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10.1007/s00153-004-0237-z **Mathematical Logic**

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An approach to infinitary temporal proof theory

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Abstract. Aim of this work is to investigate from a proof-theoretic viewpoint a propositional and a predicate sequent calculus with an ω–type schema of inference that naturally interpret the propositional and the predicate *until*–free fragments of Linear Time Logic LTL respectively. The two calculi are based on a natural extension of ordinary sequents and of standard modal rules. We examine the pure propositional case (no extralogical axioms), the propositional and the first order predicate cases (both with a possibly infinite set of extralogical axioms). For each system we provide a syntactic proof of cut elimination and a proof of completeness.

1. Introduction

The importance of Linear Temporal Logic LTL [Em90] is twofold: theoretical and in applications to computer science. As for applications, Pnueli [Pnu77] was the first to use temporal systems to prove correctness of sequential and concurrent programs.

Despite its name LTL is not a logic of time: it is rather a modal logic whose Kripke models, based on the frame of natural numbers, describe the evolution of computations. The basic modal operators $\circ$ (*next*) and $\Box$ (*always*) of LTL are interpreted as *"at the next state of computation"* and *"at each state from the current state onwards"* respectively.

So far LTL has been thoroughly investigated by using model– or automata–theoretic techniques [Em90, VaWo86]. Expressive power [Kam68] and computational complexity [SC85] have been also studied for sake of using LTL as a specification language.

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Aim of this work is to investigate from a proof-theoretic viewpoint a propositional and a predicate sequent calculus with an ω–type schema of inference that naturally interpret the *until*–free propositional and predicate fragments of LTL respectively. Hence we are interested in proving properties like *cut–elimination, subformula property, consistency* in a syntactic way.

Although a thorough proof–theoretic investigation has not been accomplished yet, it is known that *finitary* presentations of LTL suffer intrinsic limitations. Among them: lack of completeness in presence of an infinite set of extralogical axioms and lack of a recursive axiomatization for the first order predicate version (see [GHR94]). It is also known that a way to overcome those limitations is to allow infinitary rules.

In [BM03] we accomplished a proof-theoretic investigation of a finitary calculus for the *until*–free fragment LTL^- of LTL. Being sufficiently expressive, LTL^- is extensively studied.

In [BM03] we introduced a natural deduction system based on an extension of modal formulas by means of a space component. Our work pointed out similarities between LTL^- and first order Peano Arithmetic PA from a proof–theoretic viewpoint. Same as for PA we proved the following for LTL^-: 1. a strong normalization theorem; 2. strong normalization yields a syntactic consistency proof; 3. introduction of an induction rule corresponding to the schema $A \rightarrow (\Box(A \rightarrow \circ A) \rightarrow \Box A)$ makes full subformula property fail.

We recall that Schütte [Sch77] proved cut elimination and full subformula property for PA by introducing infinitary ω–rules and by restricting to recursive proofs (see also [Gir83]). As previously mentioned, aim of this work is to show that similar results hold for sequent calculi with an ω–type schema of inference in which LTL^- can be naturally interpreted.

Let PA_ω be first order Peano arithmetic with ω–rule. We point out that operators $\circ$, $\Box$ and $\Diamond$ behave in LTL^- "in the same way" as *the successor function*, $\forall$ and $\exists$ in PA_ω respectively. As a consequence, the deduction rules governing the modal operators in LTL^- are very intuitive. Such a close correspondence between LTL^- and PA_ω is lost when introducing operator *until*: for this reason we leave to future works a possible extension to LTL of the results obtained in this paper.

The features of the present work are: 1. use of a natural extension (*2–sequents*) of ordinary sequents; 2. formulation of very intuitive inference rules; 3. presence of natural ω–type schemata of inference; 4. interpretability within our systems of the *until*–free fragments of propositional and predicate LTL (see, for instance, Corollary 3.6); 5. provability of cut–elimination in a syntactic way for all the proof systems (even in presence of infinitely many extralogical axioms).

In the sequel we briefly comment on the previous points.

Point 1: the use of "enriched" formulas is not a novelty: see, for instance, the *labelled formulas* for modal logics in [BMV98] or the *prefixed formulas* for LTL–like logics in [CM98]. Contrary to those approaches, we avoid any syntactic treatment of the accessibility relation. More precisely: no deduction rule explicitly deals with the order relation on natural numbers.

Point 2: 2–sequents allow rewriting of modal inference rules simply by adding a space component. As already remarked, this forces a formulation of the rules in strict analogy with those of PA. We remark that having intuitive inference rules is by no means a novelty of our systems: it is a property shared by most calculi that deal with "enriched formulas" and have a simple semantics.

Point 3 : since our ω–type schemata of inference are very similar to the ω–rules of arithmetic, throughout the paper we can use variants of standard techniques (see, for instance, [Gir83, Chapter 6]).

We remark that, due to the already mentioned limitations of finitary systems, the use of ω–rules (or more general infinitary rules) is quite common. See [Seg94] for a model-theoretic approach to infinitary systems. We just mention two more works.

In [Sza87], Szalas provides a recursive axiomatization in Hilbert style of first order predicate LTL in a language with a binary operator different form the usual *until*. By introducing an infinitary rule (somehow reminescent of an ω–rule), the author proves completeness for his system by using algebraic methods.

An orthogonal approach to ours is pursued in [Sza96]: the paper is devoted to axiomatizing classes of modal logics (*fixpoint logics*) whose modalities are defined by least fixpoints of equations on formulas. In this regard we recall that all temporal operators can be defined in terms of $\circ$ and fixpoints [EC81], hence Szalas' approach is quite general. For each fixpoint logic (defined by a set of modalities, a class of interpretations and a notion of satisfiability), the author provides an infinitary proof system that he proves to be sound and complete. Each infinitary rule can roughly be viewed as an ω–rule that describes the semantics of a modality by "approximations". Approximations are obtained from the definition by a fixpoint of the modality and they appear in the premises of the rule.

Szalas also shows how the infinitary proof system can be reduced to a finitary one that is sound and complete relative to a suitably chosen restriction of the class of interpretations. Both systems are cut–free in the sense that cut rule *is not included* in any of them with the motivation that implementation of the proof systems and proof–search are easier. Once more we stress the point that our main goal is proving cut elimination.

Point 4 : loosely speaking, the meaning of Corollary 3.6 is that our propositional system is a conservative extension of LTL^-.

Point 5 : we just draw attention to the importance of syntactic cut elimination in proof–theory.

We recall that LTL^- naturally embeds into systems for program verification like *Modal μ–Calculus* [AKM95], *Propositional Dynamic Logic* [Ha84] and *CTL** [Em90]. Being those systems more powerful, it is not clear how to make a proof–theoretic comparison with LTL. In particular we do not know of any presentation of *CTL** as a sequent calculus for which cut elimination holds (indeed the first axiomatization in Hilbert style of *CTL** appeared only recently [Rey01]).

To summarize, the main results in this paper are: a *syntactic* proof of cut elimination in the propositional case. Such a proof immediately yields full *subformula property* and *consistency*. Cut elimination and completeness hold for the propositional system even in presence of extralogical axioms. The propositional system is then extended to a predicate system for which syntactic cut elimination, subformula principle and completeness with respect to an arbitrary set of extralogical axioms still hold.

We finish this section with a brief outline of the paper.

In Section 2 we introduce 2–sequents and the propositional system $2S_\omega$. For $2S_\omega$ we prove cut elimination and, as immediate corollary, subformula property.

In Section 3 we show that $2S_\omega$ has at least the deductive power of LTL^- and we relate provability in the two systems.

In Section 4 we extend $2S_\omega$ by means of an arbitrary set of extralogical axioms. We show that cut elimination (for cuts whose cut formula does not occur in any of the axioms) and completeness still hold.

In Section 5 we introduce a predicate extension $2SP^\Phi_\omega$ of $2S_\omega$, where Φ is an arbitrary set of extralogical axioms. For $2SP^\Phi_\omega$ we prove cut elimination (formulated as in Section 4) and a completeness theorem.

2. 2–sequent calculus with ω-rule

2.1. 2–sequents

2–sequents were introduced in [Mas92, MM96, GMM98] for sake of providing a proof-theoretic treatment of basic modalities like modal operators of modal logic $\mathsf{S4}$, exponentials of Linear Logic and others. The idea is to enrich the structure of ordinary sequents by adding a space dimension. The definition of 2–sequent is based on the notion of *2–sequence*. The latter is an expression of the form $\begin{matrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_n\end{matrix}$, where each α_i is a finite (possibly empty) sequence of formulas. We say that formulas occurring in the sequence α_i are *at level* i.

Finally, a 2–sequent is an expression of the form $\Gamma \vdash \Delta$ where Γ and Δ are 2–sequences.

The vertical structure of a 2–sequent is needed in order to have good introduction rules for modal operators. Formulas are allowed to move from one level to another only as effect of application of a modal rule.

One can also represent a 2–sequent by means of an ordinary sequent made of *indexed formulas*. By an indexed formula we mean a formula decorated with its own level. If formula A is at level i, we keep track of that by writing A^i. Therefore a 2–sequent can be represented as an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are sequences of indexed formulas.

2.2. The 2-sequent calculus

We begin by describing the alphabet of a propositional modal language $\mathcal{L}_0$:

- *proposition symbols* $p_0, p_1, \ldots$ from a countably infinite set At;
- the *propositional connectives* $\vee, \wedge, \rightarrow, \neg$;
- the *modal operators* $\circ, \Box, \diamond$;
- the *auxiliary symbols* (and).

Definition 2.1. *The set of propositional* modal formulas *of $\mathcal{L}_0$ is the least set that contains the proposition symbols and is closed under applications of the propositional connectives and the modal operators.*

As described above, in a 2–sequent each modal formula is equipped with a *level*. Levels range on natural numbers. We use $i, j, k \dots$ (possibly indexed) for levels.

For sake of clarity, we state the following:

Definition 2.2. *1. An* indexed formula *(briefly: formula) is an expression of the form A^i, where A is a modal formula and i is a natural number.*

2. A 2-sequent *is an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are* finite *sequences of formulas.*

Warning: from now on we will use the word "sequent" meaning "2-sequent", when no ambiguity arises.

Given an index i and a non atomic modal formula, say $A \wedge B$, we write $A \wedge B^i$ for $(A \wedge B)^i$ (similarly in other cases). We feel free to use braces to increase readability.

The 2–sequent calculus with ω-rule is given by the following set of rules. We name each rule: this will be needed in the sequel.

2.3. Rules

Identity rules

$$A^i \vdash A^i \quad Ax \qquad\qquad \frac{\Gamma_1 \vdash A^i, \Delta_1 \quad \Gamma_2, A^i \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \; Cut$$

Formula A^i occurring in Cut rule is called the *cut formula.*

Structural rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, A^i \vdash \Delta} \; W\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A^i, \Delta} \; \vdash W$$

$$\frac{\Gamma, A^i, A^i \vdash \Delta}{\Gamma, A^i \vdash \Delta} \; C\vdash \qquad \frac{\Gamma \vdash A^i, A^i, \Delta}{\Gamma \vdash A^i, \Delta} \; \vdash C$$

$$\frac{\Gamma_1, A^i, B^j, \Gamma_2 \vdash \Delta}{\Gamma_1, B^j, A^i, \Gamma_2 \vdash \Delta} \; Exc\vdash \qquad \frac{\Gamma \vdash \Delta_1, A^i, B^j, \Delta_2}{\Gamma \vdash \Delta_1, B^j, A^i, \Delta_2} \; \vdash Exc$$

Propositional rules

$$\frac{\Gamma \vdash A^i, \Delta}{\Gamma, \neg A^i \vdash \Delta} \; \neg\vdash \qquad \frac{\Gamma, A^i \vdash \Delta}{\Gamma \vdash \neg A^i, \Delta} \; \vdash\neg$$

$$\frac{\Gamma, A^i \vdash \Delta}{\Gamma, A \wedge B^i \vdash \Delta} \; \wedge_1\vdash \qquad \frac{\Gamma, B^i \vdash \Delta}{\Gamma, A \wedge B^i \vdash \Delta} \; \wedge_2\vdash$$

$$\frac{\Gamma_1 \vdash A^i, \Delta_1 \quad \Gamma_2 \vdash B^i, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B^i, \Delta_1, \Delta_2} \; \vdash\wedge$$

$$\frac{\Gamma_1, A^i \vdash \Delta_1 \quad \Gamma_2, B^i \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B^i \vdash \Delta_1, \Delta_2} \vee\vdash$$

$$\frac{\Gamma \vdash A^i, \Delta}{\Gamma \vdash A \vee B^i, \Delta} \vdash\vee_1 \qquad \frac{\Gamma \vdash B^i, \Delta}{\Gamma \vdash A \vee B^i, \Delta} \vdash\vee_2$$

$$\frac{\Gamma_1, B^i \vdash \Delta_1 \quad \Gamma_2 \vdash A^i, \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B^i \vdash \Delta_1, \Delta_2} \rightarrow\vdash \qquad \frac{\Gamma, A^i \vdash B^i, \Delta}{\Gamma \vdash A \rightarrow B^i, \Delta} \vdash\rightarrow$$

Modal rules

$$\frac{\Gamma, A^{i+1} \vdash \Delta}{\Gamma, \circ A^i \vdash \Delta} \circ\vdash \qquad \frac{\Gamma \vdash A^{i+1}, \Delta}{\Gamma \vdash \circ A^i, \Delta} \vdash\circ$$

$$\frac{\Gamma, A^{i+k} \vdash \Delta}{\Gamma, \Box A^i \vdash \Delta} \Box\vdash \qquad \frac{\{\Gamma \vdash A^{i+j}, \Delta\}_{j\in\omega}}{\Gamma \vdash \Box A^i, \Delta} \vdash\Box$$

$$\frac{\{\Gamma, A^{i+j} \vdash \Delta\}_{j\in\omega}}{\Gamma, \diamond A^i \vdash \Delta} \diamond\vdash \qquad \frac{\Gamma \vdash A^{i+k}, \Delta}{\Gamma \vdash \diamond A^i, \Delta} \vdash\diamond$$

We introduce now the notion of ω–proof. We follow the presentation of ω–logic proposed in [Gir83].

Let S^* be the set of finite sequences of natural numbers. We denote the empty sequence by $< >$ and an arbitrary sequence by $< n_0, \dots, n_k >$. We use $*$ for concatenation of sequences.

We define a partial ordering $\leq$ on S^* as follows: $t \leq < >$ for all $t \in S^*$ and $< n_0, \dots, n_k > \leq < m_0, \dots, m_l >$ if and only if $l \leq k$ and $n_i = m_i$ for all $0 \leq i \leq l$. We denote by $<$ the associated strict order.

Definition 2.3. *1. A* tree T *is a subset of* S^* *satisfying the property that whenever* $t \in T$ *and* $t \leq s$ *then* $s \in T$. *Elements of* T *are called* nodes. *A* leaf *is a node with no successors.*

2. A well-founded *tree (briefly: wf-tree) is a tree with no infinite strictly decreasing sequence* $\dots < t_n < \dots < t_1 < t_0$.

Given a tree T and $s \in T$, we let T_s the tree defined by:

$$s' \in T_s \Leftrightarrow s * s' \in T.$$

Notice that $T_{< >} = T$.

Definition 2.4. *The* height $\mathrm{ht}(T)$ *of a tree* T *is the ordinal recursively defined as follows:*

$$\mathrm{ht}(T) = \begin{cases} 0 & \text{if } T = \langle\,\rangle; \\ \sup\{\mathrm{ht}(T_{<i>}) + 1 :< i > \in T\} & \text{otherwise.} \end{cases}$$

We always assume that the set of immediate successors of an arbitrary node t in a tree T is of the form $\{t* < i >\}_{i\in I}$, for some initial segment I of ω (possibly the whole ω).

In the graphical representation of a tree, we place $t* < i >$ to the left of $t* < j >$ if $i < j$.

Let $\mathcal{R}$ be the set of rule names and let $\mathcal{S}$ be the set of 2–sequents. Let also $g : \mathcal{R} \cup \mathcal{S} \to \mathbf{N}$ be an effectively computable coding bijection with effectively computable inverse.

Definition 2.5. *1. An ω–*proof *is an ordered pair (T, ℓ), where T is a wf-tree and $\ell : T \to g(\mathcal{R}) \times g(\mathcal{S})$ is a so-called* labelling *function with the following properties (we denote by π_2 the projection onto the second factor of a cartesian product):*

(a) if $\ell(t) =< g(Ax), n >$ then n is $g(A^i \vdash A^i)$ for some formula A^i and t is a leaf of T;

(b) if $\ell(t) =< m, n >$, with $m = g(r)$ for some $r \neq Ax$ in $\mathcal{R}$ and $n = g(\Gamma \vdash \Delta)$, then t is not a leaf, and if $\{t* < i >\}_{i \in I}$ is the set of immediate successors of t then $\dfrac{\{g^{-1}(\pi_2\ell(t* < i >))\}_{i\in I}}{\Gamma \vdash \Delta}$ is an instance of rule r.

2. The conclusion *of the ω-proof (T, ℓ) is the sequent $g^{-1}(\pi_2\ell(\langle\,\rangle))$.*

The height $\mathrm{ht}(\Pi)$ of an ω-proof $\Pi = (T, \ell)$ is the height of tree T.

Definition 2.6. *We say that a sequent $\Gamma \vdash \Delta$ is provable in $2\mathrm{S}_\omega$ if there exists an ω-proof whose conclusion is $\Gamma \vdash \Delta$.*

By coding the nodes of a tree by means of natural numbers in an effective manner, one can easily make sense of the notion of *recursive* tree and give the following:

Definition 2.7. *A* (primitive) recursive *ω-proof is an ω–proof (T, ℓ) such that T is (primitive) recursive and ℓ is an effectively computable function.*

In practice it is convenient to regard an ω-proof as a wf-tree whose nodes are each labelled by a sequent and by the rule name that led to that sequent. This gives the possibility of less formal, but simpler, definitions that are in accordance with the usual representation of sequent proofs.

For instance we will write Π^{+k} to indicate the labelled wf-tree obtained from the ω-proof Π by replacing each formula A^i occurring in Π with A^{i+k}. It is straightforward to check that Π^{+k} is still an ω-proof.

An ω-proof free from rules with infinitely many premises will be called simply a *proof*.

In order to simplify the graphical representation of ω–proofs, we will use a double deduction line to indicate application of a rule preceded or followed by a sequence of structural rules. So we will write

$$\frac{\Gamma \vdash \Delta}{\overline{\Sigma \vdash \Theta}}\, r$$

when the sequent $\Sigma \vdash \Theta$ has been obtained from $\Gamma \vdash \Delta$ by means of an application of rule r and of a finite number of structural rules.

Definition 2.8 (subformula). *The set $Sub(A^i)$ of* subformulas *of a formula A^i is recursively defined as follows:*

$Sub(p^k) = \{p^k\}$ *if p is a proposition symbol;*
$Sub(\neg A^k) = \{\neg A^k\} \cup Sub(A^k)$*;*
$Sub(A\#B^k) = \{A\#B^k\} \cup Sub(A^k) \cup Sub(B^k)$, *when* $\# \in \{\rightarrow, \vee, \wedge\}$;
$Sub(\circ A^k) = \{\circ A^k\} \cup Sub(A^{k+1})$;
$Sub(\#A^k) = \{\#A^k\} \cup \{Sub(A^{k+i}) : i \in \omega\}$, *when* $\# \in \{\Box, \Diamond\}$.

Definition 2.9. *1. The* degree $\deg(A)$ *of a modal formula A is recursively defined as:*
(a) $\deg(p) = 0$ *if p is a proposition symbol;*
(b) $\deg(\neg A) = \deg(\circ A) = \deg(\Box A) = \deg(\Diamond A) = \deg(A) + 1$*;*
(c) $\deg(A \wedge B) = \deg(A \vee B) = \deg(A \rightarrow B) = \max\{\deg(A), \deg(B)\} + 1$.
2. The degree $\deg(A^i)$ *of formula* A^i *is just* $\deg(A)$.

Definition 2.10. *The* degree $\delta[\Pi]$ *of an* ω*–proof* Π *is the ordinal defined as follows:*

$$\delta[\Pi] = \begin{cases} 0 & \text{if } \Pi \text{ is cut-free;} \\ \sup\{\deg(A^i) + 1 : A^i \text{ is a cut formula in } \Pi\} & \text{otherwise.} \end{cases}$$

We are now ready to prove a cut-elimination theorem for $2S_\omega$. We employ ideas and techniques introduced in [Gir83].

Let Γ be a sequence of formulas. We denote by $\Gamma - A^k$ the sequence obtained by removing all occurrences of A^k in Γ. When writing $\Gamma, \Gamma' - A^k$ we actually mean $\Gamma, (\Gamma' - A^k)$.

In the sequel ordered pairs of ordinals are intended to be lexicographically ordered. Hence one can make proofs by induction on pairs of ordinals.

The next result is crucial for proving cut-elimination.

Lemma 2.11. *Let* $n \in \mathbf{N}$. *Let* A^k *be a formula of degree n and let* $< \Pi, \Pi' >$ *be a pair of recursive* ω*–proofs of sequents* $\Gamma \vdash \Delta$ *and* $\Gamma' \vdash \Delta'$ *respectively, satisfying the property* $\delta[\Pi], \delta[\Pi'] \le n$. *Then one can obtain in an effective way from* $< \Pi, \Pi' >$ *a recursive* ω*–proof* $\mathsf{Mix}(\Pi, \Pi')$ *of sequent* $\Gamma, \Gamma' - A^k \vdash \Delta - A^k, \Delta'$ *satisfying the property* $\delta[\mathsf{Mix}(\Pi, \Pi')] \le n$.

Proof. By induction on the pair $< \mathrm{ht}(\Pi), \mathrm{ht}(\Pi') >$.

Let Π and Π' be

$$\frac{\left\{ \begin{array}{c} \Pi_i \\ \Gamma_i \vdash \Delta_i \end{array} \right\}_{i \in I}}{\Gamma \vdash \Delta}\, r \qquad \text{and} \qquad \frac{\left\{ \begin{array}{c} \Pi'_j \\ \Gamma'_j \vdash \Delta'_j \end{array} \right\}_{j \in I'}}{\Gamma' \vdash \Delta'}\, r'$$

respectively, where I and I' are $\emptyset$ (in case of an axiom), $\{1\}$, $\{1, 2\}$ or ω.

We proceed by cases.

1. r is Ax.
   If $\Gamma \vdash \Delta$ is $A^k \vdash A^k$, then one gets $\mathsf{Mix}(\Pi, \Pi')$ from Π' by means of a suitable sequence of structural rules.
   If $\Gamma \vdash \Delta$ is $B^i \vdash B^i$, for $B \neq A$ or $i \neq k$, then one gets $\mathsf{Mix}(\Pi, \Pi')$ by means of a suitable sequence of structural rules.

2. r' is Ax.
   This case is symmetric to case 1.
3. r is a structural rule.
   Apply induction hypothesis to the pair $< \Pi_1, \Pi' >$ and then apply a suitable sequence of structural rules to get the conclusion.
4. r' is a structural rule
   This case is symmetric to 3.
5. r is a cut or a logical rule not introducing A^k to the right.
   Apply the induction hypothesis to each pair $< \Pi_i, \Pi' >$, so obtaining the ω–proof $\mathsf{Mix}(\Pi_i, \Pi')$, for $i \in I$. The ω–proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\dfrac{\left\{ \begin{array}{c} \mathsf{Mix}(\Pi_i, \Pi') \\ \Gamma_i, \Gamma' - A^k \vdash \Delta_i - A^k, \Delta' \end{array} \right\}_{i \in I}}{\Gamma, \Gamma' - A^k \vdash \Delta - A^k, \Delta'}\, r$$

6. r' is a cut or a logical rule not introducing A^k to the left.
   This case is symmetric to 5.
7. r is a logical rule introducing A^k to the right and r' is a logical rule introducing A^k to the left.
   (a) r is a propositional rule.
   This subcase is treated exactly as in the first order case (see, for instance, [Gir83] or [Tak75]). Here we show only the case when A is of the form $B \to C$. Let Π and Π' be

$$\dfrac{\begin{array}{c} \Pi_1 \\ \Gamma, B^k \vdash C^k, \Delta_1 \end{array}}{\Gamma \vdash B \to C^k, \Delta_1} \quad \text{and} \quad \dfrac{\begin{array}{c} \Pi'_1 \\ \Gamma'_1, C^k \vdash \Delta'_1 \end{array} \quad \begin{array}{c} \Pi'_2 \\ \Gamma'_2 \vdash B^k, \Delta'_2 \end{array}}{\Gamma'_1, \Gamma'_2, B \to C^k \vdash \Delta'_1, \Delta'_2}$$

   respectively. Apply the induction hypothesis to the pairs of ω–proofs $< \Pi, \Pi'_2 >$, $< \Pi_1, \Pi' >$ and $< \Pi, \Pi'_1 >$, obtaining $\mathsf{Mix}(\Pi, \Pi'_2)$, $\mathsf{Mix}(\Pi_1, \Pi')$ and $\mathsf{Mix}(\Pi, \Pi'_1)$ respectively. The ω–proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\dfrac{\dfrac{\dfrac{\mathsf{Mix}(\Pi, \Pi'_2)}{\Gamma, \Gamma'_2 - A^k \vdash B^k, \Delta_1 - A^k, \Delta'_2} \quad \dfrac{\mathsf{Mix}(\Pi_1, \Pi')}{\Gamma, \Gamma'_1 - A^k, \Gamma'_2 - A^k, B^k \vdash C^k, \Delta_1 - A^k, \Delta'_1, \Delta'_2}}{\Gamma, \Gamma'_1 - A^k, \Gamma'_2 - A^k \vdash C^k, \Delta_1 - A^k, \Delta'_1, \Delta'_2}\, Cut \quad \begin{array}{c} \mathsf{Mix}(\Pi, \Pi'_1) \\ \Gamma, \Gamma'_1 - A^k, C^k \vdash \Delta_1 - A^k, \Delta'_1 \end{array}}{\Gamma, \Gamma'_1 - A^k, \Gamma'_2 - A^k \vdash \Delta_1 - A^k, \Delta'_1, \Delta'_2}\, Cut$$

   (b) A is $\circ B$.
   Let Π and Π' be

$$\dfrac{\begin{array}{c} \Pi_1 \\ \Gamma \vdash B^{k+1}, \Delta_1 \end{array}}{\Gamma \vdash A^k, \Delta_1} \quad \text{and} \quad \dfrac{\begin{array}{c} \Pi'_1 \\ \Gamma'_1, B^{k+1} \vdash \Delta' \end{array}}{\Gamma'_1, A^k \vdash \Delta'}$$

respectively. Apply the induction hypothesis to the pairs of ω–proofs $< \Pi_1, \Pi' >$ and $< \Pi, \Pi'_1 >$, obtaining $\mathsf{Mix}(\Pi_1, \Pi')$ and $\mathsf{Mix}(\Pi, \Pi'_1)$ respectively. The ω–proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\dfrac{\dfrac{\dfrac{\mathsf{Mix}(\Pi_1, \Pi')}{\Gamma, \Gamma'_1 - A^k \vdash B^{k+1}, \Delta_1 - A^k, \Delta'} \quad \dfrac{\mathsf{Mix}(\Pi, \Pi'_1)}{\Gamma, \Gamma'_1 - A^k, B^{k+1} \vdash \Delta_1 - A^k, \Delta'}}{\Gamma, \Gamma'_1 - A^k, \Gamma, \Gamma'_1 - A^k \vdash \Delta_1 - A^k, \Delta', \Delta_1 - A^k, \Delta'}\,Cut}{\Gamma, \Gamma'_1 - A^k \vdash \Delta_1 - A^k, \Delta'}$$

(c) A is $\Box B$.
Let Π and Π' be

$$\dfrac{\left\{ \begin{array}{c} \Pi_j \\ \Gamma \vdash B^{k+j}, \Delta_1 \end{array} \right\}_{j \in \omega}}{\Gamma \vdash A^k, \Delta_1} \qquad \text{and} \qquad \dfrac{\begin{array}{c} \Pi'_1 \\ \Gamma'_1, B^{k+n} \vdash \Delta' \end{array}}{\Gamma'_1, A^k \vdash \Delta'}$$

respectively. Apply the induction hypothesis to the pairs of ω–proofs $< \Pi_n, \Pi' >$ and $< \Pi, \Pi'_1 >$, obtaining $\mathsf{Mix}(\Pi_n, \Pi')$ and $\mathsf{Mix}(\Pi, \Pi'_1)$ respectively. The ω–proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\dfrac{\dfrac{\dfrac{\mathsf{Mix}(\Pi_n, \Pi')}{\Gamma, \Gamma'_1 - A^k \vdash B^{k+n}, \Delta_1 - A^k, \Delta'} \quad \dfrac{\mathsf{Mix}(\Pi, \Pi'_1)}{\Gamma, \Gamma'_1 - A^k, B^{k+n} \vdash \Delta_1 - A^k, \Delta'}}{\Gamma, \Gamma'_1 - A^k, \Gamma, \Gamma'_1 - A^k \vdash \Delta_1 - A^k, \Delta', \Delta_1 - A^k, \Delta'}\,Cut}{\Gamma, \Gamma'_1 - A^k \vdash \Delta_1 - A^k, \Delta'}$$

(d) A is $\diamond B$. This subcase is symmetric to 7c.

Notice that, in all cases, being the additional cuts performed on subformulas of A^k, from the assumptions $\deg(A^k) = n$ and $\delta[\Pi], \delta[\Pi'] \leq n$ we immediately get $\delta[\mathsf{Mix}(\Pi, \Pi')] \leq n$. □

Theorem 2.12 (Cut elimination). *Let* Π *be a recursive* ω*–proof of* $\Gamma \vdash \Delta$. *Then there exists a recursive cut-free* ω*–proof* Π^* *of* $\Gamma \vdash \Delta$.

Proof. By induction on the pair $< \delta[\Pi], \mathrm{ht}(\Pi) >$. Suppose Π is not cut-free and let r be the last rule applied in Π. We distinguish two cases:

1. r is not a cut.
   Let Π be

$$\dfrac{\left\{ \begin{array}{c} \Pi_i \\ \Gamma_i \vdash \Delta_i \end{array} \right\}_{i \in I}}{\Gamma \vdash \Delta}\, r,$$

   where I is one of $\{1\}$, $\{1, 2\}$, ω. Apply the induction hypothesis to each Π_i, obtaining recursive cut-free ω-proofs Π^*_i, for $i \in I$. A cut-free ω–proof Π^* of $\Gamma \vdash \Delta$ is then

$$\dfrac{\left\{ \begin{array}{c} \Pi^*_i \\ \Gamma_i \vdash \Delta_i \end{array} \right\}_{i \in I}}{\Gamma \vdash \Delta}\, r$$

2. r is a cut
   Let Π be

$$\frac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ \Gamma_1 \vdash A^k, \Delta_1 & \Gamma_2, A^k \vdash \Delta_2\end{array}}{\Gamma \vdash \Delta}\ Cut$$

   Apply the induction hypothesis to Π_1 and Π_2 to obtain recursive cut-free ω-proofs Π_1^* and Π_2^* of $\Gamma_1 \vdash A^k, \Delta_1$ and $\Gamma_2, A^k \vdash \Delta_2$ respectively.
   Applying Lemma 2.11 to the pair $< \Pi_1^*, \Pi_2^* >$, one gets a recursive ω–proof Π_0 of sequent $\Gamma_1, \Gamma_2 - A^k \vdash \Delta_1 - A^k, \Delta_2$ such that $\delta[\Pi_0] \leq \deg(A^k) < \delta[\Pi]$. Finally one gets a recursive cut-free ω-proof of $\Gamma_1, \Gamma_2 - A^k \vdash \Delta_1 - A^k, \Delta_2$ from Π_0 by induction hypothesis and, from it, a recursive cut-free ω-proof of $\Gamma \vdash \Delta$ by application of a suitable sequence of structural rules. □

Immediate consequences of cut elimination are:

Corollary 2.13. *Each formula occurring in a cut-free ω-proof Π is a subformula of some formula occurring in the conclusion of Π.*

Corollary 2.14. $2S_\omega$ *is consistent, namely there is no ω-proof in* $2S_\omega$ *of the empty sequent* $\vdash$.

3. On the deductive power of $2S_\omega$

In this section we compare the deductive power of $2S_\omega$ with that of LTL^-.

Firstly, we remark that the introduction of ω–rule in first order arithmetic allows a purely logical treatment of the induction schema (see [Gir83]). Similarly, we show that the use of ω–rule in temporal systems makes possible a purely logical proof of the temporal induction schema

$$A \to (\Box(A \to \circ A) \to \Box A).$$

Let us begin with a Hilbert style presentation LTL^- (see [GPSS80]):

Axioms

A0 All modal instances of propositional tautologies.
A1 $\circ(A \to B) \to (\circ A \to \circ B)$
A2 $\neg \circ A \to \circ \neg A$
A3 $\Box(A \to B) \to (\Box A \to \Box B)$
A4 $\Box A \to A$
A5 $\Box A \to \circ \Box A$
A6 $A \to (\Box(A \to \circ A) \to \Box A)$

Rules

$$\frac{\vdash A \quad \vdash A \to B}{\vdash B}\ \mathrm{MP}; \qquad \frac{\vdash A}{\vdash \circ A}\ \circ\mathrm{G}; \qquad \frac{\vdash A}{\vdash \Box A}\ \Box\mathrm{G}.$$

We write $\vdash_{LTL^-}$ for the provability relation in LTL^-.

In order to investigate the deductive power of $2S_\omega$ we provide recursive ω–proofs of a natural 2–sequent translation of some LTL^- axioms. We just present two interesting cases.

A3 $\Box(A \to B) \to (\Box A \to \Box B)$

$$\dfrac{\left\{\dfrac{\dfrac{\dfrac{B^k \vdash B^k \qquad A^k \vdash A^k}{A^k, A \to B^k \vdash B^k}\ \to\vdash}{A^k, \Box(A \to B)^0 \vdash B^k}\ \Box\vdash}{\Box A^0, \Box(A \to B)^0 \vdash B^k}\ \Box\vdash\right\}_{k\in\omega}}{\dfrac{\dfrac{\Box A^0, \Box(A \to B)^0 \vdash \Box B^0}{\Box(A \to B)^0 \vdash \Box A \to \Box B^0}\ \vdash\to}{\vdash \Box(A \to B) \to (\Box A \to \Box B)^0}\ \vdash\to}\ \vdash\Box$$

A6 $A \to (\Box(A \to \circ A) \to \Box A)$

Let us inductively prove that, for all $k \in \mathbf{N}$, there exists a recursive ω–proof Π_k of sequent $A^0, \Box(A \to \circ A)^0 \vdash A^k$.

basis:

$$\dfrac{A^0 \vdash A^0}{A^0, \Box(A \to \circ A)^0 \vdash A^0}\ W\vdash$$

induction:

$$\dfrac{\dfrac{\dfrac{\dfrac{A^{k+1} \vdash A^{k+1}}{\circ A^k \vdash A^{k+1}}\ \circ\vdash \qquad \begin{array}{c}\Pi_k\\ A^0, \Box(A \to \circ A)^0 \vdash A^k\end{array}}{A^0, \Box(A \to \circ A)^0, A \to \circ A^k \vdash A^{k+1}}\ \to\vdash}{A^0, \Box(A \to \circ A)^0, \Box(A \to \circ A)^0 \vdash A^{k+1}}\ \Box\vdash}{A^0, \Box(A \to \circ A)^0 \vdash A^{k+1}}\ C\vdash$$

The required ω–proof is therefore:

$$\dfrac{\left\{\begin{array}{c}\Pi_k\\ A^0, \Box(A \to \circ A)^0 \vdash A^k\end{array}\right\}_{k\in\omega}}{\dfrac{\dfrac{A^0, \Box(A \to \circ A)^0 \vdash \Box A^0}{A^0 \vdash \Box(A \to \circ A) \to \Box A^0}\ \vdash\to}{\vdash (A \to (\Box(A \to \circ A) \to \Box A))^0}\ \vdash\to}\ \vdash\Box$$

Proposition 3.1. *If* $\vdash_{LTL^-} A$ *then there exists a recursive* ω*–proof of* $\vdash A^0$.

Proof. By induction on a proof of A in LTL^-.

We have partly shown already that there exist recursive ω–proofs of the 2–sequent translations of all LTL^- axioms (see above).

Furthermore the following piece of ω-proof shows that $2S_\omega$ is closed under modus ponens:

$$\dfrac{\vdash A \to B^0 \qquad \dfrac{B^0 \vdash B^0 \qquad \vdash A^0}{A \to B^0 \vdash B^0}\ \to\vdash}{\vdash B^0}\ Cut$$

Finally, we prove that $2S_\omega$ is closed under application of rules $\circ G$ and $\Box G$. In Section 2 we have already noticed that if A is any modal formula and Π is a recursive ω–proof of $\vdash A^0$ then Π^{+k} is a recursive ω–proof of $\vdash A^k$, for all natural numbers k. Therefore if there exists a recursive ω–proof of the sequent $\vdash A^0$, then there is also one of $\vdash \circ A^0$ and of $\vdash \Box A^0$. □

Notice that a syntactic consistency proof of LTL^- follows as straightforward corollary of the previous proposition and Corollary 2.14.

3.1. Semantics

Let the language $\mathcal{L}_0$ defined in Section 2 also be the language of LTL^-. A *Kripke model* (briefly: a *model*) for LTL^- is a pair $\mathbf{N}_a =< \mathbf{N}, a : \mathbf{N} \to 2^{At} >$.

The satisfiability relation of a modal formula A at time m in the model $\mathbf{N}_a$ (notation: $\mathbf{N}_a \models_m A$) is inductively defined as usual. For sake of completeness we recall the following cases:

$\mathbf{N}_a \models_m B \to C \;\Leftrightarrow\; (\mathbf{N}_a \models_m B \;\Rightarrow\; \mathbf{N}_a \models_m C)$;
$\mathbf{N}_a \models_m \Box B \;\Leftrightarrow\; \mathbf{N}_a \models_n B$ for all $n \geq m$;
$\mathbf{N}_a \models_m \circ B \;\Leftrightarrow\; \mathbf{N}_a \models_{m+1} B$.

A well–known result for LTL^- is the following (see [Em90]):

Theorem 3.2. *For every modal formula A*

$$\vdash_{LTL^-} A \;\Leftrightarrow\; \mathbf{N}_a \models_0 A \textit{ for all models } \mathbf{N}_a.$$

The semantics of $2S_\omega$ is defined in terms of the Kripke semantics. The models of $2S_\omega$ are those of LTL^-.

Definition 3.3. *A model $\mathbf{N}_a$* satisfies *formula A^i (notation: $\mathbf{N}_a \models A^i$) if and only if $\mathbf{N}_a \models_i A$.*

Definition 3.4. *Let Γ and Δ be sets of formulas. We write $\Gamma \models \Delta$ if, whenever model $\mathbf{N}_a$ is such that $\mathbf{N}_a \models A^i$ for all $A^i \in \Gamma$, there exists $B^k \in \Delta$ such that $\mathbf{N}_a \models B^k$.*

Theorem 3.5 (Soundness). *If the sequent $\Gamma \vdash \Delta$ is provable in $2S_\omega$ then $\Gamma \models \Delta$.*

Proof. By straightforward induction on the height of an ω–proof of $\Gamma \vdash \Delta$. □

Putting together the previous results we can relate provability in LTL^- and in $2S_\omega$:

Corollary 3.6. $\vdash_{LTL^-} A$ *if and only if there exists a recursive ω–proof of the sequent $\vdash A^0$ in the system $2S_\omega$.*

Notice that a full completeness theorem for $2S_\omega$ follows from Theorem 4.5 below.

4. Provability from axioms

It is well–known that, no matter what kind of presentation we choose, a completeness theorem for LTL^- in the vein of Theorem 3.2 fails for provability from infinite sets of axioms. Here is an easy counterexample to completeness: let

$$\Phi = \{q_i \to (\circ q_{i+1}) \wedge q : i \in \omega\},$$

where q_i and q are proposition symbols such that $q \neq q_i$ and $q_i \neq q_j$ for all $i, j \in \omega$, $i \neq j$.

A model of Φ is a model $\mathbf{N}_a$ as in Section 3.1 such that $\mathbf{N}_a \models_n A$ for all $n \in \mathbf{N}$ and all $A \in \Phi$. It is easy to see that each model $\mathbf{N}_a$ of Φ has the property that $\mathbf{N}_a \models_0 q_0 \to \Box q$.

For sake of contradiction, suppose there is a proof in LTL^- plus Φ of $q_0 \to \Box q$. Then the latter is provable in LTL^- plus $\Phi_n = \{q_i \to (\circ q_{i+1}) \wedge q : i \leq n\}$, for some $n \in \omega$. Consider the model $\mathbf{N}_a$ where

$$a(m) = \begin{cases} \{q, q_m\} & \text{if } m \leq n+1; \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly $\mathbf{N}_a$ is a model of Φ_n. On the other hand $\mathbf{N}_a \not\models_0 q_0 \to \Box q$, contradicting to soundness.

In the sequel we want to show that a completeness theorem holds for $2\mathrm{S}_\omega$ even in presence of axioms.

Definition 4.1. *Let Φ be a set of modal formulas. We let*

$$\Phi_{ax} = \{\vdash A^i : A \in \Phi \textit{ and } i \in \omega\}.$$

1. *The system* $2\mathrm{S}_\omega^\Phi$ with proper axioms *in Φ_{ax} is obtained by adding all the sequents in Φ_{ax} to the system $2\mathrm{S}_\omega$.*
   We label each proper axiom by Pax.
2. *An ω–proof in $2\mathrm{S}_\omega^\Phi$ is an ω-proof (as in Definition 2.5, with the function g extended to Pax) satisfying the additional clause*
   (a1) if $\ell(t) = \langle g(Pax), n \rangle$ then $n = g(\vdash A^i)$ for some $A^i \in \Phi_{ax}$ and t is a leaf of T.

Theorem 2.12 extends to the system $2\mathrm{S}_\omega^\Phi$. In order to do that we first define the *degree* $\delta^\Phi[\Pi]$ of an ω-proof Π in $2\mathrm{S}_\omega^\Phi$:

$$\delta^\Phi[\Pi] = \begin{cases} 0 & \text{if } \Pi \text{ is cut-free or each cut formula in } \Pi \text{ occurs in } \Phi_{ax}; \\ \sup\{\deg(A^i) + 1 : A^i \text{ is a cut formula in } \Pi \text{ and } A^i \text{ does not occur in } \Phi_{ax}\} & \text{otherwise.} \end{cases}$$

We reformulate Lemma 2.11 as follows:

Lemma 4.2. *Let $n \in \mathbf{N}$ and let A^k be a formula of degree n not occurring in Φ_{ax}. Let $< \Pi, \Pi' >$ be a pair of recursive ω–proofs of sequents $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$ respectively, satisfying the property $\delta^{\Phi}[\Pi], \delta^{\Phi}[\Pi'] \leq n$. Then one can obtain in an effective way from $< \Pi, \Pi' >$ a recursive ω-proof $\mathsf{Mix}(\Pi, \Pi')$ of sequent $\Gamma, \Gamma' - A^k \vdash \Delta - A^k, \Delta'$ satisfying the property $\delta^{\Phi}[\mathsf{Mix}(\Pi, \Pi')] \leq n$.*

The proof of Lemma 4.2 goes in the same way as the proof of Lemma 2.11.
Theorem 2.12 can be restated as follows:

Theorem 4.3 (Cut elimination). *Let Φ be a recursive set of modal formulas. Suppose there exists a (recursive) ω-proof in $2S_{\omega}^{\Phi}$ of sequent $\Gamma \vdash \Delta$. Then there exists a (recursive) ω-proof in $2S_{\omega}^{\Phi}$ of $\Gamma \vdash \Delta$ satisfying the property that each cut formula occurs in some sequent in Φ_{ax}.*

Proof. A straightforward extension of the proof of Theorem 2.12: replace δ with δ^{Φ} and restate case 1. and case 2. as follows:

1′. r is not a cut or r is a cut with cut formula occurring in Φ_{ax};
2′. r is a cut with cut formula not occurring in Φ_{ax}. □

Concerning the semantics, when dealing with proper axioms we restrict the class of models to those satisfying the axioms:

Definition 4.4. *Let Φ be a recursive set of modal formulas. A Φ–model is a model $\mathbf{N}_a$ such that $\mathbf{N}_a \models A^i$ for all $A \in \Phi$ and all $i \in \mathbf{N}$.*

Let Γ and Δ be sets of formulas and let Φ be a recursive set of modal formulas. We write $\Gamma \models_{\Phi} \Delta$ for $\Gamma \cup \Phi_{ax} \models \Delta$. Therefore $\models_{\Phi}$ is the restriction of $\models$ to the class of Φ–models.

Theorem 4.5 (Soundness and Completeness). *Let Φ be a (primitive) recursive set of modal formulas and let $\Gamma \vdash \Delta$ be a nonempty sequent. Then $\Gamma \models_{\Phi} \Delta$ if and only if there exists a (primitive) recursive ω–proof in $2S_{\omega}^{\Phi}$ of $\Gamma \vdash \Delta$.*

Proof. $\Leftarrow$ By straightforward induction on the height of a (primitive) recursive ω–proof in $2S_{\omega}^{\Phi}$ of $\Gamma \vdash \Delta$.

$\Rightarrow$ Following [Gir83], we say that a *pre–ω–proof* is an ω–proof whose associated tree may not be well-founded.
We prove that it is possible to construct a (primitive) recursive pre–ω–proof of $\Gamma \vdash \Delta$ in $2S_{\omega}^{\Phi}$ whose associated tree is well–founded if and only if $\Gamma \models_{\Phi} \Delta$.
We shall construct the pre–ω–proof step-by-step, starting from the conclusion.
We closely follow the construction outlined in [Gir83] and we simply point out the major differences.
We fix a recursive enumeration $\mathcal{E}$ of formulas.
At step n the portion of proof will look like

$$\begin{array}{c}\{\Gamma_i \vdash \Delta_i\}_{i \in I_n} \\ \vdots \\ \Gamma \vdash \Delta\end{array}$$

where $\{\Gamma_i \vdash \Delta_i\}_{i \in I_n}$ is a (possibly infinite) set of hypothesis. We extend the proof upwards by adding on top of each $\Gamma_i \vdash \Delta_i$ a portion of proof. In order to do that we need the following data:

- the sequent $\Gamma_i \vdash \Delta_i$;
- the step n;
- a *marked* formula $E_i^{k_i}$ in $\Gamma_i \vdash \Delta_i$.

We must consider the following cases:

(a) $\Gamma_i \vdash \Delta_i$ is a weakening of a sequent $\Gamma_i' \vdash \Delta_i'$ of one of the following forms: $A^k \vdash A^k$ (logical axiom) or $\vdash B^l$, for some $B^l \in \Phi_{ax}$ (proper axiom). Then we add on top of $\Gamma_i \vdash \Delta_i$ the following portion of proof:

$$\begin{array}{c}\Gamma_i' \vdash \Delta_i' \\ \vdots \\ \Gamma_i \vdash \Delta_i\end{array}$$

where the vertical dots represent a sequence of applications of weakening and exchange rules.

Notice that if, at some step n, each sequent $\Gamma_i' \vdash \Delta_i'$ with $i \in I_n$ is either a logical axiom or a proper axiom, we have obtained an ω–proof of $\Gamma \vdash \Delta$.

(b) Case (a) does not hold and the marked formula $E_i^{k_i}$ occurs in Δ_i.

Then we have the following cases:

i. $E_i^{k_i}$ is atomic. Let A^l be the first formula in the enumeration $\mathcal{E}$ not occurring in $\Gamma_i \vdash \Delta_i$. We extend the proof as follows:

$$\dfrac{\Gamma_i \vdash A^l, \Delta_i \qquad \Gamma_i, A^l \vdash \Delta_i}{\Gamma_i \vdash \Delta_i}$$

We have replaced the hypothesis $\Gamma_i \vdash \Delta_i$ by two sets of hypothesis. We will describe the marking mechanism after presenting all the cases.

ii. The cases relative to the propositional connectives are treated as in [Gir83].

iii. $E_i^{k_i}$ is of the form $\Box A^l$.

The portion of proof above $\Gamma_i \vdash \Delta_i$ is:

$$\dfrac{\{\Gamma_i \vdash A^{l+j}, \Delta_i\}_{j \in \omega}}{\Gamma_i \vdash \Delta_i}$$

iv. $E_i^{k_i}$ is of the form $\Diamond A^l$.

The portion of proof above $\Gamma_i \vdash \Delta_i$ is:

$$\dfrac{\dfrac{\dfrac{\Gamma_i \vdash A^l, \ldots, A^{l+n}, \Delta_i}{\Gamma_i \vdash A^{l+1}, \ldots, A^{l+n}, \Delta_i}}{\vdots}}{\dfrac{\Gamma_i \vdash A^{l+n}, \Delta_i}{\Gamma_i \vdash \Delta_i}}$$

v. $E_i^{k_i}$ is of the form $\circ A^l$.
The portion of proof above $\Gamma_i \vdash \Delta_i$ is:

$$\frac{\Gamma_i \vdash A^{l+1}, \Delta_i}{\Gamma_i \vdash \Delta_i}$$

(c) Case (a) does not hold and the marked formula $E_i^{k_i}$ occurs in Γ_i.
Then we have the following cases:
i. $E_i^{k_i}$ is atomic: proceed as in case (b)i.
ii. The cases relative to the propositional connectives are treated as in [Gir83].
iii. $E_i^{k_i}$ is of the form $\Box A^l$.
The portion of proof above $\Gamma_i \vdash \Delta_i$ is:

$$\cfrac{\cfrac{\Gamma_i, A^l, \ldots, A^{l+n} \vdash \Delta_i}{\Gamma_i, A^{l+1}, \ldots, A^{l+n} \vdash \Delta_i}}{\begin{array}{c}\vdots \\ \cfrac{\Gamma_i, A^{l+n} \vdash \Delta_i}{\Gamma_i \vdash \Delta_i}\end{array}}$$

iv. $E_i^{k_i}$ is of the form $\diamond A^l$.
The portion of proof above $\Gamma_i \vdash \Delta_i$ is:

$$\frac{\{\Gamma_i, A^{l+j} \vdash \Delta_i\}_{j \in \omega}}{\Gamma_i \vdash \Delta_i}$$

v. $E_i^{k_i}$ is of the form $\circ A^l$.
The portion of proof above $\Gamma_i \vdash \Delta_i$ is:

$$\frac{\Gamma_i, A^{l+1} \vdash \Delta_i}{\Gamma_i \vdash \Delta_i}$$

We still have to describe the marking mechanism.

At step 0 we mark the leftmost formula in $\Delta_0 = \Delta$ if $\Delta \neq \emptyset$; the rightmost formula in $\Gamma_0 = \Gamma$ otherwise.

Now we indicate the formula that has to be marked in every hypothesis of the portion of proof constructed above $\Gamma_i \vdash \Delta_i$ in case (b) or (c). In either case the hypothesis are of the form $\Gamma' \vdash \Delta'$, with $\Gamma' = \Gamma_i, \Gamma'_i$ and $\Delta' = \Delta'_i, \Delta_i$.

1. If $E_i^{k_i}$ is formula $D_j^{i_j}$ in $\Delta_i = D_1^{i_1}, \ldots, D_k^{i_k}$, we mark:
   - formula $D_{j+1}^{i_{j+1}}$ if $j < k$;
   - the rightmost formula in Γ' if $j = k$ and $\Gamma' \neq \emptyset$;
   - the leftmost formula in Δ' if $j = k$ and $\Gamma' = \emptyset$;

2. If $E_i^{k_i}$ is formula $C_j^{i_j}$ in $\Gamma_i = C_1^{i_1}, \ldots, C_l^{i_l}$, we mark:
   - formula $C_{j-1}^{i_{j-1}}$ if $j > 1$;
   - the leftmost formula in Δ' if $j = 1$ and $\Delta' \neq \emptyset$;
   - the rightmost formula in Γ' if $j = 1$ and $\Delta' = \emptyset$.

The procedure that we have just described produces a recursive pre–ω–proof Π of $\Gamma \vdash \Delta$ in $2S_\omega^\Phi$.

By Soundness, if Π is an ω–proof then $\Gamma \models_\Phi \Delta$.

If Π is not an ω–proof then we can find a sequence $(\Gamma_i \vdash \Delta_i)_{i\in\omega}$ such that

i. $\Gamma_0 = \Gamma$ and $\Delta_0 = \Delta$

and, for all i,

ii. $\Gamma_i \vdash \Delta_i$ is a hypothesis at step i;
iii. $\Gamma_{i+1} \vdash \Delta_{i+1}$ is a hypothesis of the portion of proof above $\Gamma_i \vdash \Delta_i$.

Lemma 4.6 below ensures that the following model is well-defined:

$$\mathbf{N}_a =< \mathbf{N}, a : \mathbf{N} \to 2^{At} >,$$

where, for all $p \in At$,

$p \in a(i)$ if p^i occurs in Γ_n for some n;
$p \notin a(i)$ if p^i occurs in Δ_n for some n.

By Lemma 4.7 below, we get $\mathbf{N}_a \models \Gamma$ and $\mathbf{N}_a \not\models \Delta$, since $\Gamma_0 = \Gamma$ and $\Delta_0 = \Delta$.

Furthermore, if B^l is any formula in Φ_{ax} then B^l does not occur in any Δ_i (otherwise the sequence of $\Gamma_i \vdash \Delta_i$'s would be finite). Hence, by Lemma 4.6, B^l occurs in Γ_n for some n and so $\mathbf{N}_a \models B^l$, again by Lemma 4.7. It follows that $\mathbf{N}_a \models \Phi_{ax}$. Finally we get $\Gamma \not\models_\Phi \Delta$.

Therefore if $\Gamma \models_\Phi \Delta$ the pre–ω–proof Π constructed above is indeed an ω–proof of $\Gamma \vdash \Delta$. Furthermore Π is (primitive) recursive. □

We are left with the proofs of the two lemmas that have been used in the previous proof.

Lemma 4.6. *Let* $(\Gamma_i \vdash \Delta_i)_{i\in\omega}$ *be a sequence satisfying the properties i.–iii. as in the proof of Theorem 4.5. Then for every formula* B^j *there exists a natural number* l *such that* B^j *occurs in* $\Gamma_l \vdash \Delta_l$ *but not on both sides of the sequent.*

Proof. It is clear that there cannot be any natural numbers m, n such that B^j occurs in both Γ_m and Δ_n because, otherwise, B^j occurs in both $\Gamma_{\max\{m,n\}}$ and $\Delta_{\max\{m,n\}}$, contradicting to the assumption that the sequence of $\Gamma_i \vdash \Delta_i$'s is infinite.

Let the fixed enumeration $\mathcal{E}$ of formulas be

$$E_0{}^{k_0}, \ldots, E_n{}^{k_n}, \ldots$$

Suppose that, for all $m < n$, , $E_m{}^{k_m}$ occurs in $\Gamma_{l_m} \vdash \Delta_{l_m}$ with $l_i \leq l_j$ if $i \leq j$. Then $E_m{}^{k_m}$ occurs in $\Gamma_{l_{n-1}} \vdash \Delta_{l_{n-1}}$ for all $m < n$.

By construction of the pre–ω–proof and by definition of the marking mechanism there exists a natural number $l \geq l_{n-1}$ such that the marked formula in $\Gamma_l \vdash \Delta_l$ is atomic. Then $E_n{}^{k_n}$ occurs in $\Gamma_{l+1} \vdash \Delta_{l+1}$. □

Lemma 4.7. *Let* $(\Gamma_i \vdash \Delta_i)_{i\in\omega}$ *be a sequence satisfying the properties i.–iii. as in the proof of Theorem 4.5 and let* $\mathbf{N}_a$ *be the model defined in the proof of the same theorem. Then, for each formula* A^l,

(a) $\mathbf{N}_a \models A^l$ *if* A^l *occurs in* Γ_n *for some* n;
(b) $\mathbf{N}_a \models \neg A^l$ *if* $\neg A^l$ *occurs in* Δ_n *for some* n.

Proof. 1. A is a proposition symbol: (a) and (b) hold by definition of $\mathbf{N}_a$.

2. the propositional cases are straightforward.
3. A is $\Box B$, for some modal formula B: suppose first that A^l occurs in Γ_n, for some n. Let $k \in \mathbf{N}$. Then there exists $m \geq \max\{n, k\}$ such that the marked formula in $\Gamma_m \vdash \Delta_m$ is A^l. Therefore Γ_{m+1} is $\Gamma_m, B^l, \ldots, B^{l+m}$. By induction hypothesis we get $\mathbf{N}_a \models B^{l+k}$. Being k arbitrary, we finally get $\mathbf{N}_a \models A^l$.
   Suppose now that A^l occurs in Δ_n, for some n. There exists $m \geq n$ such A^l is the marked formula in Δ_m. Therefore Δ_{m+1} is B^{l+k}, Δ_m, for some $k \in \omega$. By induction hypothesis we get $\mathbf{N}_a \models \neg B^{l+k}$. Hence $\mathbf{N}_a \models \neg\Box B^l$.
4. A is $\diamond B$, for some modal formula B: suppose first that A^l occurs in Γ_n, for some n. There exists $m \geq n$ such A^l is the marked formula in Γ_m. Therefore Γ_{m+1} is Γ_m, B^{l+k}, for some $k \in \omega$. By induction hypothesis we get $\mathbf{N}_a \models B^{l+k}$. Hence $\mathbf{N}_a \models \diamond B^l$.
   Suppose now that A^l occurs in Δ_n, for some n. Let $k \in \mathbf{N}$. Then there exists $m \geq \max\{n, k\}$ such that the marked formula in $\Gamma_m \vdash \Delta_m$ is A^l. Therefore Δ_{m+1} is $B^l, \ldots, B^{l+m}, \Delta_m$. By induction hypothesis we get $\mathbf{N}_a \models \neg B^{l+k}$. Being k arbitrary, we finally get $\mathbf{N}_a \models \neg A^l$.
5. A is $\circ B$, for some modal formula B: suppose first that A^l occurs in Γ_n, for some n. There exists $m \geq n$ such A^l is the marked formula in Γ_m. Therefore Γ_{m+1} is Γ_m, B^{l+1}. By induction hypothesis we get $\mathbf{N}_a \models B^{l+1}$. Hence $\mathbf{N}_a \models A^l$.
   Suppose now that A^l occurs in Δ_n, for some n. There exists $m \geq n$ such A^l is the marked formula in Δ_m. Therefore Δ_{m+1} is B^{l+1}, Δ_m. By induction hypothesis we get $\mathbf{N}_a \models \neg B^{l+1}$. Hence $\mathbf{N}_a \models \neg A^l$. □

5. Predicate modal extension

In this section we deal with the predicate case. We extend the propositional language $\mathcal{L}_0$ of Section 2 to a countable predicate language with equality $\mathcal{L}$, and we define predicate modal formulas in the usual way. We denote by Var the countably infinite set of $\mathcal{L}$-variables.

As in [Gir83], we regard predicate formulas as equivalence classes by identifying two formulas that differ only by the names of their bound variables. Also, when substituting a term t for a variable x in a formula $A(x)$, we always assume that we are indeed considering $A'(t)$, where $A'(x)$ is a formula equivalent to $A(x)$ such that no free variable of t occurs bound in $A'(t)$.

Predicate indexed formulas (briefly: *formulas*) are defined following Definition 2.2.

We extend the system $2S_\omega$ by adding the following:

Rules for quantifiers

$$\frac{\Gamma \vdash A^i, \Delta}{\Gamma \vdash \forall x A^i, \Delta} \vdash \forall \qquad \frac{\Gamma, A(t)^i \vdash \Delta}{\Gamma, \forall x A(x)^i \vdash \Delta} \forall \vdash$$
$$\frac{\Gamma \vdash A(t)^i, \Delta}{\Gamma \vdash \exists x A(x)^i, \Delta} \vdash \exists \qquad \frac{\Gamma, A^i \vdash \Delta}{\Gamma, \exists x A^i \vdash \Delta} \exists \vdash$$

In rules $\vdash \forall$ and $\exists \vdash$, variable x must not occur free in $\Gamma \vdash \Delta$. In rules $\vdash \exists$ and $\forall \vdash$, t is an arbitrary $\mathcal{L}$-term.

Notice also that rules are given up to variable renaming of bound variables.

Axioms for equality Equality is an equivalence relation satisfying the property of *indiscernibility of identicals.* Hence, for all $\mathcal{L}$-terms s, t; all *atomic* formulas A and all $n \in \mathbf{N}$, we introduce the axioms

1. $\vdash (t = t)^n$;
2. $(s = t)^n, A(s)^n \vdash A(t)^n$.

We also add, for all $\mathcal{L}$-terms s, t and all $n, k \in \mathbf{N}$, the axiom

3. $(s = t)^n \vdash (s = t)^k$,

expressing the semantic property that models have *constant domains* and *rigid interpretation of terms.*

Notice that, thanks to axiom 3., one can prove sequent $(s = t)^n, A(s)^n \vdash A(t)^n$ for an arbitrary formula A (by induction on the degree of A).

The same notation as in Section 4 is in force, with a major difference:

Definition 5.1. *Let Φ be a set of predicate modal formulas. We let*

$$\Phi_{ax} = \{\vdash A^i \ : \ A \in \Phi \text{ and } i \in \omega\} \cup \{\textit{axioms for equality}\}.$$

We write $2SP^\Phi_\omega$ to denote the predicate correspondent of $2S^\Phi_\omega$, when Φ is a set of proper axioms. We write $2SP_\omega$ for $2SP^\emptyset_\omega$.

We extend the definition of (pre)-ω-proof in accordance with the new axioms and rules.

One can easily show that the correspondent of $\forall x \Box A \to \Box \forall x A$ (*Barcan formula*) and $\Box \forall x A \to \forall x \Box A$ (*Converse Barcan formula*) are provable in $2SP_\omega$.

Concerning the third axiom for equality, notice that $\{(s = t)^n \vdash (s = t)^k : n, k \in \mathbf{N}\}$ is provably equivalent in $2SP_\omega \setminus$ {axioms for equality} to

$$\{(s = t)^n \vdash \Box(s = t)^n, \ \neg(s = t)^n \vdash \Box\neg(s = t)^n \ : n \in \mathbf{N}\},$$

for all terms s, t.

After extending the definition of degree of a formula to the predicate case by letting

$$\deg(\forall x A) = \deg(\exists x A) = \deg(A) + 1,$$

one defines the degree $\delta^\Phi[\Pi]$ of an ω–proof Π as in Section 4 and proves that Lemma 4.2 holds in the predicate case as well:

Lemma 5.2. *Let $n \in \mathbf{N}$ and let A^k be a formula of degree n not occurring in Φ_{ax}. Let $< \Pi, \Pi' >$ be a pair of recursive ω–proofs of sequents $\Gamma \vdash \Delta$ and $\Gamma' \vdash \Delta'$ respectively, satisfying the property $\delta^{\Phi}[\Pi], \delta^{\Phi}[\Pi'] \leq n$. Then one can obtain in an effective way from $< \Pi, \Pi' >$ a recursive ω-proof $\mathsf{Mix}(\Pi, \Pi')$ of sequent $\Gamma, \Gamma' - A^k \vdash \Delta - A^k, \Delta'$ satisfying the property $\delta^{\Phi}[\mathsf{Mix}(\Pi, \Pi')] \leq n$.*

Proof. Same as the proof of Lemma 2.11. Just add to case 7. the subcases

(e) A is $\forall x B(x)$.

Let Π and Π' be

$$\dfrac{\begin{array}{c}\Pi_1(x)\\ \Gamma \vdash B(x)^k, \Delta_1\end{array}}{\Gamma \vdash \forall x B(x)^k, \Delta_1} \qquad \text{and} \qquad \dfrac{\begin{array}{c}\Pi'_1\\ \Gamma'_1, B(t)^k \vdash \Delta'\end{array}}{\Gamma'_1, \forall x B(x)^k \vdash \Delta'}$$

respectively. Apply the induction hypothesis to the pairs of proofs $< \Pi_1(t), \Pi' >$ and $< \Pi, \Pi'_1 >$, obtaining $\mathsf{Mix}(\Pi_1(t), \Pi')$ and $\mathsf{Mix}(\Pi, \Pi'_1)$ respectively. The proof $\mathsf{Mix}(\Pi, \Pi')$ is then

$$\dfrac{\dfrac{\begin{array}{c}\mathsf{Mix}(\Pi_1(t), \Pi')\\ \Gamma, \Gamma'_1 - A^k \vdash B(t)^k, \Delta_1 - A^k, \Delta'\end{array} \qquad \begin{array}{c}\mathsf{Mix}(\Pi, \Pi'_1)\\ \Gamma, \Gamma'_1 - A^k, B(t)^k \vdash \Delta_1 - A^k, \Delta'\end{array}}{\Gamma, \Gamma'_1 - A^k, \Gamma, \Gamma'_1 - A^k \vdash \Delta_1 - A^k, \Delta', \Delta_1 - A^k, \Delta'}\ Cut}{\Gamma, \Gamma'_1 - A^k \vdash \Delta - A^k, \Delta'}$$

(f) A is $\exists x B(x)$.

This case is symmetric to the previous one. □

From the previous lemma we get:

Theorem 5.3 (Cut elimination). *Let Φ be a recursive set of modal formulas. Suppose there exists a (recursive) ω-proof in $2SP^{\Phi}_{\omega}$ of sequent $\Gamma \vdash \Delta$. Then there exists a (recursive) ω-proof of $\Gamma \vdash \Delta$ satisfying the property that each cut formula occurs in some sequent in Φ_{ax}.*

Proof. Same as the proof of Theorem 4.3. □

Corollary 5.4. *Suppose there exists a (recursive) ω-proof of sequent $\Gamma \vdash \Delta$ in $2SP_{\omega}$. Then there exists a (recursive) ω-proof Π^* of $\Gamma \vdash \Delta$ in $2SP_{\omega}$ satisfying the property that each cut formula in Π^* is atomic.*

5.1. Predicate semantics

Many different choices can be made to define the semantics of predicate modal logic (see [Gar84]). None of the choices is free from restrictions and/or drawbacks.

In this section we present a semantics for $2SP_{\omega}$ closely connected to the modal semantics prescribing *constant domains* and *rigid designators*.

A model $\mathcal{M}$ is a triple $< \mathbf{N}, (\mathbf{M}_n)_{n\in\omega}, \sigma >$ such that:

1. $\mathbf{N}$ is the frame of natural numbers;
2. $(\mathbf{M}_n)_{n\in\omega}$ is a sequence of $\mathcal{L}$-structures with common universe M such that the interpretation of each constant or function symbol of $\mathcal{L}$ is the same in all structures;
3. $\sigma : Var \to M$ is an assignment of values to variables in M.

The definition of $\mathcal{M} \models A^n$ is the standard inductive one, after reading $\mathcal{M} \models A^n$ as "A is true at world n." For instance

$$< \mathbf{N}, (\mathbf{M}_n)_{n\in\omega}, \sigma > \models_n \forall x A \ \Leftrightarrow \ \text{for all } a \in M \ \ < \mathbf{N}, (\mathbf{M}_n)_{n\in\omega}, \sigma_{x/a} > \models_n A,$$

where $\sigma_{x/a}$ is the same as σ apart from $\sigma_{x/a}(x) = a$.

The relations $\Gamma \models \Delta$ and $\Gamma \models_\Phi \Delta$ defined in the previous sections extend in a natural way to the current setting.

Notice that if $\Gamma \vdash \Delta$ is an axiom for equality then $\Gamma \models \Delta$. Hence $\Gamma \models \Delta$ is the same as $\Gamma \models_\emptyset \Delta$. (Recall the definition of Φ_{ax}.)

We claim that the following extension of Theorem 4.5 holds:

Theorem 5.5 (Soundness and Completeness). *Let Φ be a (primitive) recursive set of modal formulas and let $\Gamma \vdash \Delta$ be a nonempty sequent. Then $\Gamma \models_\Phi \Delta$ if and only if there exists a (primitive) recursive ω–proof in $2\mathrm{SP}^\Phi_\omega$ of $\Gamma \vdash \Delta$.*

Proof. We just give a sketch of the completeness proof. We closely follow the proof of Theorem 4.5. We first describe the additional steps in the construction of a pre-ω-proof Π of $\Gamma \vdash \Delta$.

Nothing changes in the marking mechanism of formulas.

In addition to a recursive enumeration $\mathcal{E}$ of $\mathcal{L}$-formulas we also fix a recursive enumeration $\mathcal{T}$ of $\mathcal{L}$-terms.

When describing how to extend upwards the portion of proof obtained at step n of the construction of a pre-ω-proof, we take into account the additional cases due to quantifiers:

Case (a) We include among the axioms those for equality.
Case (b) We add the subcases:

vi. The marked formula ${E_i}^{k_i}$ is of the form $\forall x A^l$. The portion of proof to be added above $\Gamma_i \vdash \Delta_i$ is then

$$\frac{\Gamma_i \vdash A(y)^l, \Delta_i}{\Gamma_i \vdash \Delta_i},$$

where y is a variable not occurring in $\Gamma_i \vdash \Delta_i$.

vii. The marked formula ${E_i}^{k_i}$ is of the form $\exists x A^l$. Let $t_0, \ldots, t_n$ be the first $n+1$ terms in the enumeration $\mathcal{T}$ (here and in the sequel think as if we are performing step n of the construction).

The portion of proof to be added above $\Gamma_i \vdash \Delta_i$ is then

$$\cfrac{\cfrac{\Gamma_i \vdash A(t_0)^l, \ldots, A(t_n)^l, \Delta_i}{\Gamma_i \vdash A(t_1)^l, \ldots, A(t_n)^l, \Delta_i}}{\vdots} \qquad \cfrac{\Gamma_i \vdash A(t_n)^l, \Delta_i}{\Gamma_i \vdash \Delta_i}$$

Case (c) We add the subcases:

vi. ${E_i}^{k_i}$ is of the form $\forall x A^l$. Let $t_0, \ldots, t_n$ be the first $n+1$ terms in the enumeration $\mathcal{T}$. The portion of proof above $\Gamma_i \vdash \Delta_i$ is:

$$\cfrac{\cfrac{\Gamma_i, A(t_0)^l, \ldots, A(t_n)^l \vdash \Delta_i}{\Gamma_i, A(t_1)^l, \ldots, A(t_n)^l \vdash \Delta_i}}{\vdots} \qquad \cfrac{\Gamma_i, A(t_n)^l \vdash \Delta_i}{\Gamma_i \vdash \Delta_i}$$

vii. $E_i^{k_i}$ is of the form $\exists x A^l$. The portion of proof above $\Gamma_i \vdash \Delta_i$ is:

$$\cfrac{\Gamma_i, A(y)^l \vdash \Delta_i}{\Gamma_i \vdash \Delta_i},$$

where y is a variable not occurring in $\Gamma_i \vdash \Delta_i$.

As in the proof of Theorem 4.5, we obtain a recursive pre–ω–proof Π of $\Gamma \vdash \Delta$ in $2SP_\omega^\Phi$. If Π is not an ω–proof then we can find a sequence $(\Gamma_i \vdash \Delta_i)_{i\in\omega}$ such that

i. $\Gamma_0 = \Gamma$ and $\Delta_0 = \Delta$

and, for all i,

ii. $\Gamma_i \vdash \Delta_i$ is a hypothesis at step i;
iii. $\Gamma_{i+1} \vdash \Delta_{i+1}$ is a hypothesis of the portion of proof above $\Gamma_i \vdash \Delta_i$.

Remark 5.1. One can easily check that Lemma 4.6 still holds.

We now define a model $\mathcal{M} = < \mathbf{N}, (\mathbf{M}_n)_{n\in\omega}, \sigma >$. We let $\sim$ be the equivalence relation on $\mathcal{L}$–terms given by:

$$s \sim t \Leftrightarrow \forall n \in \mathbf{N}\, \exists l \in \mathbf{N} \text{ such that } (s = t)^n \text{ occurs in } \Gamma_l.$$

To check that $\sim$ is actually an equivalence relation, use Remark 5.1, the first two axioms for equality and recall how the sequence $(\Gamma_i \vdash \Delta_i)$ has been constructed.

We denote by $[t]$ the $\sim$–equivalence class of term t.

For each constant symbol c, we let its interpretation c^n in $\mathbf{M}_n$ be $[c]$.

If f is a k–ary function symbol, $k > 0$, we let its interpretation f^n in $\mathbf{M}_n$ be given by: $f^n([t_1], \ldots, [t_k]) = [f(t_1, \ldots, t_k)]$ for all terms $t_1, \ldots, t_k$.

If P is a k–ary predicate symbol, we let its interpretation P^n in $\mathbf{M}_n$ be given by:

$$([t_1], \ldots, [t_k]) \in P^n \Leftrightarrow \text{there exists } l \in \mathbf{N} \text{ such that } P(t_1, \ldots, t_k)^n \in \Gamma_l.$$

As usual, one has to check that f^n and P^n are well–defined: as before, keeping in mind the axioms for equality, how the sequence $(\Gamma_i \vdash \Delta_i)$ has been constructed and Remark 5.1, this is a matter of routine.

Finally we let $\sigma(x) = [x]$ for all $x \in Var$. This ends the definition of $\mathcal{M}$.

A straightforward argument shows that $t^n = [t]$ for all terms t and all $n \in \omega$.

Arguing as at the end of the proof of Theorem 4.5, by Lemma 5.6 below we get $\Gamma \not\models_\Phi \Delta$.

Therefore, assuming that $\Gamma \models_\Phi \Delta$, we obtain that the pre–ω–proof Π whose construction has been sketched above is indeed a (primitive) recursive ω–proof of $\Gamma \vdash \Delta$. □

We are left with the proof of the following:

Lemma 5.6. *Let $(\Gamma_i \vdash \Delta_i)_{i\in\omega}$ be a sequence satisfying the properties i.–iii. as in the proof of Theorem 5.5 and let $\mathcal{M}$ be the model defined in the proof of the same theorem. Then, for each formula A^k,*

$$\mathcal{M} \models A^k \Leftrightarrow \textit{there exists } l \in \mathbf{N} \textit{ such that } A^k \textit{ occurs in } \Gamma_l.$$

Proof. As in Lemma 4.7 the proof goes by induction on $\deg(A)$. In addition to the cases treated in the proof of that lemma, we only examine the case when A is of the form $\forall x B(x)$.

Suppose that A^k occurs in Γ_l, for some l, and let t be an arbitrary term. Without loss of generality we can assume that no variable occurring in t is bound in A (recall that A is an equivalence class rather than a formula).

There exists a natural number j such that $B(t)^k$ occurs in Γ_j (for there are arbitrarily large indices i such that A^k is the marked formula in $\Gamma_i \vdash \Delta_i$ so the claim follows by Case (c) vi).

By induction hypothesis $\mathcal{M} \models B(t)^k$. The latter is equivalent to saying that $<\mathbf{N}, (\mathbf{M}_n)_{n\in\omega}, \sigma_{x/[t]}> \models B(x)^k$, where $\sigma_{x/[t]}$ is the same as σ apart from $\sigma_{x/[t]}(x) = [t]$. Being each element in M of the form $[t]$, for some term t, we get $\mathcal{M} \models A^k$.

Concerning the inverse implication, by Remark 5.1 suppose that A^k occurs in Δ_l, for some l. Then $B(y)^k$ occurs in Δ_j for some natural number j and some variable y (see Case (b) vi). By induction hypothesis $\mathcal{M} \not\models B(y)^k$, namely $< \mathbf{N}, (\mathbf{M}_n)_{n\in\omega}, \sigma_{x/[y]} > \not\models B(x)^k$ (we can assume that y does not occur bound in $B(x)$). Hence $\mathcal{M} \not\models A^k$.

The case when A is of the form $\exists x B(x)$ is symmetric. □

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