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Density of the Medvedev lattice of Π_1^0 classes

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Abstract. The partial ordering of Medvedev reducibility restricted to the family of Π_1^0 classes is shown to be dense. For two disjoint computably enumerable sets, the class of separating sets is an important example of a Π_1^0 class, which we call a "c.e. separating class". We show that there are no non-trivial meets for c.e. separating classes, but that the density theorem holds in the sublattice generated by the c.e. separating classes.

The Medvedev lattice was introduced in [5] to classify problems according to their degree of difficulty. A mass problem is a set of functions f mapping natural numbers to natural numbers and is thought of as representing the set of solutions to some problem. For example, we might consider the set of 4-colorings of a given countably infinite graph G as a set of functions each mapping ω into $\{1, 2, 3, 4\}$. One such set P is reducible to another set Q (written $P \leq_M Q$) iff there is a partial computable functional Φ which maps Q into P. Thus if we have a solution in Q, then we can use Φ to compute a solution in P. As usual, $P \equiv_M Q$ means that both $P \leq_M Q$ and $Q \leq_M P$, $P <_M Q$ means $P \leq_M Q$ but not $Q \leq_M P$, and the Medvedev degree $\mathbf{dg}_M(P)$ of P is the class of all sets Q such that $P \equiv_M Q$. We will see below that the set of Medvedev degrees is a lattice with meet and join given by the natural operations of direct product and disjoint union. For more on the general notion of Medvedev degrees, see the survey by Sorbi [9].

In this paper, we will examine the sublattice $\mathcal{P}_M$ of degrees of Π_1^0 classes of sets, that is, nonempty subclasses of $\{0, 1\}^\omega$. (We will refer to elements of $\{0, 1\}^\omega$ simply as *sets*.) The main result of this paper is that the partial ordering $\leq_M$ restricted to this sublattice is dense.

We first introduce some notation. For a finite sequence $\sigma \in \{0, 1\}^n$, we let $|\sigma| = n$ denote the length of σ. For $\sigma \in \{0, 1\}^n$ and $X \in \{0, 1\}^\omega$, we say that σ is an *initial segment* of X (written $\sigma \prec X$) if $X(i) = \sigma(i)$ for all $i \leq |\sigma|$. The *interval* $I(\sigma)$ determined by σ is $\{X \in \{0, 1\}^\omega : \sigma \prec X\}$. These intervals form a basis for

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the usual product topology on $\{0,1\}^\omega$. For $\sigma \in \{0,1\}^n$, $\sigma^\frown k$ is the extension of σ to a finite sequence of length $n+1$ with last component k. We sometimes interpret σ as coding a binary number and write, for example, $\sigma < n$.

Let $\mathcal{B}$ be the Boolean algebra of clopen subsets of $\{0,1\}^\omega$. Then each interval is in $\mathcal{B}$ and every clopen set is a finite union of intervals. Thus we can define a length $|b|$ for each clopen set $b = I(\sigma_0) \cup \cdots \cup I(\sigma_{k-1})$ to be the maximum of $\{|\sigma_i| : i < k\}$.

A Π^0_1 class $P \subseteq \{0,1\}^\omega$ may be viewed as the set $[T]$ of infinite paths X through a computable tree $T \subseteq \{0,1\}^{<\omega}$. We say that $\sigma \in T$ is *extendible* if there is an infinite path $X \in P$ such that $\sigma \prec X$; let $Ext(T)$ be the set of extendible nodes of T. Then $[T] = [Ext(T)]$ and if T is computable, $Ext(T)$ is a Π^0_1 tree with no *dead ends*. Note that in fact $Ext(T)$ depends only on P, since $Ext(T) = \{\sigma : (\exists X \in P)\ \sigma \prec X\}$ and we often denote it T_P. There is an enumeration P_e of the Π^0_1 classes as $P_e = [T_e]$, where the relation $\sigma \in T_e$ is primitive recursive and the relation $\sigma \in Ext(T_e)$ is Π^0_1 – see [2].

We begin with some background on the Medvedev reducibility of Π^0_1 classes. First we show that only total functionals are needed.

Lemma 1. *For any Π^0_1 subclasses P and Q of ω^ω, if $P \leq_M Q$, then there exists a total computable functional $F : \omega^\omega \to \omega^\omega$ such that $F[Q] \subseteq P$.*

Proof. Given that $P \leq_M Q$, there is a partial computable functional Φ which maps Q into P. This means that there is a partial computable function ϕ mapping finite sequences to finite sequences such that $\Phi(X) = \cup_n \phi(X \restriction n)$ and with the property that $\sigma \prec \tau$ implies $\phi(\sigma) \prec \phi(\tau)$. Now Q may be expressed as the set of infinite paths through some computable tree T. Then we can extend the mapping Φ from Q to a total mapping F on $\{0,1\}^\omega$ with representing function f defined recursively as follows. Let $f(\emptyset) = \emptyset$. Then for any finite sequence σ and any n, define $f(\sigma^\frown n)$ in two cases. If $\sigma^\frown n \in T$, let $f(\sigma^\frown n) = \phi(\sigma^\frown n)$, which must be defined. If $\sigma^\frown n \notin T$, let $f(\sigma^\frown n) = f(\sigma)^\frown 0$. □

The lattice operation of $\mathcal{P}_M$ are provided by sum and product operations defined as follows. For $i = 0$ or $i = 1$, let $Y = (i)X$ mean that $Y(0) = i$ and $Y(n+1) = X(n)$ for all n. Then the direct sum, or disjoint union of P and Q is given by

$$P \oplus Q = \{(0)X : X \in P\} \cup \{(1)Y : Y \in Q\}.$$

For two elements $X, Y \in \{0,1\}^\omega$, let $\langle X, Y\rangle = Z$, where $Z(2n) = X(n)$ and $Z(2n+1) = Y(n)$. Then $P \otimes Q = \{\langle X, Y\rangle : X \in P \;\&\; Y \in Q\}$ and is easily seen to be a Π^0_1 class.

We summarize here some basic facts about these meet and join operations.

Proposition 2. *For any Π^0_1 classes P, Q and R,*

(i) $P \oplus Q \equiv_M Q \oplus P$ and $P \otimes Q \equiv_M Q \otimes P$;
(ii) $P \otimes (Q \oplus R) \equiv_M (P \otimes Q) \oplus (P \otimes R)$ and $P \oplus (Q \otimes R) \equiv_M (P \oplus Q) \otimes (P \oplus R)$.
(iii) The Medvedev degree of $P \oplus Q$ is the meet, or greatest lower bound, of the Medvedev degrees of P and Q;

(iv) The Medvedev degree of $P \otimes Q$ is the join, or least upper bound, of the Medvedev degrees of P and Q
(v) If $P \leq_M Q$, then, for any R, $(P \otimes R) \oplus Q \equiv_M P \otimes (Q \oplus R)$.

Proof. We prove only the second part of (ii) and (v), which we will need for the proof of the Density Theorem.

To see that $P \oplus (Q \otimes R) \equiv_M (P \oplus Q) \otimes (P \oplus R)$, we define computable functionals in each direction. First define $\Phi : P \oplus (Q \otimes R) \to (P \oplus Q) \otimes (P \oplus R)$ by

$$\Phi((0)X) = \langle (0)X, (0)X \rangle$$

and

$$\Phi((1)\langle Y, Z \rangle) = \langle (1)Y, (1)Z \rangle.$$

Then define $\Psi : (P \oplus Q) \otimes (P \oplus R) \to P \oplus (Q \otimes R)$ as follows. Given $Z = \langle V, W \rangle \in (P \oplus Q) \otimes (P \oplus R)$, there are three cases.

$$\begin{aligned} \text{If } V = (0)X, &\quad \text{let } \Psi(Z) = V; \\ \text{if } V = (1)Y \text{ and } W = (0)X, &\quad \text{let } \Psi(Z) = W; \\ \text{if } V = (1)Y \text{ and } W = (1)Z, &\quad \text{let } \Psi(Z) = (1)\langle Y, Z \rangle. \end{aligned}$$

(v) Since $P \leq_M Q$, we have $P \oplus Q \equiv_M P$ and $P \otimes Q \equiv_M Q$. Then $(P \otimes R) \oplus Q \equiv_M (P \oplus Q) \otimes (R \oplus Q) \equiv_M P \otimes (Q \oplus R)$. □

Next we observe that $\mathcal{P}_M$ has both a least and a greatest element. The least element **0** consists of all classes P which contain a computable element. To see this, just let X_0 be a computable element of P and define $F(X) = X_0$ for any X. Then F maps any class Q into P, so that $P \leq_M Q$. In particular, the classes $\{0, 1\}^\omega$ and $\{0^\omega\}$ are both in **0**.

We can define arbitrary finite products by iteration. Let $[m, n] = \frac{1}{2}((m+n)^2 + 3m + n)$ be the usual coding of pairs of natural numbers which maps $\omega \times \omega$ 1-1 and onto ω. For an infinite sequence $X_0, X_1, \dots$ of sets, let $\langle X_0, X_1, \dots \rangle = Z$, where $Z([m, n]) = X_m(n)$. For an infinite sequence $Q_0, Q_1, \dots$ of Π^0_1 classes, let

$$\Pi_{i=0}^{\infty} Q_i = \{\langle X_0, X_1, \dots \rangle : X_i \in Q_i \text{ for each } i\}.$$

Let U be the product $\Pi_{e=0}^{\infty} P_e$. Then $P \leq_M U$ for any Π^0_1 class $P = P_e$ via the map F which takes $\langle X_0, X_1, \dots \rangle$ to X_e, that is, $F(X) = Y$, where $Y(n) = X([e, n])$. Thus the Medvedev degree of U is **1**.

The Medvedev degree is closely related both to the Turing degree and to the lattice of Π^0_1 classes under inclusion. Whenever $Q \subseteq P$, we always have $P \leq_M Q$, by the natural injection of Q into P. Conversely, using the meet operation, whenever $P \leq_M Q$, there are classes $P' \equiv_M P$ and $Q' \equiv_M Q$ with $Q' \subseteq P'$. To show that P is not Medvedev reducible to Q, it suffices to find an element X of Q such that no element of P is Turing reducible to X, since if F maps Q into P, then $F(X) \in P$ and $F(X) \leq_T X$.

With this in mind, we can find plenty of intermediate degrees, using the result of Jockusch and Soare [4] that there is a Π^0_1 class P such that any two members have incomparable Turing degree. Now such a class has no computable element and thus is uncountable and in fact perfect (see [1], p. 57). Thus we can partition P into two uncountable subclasses, Q and R, such that each member of Q is incomparable with each member of R. It follows that Q and R are Medvedev incomparable. It is not hard to obtain an infinite family of incomparable sets in this way. Binns and Simpson [8] have greatly improved this observation by showing that the free countable distributive lattice can be embedded into $\mathcal{P}_M$ below any nonzero degree.

There are two types of classes which are of special interest. For any disjoint computably enumerable (c.e.) sets A and B, define the class of separating sets as follows, where $\overline{B}$ denotes the complement of B.

$$S(A, B) = \{X : A \subseteq X \subseteq \overline{B}\}.$$

Then $S(A, B)$ is always a Π^0_1 class; we call $S(A, B)$ a *c.e. separating class* and we call the Medvedev degree of $S(A, B)$ a *c.e. separating degree*.

In particular, both **0** and **1** are c.e. separating degrees. For **0**, let A_0 be the set of even numbers and B_0 the set of odd numbers. For **1**, let A_1 be the set of theorems of Peano Arithmetic and B_1 the set of negations of theorems. Applying recent results of Simpson [7], we will sketch an argument that the Medvedev degree of $S(A_1, B_1)$ is **1**.

Simpson defined the notion of a *productive* Π^0_1 class and showed in [7] that any productive class is Medvedev complete. P is productive if there is a *splitting function* $g : \omega \to \mathcal{B}$ such that, for all e, if $P_e \subseteq P$ and P_e is nonempty, then $P_e \cap g(e)$ and $P_e \setminus g(e)$ are both nonempty. Thus it suffices to show that $S(A_1, B_1)$ is productive. Now it is well known that A_1 and B_1 are *effectively inseparable*–see Odifreddi [6], p 356. This means that there is a recursive function ϕ such that, for any x and y, if $A_1 \subseteq W_x$ and $B_1 \subseteq W_y$ and $W_x \cap W_y = \emptyset$, then $\phi(x, y) \notin W_x \cup W_y$. The following lemma thus implies that $S(A_1, B_1)$ has Medvedev degree **1**.

Proposition 3. *If A are B are effectively inseparable c.e. sets, then $S(A, B)$ is a productive Π^0_1 class.*

Proof. Let $P = S(A, B)$ where A and B are effectively inseparable c.e. sets and and let ϕ be given as above. Define $W_{f(e)} = \{n : (\forall X \in P_e) n \in X\}$ and $W_{h(e)} = \{n : (\forall X \in P_e) n \notin X\}$. To see that these are indeed c.e. sets, note that $W_{f(e)}$ has an alternate definition, that is,

$$n \in W_{f(e)} \iff (\forall \sigma \in \{0, 1\}^{n+1})(\sigma \in Ext(T_e) \implies \sigma(n) = 1),$$

where P_e is the set of infinite paths throught the e-th primitive recursive tree T_e. Clearly $W_{f(e)} \cap W_{h(e)} = \emptyset$, and if $P_e \subseteq P$, then $A \subseteq W_{f(e)}$ and $B \subseteq W_{h(e)}$. Thus $\phi(f(e), h(e)) = n \notin W_{f(e)} \cup W_{h(e)}$. Hence there exist X and Y in P_e such that $n \in X$ and $n \notin Y$. The splitting function for P can thus be defined by $g(e) = \{X : \phi(f(e), h(e)) \in X\}$. □

Let us say that c.e. sets A and B are *weakly effectively inseparable* if there is a computable function F, mapping ω^2 into the family of finite sets of natural numbers, such that, for any x and y, if $A \subseteq W_x$ and $B \subseteq W_y$ and $W_x \cap W_y = \emptyset$, then $F(x, y)$ contains at least one element which is not in $W_x \cup W_y$. Of course, effectively inseparable sets are also weakly effectively inseparable, simply by taking the singleton set $\{\phi(x, y)\}$.

We now give a weakened form of the converse of Proposition 3.

Proposition 4. *For any disjoint c.e. sets A and B, if $S(A, B)$ is productive, then A and B are weakly effectively inseparable.*

Proof. Let $P = S(A, B)$ be productive and let g be given as above. Given x and y, we can define the Π^0_1 class $P_e = P_{f(x,y)} = S(W_x, W_y)$ and from that obtain the clopen set $G = g(f(x, y))$. Finally, let $F(x, y) = \{0, 1, \ldots, |g(f(x, y))|\}$. To see that this works, suppose that in fact $A \subseteq W_x$, $B \subseteq W_y$, and $W_x \cap W_y = \emptyset$. Then $S(W_x, W_y)$ is a nonempty subclass of $S(A, B)$. Thus both $P_e \cap G$ and $P_e \setminus G$ are nonempty. Choose $X \in P_e \cap G$ and $Y \in P_e \setminus G$. Then by the definition of $F(x, y)$, there exist disjoint intervals $I(\sigma)$ and $I(\tau)$ with $|\sigma| = |\tau| \in F(x, y)$ such that $\sigma \prec X$ and $\tau \prec Y$. Thus there must be some $n \in F(x, y)$ such that $X(n) \neq Y(n)$ and it follows that $n \notin W_x \cup W_y$. □

The family of c.e. separating degrees is closed under join, since

$$S(A, B) \otimes S(C, D) = S(\langle A, C\rangle, \langle B, D\rangle).$$

However, the meet of two incomparable c.e. separating degrees is never a c.e. separating degree, as shown by the following.

Lemma 5. *For any Π^0_1 class P and any clopen sets G and H, if $P \cap G \leq_M P \cap H$, then $P \cap G \equiv_M P \cap (G \cup H)$.*

Proof. First, $P \cap (G \cup H) \leq_M P \cap G$ via the identity map. Fix a computable functional $\Phi : P \cap H \to P \cap G$ and define $\Psi : P \cap (G \cup H) \to P \cap G$ by

$$\Psi(X) := \begin{cases} X, & \text{if } X \in G; \\ \Phi(X), & \text{otherwise.} \end{cases}$$

Note that Ψ is computable since clopen sets are simply finite unions of intervals. □

Lemma 6. *For any c.e. separating class P and any clopen set G, if $P \cap G \neq \emptyset$, then $P \cap G \equiv_M P$.*

Proof. By Lemma 5, it suffices to prove this for intervals, and we proceed by induction on the length n of σ. If $n = 0$, then $I(\sigma) = 2^\omega$, so $P \cap I(\sigma) = P$. Assume as induction hypothesis that $P \cap I(\sigma) \equiv_M P$ for some σ of length n, and suppose that $P \cap I(\sigma^\frown e) \neq \emptyset$. If $P \cap I(\sigma^\frown 1-e) = \emptyset$, then $P \cap I(\sigma^\frown e) = P$. Otherwise, $P \cap I(\sigma^\frown e) \equiv_M P \cap I(\sigma^\frown 1 - e)$ via the computable maps $X \mapsto X \cup \{n\}$ and $X \mapsto X \setminus \{n\}$. Then by Lemma 5 again,

$$P \cap I(\sigma^\frown e) \equiv_M P \cap \big(I(\sigma^\frown e) \cup I(\sigma^\frown 1 - e)\big) = P. \qquad \square$$

Proposition 7. *For any Π^0_1 classes P and Q and any c.e. separating class R, if $P \oplus Q \leq_M R$, then either $P \leq_M R$ or $Q \leq_M R$.*

Proof. Fix a computable functional $\Phi : R \to P \oplus Q$ and set $G := \{X : \Phi(X) \in I((0))\}$. G is clopen as the continuous inverse image of an interval. $P \leq_M R \cap G$ via the map $X \mapsto \big(k \mapsto \Phi(X)(k+1)\big)$. If $R \cap G \neq \emptyset$, then by Lemma 6 $R \cap G \equiv_M R$, so $P \leq_M R$. Otherwise $R \setminus G \neq \emptyset$ and we have similarly $Q \leq_M R$. □

This suggests that we should consider the sublattice of $\mathcal{P}_M$ generated by the family of c.e. separating degrees. This turns out to have a simple direct characterization.

Definition 8. *For any tree $T \subseteq \{0, 1\}^{<\omega}$ and any Π^0_1 class $P \subseteq \{0, 1\}^{\omega}$,*

(i) T is homogeneous *iff* $(\forall \sigma, \tau \in T)(\forall i < 2)$,

$$|\sigma| = |\tau| \Longrightarrow (\sigma^\frown i \in T \iff \tau^\frown i \in T);$$

(ii) T is almost homogeneous *iff* $\exists n(\forall \sigma, \tau \in T)(\forall i < 2)$,

$$n \leq |\sigma| = |\tau| \wedge \sigma \restriction n = \tau \restriction n \Longrightarrow (\sigma^\frown i \in T \iff \tau^\frown i \in T);$$

The least such n is called the modulus *of T;*

(iii) P is (almost) homogeneous *iff T_P is (almost) homogeneous; a Medvedev degree is* (almost) homogeneous *iff it contains an (almost) homogeneous class;* **AH** *denotes the family of almost homogeneous degrees.*

Proposition 9. *For any Π^0_1 class P,*

$$P \text{ is homogeneous} \iff P \text{ is a c.e. separating class.}$$

Proof. If $P = S(A, B)$ for c.e. sets A and B, then

$$T_P = \Big\{\sigma : (\forall i < |\sigma|)\big[\sigma(i) = 0 \wedge i \notin A) \vee (\sigma(i) = 1 \wedge i \notin B)\big]\Big\}.$$

This is clearly a homogeneous tree. Conversely, if T_P is homogeneous, then $P = S(A, B)$ for

$$A = \{n : 0^{n\frown}0 \notin T_P\} \quad \text{and} \quad B = \{n : 0^{n\frown}1 \notin T_P\}.$$

Corollary 10. *For any Π^0_1 class P, if P is almost homogeneous with modulus n, then P is the disjoint union of 2^n c.e. separating classes.*

Proof. Given $P \in$ **AH** with modulus n, for each sequence σ of length n, let $P[\sigma] := \{X \in P : \sigma \prec X\}$. Each $P[\sigma]$ is homogeneous, so is a c.e. separating class, and clearly P is the disjoint union of the $P[\sigma]$.

Proposition 11. *For any Π^0_1 classes P and Q, if P and Q are almost homogeneous, then also $P \oplus Q$ and $P \otimes Q$ are almost homogeneous.*

Proof. If P and Q are almost homogeneous with moduli m and n, respectively, then easily $P \oplus Q$ is almost homogeneous with modulus $\max\{m, n\} + 1$ and $P \otimes Q$ is almost homogeneous with modulus $2\max\{m, n\}$.

Theorem 12. **AH** *is the smallest sublattice of* $\mathcal{P}_M$ *which includes the family of c.e. separating degrees.*

Proof. By the preceding two propositions, **AH** is a sublattice of $\mathcal{P}_M$ which includes the family of c.e. separating degrees. Let L be any other such lattice; we prove by induction that for all n,

$$P \text{ is almost homogeneous with modulus } n \quad \Longrightarrow \quad \mathbf{dg}_M(P) \in L.$$

For $n = 0$ this is true by Proposition 9, so assume as induction hypothesis that it holds for n and that P is almost homogeneous with modulus $n+1$. Then if for $i < 2$ we set $P_i := \{X : (i)X \in P\}$, P_i is almost homogeneous with modulus n, so $\mathbf{dg}_M(P_i) \in L$ and clearly $P = P_0 \oplus P_1$ so also $\mathbf{dg}_M(P) \in L$. □

Classes of positive measure are also of interest. We will say that a Medvedev degree has positive measure if it contains some class of positive measure. Thus **0** has positive measure, since 2^ω has Medvedev degree **0**. On the other hand, it is a classic result ([3], p. 110) that the computable sets are not a basis for the Π_1^0 classes of positive measure, so that there is a nonzero Medvedev degree of positive measure. It is not hard to see that the Medvedev degrees of positive measure form an ideal of $\mathcal{P}_M$. The precise positive measure is not important here, since it is easy to see that for any Π_1^0 class P of positive measure and any $\epsilon > 0$, we can find a Π_1^0 class $Q \equiv_M P$ with measure $> 1 - \epsilon$ and a second Π_1^0 class $R \equiv_M P$ with measure $< \epsilon$.

It turns out that **0** is the only Medvedev degree which is both an almost homogeneous degree and has positive measure.

Theorem 13. *For any* Π_1^0 *class* P *of positive measure and any almost homogeneous class* $Q >_M \mathbf{0}$, $Q \not\leq_M P$.

Proof. Suppose first that $Q = S(A, B)$, where A and B are recursively inseparable c.e. sets, and let P have positive measure. Jockusch and Soare ([4], p. 50) proved that the collection $U(Q)$, of all sets X such that some $Y \in Q$ is Turing reducible to X, has measure 0. Now suppose by way of contradiction that $Q \leq_M P$. Then there would be a recursive functional Φ mapping P into Q, so that for each $X \in P$, $Y = \Phi(X)$ is in Q and is Turing reducible to X. Thus $P \subseteq U(Q)$ and hence has measure zero.

Now if Q is almost homogeneous, say with modulus n, then by Corollary 10, Q is the disjoint union of 2^n many c.e. separating sets $Q[\sigma]$. If there is a recursive functional Φ mapping P into Q, then P is the disjoint union of the sets $\Phi^{-1}(Q[\sigma])$. Each of these is of measure 0 by the first part of the proof, hence so is P. □

It follows in particular that no class of positive measure has degree **1**. We now present the main theorem of the paper.

Theorem 14. *(Density Theorem) For any* Π_1^0 *classes* P *and* Q, *if* $P <_M Q$, *then there exists a* Π_1^0 *class* S *such that* $P <_M S <_M Q$.

Proof. Fix Π^0_1 classes $P <_M Q$ and corresponding Π^0_1 trees T_P and T_Q with no dead ends. We shall construct a Π^0_1 class R such that

$$Q \oplus R \not\leq_M P; \tag{1}$$
$$Q \not\leq_M P \otimes R; \tag{2}$$

and take, using Proposition 2(v),

$$S := (P \otimes R) \oplus Q \equiv_M P \otimes (Q \oplus R).$$

Then $P <_M S <_M Q$ as required because of the following four facts:

$P \leq_M S$	because	S is of the form $P \otimes P'$;	
$S \leq_M Q$	because	S is of the form $Q' \oplus Q$;	
$S \not\leq_M P$	because otherwise	$Q \oplus R \leq_M S \leq_M P$	contrary to (1);
$Q \not\leq_M S$	because otherwise	$Q \leq_M S \leq_M P \otimes R$	contrary to (2).

The class R will be a c.e. separating class $S(A, B)$ and we shall establish (1) by satisfying for all a,

$$\text{not} \quad \forall X \in P\Big(\{a\}^X \in Q \oplus R\Big). \tag{1a}$$

For (2) it will suffice to satisfy for all a,

$$\text{not} \quad \forall X \in P\Big(\{a\}^{X,A} \in Q\Big), \tag{2a}$$

because from this it follows that $Q \not\leq_M P \otimes \{A\}$, which implies (2).

The strategy for satisfying (1a) is a variant of the Sacks coding strategy for the density of the c.e. Turing degrees. First note that if (1a) fails, then for all $X \in P$, $\{a\}^X$ is of one of the forms $(0)Y$ for some $Y \in Q$ or $(1)Z$ for some $Z \in S(A, B)$. Thus we may think of $\{a\}$ as the union of a map $\{a_0\} : P_0 \to Q$ and a map $\{a_1\} : P_1 \to S(A, B)$, where P_0 and P_1 are two disjoint Π^0_1 subclasses of P whose union is P. The construction involves the enumeration of certain markers $\mathsf{m}^a_{\sigma,t}$ into A and B. We shall arrange that under the hypothesis that (1a) fails that there exists a recursive function g such that for all $\sigma \in T_Q$,

$$\sigma^\frown 0 \notin T_Q \implies \mathsf{m}^a_{\sigma,g(\sigma)} \in A \quad \text{and} \quad \sigma^\frown 1 \notin T_Q \implies \mathsf{m}^a_{\sigma,g(\sigma)} \in B.$$

Since T_Q has no dead ends, this ensures that A and B are disjoint. Then there exists an index a_2 such that for all $X \in P_1$ and all y,

$$\{a_2\}^X(y) = \begin{cases} 1, & \text{if } \mathsf{m}^a_{\sigma_y, g(\sigma_y)} \in \{a_1\}^X; \\ 0, & \text{otherwise,} \end{cases}$$

where σ_y denotes $\{a_2\}^X \restriction y$. Now we can show by induction on y that

$$\{a_1\}^X \in S(A, B) \implies \sigma_y \in T_Q,$$

from which it follows that $\{a_2\}^X \in Q$ — thus $\{a_2\} : P_1 \to Q$. This is trivially true for $y = 0$, so assume it for y as induction hypothesis. If both $\sigma_y^\frown 0$ and $\sigma_y^\frown 1$ belong to T_Q, then certainly $\sigma_{y+1} \in T_Q$. Otherwise, either $\sigma_y^\frown 0 \notin T_Q$, so

$$\mathsf{m}^a_{\sigma_y, g(\sigma_y)} \in A \subseteq \{a_1\}^X \implies \{a_2\}^X(y) = 1 \implies \sigma_{y+1} = \sigma_y^\frown 1 \in T_Q,$$

or $\sigma_y^\frown 1 \notin T_Q$, so

$$\mathsf{m}^a_{\sigma_y, g(\sigma_y)} \in B \subseteq \overline{\{a_1\}^X} \implies \{a_2\}^X(y) = 0 \implies \sigma_{y+1} = \sigma_y^\frown 0 \in T_Q.$$

The last implication in each case follows from the hypothesis that T_Q has no dead ends. Now, combining indices a_0 and a_2 produces a recursive mapping $\{b_1\} : P \to Q$ — that is, $Q \leq_M P$, contrary to hypothesis.

The strategy for satisfying $(2a)$ relies on restraints imposed on the enumeration of markers into A and B. The result of these restraints, described below, is to establish the existence of a recursive functional H such that if $(2a)$ fails, then for all $X \in P$ and all y,

$$\{a\}^{X,A}(y) \simeq \{a\}^{X, A_{H(X,y)}}_{H(X,y)}(y).$$

It follows that there is an index b_2 such that for all $X \in P$, $\{b_2\}^X = \{a\}^{X,A} \in Q$ — that is, $\{b_2\}$ witnesses that $Q \leq_M P$, contrary to hypothesis.

Before we can continue with the details of the proof, we need to develop some machinery. The basic tools of the proof are the so-called *hat trick* and the notion of a *length of agreement* function, which we shall adapt in several ways to the present context.

Definition 15. *For any tree T and any s, T^s denotes the set of members of T of length s.*

Since T_P is Π^0_1, it may be represented as the intersection of a decreasing sequence $\langle T_{P,s} : s \in \omega \rangle$ of recursive trees with the property that $\lim_{t\to\infty} T^s_{P,t} = T^s_P$.

We write $\{a\}^\sigma_s(y) \simeq i$ to mean that the oracle computation with index a applied to argument y asks questions of the oracle only for $z < |\sigma|$ and converges in at most s steps with value i. Similarly, $\{a\}^\sigma_s \upharpoonright y \in T$ means that for all $z < y$, there is some i_z such that $\{a\}^\sigma_s(z) \simeq i_z$ and $\langle i_0, i_1, \dots, i_{y-1} \rangle \in T$. The basic properties of computations yield immediately the following facts.

Proposition 16. *For all values of the variables,*

(i) $\{a\}^X(y) \simeq i \iff \exists s \left[\{a\}^{X\upharpoonright s}_s(y) \simeq i \right]$;

(ii) $\{a\}^X \upharpoonright y \in T \implies \exists s \left[\{a\}^{X\upharpoonright s}_s(y) \in T \right]$;

(iii) $\{a\}^\sigma_s(y) \simeq i \implies$

$$(\forall \tau \succeq \sigma)(\forall t \geq s)\{a\}^\tau_t(y) \simeq i \quad \textit{and} \quad (\forall X \succ \sigma)\{a\}^X(y) \simeq i;$$

(iv) $\{a\}^\sigma_s \upharpoonright y \in T \implies$

$$(\forall \tau \succeq \sigma)(\forall t \geq s)\{a\}^\tau_t \upharpoonright y \in T \quad \textit{and} \quad (\forall X \succ \sigma)\{a\}^X \upharpoonright y \in T.$$ □

If P and R are two Π^0_1 classes with associated trees T_P and T_R, an index a witnesses that $R \leq_M P$ iff $\{a\} : P \to R$ — that is, for all $X \in P$, $\{a\}^X \in R$ or equivalently

$$\forall y(\forall X \in P)\{a\}^X \restriction y \in T_R.$$

It will be useful to note an equivalent condition.

Proposition 17. *For any Π^0_1 classes P and R and any a and y,*

$$(\forall X \in P)\Big[\{a\}^X \restriction y \in T_R\Big] \iff \exists s(\forall \sigma \in T^s_{P,s})\Big[\{a\}^\sigma_s \restriction y \in T_R\Big].$$

Hence,

$$\{a\} : P \to R \iff \forall y \exists s(\forall \sigma \in T^s_{P,s})\Big[\{a\}^\sigma_s \restriction y \in T_R\Big].$$

Proof. By Proposition 16, from the left-hand side it follows that

$$(\forall X \in P)\ \exists s\Big[\{a\}^{X\restriction s}_s \restriction y \in T_R\Big], \tag{1}$$

and hence, by König's Lemma (compactness)

$$\exists s(\forall X \in P)\Big[\{a\}^{X\restriction s}_s \restriction y \in T_R\Big], \tag{2}$$

since otherwise, $\{\sigma \in T_P : \{a\}^\sigma_{|\sigma|} \restriction y \notin T_R\}$ is an infinite subtree of the finitely branching tree T_P, hence has an infinite path contrary to (1). Now by (2), fix s such that for all $X \in P$, $\{a\}^{X\restriction s}_s \restriction y \in T_R$. For some $t \geq s$, $T^s_{P,t} = T^s_P$, so for each $\tau \in T^t_{P,t}$, $\tau \restriction s \in T^s_P$. Since T_P has no dead ends, for each $\tau \in T^t_{P,t}$ there is an $X \in P$ such that $X \restriction s = \tau \restriction s$ and hence $\{a\}^{\tau\restriction s}_s \restriction y \in T_R$. Then by Proposition 16, $\{a\}^{\tau\restriction s}_s \restriction y = \{a\}^\tau_t \restriction y$ and the right-hand side holds with t for s. Conversely, given the right-hand side, fix s such that for all $\sigma \in T^s_{P,s}$, $\{a\}^\sigma_s \restriction y \in T_R$. Then for each $X \in P$, $X \restriction s \in T^s_P \subseteq T^s_{P,s}$, so $\{a\}^X \restriction y = \{a\}^{X\restriction s}_s \restriction y \in T_R$. Hence the left-hand side holds. □

We introduce next some functions which measure the extent to which the partial recursive function with index a maps one Π^0_1 class P into another R.

Definition 18. *For any Π^0_1 classes P and R and any a and s,*

$$\ell^{P,R}(a) = \begin{cases} \infty, & \textit{if } \{a\} : P \to R; \\ \textit{least } y\Big[(\exists X \in P)\ \{a\}^X \restriction (y+1) \notin T_R\Big], & \textit{otherwise;} \end{cases}$$

$$\ell^{P,R}(a,s) = \textit{least } y\left[(\exists \sigma \in T^s_{P,s})\ \{a\}^\sigma_s \restriction (y+1) \notin T_{R,s}\right];$$

$$\ell^{+P,R}(a,s) = \max_{s' \leq s}\left[\ell^{P,R}(a,s')\right].$$

The notation should be interpreted to mean that $a^X \upharpoonright (y+1) \notin T_R$ holds also if for some $z \leq y$, $\{a\}^X(z) \uparrow$. Thus $\{a\} : P \to R$ iff $\ell^{P,R}(a) = \infty$ and $\ell^{+P,R}(a, s)$ approximates $\ell^{P,R}(a)$ in the following sense.

Proposition 19. *For any Π^0_1 classes P and R,*

(i) if $\ell^{P,R}(a) = \infty$, then $\lim_{s\to\infty} \ell^{+P,R}(a, s) = \infty$;
(ii) if $\ell^{P,R}(a) < \infty$, then for some number $\ell^{+P,R}(a) \geq \ell^{P,R}(a)$, $\lim_{s\to\infty} \ell^{+P,R}(a, s) = \ell^{+P,R}(a)$;
(iii) for all $s \leq t$, $\ell^{+P,R}(a, s) \leq \ell^{+P,R}(a, t)$.

Proof. Part (i) is simply a translation of Proposition 17. For (ii), if $\ell^{P,R}(a) < \infty$, then for some $X \in P$, $\{a\}^X \upharpoonright (\ell^{P,R}(a) + 1) \notin T_R$. Let

$$\overline{y} := \max\{y \leq \ell^{P,R}(a) : (\forall X \in P)(\forall z \leq y)\{a\}^X(z) \downarrow\}.$$

If $\overline{y} < \ell^{P,R}(a)$, then easily $\lim_{s\to\infty} \ell^{+P,R}(a, s) \leq \overline{y} + 1$. If $\overline{y} = \ell^{P,R}(a)$, then for some $\overline{s}$ and some $\sigma \in T^{\overline{s}}_P$,

$$(\forall z \leq \ell^{P,R}(a))\{a\}^{\sigma}_{\overline{s}}(z) \downarrow \quad \text{but} \quad \{a\}^{\sigma}_{\overline{s}} \upharpoonright (\ell^{P,R}(a) + 1) \notin T_{R,\overline{s}}.$$

Hence, for all $s \geq \overline{s}$,

$$\exists \sigma \in T^s_{P_s} \left[\{a\}^{\sigma}_s \upharpoonright (\ell^{P,R}(a) + 1) \notin T_{R,s}\right],$$

so $\ell^{P,R}(a, s) \leq \ell^{P,R}(a)$. Furthermore, by the same argument as for (i), there exist s such that $\ell^{P,R}(a, s) = \ell^{P,R}(a)$ and thus

$$\lim_{s\to\infty} \ell^{+P,R}(a, s) = \max\{\ell^{P,R}(a), \ell^{+P,R}(a, \bar{s})\} =: \ell^{+P,R}(a).$$

Part (iii) is immediate from the definition. □

As part of the proof below we shall need to consider also mappings of the form $\{b\} : P \otimes \{A\} \to Q$, where A is a c.e. set given by a *recursive stage enumeration* $\langle A_s : s \in \omega\rangle$ — that is an increasing chain of finite sets with union A such that the relation $\{\langle x, s\rangle : x \in A_s\}$ is recursive. We recall first the "hat trick", adapted to the current setting. For any computation of the form $\{b\}^{\sigma,A}_s(x)$, we denote by $\mathbf{u}(A_s; \sigma, b, x, s)$ the actual A_s-use of the computation — that is, the smallest number which properly bounds all oracle queries to A_s. In the following, σ may denote either a finite or infinite sequence.

Definition 20. *For any recursive stage enumeration $\langle A_s : s \in \omega\rangle$ of a set A and any b and σ, set*

$$p_s := \begin{cases} \textit{least } p\ [p \in A_s \setminus A_{s-1}], & \textit{if } A_s \setminus A_{s-1} \neq \emptyset; \\ \max\ A_s \cup \{s\}, & \textit{otherwise;} \end{cases}$$

$$\widehat{\{b\}}^{\sigma,A_s}_s(x) \simeq \begin{cases} \{b\}^{\sigma,A_s}_s(x), & \textit{if } \mathbf{u}(A_s; \sigma, b, x, s) \leq p_s; \\ \uparrow, & \textit{otherwise;} \end{cases}$$

$$\hat{\mathbf{u}}(A_s; \sigma, b, x, s) := \begin{cases} \mathbf{u}(A_s; \sigma, b, x, s), & \textit{if } \widehat{\{b\}}^{\sigma,A_s}_s(x) \downarrow; \\ 0, & \textit{otherwise.} \end{cases}$$

We say that $\widehat{\{b\}}_s^{\sigma,A_s}(x) \downarrow$ correctly *iff* $\widehat{\{b\}}_s^{\sigma,A_s}(x) \downarrow$ *and the computation is A-correct in the sense that* $A_s \restriction \mathbf{u}(A_s;\sigma,b,x,s) = A \restriction \mathbf{u}(A_s;\sigma,b,x,s)$. *s is a* true stage *in the stage enumeration* $\langle A_s : s \in \omega\rangle$ *of a set A iff* $A_s \restriction p_s = A \restriction p_s$. $\mathbf{V}^A$ *denotes the set of true stages.*

Some familiar properties of computations carry over to this context.

Lemma 21. *(Correctness Lemma) For any stage enumeration* $\langle A_s : s \in \omega\rangle$ *of a set A, any* Π_1^0 *class P, and any X, σ, b, s and x,*

(i) $\{b\}^{X,A}(x)\downarrow \iff \exists s\ \widehat{\{b\}}_s^{X\restriction s,A_s}(x)\downarrow$ *correctly;*

(ii) *if* $\widehat{\{b\}}_s^{\sigma,A_s}(x) \simeq z$ *correctly, then for all* $t \geq s$ *and* $X \supseteq \tau \supseteq \sigma$, $\widehat{\{b\}}_t^{\tau,A_t}(x) \simeq z$ *correctly and* $\{b\}^{X,A}(x) \simeq z$;

(iii) $(\forall X \in P)\{b\}^{X,A}(x)\downarrow \iff \exists s\ (\forall \sigma \in T_{P,s}^s)\ \widehat{\{b\}}_s^{\sigma,A_s}(x)\downarrow$ *correctly;*

(iv) *if for all* $\sigma \in T_{P,s}^s$, $\widehat{\{b\}}_s^{\sigma,A_s}(x) \simeq z_\sigma$ *correctly, then for all* $t \geq s$ *and all* $\tau \in T_{P,t}^t$, $\widehat{\{b\}}_t^{\tau,A_t}(x) \simeq z_{\tau\restriction s}$ *correctly, and for all* $X \in P$, $\{b\}^{X,A}(x) \simeq z_{X\restriction s}$;

(v) *if* $s \in \mathbf{V}^A$ *and* $\widehat{\{b\}}_s^{\sigma,A_s}(x)\downarrow$, *then* $\widehat{\{b\}}_s^{\sigma,A_s}(x)\downarrow$ *correctly.*

Proof. Parts (i) and (ii) are simple consequences of the definitions and furthermore are special cases of (iii) and (iv). For (iii) ($\Rightarrow$), suppose that $(\forall X \in P)\ \{b\}^{X,A}(x)\downarrow$. Arguing as in the proof of Proposition 17, there is some t such that for all $\tau \in T_{P,t}^t$, $\{b\}^{\tau,A}(x)\downarrow$. Let

$$u := \max\{\mathbf{u}(A;\tau,b,x) : \tau \in T_{P,t}^t\}$$

and choose $s \geq t$ such that $A \restriction u = A_s \restriction u$. Then for each $\sigma \in T_{P,s}^s$,

$$\widehat{\{b\}}_s^{\sigma,A_s}(x) \simeq \{b\}^{\sigma\restriction t,A}(x)\downarrow,$$

since $\sigma \restriction t \in T_{P,s} \subseteq T_{P,t}$, and by the choice of s, these computations are correct.

Now suppose that s is such that for all $\sigma \in T_{P,s}^s$, $\widehat{\{b\}}_s^{\sigma,A_s}(x) \simeq z_\sigma$ correctly. Then for $u_\sigma := \mathbf{u}(A_s;\sigma,b,x,s)$, for each $\sigma \in T_{P,s}^s$, $A \restriction u_\sigma = A_s \restriction u_\sigma$, so for all $t \geq s$, $A \restriction u_\sigma = A_t \restriction u_\sigma$. Hence, for each $\tau \in T_{P,t}^t$,

$$\widehat{\{b\}}_t^{\tau,A_t}(x) \simeq \widehat{\{b\}}_s^{\tau\restriction s,A_s}(x) \simeq z_{\tau\restriction s}$$

since $\tau \restriction s \in T_{P,t} \subseteq T_{P,s}$, and this computation is correct. Similarly, for $X \in P$, $\{b\}^{X,A}(x) \simeq \widehat{\{b\}}_s^{X\restriction s,A_s}(x) \simeq z_{X\restriction s}$. This establishes (iv) as well as (iii) ($\Leftarrow$). Finally, (v) is immediate from the definitions. □

The associated length of agreement functions are

Definition 22. *For any* Π_1^0 *classes P and Q, any recursive stage enumeration* $\langle A_s : s \in \omega\rangle$ *of a set A and any a, set*

$$\ell^{P\times A,Q}(a) := \begin{cases} \infty, & \text{if } \{a\} : P \otimes \{A\} \to Q; \\ \text{least } y\left[(\exists X \in P)\ \{a\}^{X,A} \restriction (y+1) \notin T_Q\right], & \text{otherwise.} \end{cases}$$

As recursive approximations to $\ell^{P\times A,Q}$ *we set*

$$\ell^{P\times A,Q}(a,s) := \textit{least } y\left[(\exists\sigma\in T^s_{P,s})\ \widehat{\{a\}}^{\sigma,A_s}_s\restriction(y+1)\notin T_{Q,s}\right],$$

and

$$\ell^{P\times A,Q}(X;a,s) := \textit{least } y\left[\widehat{\{a\}}^{X\restriction s,A_s}_s\restriction(y+1)\notin T_{Q,s}\right],$$

For any y*, we say that* $\ell^{P\times A,Q}(a,s)\geq y$ correctly *iff all of the following hold:*

(i) $\ell^{P\times A,Q}(a,s)\geq y$
(ii) for all $\sigma\in T^s_{P,s}$ *and all* $z<y$, $\widehat{\{a\}}^{\sigma,A_s}_s(z)\downarrow$ *correctly;*
(iii) for all $\sigma\in T^s_{P,s}$, $\widehat{\{a\}}^{\sigma,A_s}_s\restriction y\in T_Q$.

Similarly, $\ell^{P\times A,Q}(X;a,s)\geq y$ correctly *iff all of the following hold:*

(iv) $\ell^{P\times A,Q}(X;a,s)\geq y$
(v) for all $z<y$, $\widehat{\{a\}}^{X\restriction s,A_s}_s(z)\downarrow$ *correctly;*
(vi) $\widehat{\{a\}}^{X\restriction s,A_s}_s\restriction y\in T_Q$.

The key properties of these functions are contained in the following

Lemma 23. *(Correctness Lemma for Length CLL) For any* Π^0_1 *classes* P *and* Q*, any recursive stage enumeration* $\langle A_s : s\in\omega\rangle$ *of a set* A*, and any* a*,* y *and* s*,*

(i) if $y\leq\ell^{P\times A,Q}(a)$*, there exists* s *such that* $\ell^{P\times A,Q}(a,s)\geq y$ *correctly;*
(ii) if $\ell^{P\times A,Q}(a,s)\geq y$ *correctly, then* $y\leq\ell^{P\times A,Q}(a)$ *and for all* $t\geq s$, $\ell^{P\times A,Q}(a,t)\geq y$ *correctly;*
(iii) if $\ell^{P\times A,Q}(a)\geq y$, $\ell^{P\times A,Q}(X;a,s)\geq y$ *and for all* $z<y$, $\widehat{\{a\}}^{X\restriction s,A_s}_s(z)\downarrow$ *correctly — in particular, if* $s\in\mathbf{V}^A$ *— then* $\ell^{P\times A,Q}(X;a,s)\geq y$ *correctly.*

Proof. Part (i) follows by the same methods as in the proof of Proposition 17. For (ii), assume that $\ell^{P\times A,Q}(a,s)\geq y$ correctly. Then for all $\sigma\in T^s_{P,s}$ and $z<y$, $\widehat{\{a\}}^{\sigma,A_s}(z)\downarrow$ correctly, so by 21(ii), for all $\sigma\in T^s_{P,s}$ and all $t\geq s$,

$$\widehat{\{a\}}^{\sigma,A_t}_t\restriction y\simeq\widehat{\{a\}}^{\sigma,A_s}_s\restriction y\simeq\{a\}^{\sigma,A}\restriction y\in T_Q\subseteq T_{Q,t}.$$

Hence for all $\sigma\in T^s_{P,s}$, $\{a\}^{\sigma,A}\restriction y\in T_Q$. Then on the one hand, for all $X\in P$,

$$\{a\}^{X,A}\restriction y\simeq\widehat{\{a\}}^{X\restriction s,A_s}_s\restriction\in T_Q,\quad\text{so}\quad\ell^{P\times A,Q}(a)\geq y,$$

and on the other for all $t\geq s$ and $\tau\in T^t_{P,t}$,

$$\widehat{\{a\}}^{\tau,A_t}_t\restriction y\simeq\widehat{\{a\}}^{\tau\restriction s,A_s}_s\restriction y\in T_{Q,t},\quad\text{so}\quad\ell^{P\times A,Q}(a,t)\geq y.$$

For (iii), given the hypotheses, we have

$$\widehat{\{a\}}^{X\restriction s,A_s}_s\restriction y\simeq\{a\}^{X,A}\restriction y\in T_Q,$$

from which it follows that $\ell^{P\times A,Q}(a,s)\geq y$ correctly. □

Corollary 24. *For any Π^0_1 classes P and Q, any recursive stage enumeration $\langle A_s : s \in \omega\rangle$ of a set A, and any a, y and s,*

(i) if $\ell^{P\times A,Q}(a) = \infty$, then $\lim_{s\to\infty} \ell^{P\times A,Q}(a,s) = \infty$, and for all $X \in P$, $\lim_{s\to\infty} \ell^{P\times A,Q}(X; a, s) = \infty$;
(ii) if $\ell^{P\times A,Q}(a) < \infty$, then for all sufficiently large s, $\ell^{P\times A,Q}(a,s) \geq \ell^{P\times A,Q}(a)$ and for all sufficiently large $s \in \mathbf{V}^A$, $\ell^{P\times A,Q}(a,s) = \ell^{P\times A,Q}(a)$.

Proof. Part (i) and the first part of (ii) are immediate from Lemma 23. Choose t large enough that $T_{P,t}^{\ell^{P\times A,Q}(a)+1} = T_P^{\ell^{P\times A,Q}(a)+1}$ and suppose, towards a contradiction, that for some $s \geq t$ with $s \in \mathbf{V}^A$ that $\ell^{P\times A,Q}(a,s) \geq \ell^{P\times A,Q}(a) + 1$. Then for all $\sigma \in T^s_{P,s}$,

$$\widehat{\{a\}}_s^{\sigma,A_s} \restriction (\ell^{P\times A,Q}(a) + 1) \in T_{Q,s}.$$

Since $s \in \mathbf{V}^A$, the computations are all correct, and by the choice of t, we have

$$\{a\}^{\sigma,A} \restriction (\ell^{P\times A,Q}(a) + 1) \in T_Q.$$

Hence, for all $X \in P$, $\{a\}^{X,A} \restriction (\ell^{P\times A,Q}(a) + 1) \in T_Q$ contrary to the definition of $\ell^{P\times A,Q}(a)$. □

We are now ready to continue with the proof of the Density Theorem. The overall structure of the proof is an induction on a to establish $(1a)$ and $(2a)$ simultaneously. To describe the construction, let

$$r^{P\times A,Q}(b,s) := \max\{\hat{\mathbf{u}}(A_s; \sigma, b, s, z) : \sigma \in T^s_{P,s} \text{ and } z \leq \ell^{P\times A,Q}(b,s)\};$$
$$R_s^{P\times A,Q}(a) := \max\{r^{P\times A,Q}(b,s) : b < a\}.$$

For the markers we take $\mathsf{m}^a_{\sigma,t} := \langle a, \langle \sigma, t\rangle\rangle$. We say that $\mathsf{m}^a_{\sigma,t}$ is *qualified at* stage $s \geq t$ iff $\sigma < \ell^{+P,R}(a,t)$ and further

$$\text{0-qualified at } s \iff \mathsf{m}^a_{\sigma,t} \notin B_s \text{ and } \sigma^\frown 0 \notin T_{Q,s} \text{ and } \mathsf{m}^a_{\sigma,t} > R_s^{P\times A,Q}(a);$$
$$\text{1-qualified at } s \iff \sigma^\frown 0 \in T_{Q,s} \text{ and } \sigma^\frown 1 \notin T_{Q,s} \text{ and } \mathsf{m}^a_{\sigma,t} > R_s^{P\times A,Q}(a).$$

Now the construction is as follows: at stage s, for all $a, \sigma, t < s$,

(i) enumerate into A_{s+1} all markers $\mathsf{m}^a_{\sigma,t}$ which are 0-qualified at s;
(ii) enumerate into B_{s+1} all markers $\mathsf{m}^a_{\sigma,t}$ which are 1-qualified at s.

We define as usual

$$A^{[a]} := \{\langle a, y\rangle : \langle a, y\rangle \in A\} \quad \text{(the } a\text{-th column of } A);$$
$$A^{[\leq a]} := \bigcup_{b\leq a} A^{[b]};$$
$$\mathbf{V}^A_a := \{s : A_s^{[\leq a]} \restriction p_s = A^{[\leq a]} \restriction p_s\};$$
$$\mathbf{V}^A_{<a} := \bigcap_{b<a} \mathbf{V}^A_b.$$

Before addressing directly the conditions ($1a$) and ($2a$), we derive some consequences of the construction. We say that $\ell^{P\times A,Q}(b,s) \geq y$ *very correctly* iff $\ell^{P\times A,Q}(b,s) \geq y$ correctly and

$$\text{for each } \sigma \in T^s_{P,s}, \text{ if } \widehat{\{b\}}^{\sigma,A_s}_s(y)\downarrow, \text{ then } \widehat{\{b\}}^{\sigma,A_s}_s(y)\downarrow \text{ correctly.}$$

Similarly, $\ell^{P\times A,Q}(X;b,s) \geq y$ *very correctly* iff $\ell^{P\times A,Q}(X;b,s) \geq y$ correctly and

$$\text{if } \widehat{\{b\}}^{X\restriction s,A_s}_s(y)\downarrow, \text{ then } \widehat{\{b\}}^{X\restriction s,A_s}_s(y)\downarrow \text{ correctly.}$$

Then, for all a, b, s, and y, and all $X \in P$

(A1) if $s \in \mathbf{V}^A_b$, $T^y_{Q,s} = T^y_Q$ and $\ell^{P\times A,Q}(b,s) \geq y$, then $\ell^{P\times A,Q}(b,s) \geq y$ very correctly;

(A2) if $s \in \mathbf{V}^A_b$, $y \leq \ell^{P\times A,Q}(b)$ and $\ell^{P\times A,Q}(X;b,s) \geq y$, then $\ell^{P\times A,Q}(X;b,s) \geq y$ very correctly;

(B1) $\lim_{s\in\mathbf{V}^A_b} \ell^{P\times A,Q}(b,s) = \ell^{P\times A,Q}(b)$;

(B2) if for all $b < a$, $\ell^{P\times A,Q}(b) < \infty$, then $\lim_{s\in\mathbf{V}^A_{<a}} R^{P\times A,Q}_s(a) =: R^{P\times A,Q}(a)$ exists and is finite.

For (A1), assume that $s \in \mathbf{V}^A_b$; we prove by induction on y that

$$\ell^{P\times A,Q}(b,s) \geq y \quad \text{and} \quad T^y_{Q,s} = T^y_Q \Longrightarrow \ell^{P\times A,Q}(b,s) \geq y \text{ very correctly.}$$

Assume as induction hypothesis that this holds for y and suppose that $\ell^{P\times A,Q}(b,s) \geq y+1$, hence $\ell^{P\times A,Q}(b,s) \geq y$ very correctly (The basis case $y = 0$ is identical without any use of an induction hypothesis). Hence, for all $\sigma \in T_{P,s}$, $\widehat{\{a\}}^{\sigma,A_s} \restriction (y+1) \in T^{y+1}_{Q,s} \subseteq T_Q$ via correct computations, so $\ell^{P\times A,Q}(b,s) \geq y+1$ correctly, and it suffices to prove that for all $\sigma \in T^s_{P,s}$, if $u_\sigma := \hat{\mathbf{u}}(A_s;\sigma,b,y+1,s)$, then for all $t \geq s$, $A_t \restriction u_\sigma = A_s \restriction u_\sigma$. This is immediate for $t = s$, so assume as induction hypothesis that it holds for t. By the construction, any element $x \in A_{t+1} \setminus A_t$ is of the form $x = \langle c, z\rangle$ with $x > R^{P\times A,Q}_t(c)$. If $c \leq b$, then

$$x \in A^{[\leq b]} \setminus A^{[\leq b]}_s \quad \text{so} \quad x \geq p_s \geq u$$

because $s \in \mathbf{V}^A_b$. If $c > b$, then

$$\begin{aligned} x > R^{P\times A,Q}_t(c) &\geq r^{P\times A,Q}(b,t) \\ &\geq \hat{\mathbf{u}}(A_t;\sigma,b,y+1,t) \quad \text{since by 23(ii), } \ell^{P\times A,Q}(b,t) \geq y+1 \\ &\geq u_\sigma. \end{aligned}$$

Hence, in either case $A_{t+1} \restriction u_\sigma = A_t \restriction u_\sigma = A_s \restriction u_\sigma$ as desired.

For (A2), for $s \in \mathbf{V}^A_b$ we prove similarly by induction on $y \leq \ell^{P\times A,Q}(b)$ that

$$\ell^{P\times A,Q}(X;b,s) \geq y \quad \Longrightarrow \quad \ell^{P\times A,Q}(X;b,s) \geq y \text{ very correctly.}$$

Assume as induction hypothesis that this holds for y and suppose that $\ell^{P\times A,Q}(X;b,s) \geq y+1$, hence $\ell^{P\times A,Q}(X;b,s) \geq y$ very correctly. It follows from Lemma 23(iii) that $\ell^{P\times A,Q}(X;b,s) \geq y+1$ correctly, and it suffices to prove that if $u := \hat{\mathbf{u}}(A_s; X \upharpoonright s, b, y+1, s)$, then for all $t \geq s$, $A_t \upharpoonright u = A_s \upharpoonright u$. This is done exactly as in the proof of (A1).

(B1) is immediate from the Corollary to 23 in case $\ell^{P\times A,Q}(b) = \infty$. If $\ell^{P\times A,Q}(b) < \infty$, then by the same Corollary, for all sufficiently large s, $\ell^{P\times A,Q}(b,s) \geq \ell^{P\times A,Q}(b)$. Furthermore, using (A1), by a proof parallel to the proof of the second half of part (ii) of that Corollary, for all sufficiently large $s \in \mathbf{V}_b^A$, $\ell^{P\times A,Q}(b,s) = \ell^{P\times A,Q}(b)$.

Now (B2) follows, since for sufficiently large $s \in \mathbf{V}_b^A$, if $\ell^{P\times A,Q}(b) < \infty$,

$$\begin{aligned} r^{P\times A,Q}(b,s) &= \max\{\hat{\mathbf{u}}(A_s;\sigma,b,z,s) : \sigma \in T_{P,s}^s \quad \text{and} \quad z \leq \ell^{P\times A,Q}(b,s)\} \\ &= \max\{\mathbf{u}(A;\sigma,b,z) : \sigma \in T_P \quad \text{and} \quad z \leq \ell(b)\} \\ &=: r^{P\times A,Q}(b). \end{aligned}$$

Thus under the hypothesis of (B2), for sufficiently large $s \in \mathbf{V}_{<a}^A$, $R_s^{P\times A,Q}(a)$ has the constant value $R^{P\times A,Q}(a) := \max\{r^{P\times A,Q}(b) : b < a\}$.

We now proceed to the proof of $(1a)$ and $(2a)$ along with

$$A^{[a]} \quad \text{and} \quad \mathbf{V}_a^A \quad \text{are recursive} \tag{3a}$$

by induction on a. Assume as induction hypothesis that $(1b)$, $(2b)$ and $(3b)$ hold for all $b < a$. Hence for all $b < a$, $\ell^{P\times A,Q}(b) < \infty$ and thus by (B2), $\lim_{s\in\mathbf{V}_{<a}^A} R_s^{P\times A,Q}(a) = R^{P\times A,Q}(a)$. Suppose towards a contradiction that $(1a)$ fails, so

$$\ell^{P,Q+R}(a) = \infty \quad \text{and thus} \quad \lim_{s\to\infty} \ell^{+P,Q+R}(a,s) = \infty$$

by Proposition 19(i). By (iii) of this Proposition, if

$$g(\sigma) := \text{ least } t \left[\ell^{+P,Q+R}(a,t) > \sigma \wedge \mathsf{m}_{\sigma,t}^a > R^{P\times A,Q}(a)\right],$$

then $\mathsf{m}_{\sigma,g(\sigma)}^a$ is qualified at all $s \geq g(\sigma)$, and by (B2), for all sufficiently large $s \in \mathbf{V}_a^A$, $R_s^{P\times A,Q}(a) = R^{P\times A,Q}(a)$ so for $\sigma \in T_Q$, $\mathsf{m}_{\sigma,g(\sigma)}^a$ is 0-qualified at s iff $\sigma^\frown 0 \notin T_{Q,s}$ and 1-qualified at s iff $\sigma^\frown 1 \notin T_{Q,s}$. Hence we have

$$\sigma^\frown 0 \notin T_Q \quad \Longrightarrow \quad \exists s\left[\mathsf{m}_{\sigma,g(\sigma)}^a \in A_{s+1}\right] \quad \Longrightarrow \quad \mathsf{m}_{\sigma,g(\sigma)}^a \in A,$$

and

$$\sigma^\frown 1 \notin T_Q \quad \Longrightarrow \quad \exists s\left[\mathsf{m}_{\sigma,g(\sigma)}^a \in B_{s+1}\right] \quad \Longrightarrow \quad \mathsf{m}_{\sigma,g(\sigma)}^a \in B,$$

Thus, with a_2 as in the sketch above, the index b_1 defined by

$$\{b_1\}^X(y) \simeq \begin{cases} \{a\}^X(y+1), & \text{if } \{a\}^X(0) = 0; \\ \{a_2\}^X(y), & \text{if } \{a\}^X(0) = 1; \end{cases}$$

witnesses that $Q \leq_M P$, contrary to hypothesis. Hence $(1a)$ holds and $\ell^{P,Q+R}(a) < \infty$.

We establish next $(3a)$ and argue first that $A^{[a]}$ is recursive. Define

$$j_a(t) := \text{ least } s \geq t \left[R_s^{P\times A,Q}(a) = R^{P\times A,Q}(a) \right].$$

j_a is well-defined by Proposition 19 and (B2) and is clearly recursive. Now, let k_a be a computable function such that

$$k_a(\mathsf{m}^a_{\sigma,t}) \simeq \begin{cases} 0, & \text{if } \sigma \geq \ell^{+P,Q+R}(a); \\ A_{j_a(\sigma,t)+1}(\mathsf{m}^a_{\sigma,t}), & \text{if } \sigma < \ell^{+P,Q+R}(a) \quad \text{and} \quad t \geq s_a; \\ A(\mathsf{m}^a_{\sigma,t}), & \text{otherwise;} \end{cases}$$

where

$$s_a := \text{ least } s \left[\forall \sigma \leq \ell^{+P,Q+R}(a)(\forall i < 2)\left(\sigma^\frown i \in T_Q \iff \sigma^\frown i \in T_{Q,s}\right) \right.$$
$$\left. \wedge \quad \ell^{+P,Q+R}(a,s) = \ell^{+P,Q+R}(a) \wedge \forall t \geq s \left(R^{P\times A,Q}(a) \leq R_t^{P\times A,Q}(a) \right) \right].$$

Since the third clause has only finitely many instances, k_a is recursive and it suffices to show that for all σ and t, $k_a(\mathsf{m}^a_{\sigma,t}) = A(\mathsf{m}^a_{\sigma,t})$. Clearly $\mathsf{m}^a_{\sigma,t} \notin A \implies k_a(\mathsf{m}^a_{\sigma,t}) = 0$. If $\sigma \geq \ell^{+P,Q+R}(a)$, then $\mathsf{m}^a_{\sigma,t}$ is never qualified and hence never enumerated into A. Suppose that $\sigma < \ell^{+P,Q+R}(a)$, $t \geq s_a$, and $\mathsf{m}^a_{\sigma,t} \in A$. Then for some $s \geq t$, $\mathsf{m}^a_{\sigma,t}$ is 0-qualified at s — that is,

$$\sigma < \ell^{+P,Q+R}(a,s), \quad \sigma^\frown 0 \notin T_{Q,s} \quad \text{and} \quad \mathsf{m}^a_{\sigma,t} > R_s^{P\times A,Q}(a).$$

But since $j_a(t) \geq t \geq s_a$, also $\sigma < \ell^{+P,Q+R}(a, j_a(t))$, $\sigma^\frown 0 \notin T_{Q,j_a(t)}$ and

$$\mathsf{m}^a_{\sigma,t} > R_s^{P\times A,Q}(a) \geq R^{P\times A,Q}(a) = R_{j_a(t)}^{P\times A,Q}(a).$$

Hence $\mathsf{m}^a_{\sigma,t}$ is 0-qualified at $j_a(t)$ so $\mathsf{m}^a_{\sigma,t} \in A_{j_a(t)+1}$ and also $k_a(\mathsf{m}^a_{\sigma,t}) = 1$.

Combining this with the induction hypothesis, $A^{[\leq a]}$ is recursive and it follows immediately from its definition that also $\mathbf{V}^A_a$ is recursive.

Finally, suppose towards a contradiction that $(2a)$ is not satisfied, so $\ell^{P\times A,Q}(a) = \infty$, and define for each X and y,

$$H(X,y) \simeq \text{ least } s \left[s \in \mathbf{V}^A_a \quad \text{and} \quad \ell^{P\times A,Q}(X;a,s) \geq y+1 \right].$$

H is partial recursive, and by (A2) and Corollary 24, for all $X \in P$ and all y, $H(X,y)$ is defined and $\ell^{P\times A,Q}(X;a,H(X,y)) \geq y+1$ correctly. Thus, there is an index b_2 such that

$$\{b_2\}^X(y) \simeq \widehat{\{a\}}^{X,A_{H(X,y)}}_{H(X,y)}(y) \simeq \{a\}^{X,A}(y),$$

and $\{b_2\}$ witnesses that $Q \leq_M P$, contrary to the hypothesis. Hence $(2a)$ holds and the induction step is complete. □

Corollary 25. *The partial ordering $\leq_M$ restricted to either $\mathcal{P}_M$ or to the sublattice* **AH** *of almost homogeneous degrees is dense.*

Proof. The first assertion is immediate and the second follows from Theorem 12, since, in the notation of the preceding proof, if P and Q are almost homogeneous, then since R is constructed as a c.e. separating class, also R and hence S is almost homogeneous. □

References

1. Cenzer, D.: Π^0_1 classes in computability theory, *Handbook of Computability Theory*, Studies in Logic and the Foundations of Mathematics, **140**, 37–85 (1999)
2. Cenzer, D., Remmel, J.: Index sets for Π^0_1 classes, Ann. Pure and Appl. Logic **43**, 3–61 (1998)
3. Hinman, P.G.: *Recursion-Theoretic Hierarchies*. Perspectives in Mathematical Logic, Springer-Verlag, Berlin (1978)
4. Jockusch, C., Soare, R.: Π^0_1 classes and degrees of theories. Trans. Amer. Math. Soc. **173**, 33–56 (1972)
5. Medvedev, Yu.: Degrees of difficulty of the mass problem. Dok. Akad. Nauk SSSR **104**, 501–504 (1955)
6. Odifreddi, P.: *Classical Recursion Theory*. North-Holland (1989)
7. Simpson, S.: Π^0_1 Sets and Models of WKL_0, to appear in *Reverse Mathematics 2001*, ed. S. Simpson
8. Binns, S., Simpson, S.: Medvedev and Muchnik Degrees of Nonempty Π^0_1 Subsets of 2^ω, preprint, May 2001
9. Sorbi, A.: The Medvedev lattice of degrees of difficulty, in *Computability, Enumerability, Unsolvability*, London Math. Soc. Lecture Notes 224, Cambridge University Press, Cambridge 289–312 (1996)