overline{\bar{A}^\circ},$$

for all $A \in L^X$.

[5] In what follows, for instance, $\bar{A}^\circ$ denotes the result of applying to A, first the operator $^-$, and then the operator $\circ$; and similarly for $\bar{A}^\bullet$. On the other hand, $\overline{A^\circ}$ refers to applying first $\circ$ and then $^-$.

2. *The mapping* $\circ \mapsto \bullet$ *is not necessarily injective or surjective.*

Proof. (1): Let $A, A_1, A_2 \in L^X$. Because $\bar{A}^{\circ} \subseteq \bar{A}$, we have $A \subseteq \overline{\bar{A}^{\circ}} = A^{\bullet}$. Moreover, it is immediate that $A^{\bullet} \subseteq A^{\bullet\bullet}$. Now, $\bar{A}^{\circ} \subseteq \overline{\overline{\bar{A}^{\circ}}}$, hence $\bar{A}^{\circ} = \bar{A}^{\circ\circ} \subseteq \overline{\overline{\bar{A}^{\circ}}}^{\circ} = \overline{A^{\bullet}}^{\circ}$; it follows that $\overline{\overline{A^{\bullet}}^{\circ}} \subseteq \overline{\bar{A}^{\circ}}$, that is $A^{\bullet\bullet} \subseteq A^{\bullet}$. Finally, $S(A_1, A_2) \leq S(\overline{A_2}, \overline{A_1}) \leq S(\overline{A_2}^{\circ}, \leq A_1{}^{\circ}) \leq S(\overline{\overline{A_1}^{\circ}}, \overline{\overline{A_2}^{\circ}}) = S(A_1^{\bullet}, A_2^{\bullet})$.

(2): We consider the same truth values structures from the proof of Proposition 3.(3), take $X = \{x\}$ and define the following operators on L (that can be seen as operating on L^X):

x	**0**	**1/3**	**2/3**	**1**
x°_1}	0	0	2/3	2/3
x°_2}	0	1/3	1/3	1/3
$x^{\bullet_1}$	2/3	2/3	2/3	1
$x^{\bullet_2}$	1/3	1/3	2/3	1

One can see that:

- $\circ_1, \circ_2$ are fuzzy opening and $\bullet_1, \bullet_2$ fuzzy closure operators.
- both $\circ_1$ and $\circ_2$ map to $\bullet_1$ and nothing maps to $\bullet_2$. □

Notice that there is no canonical way to obtain a fuzzy opening operator out of a closure one, unless L is a Boolean algebra. If $\bullet$ is a fuzzy closure operator, $\circ$ given by $A^{\circ} = \overline{\bar{A}^{\bullet}}$, for all $A \in L^X$, is not necessarily a fuzzy opening operator, as one can immediately see.

A similar discussion can be made for opening and closure systems, but we do not further insist on these matters - the ideea, we think, is pretty clear: "isotone" is not dual to "antitone" in fuzzy set theory; but, quite surprisingly, these two concepts behave very similarly.

Remark 1. We have recently discovered the existence of some parallel work regarding the topics of this chapter - in [11], the authors introduce L_K-*interior operators*, a somewhat more general notion than our fuzzy opening operator, in that they also consider, besides the residuated lattice L of truth values, an order filter K, "the set of designated truth values".

5. Operators induced by fuzzy isotone Galois connections

Two universes, X and Y, shall be fixed throughout this section.

FICs gives rise to fuzzy opening and fuzzy closure operators, as one can see by quite routine check, using Lemma 1:

Proposition 10. *If* $(\uparrow, \downarrow)$ *is a FIC, then* $\uparrow\downarrow$ *is a fuzzy closure operator and* $\downarrow\uparrow$ *is a fuzzy opening operator.*

Next, in order to characterize the above operators, we give another "isotone" result that is very similar to an antitone one [5].

Definition 8. *Let* $\mathcal{O}, \mathcal{C} \subseteq L^X$. *A bijection* $\phi : \mathcal{O} \longrightarrow \mathcal{C}$ *is called* **fuzzy isomorphism** *if, for all* $A, B \in \mathcal{O}$, $S(A, B) = S(\phi(A), \phi(B))$. [6]

Proposition 11. *There exists a bijective correspondence between the set of FICs between* X *and* Y *and the set of triples* $(\bullet, \circ, \phi)$ *such that* $\bullet$ *is a fuzzy closure operator on* Y, $\circ$ *is a fuzzy opening operator on* X *and* $\phi : \mathcal{C}_\bullet \longrightarrow \mathcal{O}_\circ$ *is a fuzzy isomorphism.*

Proof. For $C = (\uparrow, \downarrow)$ an L-isotone Galois connection, define $T_C = (\uparrow\downarrow, \downarrow\uparrow, \phi)$, where, for each $A \in \mathcal{C}_{\uparrow\downarrow}$, $\phi(A) = A^{\uparrow}$. The bijetcivity of ϕ follows immediately if we notice that its inverse is $\psi : \mathcal{O} \longrightarrow \mathcal{C}$ defined by $\psi(B) = B^{\downarrow}$. Also, $S(\phi(A), \phi(A')) \geq S(A, A')$ by the definition of FIC. On the other hand, since $A = B^{\downarrow}$ for some B, we have that $\phi(A) = B^{\downarrow\uparrow\downarrow} = B^{\downarrow}$ (and same goes for A'). Thus, using again the definition, $S(\phi(A), \phi(A')) \leq S(A, A')$.

Let now $T = (\bullet, \circ, \phi)$ be a triple as above. Define $C_T = (\uparrow, \downarrow)$ by

$A^{\uparrow} = \phi(A^\bullet)$, for all $A \in L^X$;
$B^{\downarrow} = \psi(B^\circ)$, for all $B \in L^Y$,

where ψ is the inverse of ϕ.

We check that C_T is a FIC. Let $A \in L^X$ and $B \in L^Y$. Notice that, since $\phi(A^\bullet)$ is open, $\phi(A^\bullet)^\circ = \phi(A^\bullet)$, hence

$$S(\phi(A^\bullet), B) \leq S(\phi(A^\bullet)^\circ, B^\circ) = S(\phi(A^\bullet), B^\circ) \leq S(\phi(A^\bullet), B) \ ,$$

so $S(\phi(A^\bullet), B) = S(\phi(A^\bullet), B^\circ)$. Then

$$\begin{aligned} S(A^{\uparrow}, B) &= S(\phi(A^\bullet), B) = S(\phi(A^\bullet), B^\circ) \\ &= S(\psi(\phi(A^\bullet)), \psi(B^\circ)) = S(A^\bullet, \psi(B^\circ)) = S(A^\bullet, B^{\downarrow}) \ . \end{aligned}$$

But, since $B^{\downarrow}$ is closed, $S(A^\bullet, B^{\downarrow}) = S(A, B^{\downarrow})$, so

$$S(A^{\uparrow}, B) = S(A, B^{\downarrow}) \ .$$

We show $C_{T_C} = C$. Let $C = (\uparrow, \downarrow)$, $C_{T_C} = (\uparrow', \downarrow')$, $T_C = \uparrow\downarrow, \downarrow\uparrow, \phi)$. Then

$$A^{\uparrow'} = \phi(A^{\uparrow\downarrow}) = A^{\uparrow\downarrow\uparrow} = A^{\uparrow} \ .$$

Similarly, $\downarrow' = \downarrow$.

Let us prove that $T_{C_T} = T$. Let $T = (\bullet, \circ, \phi)$, $T_{C_T} = (\bullet', \circ', \phi')$, $C_T = (\uparrow, \downarrow)$. Then

$$A^{\bullet'} = A^{\uparrow\downarrow} = \psi\left(\phi(A^\bullet)^\circ\right) = \psi\left(\phi(A^\bullet)\right) = A^\bullet \ .$$

Similarly, $\circ' = \circ$. Finally,

$$\phi'(A) = A^{\uparrow} = \phi(A) \ . \qquad \square$$

[6] This concept is actually a particular case of the one of isomorphism between two fuzzy partially ordered sets (ie. sets equiped with a fuzzy equality - here the usual equality - and a fuzzy order relative to that fuzzy equality - here the subsethood degree, see [4]).

Corollary 4. *A fuzzy closure system on X is given by an L-isotone Galois connection between X and Y iff it is isomorphic to a fuzzy opening operator on Y and vice versa.*

We are now interested in characterizing closure operators and systems comming from FICs in terms of the underlying relation R. These are actually very natural and "famous" types of closure operators in the crisp case, especially when $X = Y$ and the correponding relation is an equivalence or a partial order.[7]

Suppose R is a fuzzy relation between X and Y and consider $(\uparrow, \downarrow)$ the induced FIC and $\bullet = \uparrow\downarrow$ the corresponding fuzzy closure operator. For each $A \in L^X$,

$$A^{\bullet}(x) = \bigwedge_{y \in Y} R(x, y) \to \bigvee_{x' \in X} A(x') * R(x', y) \ .$$

As pointed out in [13], in case $X = Y$ and L is an L-preorder, the expression of $\bullet$ becomes simpler (since $\bullet = \uparrow$):

$$A^{\bullet}(x) = \bigvee_{x \in X} A(x) * R(x, y) \ .$$

In [9], there are characterized the fuzzy closure operators and systems comming from fuzzy equivalences. In case R is an arbitrary fuzzy relation, we have the following. (The **left-shift generation rule** on L^X is: from A, infer $A \to b$, where $A \in L^X$ and $b \in L$.)

Proposition 12. *Let $\mathcal{C} \subseteq L^X$ be a fuzzy closure system. Then $\mathcal{C}$ is induced (via a FIC) by some fuzzy relation between X and Y iff it is generated using left-shift and intersection by a class of fuzzy sets of cardinal at most $|Y|$.*

Proof. "only if": Let $\bullet = (\uparrow, \downarrow)$ and R the corresponding L-relation. We define the family of fuzzy sets $(\Theta_y)_{y \in Y} \subseteq L^X$ by $\Theta_y(x) = R(x, y)$ for all $y \in Y, x \in X$. Let $A \in L^X$. Then there exists $B \in L^Y$ such that $A = B^{\downarrow}$. We have that:

$$A(x) = B^{\downarrow}(x) = \bigwedge_{y \in Y} R(x, y) \to B(y) = \left(\bigcup_{y \in Y} \Theta_y \to B(y) \right)(x) \ .$$

Thus $A = \bigcup_{y \in Y} \Theta_y \to B(y)$.

[7] When it is a partial order, the closing operator assigns to each set the generated order ideal. In case we deal with an equivalence, it assigns to each set the union of all the equivalence classes that cross it.

"if": Suppose $(\Theta_y)_{y\in Y}$ generates $\mathcal{C}$ using left-shift and intersection. Define $R(x, y) = \Theta_y(x)$ for all x, y. Let $A \in L^X$.

$$A^\bullet(x) = \left(\bigcap_{A\subseteq\Theta_y\to B} \Theta(y) \to B\right)(x) = \bigwedge_{A*\Theta_y\subseteq b} \Theta_y(x) \to b$$

$$= \bigwedge_{\forall x'\in X, A(x')*\Theta_y(x')\leq b} \Theta_y \to b = \bigwedge_{\bigvee_{x'\in X} A(x')*\Theta_y(x')\leq b} \Theta_y \to b$$

$$= \bigwedge_{y\in Y} \Theta_y \to \left(\bigvee_{x'\in X} A(x') * \Theta_y(x)\right)$$

(the last equality holds because $\to$ is isotone on the second argument). The proved equality shows that $\bullet$ is induced by R. □

If we care about fuzzy closure operators induced by antitone Galois connections, we have a much nicer, but similarly provable, characterization result. Consider, on L^X, for each $a \in L$, the unary operation "a-left-shift", that assigns to each $A \in L^X$ the fuzzy set $a \to A$. Also, consider intersection as a family of infinitary operations, indexed on the cardinal of the set to which we take the intersetcion. In [5], the fuzzy closure systems on a set X are proven to be exactly the subalgebras of L^X together with the above operations. In this context, we have the following (not quite parallel result) for the antiotone case.

Proposition 13. *A fuzzy closure system on X is induced (via a FAC) by some fuzzy relation iff, as a subalgebra, it is generated by some set of cardinal* $|Y|$.

The proof, similar to the one of Proposition 12, is left as an exercise to the reader.

6. Operators induced by fuzzy conjugated pairs

Let $(\uparrow, \downarrow)$ be a FCP. Remember that it is always given by a fuzzy relation, that we denote R, by

$$A^\uparrow(y) = \bigvee_{x\in X} A(x) * R(x, y) \ , \quad B^\downarrow(x) = \bigvee_{y\in Y} B(y) * R(x, y) \ .$$

For an arbitrary operator $\Delta : L^X \longrightarrow L^X$ (or $\Delta : L^Y \longrightarrow L^Y$), we denote by Δ^i, with $i \in \mathbb{N}$, the operator $\Delta \circ \Delta \circ \ldots \circ \Delta$, i times. In particular Δ° is the identity.

Define $\bullet : L^X \longrightarrow L^X$ and $\spadesuit : L^Y \longrightarrow L^Y$ by

$$A^\bullet = \bigcup_{i\in\mathbb{N}} A^{(\uparrow\downarrow)^i} \ , \quad B^\spadesuit = \bigcup_{i\in\mathbb{N}} B^{(\downarrow\uparrow)^i} \ ,$$

for all $A \in L^X$, $B \in L^Y$.

Lemma 4. *For each* $A, A_1, A_2 \in L^X$, $B, B_1, B_2 \in L^Y$, $(A_i)_{i\in I} \subseteq L^X$, $(B_i)_{i\in I} \subseteq L^Y$,
(1) $\left(\bigcup_{i\in I} A_i\right)^{\uparrow} = \bigcup_{i\in I} A_i^{\uparrow}$, $\left(\bigcup_{i\in I} B_i\right)^{\uparrow} = \bigcup_{i\in I} B_i^{\uparrow}$;
(2) $S(A_1, A_2) \le S(A_1^{\uparrow}, A_2^{\uparrow})$, $S(B_1, B_2) \le S(B_1^{\downarrow}, B_2^{\downarrow})$;
(3) $A^{\bullet} = A \vee A^{\bullet\uparrow\downarrow}$, $B^{\spadesuit} = B \vee B^{\spadesuit\downarrow\uparrow}$.

Proof.

(1) is obvious, looking at the definition of $\uparrow$, $\downarrow$ using R.
(2) We use Lemma 1.(9 and 8).

$$
\begin{aligned}
S(A_1^{\uparrow}, A_2^{\uparrow}) &= \bigwedge_{y\in Y} A_1^{\uparrow}(y) \to A_2^{\uparrow}(y)\\
&= \bigwedge_{y\in Y} \left(\bigvee_{x\in X} A_1(x) * R(x,y)\right) \to \left(\bigvee_{x\in X} A_2(x) * R(x,y)\right)\\
&\ge \bigwedge_{y\in Y}\bigwedge_{x\in X} (A_1(x) * R(x,y)) \to (A_2(x) \to R(x,y))\\
&\ge \bigwedge_{y\in Y}\bigwedge_{x\in X} A_1(x) \to A_2(x) = S(A_1, A_2)\ .
\end{aligned}
$$

The other inequality follows analogously.
(3) We use point (1):

$$
A^{\bullet} = \bigcup_{i\in\mathbb{N}} A^{(\uparrow\downarrow)^i} = A \vee \bigcup_{i\ge 1} A^{(\uparrow\downarrow)^i} = A \vee \left(\bigcup_{i\in\mathbb{N}} A^{\uparrow\downarrow}\right)^{\uparrow\downarrow} = A \vee A^{\bullet\uparrow\downarrow}\ .
$$

A similar proof gives the other equality. □

Proposition 14. $\bullet$ *and* $\spadesuit$ *are fuzzy closure operators.*

Proof. $A \subseteq A^{\bullet}$ follows from $A^{(\uparrow\downarrow)^0} = A$. Let $A_1, A_2 \in L^X$, We rewrite $S(A_1^{\bullet}, A_2^{\bullet})$ using Lemma 1.(8) and Lemma 4.(2):

$$
\begin{aligned}
S(A_1^{\bullet}, A_2^{\bullet}) &= \bigwedge_{x\in X} A_1^{\bullet}(x) \to A_2^{\bullet}(x)\\
&= \bigwedge_{x\in X} \left(\bigvee_{i\in\mathbb{N}} A_1^{(\uparrow\downarrow)^i}(x)\right) \to \bigvee_{i\in\mathbb{N}} \left(A_2^{(\uparrow\downarrow)^i}(x)\right)\\
&\ge \bigwedge_{x\in X}\bigwedge_{i\in\mathbb{N}} A_1^{(\uparrow\downarrow)^i}(x) \to A_2^{(\uparrow\downarrow)^i}(x)\\
&\ge \bigwedge_{x\in X}\bigvee_{i\in\mathbb{N}} A_1(x) \to A_2(x) = S(A_1, A_2)\ .
\end{aligned}
$$

We now prove $A^{\bullet} = A^{\bullet\bullet}$. That $A^{\bullet} \subseteq A^{\bullet\bullet}$ we already know. For the converse inclusion, apply Lemma 4.(3):

$$
A^{\bullet\bullet} = \bigcup_{i\in\mathbb{N}} (A^{\bullet})^{(\uparrow\downarrow)^i} \subseteq \bigcup_{i\in\mathbb{N}} A^{\bullet} = A^{\bullet}\ .
$$

That $\spadesuit$ is a fuzzy closure operator follows similarly. □

For a fuzzy relaton $P \in L^{X \times Y}$, define its **transitive closure** to be $P^* \in L^{X \times Y}$,

$$P^*(x, y) = \bigvee_{\substack{i \in \mathbb{N} \\ x_1,\dots x_i \in X \\ y_1,\dots y_i \in Y}} P(x, y_1) * P(x_1, y_1) * P(x_1, y_2) * P(x_2, y_2) * \dots * P(x_{i-1}, y_i) * P(x_i, y) \,,$$

for all $x \in X$, $y \in Y$. (In case $i = 0$, we agree to identify $P(x, y_1) * \dots * P(x_i, y)$ with $P(x, y)$.)

The name "transitive closure" is justified by the fact that P^* is the least fuzzy relation Q such that:

- $P \subseteq Q$;
- $Q(x, y') * Q(x', y') * Q(x', y) \le Q(x, y)$, for all $x, x' \in X$, $y, y' \in Y$ (call this last property **transitivity**[8]).

Transitive closures can be put in connection to the above induced operators. Denote, as before, by $\uparrow, \downarrow, \bullet, \spadesuit$ (without any subscript) the entities associated to R, and by $\uparrow_{R^*}, \downarrow_{R^*}, \bullet_{R^*}, \spadesuit_{R^*}$ those associated to R^*.

The transitive closure R^* is an extension of R that does not change the induced operators, but expresses them more compactly; moreover, it is, in some sense, optimal:

Proposition 15. *(1) $\bullet \uparrow = \uparrow \spadesuit = \uparrow_{R^*}$; $\downarrow \bullet = \spadesuit \downarrow = \downarrow_{R^*}$;*
(2) $\bullet_R = \bullet_{R^*} = 1_{L^X} \vee (\uparrow_{R^*} \downarrow_{R^*})$; $\spadesuit_R = \spadesuit_{R*} = 1_{L^Y} \vee (\downarrow_{R^*} \uparrow_{R^*})$;
(3) R^* is the smallest fuzzy relation Q such that:
- *$R \subseteq Q$;*
- *$\bullet_Q \uparrow_Q = \uparrow_Q$.*

Proof.

(1): An easy check by induction on i shows that, for each $i \ge 1$ and $A \in L^X$,

$$A^{(\uparrow\downarrow)^{i-1}\uparrow}(y) = \bigvee_{\substack{x_1,\dots x_i \in X \\ y_1,\dots y_i \in Y}} A(x_1) * R(x_1, y_1) * R(x_1, y_2) * R(x_2, y_2) * \dots * R(x_{i-1}, y_i) * R(x_i, y) \,.$$

Hence,

$$A^{\uparrow\spadesuit}(y) = \left(\bigcup_{i \ge 1} A^{\uparrow(\downarrow\uparrow)^{i-1}} \right)(y) = \left(\bigcup_{i \ge 1} A^{(\uparrow\downarrow)^{i-1}\uparrow} \right)(y)$$
$$= \bigvee_{i \ge 1} \bigvee_{\substack{x_1,\dots x_i \in X \\ y_1,\dots y_i \in Y}} A(x_1) * R(x_1, y_1)$$

[8] Notice that, if $X = Y$, a reflexive fuzzy relation R from $L^{X \times Y}$ is transitive in our acceptance if and only if it is transitive in the usual sense.

$$*R(x_1, y_2) * R(x_2, y_2) * \ldots * R(x_{i-1}, y_i) * R(x_i, y)$$

$$= \bigvee_{x \in X} A(x) * R^*(x, y) = A^{\uparrow_{R^*}}(y) .$$

Thus $\uparrow\spadesuit = \uparrow_{R^*}$. That $\bullet\uparrow = \uparrow\spadesuit$ follows easily by Lemma 4.(1). So $\bullet\uparrow = \uparrow\spadesuit = \uparrow_{R^*}$. The other triple equality follows similarly.

(2): Let $B \in L^Y$. By Lemma 4.(3),

$$B^{\spadesuit}(y) = B(y) \vee B^{\uparrow\downarrow\spadesuit}(y) = B(y) \vee (B^{\downarrow})^{\uparrow\spadesuit}(y) = B(y) \vee B^{\downarrow\uparrow_{R*}} .$$

This means that $\spadesuit = 1_{L^Y} \vee (\downarrow\uparrow_{R^*})$. Similarly, $\bullet = 1_{L^X} \vee (\uparrow\downarrow_{R^*})$. Applying these two equalities for R^* instead of R, we get

$$\spadesuit_{R^*} = 1_{L^Y} \vee (\downarrow_{R^*}\uparrow_{R^*}) , \quad \bullet_{R^*} = 1_{L^X} \vee (\uparrow_{R^*}\downarrow_{R^*}) .$$

In order to finish the proof of (2), it suffices to show $\downarrow\uparrow_{R^*} = \downarrow_{R^*}\uparrow_{R^*}$ and $\downarrow\uparrow_{R^*} = \downarrow_{R^*}\uparrow_{R^*}$. We only prove the first, the other following similarly. Let $B \in L^Y$ and $y \in Y$. We need

$$\bigvee_{x \in X, y' \in Y} B(y') * R(x, y') * R^*(x, y)$$

$$= \bigvee_{x \in X, y' \in Y} B(y') * R^*(x, y') * R^*(x, y)$$

It suffices that, for each $y' \in Y$,

$$\bigvee_{x \in X} R(x, y') * R^*(x, y) = \bigvee_{x \in X} R^*(x, y') * R^*(x, y) .$$

One inequality is immediate. For the other, we rewrite the righthand side: $\bigvee_{x \in X} R^*(x, y') * R^*(x, y)$ is the supremum of

$$R(x, y_1) * R(x_1, y_1) * R(x_1, y_2) * \ldots R(x_n, y_n) * R(x_n, y')$$

$$*R(x, z_1) * R(u_1, z_1) * R(u_1, z_2) * \ldots R(u_m, z_m) * R(u_m, y) ,$$

where n, m take values in $\mathbb{N}$, x_i, u_i in X y_i, z_i in Y. We want to show that any such product is lower or equal to $\bigvee_{x \in X} R(x, y') * R^*(x, y)$. If $n = 0$, the inequality is immediate; so assume $n > 0$. Regroup the product as

$$R(x_n, y') * [R(x_n, y_n) * R(x_{n-1}, y_n) * \ldots R(x, y_1) \ * R(x, z_1)$$
$$*R(u_1, z_1) * \ldots * R(u_m, y)]$$

and notice that this is lower or equal to $R(x_n, y') * R^*(x_n, y)$. This proves the inequality.

(3): Because R^* is transitive, $\uparrow_{R^*}\downarrow_{R^*}\uparrow_{R^*} \leq \uparrow_{R^*}$. This means

$$\uparrow_{R^*} = \uparrow_{R^*} \vee \uparrow_{R^*}\downarrow_{R^*}\uparrow_{R^*} = (1_{L^X} \vee \uparrow_{R^*}\downarrow_{R^*})\uparrow_{R^*} = \bullet_{R^*}\uparrow_{R^*} .$$

Let now Q having the two properties from point (3). Then

$$\uparrow_Q\downarrow_Q\uparrow_Q \subseteq \bullet_Q \uparrow_Q = \uparrow_Q .$$

Let $x \in X$, $y \in Y$ and $A \in L^X$ defined by $A(x') = 1$ if $x = x'$ and 0 otherwise. Then

$$Q(x,y) = A^{\uparrow_Q}(y) = A^{\uparrow_Q \downarrow_Q \uparrow_Q}(y)$$
$$= \bigvee_{y' \in Y, x' \in X} Q(x,y') * Q(x',y') * Q(x',y) \, .$$

This implies the transitivity of Q, hence $R^* \subseteq Q$. □

7. Induced hierarchical structures

The lattice of fuzzy concepts, structure associated to a fuzzy binary relation, via its induced FAC, is a generalization of the concept lattice from classical case, having a similar, yet fuzzy, characterization theorem [2].

One can also construct hierarchies associated to FICs and FCPs. In the crisp case, the situation is the following: while hierarchies for FICs are perfectly dual to those for FACs (and thus one does not need to study both), those for FCPs are fundamentally different.[9] However, when fuzziness is concerned, all three differ.

To each FAC, FIC, or FCP between X and Y, one associates the lattice of fix points $\mathcal{L}_s(\uparrow, \downarrow)$ (where $s \in \{\text{FAC, FIC, FCP}\}$, consisting of all pairs (A, B) from $L^X \times L^Y$ with $A^\uparrow = B$ and $B^\downarrow = A$. The order is taken according to first component: $(A, B) \leq (A', B')$ iff $A \subseteq A'$. In fact, the second component is not neglected this way, because

- for FACs, $(A, B) \leq (A', B')$ iff $B' \subseteq B$;
- for FICs and FCPs, $(A, B) \leq (A', B')$ iff $B \subseteq B'$.

Instead of $\mathcal{L}_s(\uparrow, \downarrow)$, we can write $\mathcal{L}_s(R)$, where R is the underlying fuzzy relation. It is known that $\mathcal{L}_{\text{FAC}}(R)$ and $\mathcal{L}_{\text{FIC}}(R)$ are complete lattices that can be abstractly characterized (see [2], [23]):

Proposition 16. *I. A complete lattice W is isomorphic to $\mathcal{L}_{\text{FAC}}(R)$ if and only if there exist two functions $\gamma : L \times X \longrightarrow W$, $\mu : \mathcal{L} \times Y \longrightarrow W$ such that $\gamma(L \times X)$ is $\bigvee$-dense, $\mu(L \times Y)$ is $\bigwedge$-dense, and*

$$\gamma(a, x) \leq \mu(b, y) \text{ iff } a * b \leq R(x, y) \, .$$

II. A complete lattice W is isomorphic to $\mathcal{L}_{\text{FIC}}(R)$ if and only if there exist two functions $\gamma : L \times X \longrightarrow W$, $\mu : \mathcal{L} \times Y \longrightarrow W$ such that $\gamma(L \times X)$ is $\bigvee$-dense, $\mu(L \times Y)$ is $\bigwedge$-dense, and

$$\gamma(a, x) \leq \mu(b, y) \text{ iff } R(x, y) \leq a \to b \, .$$

[9] We do not know any study of this kind of hierarchies (for conjugated pairs that is) in classical setting.

While the elements of $\mathcal{L}_{\mathrm{FAC}}(R)$ are canonically interpretted as fuzzy concepts (pairs extent-intent), $\mathcal{L}_{\mathrm{FIC}}(R)$ can be seen as the hierarchy of *concise local context* [23]. We again emphasize the fact that, although very similar, the structures associated to FACs and FICs are not dual. In classical setting (i.e. $L = \{0, 1\}$), the only observed duality can be expressed as follows: for each relation R, $\mathcal{L}_{\mathrm{FIC}}(R) \simeq \mathcal{L}_{\mathrm{FAC}}(\bar{R})$ and $\mathcal{L}_{\mathrm{FAC}}(R) \simeq \mathcal{L}_{\mathrm{FIC}}(\bar{R})$, $\bar{R}$ being the complement of R. This is by far not the case here, where negation behaves very unsymmetrically. Take, for instance, the residuated lattice $L = \{0, 1/4, 2/4, 3/4, 1\}$, with the operations $*$, $\rightarrow$ and $\bar{\ }$ (the negation) displayed below:

$*$	0	1/4	2/4	3/4	1
0	0	0	0	0	0
1/4	0	0	1/4	1/4	1/4
2/4	0	1/4	1/4	2/4	2/4
3/4	0	1/4	2/4	2/4	3/4
1	0	1/4	2/4	3/4	1

$\rightarrow$	0	1/4	2/4	3/4	1
1	1	1	1	1	1
1/4	1/4	1	1	1	1
2/4	0	2/4	1	1	1
3/4	0	1/4	3/4	1	1
1	0	1/4	2/4	3/4	1

a	0	1/4	2/4	3/4	1
$\bar{a}$	1	1/4	0	0	0

Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, and the L-relation R between X and Y, together with its complement $\bar{R}$, as below

R	x_1	x_2
y_1	2/4	0
y_2	1	3/4
y_3	1/4	2/4

$\bar{R}$	x_1	x_2
y_1	0	1
y_2	0	0
y_3	1/4	0

We shall now list the composition and structure of each R or $\bar{R}$-lattice of fixed points for FAC and FIC, but also for FCP (which is also a complete lattice, that we shall study afterwards), in order to point out something that is in fact quite transparent: there is no duality between any of these structures.

$\mathcal{L}_{\mathrm{FAC}}(R)$

x_1	x_2	y_1	y_2	y_3
1/4	0	1	1	1
2/4	0	1	1	2/4
1/4	1/4	1/4	1	1
3/4	0	3/4	1	1/4
1	0	2/4	1	1/4
2/4	1/4	1/4	1	2/4
1/4	3/4	0	1	3/4
1	1/4	1/4	1	1/4
2/4	3/4	0	1	2/4
1	3/4	0	1	1/4
2/4	1	0	3/4	2/4
1	1	0	3/4	2/4

$\mathcal{L}_{\mathrm{FAC}}(\bar{R})$

x_1	x_2	y_1	y_2	y_3
0	0	1	1	1
1/4	0	1/4	1/4	1
0	1/4	1	1/4	1/4
2/4	0	0	0	2/4
1/4	1/4	1/4	1/4	1/4
0	1	1	0	0
1	1/4	0	0	1/4
1/4	1	1/4	0	0
1	1	0	0	0

$\mathcal{L}_{\mathrm{FIC}}(R)$

x_1	x_2	y_1	y_2	y_3
0	0	0	0	0
1/4	0	1/4	1/4	0
0	1/4	0	1/4	1/4
0	2/4	0	2/4	1/4
2/4	2/4	1/4	2/4	1/4
0	3/4	0	2/4	2/4
3/4	2/4	2/4	3/4	1/4
2/4	3/4	1/4	2/4	2/4
0	1	0	3/4	2/4
1	2/4	2/4	1	1/4
2/4	1	1	3/4	2/4
3/4	1	2/4	3/4	2/4
1	1	2/4	1	2/4

$\mathcal{L}_{\mathrm{FIC}}(\bar{R})$

x_1	x_2	y_1	y_2	y_3
1/4	0	0	0	0
1	0	0	0	1/4
1/4	1/4	1/4	0	0
1	1/4	1/4	0	1/4
1/4	2/4	2/4	0	0
1	2/4	2/4	0	1/4
1/4	3/4	3/4	0	0
1	3/4	3/4	0	1/4
1/4	1	1	0	0
1	1	1	0	1/4

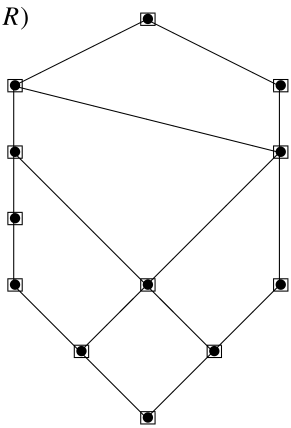
$\mathcal{L}_{\mathrm{FCP}}(R)$

x_1	x_2	y_1	y_2	y_3
0	0	0	0	0
1/4	1/4	1/4	1/4	1/4
2/4	2/4	1/4	2/4	1/4
3/4	2/4	2/4	3/4	1/4
1	3/4	2/4	1	2/4

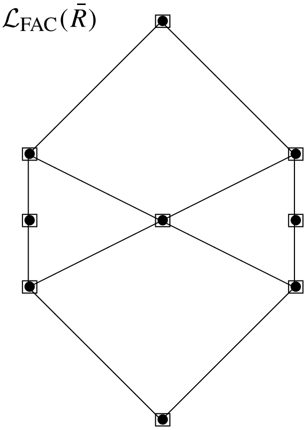
$\mathcal{L}_{\mathrm{FCP}}(\bar{R})$

x_1	x_2	y_1	y_2	y_3
0	0	0	0	0
0	1/4	1/4	0	0
0	2/4	2/4	0	0
0	3/4	3/4	0	0
0	1	1	0	0

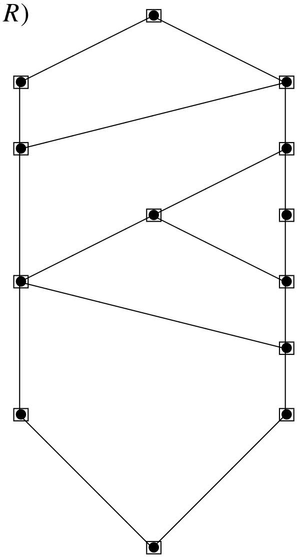
$\mathcal{L}_{\mathrm{FAC}}(R)$



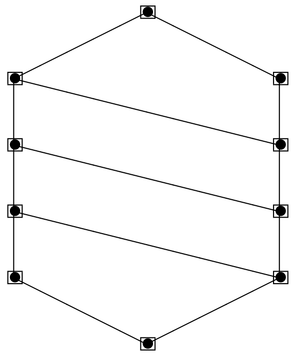
$\mathcal{L}_{\mathrm{FAC}}(\bar{R})$



$\mathcal{L}_{\mathrm{FIC}}(R)$

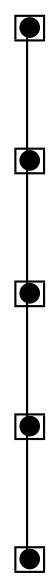


$\mathcal{L}_{\mathrm{FIC}}(\bar{R})$



$\mathcal{L}_{\mathrm{FCP}}(R)$

$\mathcal{L}_{\mathrm{FCP}}(\bar{R})$



We mention that, for a relation R, $\mathcal{L}_{\mathrm{FCP}}(R)$ is not even in the classical case totally ordered, isomorphic, or dual to $\mathcal{L}_{\mathrm{FCP}}(\bar{R})$, as shown by the following example, where L is the Boolean algebra $\{1, 2\}$, $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$, and R, $\bar{(R)}$ as below.

R	x_1	x_2	x_3
y_1	0	0	1
y_2	1	1	0
y_3	0	1	0

$\bar{R}$	x_1	x_2	x_3
y_1	1	1	0
y_2	0	0	1
y_3	1	0	1

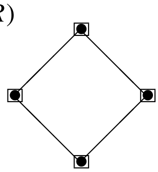
$\mathcal{L}_{\mathrm{FCP}}(R)$

x_1	x_2	x_3	y_1	y_2	y_3
0	0	0	0	0	0
0	0	1	1	0	0
1	0	0	0	1	0
1	1	1	1	1	1

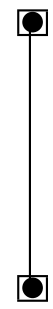
$\mathcal{L}_{\mathrm{FCP}}(\bar{R})$

x_1	x_2	x_3	y_1	y_2	y_3
0	0	0	0	0	0
1	1	1	1	1	1

$\mathcal{L}_{\mathrm{FCP}}(R)$



$\mathcal{L}_{\mathrm{FCP}}(\bar{R})$



Let us now focus all our attention to the new structure $\mathcal{L}_{\mathrm{FCP}}$. As usual, we look at X as the universe of objects, at Y as that of properties, and at R as the relation "to have" between objects and attributes. A fixed point (A, B) can be seen as a **fuzzy subcontext**, since the whole initial contextual framework is preserved between A and B: all the properties of any object from A (and no other) are in B ($A^{\uparrow} = B$) and vice versa ($B^{\downarrow} = A$). The inductive definitions of operators $\bullet, \spadesuit$ constitute some closing process, including, at each step, the whole R-available "space" for a set of objects (or of attributes): A takes for itself all the needed properties, $A^{\uparrow}$, while $A^{\uparrow}$ calls for more objects, $A^{\uparrow\downarrow}$, and so on - this could be seen as a try to build a minimal subcontext. Unfortunately, this does not always work, since the result is not necessarily a fixed point. All we can say at this stage is that any fixed point is closed to the above construction and, consequently, the relation R can be taken to be transitive.

Proposition 17. *Let $(\uparrow, \downarrow)$, R its relation, $\bullet, \spadesuit$ its operators.*

(1) $\mathcal{L}_{\mathrm{FCP}}$ is a complete lattice in which arbitrary suprema are given by

$$\bigvee_{i\in I}(A_i, B_i) = \left(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\right) .$$

(2) If $(A, B) \in \mathcal{L}_{\mathrm{FCP}}(R)$, then $A = A^{\bullet}$ and $B = B^{\spadesuit}$;
(3) $\mathcal{L}_{\mathrm{FCP}}(R) = \mathcal{L}_{\mathrm{FCP}}(R^*)$.

Proof. 1. follows immediately from Lemma 4.(1).

2. If $B = A^{\uparrow}$ and $A = B^{\downarrow}$, then $A = A^{\uparrow\downarrow}$. This implies $A = A^{(\uparrow\downarrow)^i}$, for all $i \in \mathbb{N}$, which immediately gives $A = A^{\bullet}$. Similarly, $B = B^{\spadesuit}$.
3. It is sufficient to prove that, if A and B are closed (w.r.t. the operators induced by R, which are, by Proposition 15.(2), equal to those induced by R^*), then $A^{\uparrow} = A^{\uparrow_{R^*}}$ and $B^{\downarrow} = B^{\downarrow_{R^*}}$. We only prove the first of these. By Proposition 15.(1) $\bullet \uparrow = \uparrow_{R^*}$. Replacing R by R^*, we get $\bullet_{R^*} \uparrow_{R^*} = \uparrow_{R^*} \spadesuit_{R^*}$, hence $\bullet \uparrow = \bullet_{R^*} \uparrow_{R^*}$, which is precisely what we needed, because any closed A is equal to $A^{\bullet}$ and $A^{\bullet_{R^*}}$. □

We have already seen that operators induced by FCPs are not as smooth as the ones from FACs and FICs. This is the reason why their sets of objects or attributes that are closed to these operators do not necessarily provide fixed points. However, they do that, if we take the following additional assumption: all the objects and properties **are empirically consistent**, that is:

– for every object x, there exists a property y such that $R(x, y) = 1$;
– for every property y, there exists an object x such that $R(x, y) = 1$.

Lemma 5. *Let V, W be two complete lattices and I, J two non-empty sets. Assume that there exist the families $A = (a_i)_{i\in I}$, $B = (b_j)_{j\in J} \subseteq V$, $A' = (a'_i)_{i\in I}$, $B' = (b'_j)_{j\in J} \subseteq W$ such that*

$$\forall i \in I, J' \subseteq J, \qquad a_i \leq \bigvee_{j\in J'} b_j \text{ iff } a'_i \leq \bigvee_{j\in J'} b'_j \ ,$$

$$\forall j \in J, I' \subseteq I, \qquad b_j \leq \bigvee_{i\in I'} a_i \text{ iff } b'_j \leq \bigvee_{i\in I'} a'_i \ .$$

If A, B, A', B' are $\bigvee$-dense in V and W, then V and W are isomorphic.

Proof. Define $\phi : V \longrightarrow W$ and $\psi : W \longrightarrow V$ by

$$\phi(v) = \bigvee_{i\in I, a_i\leq v} a'_i \ ,$$

$$\psi(w) = \bigvee_{j\in J, b'_j\leq w} b_j \ .$$

These functions are obviously increasing. In order to prove that they are inverse to each other, let $w \in W$. Then $\phi\psi(w)$ is equal to

$$\bigvee\{a'_i \ / \ i \in I, a_i \leq \bigvee_{j\in J, b'_j\leq w} b_j\}.$$

Using one of the hypotheses, the last becomes

$$\phi(\psi(w)) = \bigvee\{a'_i \ / \ i \in I, \ a'_i \leq \bigvee_{j\in J, b'_j\leq w} b'_j\} \ ,$$

the last being, using the $\bigvee$-density of B' and A', equal to w. That the other composition is the identity follows similarly. □

Lemma 6. *Suppose that all the objects and properties are empirically consistent. Then, for each $(A, B) \in L^X \times L^Y$,*

$$(A, B) \in \mathcal{L}_{\mathrm{FCP}}(R) \text{ iff } [A = A^{\bullet} \text{ and } B = A^{\uparrow}] \text{ iff } [B = B^{\spadesuit} \text{ and } A = B^{\downarrow}] \ .$$

Proof. We only prove the first equivalence. One implication follows from Proposition 17.(2). For the other, assume that $A = A^{\bullet}$ and $B = A^{\uparrow}$. This implies, by Lemma 4.(3), that $A^{\uparrow\downarrow} = A^{\bullet\uparrow\downarrow} \subseteq A^{\bullet} = A$. On the other hand, using the empirical consistency of objects, for each $x \in X$,

$$\begin{aligned} A^{\uparrow\downarrow}(x) &= \bigvee_{y\in Y, x'\in X} A(x') * R(x', y) * R(x, y) \\ &\geq \bigvee_{y\in Y} A(x) * R(x, y) * R(x, y) \geq A(x) \ . \end{aligned}$$

Thus $A = A^{\uparrow\downarrow}$. This means $A = B^{\downarrow}$. □

We are now ready to characterize the lattice $\mathcal{L}_{\mathrm{FCP}}(R)$.

Proposition 18. *Suppose that all objects and properties are empirically consistent. A complete lattice W is isomorphic to $\mathcal{L}_{\mathrm{FCP}}(R)$ if and only if there exist two functions $\gamma : L \times X \longrightarrow W$, $\mu : L \times Y \longrightarrow W$ such that $\gamma(L \times X)$, $\mu(L \times Y)$ are $\bigvee$-dense and, for each $a \in L$, $G \subseteq L \times X$, $F \subseteq L \times Y$, $x \in X$, $y \in Y$,*

$$\gamma(a,x) \le \bigvee \mu(F) \quad \textit{iff} \quad a \le \bigvee_{(b,y')\in F} b * R^*(x,y') \,,$$

$$\mu(a,y) \le \bigvee \mu(G) \quad \textit{iff} \quad a \le \bigvee_{(b,x')\in G} b * R^*(x',y) \,.$$

Proof. According to Lemma 5, any two complete lattices that satisfy the above conditions are isomorphic. It suffices to show that $\mathcal{L}_{\mathrm{FCP}}$ satisfies them. Denote, for each $x \in X$ (or $x \in Y$) and $a \in L$, by $\{a|x\}$ the (singleton) function from L^X (or L^Y) which takes value a in x and 0 in the other elements.[10]

Define γ and μ by

$$\gamma(a,x) = (\{a|x\}^{\bullet}, \{a|x\}^{\bullet\uparrow}),$$
$$\mu(a,y) = (\{a|y\}^{\spadesuit\downarrow}, \{a|y\}^{\spadesuit}),$$

for all $a \in L, x \in X, y \in Y$. Let $(A,B) \in \mathcal{L}_{\mathrm{FCP}}$. Then, using Lemma 4.(1) and closure operators properties,

$$(A,B) = \left(\left(\bigcup_{x\in X}\{A(x)|x\}\right)^{\bullet}, A^{\uparrow}\right) = \left(\left(\bigcup_{x\in X}\{A(x)|x\}^{\bullet}\right)^{\bullet}, A^{\uparrow}\right) .$$

By Lemma 6, each $(\{A(x)|x\}^{\bullet}, \{A(x)|x\}^{\bullet\uparrow}$ is a fixed point; hence, by Proposition 17.(1), so is their componentwise union; this implies that $\bigcup_{x\in X}\{A(x)|x\}^{\bullet}$ is closed to $\bullet$, which means that we can further rewrite (A,B) into $\bigvee_{x\in X}\gamma(A(x),x)$. So $\gamma(L\times X)$ is $\bigvee$-dense in $\mathcal{L}_{\mathrm{FCP}}$. Similarly, $\mu(L \times Y)$ is $\bigvee$-dense. We now check the first of the remaining required properties (the other following similarly).

$$\gamma(a,x) \le \mu(F)$$

means

$$\{a|x\}^{\bullet} \subseteq \left(\bigvee_{(b,y)\in F}\{b|y\}\right)^{\spadesuit\downarrow} ,$$

that is

$$\{a|x\}^{\bullet} \subseteq \left(\bigvee_{(b,y)\in F}\{b|y\}\right)^{\downarrow\bullet} ,$$

[10] The denoted entity is different from that of Section 3, where $\{x|a\}$ denoted the hyper-singleton.

which, since the lefthand side is closed w.r.t. $\bullet$, is equivalent to

$$\{a|x\} \subseteq \left(\bigvee_{(b,y)\in F} \{b|y\} \right)^{\downarrow\bullet} ,$$

and, since, $\downarrow \bullet$ commutes with suprema, further to

$$\{a|x\} \subseteq \bigvee_{(b,y)\in F} \{b|y\}^{\downarrow\bullet} ,$$

that is

$$a \leq \bigvee_{(b,y)\in F} \{b|y\}^{\downarrow\bullet}(x) .$$

But, from Proposition 15.(1), $\downarrow \bullet = \uparrow_{R^*}$, so the above is equivalent to

$$a \leq \bigvee_{(b,y)\in F} \{b|y\}^{\uparrow_{R^*}}(x) ,$$

which means $a \leq \bigvee_{(b,y)\in F} b * R^*(x, y)$. □

8. Concluding remarks

The present paper introduced and studied fuzzy conjugated pairs, with their underlying closure operators and hierarchical structure. Isotone versions of fuzzy Galois connections and closure operators were also considered, arguing that they are not dual to the antitone ones, but providing quite similar results about them. A form of artificial common treatment of classically dual notions that fail to be fuzzily dual has concretized into the concept of (α, β)-connection, that could be seen as recovering remains of duality by means of common polynomial invariants.

A very interesting subject for future research would be a systematic study of duality at the level not of involved concepts, but of proofs regarding the concepts - the very similar techniques in proving some isotone-antitone results suggest that, if we restrict our statements to some that are well formed w.r.t. residuation property, we could achieve a productive duality in fuzzy logic.

Another open problem would be, once we have some working "concrete" fuzzy notions of Galois connection, opening and closure operator etc., to provide their suitable algebraic abstractizations (notice that, for instance, a fuzzy closure operator (between sets) is not a particular case of (abstract) closure operator, while a fuzzy Galois connection is not even a particular case of functor!).

Acknowledgements. Many thanks to the referee for helpful remarks and suggestions.

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