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Hankel Matrices Acting on the Dirichlet Space

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Abstract

The study of the infinite Hankel matrix acting on analytic function spaces dates back to the influential work of Nehari and Widom on the Hardy space H^2. Since then, it has been extensively generalized to other settings such as weighted Bergman spaces, Dirichlet type spaces, and Möbius invariant function spaces. Nevertheless, several fundamental operator-theoretic questions, including the boundedness and compactness, remain unresolved in the context of the Dirichlet space. Motivated by this, via Carleson measures, the Widom type condition, and the reproducing kernel thesis, we obtain:

(i) necessary and sufficient conditions for bounded and compact operators induced by Hankel matrices on the Dirichlet space, thereby answering a folk question in this field (Galanopoulos et al. in Result Math 78(3) Paper No. 106, 2023);
(ii) necessary and sufficient conditions for bounded and compact operators induced by Cesàro type matrices on the Dirichlet space.

As a beneficial product, we find an intrinsic function-theoretic characterization of functions with positive decreasing Taylor coefficients in the function space $\mathcal{X}$ throughly studied by Arcozzi et al. (Lond Math Soc II Ser 83(1):1–18, 2011). In addition, we

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also show that a random Dirichlet function almost surely induces a compact Hankel type operator on the Dirichlet space.

Keywords Hankel matrix · Cesàro matrix · Carleson measure · Dirichlet space · Bergman space

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1 Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. Suppose $\boldsymbol{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers. An infinite Hankel matrix $\boldsymbol{H}_{\boldsymbol{\lambda}}$ induced by the sequence $\boldsymbol{\lambda}$ is given by $\boldsymbol{H}_{\boldsymbol{\lambda}} = (\lambda_{j+k})_{j,k\in\mathbb{N}}$. The Hankel matrix $\boldsymbol{H}_{\boldsymbol{\lambda}}$ is initially defined for all finitely supported sequences in ℓ^2. The celebrated Nehari theorem [36] illustrates that $\boldsymbol{H}_{\boldsymbol{\lambda}}$ represents a bounded operator on ℓ^2 if and only if there exists a function ψ in L^∞ on the unit circle $\mathbb{T}$ such that λ_n, $n \geq 0$, is the n-th Fourier coefficient of ψ.

For $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $H(\mathbb{D})$, the class of functions analytic on the open unit disk $\mathbb{D}$, the Hankel matrix $\boldsymbol{H}_{\boldsymbol{\lambda}}$ acts on the function f via

$$\mathcal{H}_{\boldsymbol{\lambda}}(f)(z) := \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\lambda_{n+k}a_k\right)z^n, \quad z \in \mathbb{D},$$

whenever the right-hand side makes sense and defines an analytic function on $\mathbb{D}$. For a space $X \subseteq H(\mathbb{D})$ equipped with a norm $\|\cdot\|_X$, the Hankel type operator $\mathcal{H}_{\boldsymbol{\lambda}}$ is bounded on X if $\mathcal{H}_{\boldsymbol{\lambda}}(f)$ is well defined in $H(\mathbb{D})$ for any $f \in X$ and there exists a positive constant C independent of g such that $\|\mathcal{H}_{\boldsymbol{\lambda}}(g)\|_X \leq C\|g\|_X$ for all g in X.

By the Fourier transform, the Hankel matrix $\boldsymbol{H}_{\boldsymbol{\lambda}}$ represents a bounded operator on ℓ^2 if and only if $\mathcal{H}_{\boldsymbol{\lambda}}$ is bounded on the Hardy space H^2. See [38] for details. When $\boldsymbol{\lambda}$ is the moment sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of a finite positive Borel measure μ on $[0, 1)$, where $\mu_n = \int_{[0,1)} t^n d\mu(t)$, the related Hankel matrix is denoted by $\boldsymbol{H}_\mu$ in the literature. Write $\mathcal{H}_\mu$ for the corresponding Hankel type operator. Widom [45] proved that $\mathcal{H}_\mu$ is bounded on H^2 if and only if $\mu_j = O(\frac{1}{j+1})$. It is worth noting that $\{\frac{1}{j+1}\}_{j\in\mathbb{N}}$ is the moment sequence of the Lebesgure measure on $[0, 1)$, which corresponds to the classical Hilbert matrix $\boldsymbol{H} = ((j + k + 1)^{-1})_{j,k\in\mathbb{N}}$. See [12, 17–19, 29–31, 33–35, 48] for developments of the Hilbert matrix acting on analytic function spaces.

Since then, operator questions regarding the Hankel matrix have been investigated in various other function spaces of analytic functions on the open unit disk, including Hardy spaces, weighted Bergman spaces, Dirichlet type spaces, and Möbius invariant function spaces. Diamantopoulos' [16] work illustrates that Widom's condition remains true for the Dirichlet type space $\mathcal{D}_\alpha$ with $0 < \alpha < 2$. He also noted that the boundedness of $\mathcal{H}_\mu$ on the classical Dirichlet space $\mathcal{D}$ and the Bergman space A^2 coincides. Under a milder condition, using Carleson measures for $\mathcal{D}$, Galanopoulos and Peláez [21] completely characterized the boundedness of $\mathcal{H}_\mu$ on the Dirichlet space. Chatzifountas, Girela and Peláez studied the boundedness and compactness of

$\mathcal{H}_\mu$ between distinct Hardy spaces in [14]. Girela and Merchán [25] obtained a method to give complete descriptions of the boundedness and compactness of $\mathcal{H}_\mu$ on some Hardy spaces and Möbius invariant function spaces. Bao et al. [6] considered bounded $\mathcal{H}_\mu$ on analytic functions spaces in terms of so-called Hankel measures.

In their recent work, Galanopoulos et al. [23] demonstrated that a 1-logarithmic 1-Carleson measure ensures the boundedness of $\mathcal{H}_\mu$ on the Dirichlet space. However, they also pointed out that the characterization of the finite positive Borel measure μ on $[0, 1)$ for which the operator $\mathcal{H}_\mu$ is bounded on the Dirichlet space remains unresolved. Via Carleson measures, Widom type conditions and the reproducing kernel thesis, the current work answers this question.

Recall that the Dirichlet space $\mathcal{D}$ is a Hilbert space of analytic functions on $\mathbb{D}$ equipped with the Dirichlet inner product

$$\langle f, g\rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}dA(z). \tag{1.1}$$

Here $dA(z) = \pi^{-1}dxdy$ for $z = x + iy$ is the normalized Lebesgue measure on $\mathbb{D}$. A finite positive Borel measure ν on $\mathbb{D}$ is said to be a Carleson measure for the Dirichlet space if there is a positive constant C such that

$$\left(\int_{\mathbb{D}} |f(z)|^2 d\nu(z)\right)^{\frac{1}{2}} \le C\|f\|_{\mathcal{D}} \tag{1.2}$$

for all $f \in \mathcal{D}$. Denote by $CM(\mathcal{D})$ the set of Carleson measures for $\mathcal{D}$. The smallest such C in (1.2) is denoted by $\|\nu\|_{CM(\mathcal{D})}$, the Carleson measure norm of ν. If the identity map $I_d : \mathcal{D} \to L^2(\mathbb{D}, d\nu)$ is compact, then we say that ν is a vanishing Carleson measure for $\mathcal{D}$. See [2, 5, 20, 21, 44, 47] for this definition of Carleson measures for $\mathcal{D}$ and related investigations.

In this paper, we first obtain the following result.

Theorem 1.1 *Suppose $\boldsymbol{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers. Then the Hankel type operator $\mathcal{H}_{\boldsymbol{\lambda}}$ is bounded on the Dirichlet space $\mathcal{D}$ if and only if $h_{\overline{\boldsymbol{\lambda}}}$ is analytic on $\mathbb{D}$ and the measure $|h'_{\overline{\boldsymbol{\lambda}}}(z)|^2 dA(z)$ is a Carleson measure for the Dirichlet space $\mathcal{D}$, where $h_{\overline{\boldsymbol{\lambda}}}(z) = \sum_{n=0}^{\infty} \overline{\lambda_n} z^n$.*

In the case of $\mathcal{H}_\mu$, by Theorem 1.1, the operator $\mathcal{H}_\mu$ is bounded on $\mathcal{D}$ if and only if $|h'_\mu(z)|^2 dA(z)$ is a Carleson measure for $\mathcal{D}$, where

$$h_\mu(z) = \int_{[0,1)} \frac{1}{1 - tz} d\mu(t).$$

Note that in [44] Stegenga's beautiful characterization of $CM(\mathcal{D})$ is related to the logarithmic capacity of a finite union of intervals of the unit circle. Based on certain integrals involve the Carleson box and the heightened box, Arcozzi et al. [2, Theorem 1] gave a complete description of $CM(\mathcal{D})$. There still lacks a direct relation between moment sequences and the boundedness of $\mathcal{H}_\mu$ on the Dirichlet space. We will establish these relations through the use of Widom type conditions and the reproducing kernel thesis.

Corresponding to the Dirichlet inner product (1.1), the reproducing kernel of the Dirichlet space $\mathcal{D}$ at a point $w \in \mathbb{D}$ is

$$K_w(z) = 1 + \log \frac{1}{1 - z\overline{w}}$$

and its normalized reproducing kernel

$$k_w(z) = \frac{K_w(z)}{\sqrt{K_w(w)}}.$$

Now, we state our characterization of the bounded Hankel type operator $\mathcal{H}_\mu$ on the Dirichlet space as follows.

Theorem 1.2 *Suppose μ is a finite positive Borel measure on* $[0, 1)$. *Then the following conditions are equivalent.*

(i) *The Hankel type operator $\mathcal{H}_\mu$ is bounded on $\mathcal{D}$.*
(ii) *The reproducing kernel thesis holds; that is,*

$$\sup_{t \in [0,1)} \|\mathcal{H}_\mu(k_t)\|_{\mathcal{D}} < \infty,$$

where k_t is the normalized reproducing kernel of $\mathcal{D}$ at t in $[0, 1)$.
(iii) *The Widom type condition is true; that is,*

$$\sum_{n=m}^{\infty} n\mu_n^2 = O\left(\frac{1}{\log(m+2)}\right).$$

To prove Theorem 1.2, we need Cesàro type matrices. As benefit products, the necessary and sufficient condition for the boundedness of Cesàro type matrix acting on the Dirichlet space is obtained.

Let $\boldsymbol{\eta} = \{\eta_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers. Recall that the Cesàro type matrix $\boldsymbol{C}_{\boldsymbol{\eta}}$ is the following infinite lower triangular matrix:

$$\boldsymbol{C}_{\boldsymbol{\eta}} = \begin{pmatrix} \eta_0 & 0 & 0 & 0 & \cdots \\ \eta_1 & \eta_1 & 0 & 0 & \cdots \\ \eta_2 & \eta_2 & \eta_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

In the same way as operators induced by Hankel matrices, the matrix $\boldsymbol{C}_{\boldsymbol{\eta}}$ induces a Cesàro type operator $\mathcal{C}_{\boldsymbol{\eta}}$ as follows:

$$\mathcal{C}_{\boldsymbol{\eta}}(f)(z) = \sum_{n=0}^{\infty} \left(\eta_n \sum_{k=0}^{n} a_k\right) z^n, \quad z \in \mathbb{D}, \tag{1.3}$$

for $f(z)=\sum_{n=0}^{\infty} a_n z^n$ in $H(\mathbb{D})$, whenever the right-hand side of (1.3) makes sense and defines an analytic function on $\mathbb{D}$.

If $\eta_n = 1/(n+1)$ for every n, then the Cesàro type matrix $\boldsymbol{C}_{\boldsymbol{\eta}}$ is the classical Cesàro matrix $\boldsymbol{C}$. Danikas and Siskakis [15] showed that the operator induced by $\boldsymbol{C}$ is bounded from H^∞ to $BMOA$. Siskakis [42, 43] studied the action of $\boldsymbol{C}$ on Hardy spaces.

Given a finite positive Borel measure μ on [0, 1), if

$$\eta_n = \mu_n = \int_{[0,1)} t^n d\mu(t)$$

for each n, then we write $\mathcal{C}_{\boldsymbol{\eta}}$ as $\mathcal{C}_\mu$. The operator $\mathcal{C}_\mu$ is said to be the Cesàro-like operator which was introduced in [22, 27] recently. We refer to [7, 9, 23, 24] for more results about $\mathcal{C}_\mu$ on some spaces of analytic functions. From Theorem 5 in [23], if μ is a 1-logarithmic 1-Carleson measure, then $\mathcal{C}_\mu$ is bounded on $\mathcal{D}$. Conversely, if $\mathcal{C}_\mu$ is bounded on $\mathcal{D}$, then μ is a $\frac{1}{2}$-logarithmic 1-Carleson measure. In his recent work [10], Blasco studied the boundedness of Cesàro-like operators induced by complex Borel measures on $\mathbb{D}$ acting on some weighted Dirichlet spaces.

Our subsequent finding gives necessary and sufficient conditions for the boundedness of every Cesàro type operator on the Dirichlet space $\mathcal{D}$.

Theorem 1.3 *Suppose $\boldsymbol{\eta} = \{\eta_n\}_{n=0}^{\infty}$ is a sequence of complex numbers. Then the following conditions are equivalent.*

(i) *The Cesàro type operator $\mathcal{C}_{\boldsymbol{\eta}}$ is bounded on $\mathcal{D}$.*
(ii) *The reproducing kernel thesis holds; that is,*

$$\sup_{t\in[0,1)} \|\mathcal{C}_{\boldsymbol{\eta}} k_t\|_{\mathcal{D}} < \infty,$$

where k_t is the normalized reproducing kernel of $\mathcal{D}$ at t in [0, 1).
(iii) *The Widom type condition is true; that is,*

$$\sum_{n=m}^{\infty} n|\eta_n|^2 = O\left(\frac{1}{\log(m+2)}\right).$$

Following our previous work on the boundedness of Hankel type operators and Cesàro type operators on the Dirichlet space, we also establish their compactness counterparts. These results are included in Sect. 4.

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent identically distributed (i.i.d.) real random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For a sequence $\{\alpha_n\}$ of complex numbers, one formally defines a random power series

$$\varphi_\omega(z) = \sum_{n=0}^{\infty} X_n(\omega)\alpha_n z^n.$$

By Nehari's theorem, almost surely the random analytic function φ_ω induces a bounded Hankel type operator on the Hardy space H^2 if and only if the symbol function φ_ω is a *BMOA* function almost surely. The famous unsolved Anderson's question [1] is to seek the necessary and sufficient condition to characterize random *BMOA* functions. One can consult [28, 37] for recent progress and more details in this subject.

According to Theorem 1.1, the symbol function $h_{\overline{\boldsymbol{\lambda}}}$ of a bounded Hankel type operator $\mathcal{H}_{\boldsymbol{\lambda}}$ on $\mathcal{D}$ must be in the Dirichlet space. However, our next corollary illustrates that, under milder conditions, a random Dirichlet function almost surely induced a compact Hankel type operator, and hence almost surely induced a bounded Hankel type operator.

Theorem 1.4 *Suppose $\boldsymbol{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. real random variables satisfying that $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^4]$ is finite. If $h_{\boldsymbol{\lambda}} \in \mathcal{D}$, then almost surely the Hankel type operator $\mathcal{H}_{\boldsymbol{\omega}}$ is compact on $\mathcal{D}$, where $\boldsymbol{\omega} = \left\{X_n\overline{\lambda_n}\right\}_{n\in\mathbb{N}}$.*

Denote by $\mathbb{Z}$ the set of integers. Let $E \subseteq \mathbb{Z}$. As defined by Rudin [40], the integrable function f on the unite circle $\mathbb{T}$ is called an E-function if $\hat{f}(n)$, the n-th Fourier coefficient of f, is equal to zero for all $n \in \mathbb{Z} \setminus E$. Suppose $p > 2$, the set $E \subseteq \mathbb{Z}$ is called a Rudin's $\Lambda(p)$ set if there is a positive constant C such that $\|f\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^2(\mathbb{T})}$ for all trigonometric polynomials whose coefficients are equal to 0 on $\mathbb{Z}\setminus E$. The Rudin's $\Lambda(p)$ set is a natural generalization of the classical lacunary sequence (the Hadamard set) which plays important roles in many aspects of analysis (see [11, 26, 40] for more details). Recall that an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers is said to be lacunary if there exists q in $(1, \infty)$ such that $n_{k+1}/n_k > q$ for all k. It is also known that there exist Rudin's $\Lambda(p)$ sets rather than Hadamard sets.

Our next result presents a characterization of bounded and compact Hankel type operators on $\mathcal{D}$ when non-zero Fourier coefficients of the symbol function are located in a Rudin's $\Lambda(p)$ set.

Corollary 1.5 *Suppose $p > 2$, the set E is a Rudin's $\Lambda(p)$ set and $\boldsymbol{\lambda} = \{\lambda_n : n \in E\}$ is a sequence of complex numbers. Then the following statements are equivalent.*

(i) *The Hankel type operator $\mathcal{H}_{\boldsymbol{\lambda}}$ is compact on $\mathcal{D}$.*
(ii) *The Hankel type operator $\mathcal{H}_{\boldsymbol{\lambda}}$ is bounded on $\mathcal{D}$.*
(iii) *The symbol function $h_{\overline{\boldsymbol{\lambda}}}(z) = \sum_{n\in E} \overline{\lambda_n} z^n$ belongs to $\mathcal{D}$; that is,*

$$\sum_{n\in E}(n+1)|\lambda_n|^2 < \infty.$$

We end this introduction by some remarks on the boundedness of Hankel matrix on the Dirichlet space and the following function space $\mathcal{X}$ related to the Dirichlet space thoroughly studied by Arcozzi et al. [4]. Precisely, $\mathcal{X}$ is an analytic function space on $\mathbb{D}$ [4, p. 2] consisting of functions f such that

$$\|f\|_{\mathcal{X}}^2 = |f(0)|^2 + \||f'|^2 dA\|_{CM(\mathcal{D})} < \infty.$$

Denote by $\mathcal{X}_0$ the norm closure in $\mathcal{X}$ of the space of polynomials. Equivalently, $f \in \mathcal{X}_0$ if and only if $|f'|^2 dA$ is a vanishing Carleson measure for $\mathcal{D}$. We refer to [4, 39, 46, 47] for more details and applications of $\mathcal{X}$ and $\mathcal{X}_0$.

Now, Theorem 1.1 reads as that the Hankel type operator $\mathcal{H}_{\boldsymbol{\lambda}}$ is bounded on the Dirichlet space $\mathcal{D}$ if and only if $h_{\overline{\boldsymbol{\lambda}}}$ is in $\mathcal{X}$. Such a result may not be surprising to the operator theory on the function spaces. However, there is still a lack of a full function-theoretic description of functions on $\mathcal{X}$ (one can consult [4, p. 16] for more details). Hence, our Widom type conditions in Theorems 1.2 and 3.1 seem to be the first intrinsic function-theoretic characterization of functions with positive decreasing Taylor coefficients in the function space $\mathcal{X}$. For convenience, we include a separate corollary as follows.

Corollary 1.6 *Suppose $\boldsymbol{\lambda} = \{\lambda_n\}_{n=0}^{\infty}$ is a decreasing sequence of positive real numbers and $h_{\boldsymbol{\lambda}}(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ for $z \in \mathbb{D}$. Then $h_{\boldsymbol{\lambda}} \in \mathcal{X}$ if and only if*

$$\sum_{n=m}^{\infty} n\lambda_n^2 = O\left(\frac{1}{\log(m+2)}\right).$$

For an analytic function f with non-zero Fourier coefficients belonging to a Rudin's $\Lambda(p)$ set, by Corollary 1.5, it follows that $f \in \mathcal{X}$ if and only if $f \in \mathcal{D}$. This is a slice generation of Arcozzi, Rochberg, Sawyer, Wick's work [4] related to the equivalence of Parts (3.b) and (3.c) in Theorem 4. More details of discussions about the function space $\mathcal{X}$ are given in Sect. 6.

Throughout this paper, the symbol $A \approx B$ means that $A \lesssim B \lesssim A$. We say that $A \lesssim B$ if there exists a positive constant C such that $A \leq CB$.

2 Bounded Hankel Type Operators $\mathcal{H}_{\boldsymbol{\lambda}}$ on $\mathcal{D}$

In this section we will prove Theorem 1.1. We also consider the Möbius invariance of the norm of a bounded Hankel type operator on the Dirichlet space.

Given a function b in $H(\mathbb{D})$, define a Hankel type bilinear form on the Dirichlet space, initially for f, g in $\mathcal{P}$, as

$$T_b(f, g) := \langle fg, b\rangle_{\mathcal{D}}.$$

Here $\mathcal{P}$ is the space of polynomials. The norm of the bilinear form T_b is

$$\|T_b\|_{\mathcal{D}\times\mathcal{D}} = \sup\left\{|T_b(f, g)| : \|f\|_{\mathcal{D}} = \|g\|_{\mathcal{D}} = 1,\ f, g \in \mathcal{P}\right\}.$$

For a bounded bilinear form T_b on $\mathcal{D}$, T_b is said to be compact on $\mathcal{D}$ if $T_b(B_{\mathcal{D}} \times B_{\mathcal{D}})$ is precompact in $\mathbb{C}$, where $B_{\mathcal{D}} = \{f \in \mathcal{D} : \|f\|_{\mathcal{D}} \leq 1\}$. In other words, T_b is compact on $\mathcal{D}$ if and only if for all bounded sequences $\{(f_n, g_n)\} \subseteq \mathcal{D} \times \mathcal{D}$, the sequence $\{T_b(f_n, g_n)\}$ has a convergent subsequence. See [8] for more descriptions on compact bilinear operators.

With respect to the normalized basis $\{(n+1)^{-1/2}z^n\}$ of the Dirichlet space $\mathcal{D}$, the matrix representation of T_b is

$$\left(\frac{j+k}{\sqrt{j+1}\sqrt{k+1}}\overline{b_{j+k}}\right), \quad j,k=0,1,\ldots,$$

where b_{j+k} is the $(j+k)$-th Taylor coefficient of b. Arcozzi, i et al. characterized the boundedness and compactness of the bilinear form on the Dirichlet space as follows.

Theorem A [3, Theorem 1.1] *Let $b \in H(\mathbb{D})$. Then the following assertions hold.*

(i) *T_b extends to a bounded bilinear form on $\mathcal{D}$ if and only if $b \in \mathcal{X}$.*
(ii) *T_b extends to a compact bilinear form on $\mathcal{D}$ if and only if $b \in \mathcal{X}_0$.*

Proof of Theorem 1.1 Given a sequence of complex number $\boldsymbol{\lambda} = \{\lambda_n\}_{n=0}^{\infty}$, suppose $h_{\overline{\boldsymbol{\lambda}}}$ is analytic on $\mathbb{D}$ and the measure $|h'_{\overline{\boldsymbol{\lambda}}}(z)|^2 dA(z)$ is a Carleson measure for $\mathcal{D}$. Then $h_{\overline{\boldsymbol{\lambda}}} \in \mathcal{X}$. Hence, $h_{\boldsymbol{\lambda}} \in \mathcal{D}$. Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ belong to $\mathcal{D}$. Using the fact that $\mathcal{D}$ is a subset of the Hardy space H^2, we get

$$\sup_{j\in\mathbb{N}}\left|\sum_{k=0}^{\infty}\lambda_{j+k}c_k\right| < \infty.$$

Write $a_j = \sum_{k=0}^{\infty}\lambda_{j+k}c_k$. Then $\{a_j\}_{j\in\mathbb{N}}$ is a bounded sequence of complex numbers and hence $\mathcal{H}_{\boldsymbol{\lambda}}(f)(z) = \sum_{j=0}^{\infty} a_j z^j$ is analytic on $\mathbb{D}$. Note that

$$\|\mathcal{H}_{\boldsymbol{\lambda}} f\|_{\mathcal{D}}^2 = |a_0|^2 + \sum_{j=0}^{\infty}(j+1)|a_{j+1}|^2.$$

Let m be a positive integer. For any polynomial $g(z) = \sum_{j=0}^{m} b_j z^j$ with $\|g\|_{\mathcal{D}} \le 1$, we have

$$\begin{aligned}
\left|a_0\overline{b_0} + \sum_{j=0}^{m-1}(j+1)a_{j+1}\overline{b_{j+1}}\right| &\le |\lambda_0||c_0||b_0| + |b_0|\left|\sum_{k=1}^{\infty}\lambda_k c_k\right| \\
&\quad + \left|\sum_{j=0}^{m-1}\overline{b_{j+1}}(j+1)\sum_{k=0}^{\infty}\lambda_{j+1+k}c_k\right| \\
&= |\lambda_0||c_0||b_0| + |b_0|\left|\int_{\mathbb{D}}\frac{f(z)-f(0)}{z}\overline{h'_{\overline{\boldsymbol{\lambda}}}(z)}dA(z)\right| \\
&\quad + \left|\int_{\mathbb{D}} f(z)g_1'(z)\,\overline{h'_{\overline{\boldsymbol{\lambda}}}(z)}dA(z)\right| \\
&\le |\lambda_0|\|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}} + \|g\|_{\mathcal{D}} \\
&\quad \times \left(\int_{\mathbb{D}}\left|\frac{f(z)-f(0)}{z}\right|^2 |h'_{\overline{\boldsymbol{\lambda}}}(z)|^2 dA(z)\right)^{\frac{1}{2}}
\end{aligned}$$

$$+\left(\int_{\mathbb{D}} |f(z)|^2 |h'_{\overline{\lambda}}(z)|^2 dA(z)\right)^{\frac{1}{2}}$$
$$\times\left(\int_{\mathbb{D}} \left|g_1'(z)\right|^2 dA(z)\right)^{\frac{1}{2}},$$

where $g_1(z) = \sum_{j=0}^{m} \overline{b_j} z^j$. Since $|h'_{\overline{\lambda}}(z)|^2 dA(z)$ is a Carleson measure for $\mathcal{D}$, there exists a positive constant C such that

$$\left| a_0 \overline{b_0} + \sum_{j=0}^{m-1} (j+1) a_{j+1} \overline{b_{j+1}} \right| \leq C \|f\|_{\mathcal{D}}.$$

Therefore

$$\left(|a_0|^2 + \sum_{j=0}^{m-1} (j+1) |a_{j+1}|^2 \right)^{1/2} \leq C \|f\|_{\mathcal{D}}$$

and $\|\mathcal{H}_{\lambda} f\|_{\mathcal{D}} \leq C \|f\|_{\mathcal{D}}$.

On the other hand, suppose the operator $\mathcal{H}_{\lambda}$ is bounded on $\mathcal{D}$. Then, for the function $I(z) = z$ on $\mathbb{D}$, we see that $\mathcal{H}_{\lambda}(I)(z) = \sum_{n=0}^{\infty} \lambda_{n+1} z^n$ belongs to $\mathcal{D}$. This implies that $h_{\overline{\lambda}}$ is analytic on $\mathbb{D}$ and

$$\int_{\mathbb{D}} |h'_{\overline{\lambda}}(z)|^2 dA(z) < \infty.$$

The boundedness of the operator $\mathcal{H}_{\lambda}$ on $\mathcal{D}$ yields

$$|\langle \mathcal{H}_{\lambda} f, g \rangle_{\mathcal{D}}| \lesssim \|f\|_{\mathcal{D}} \|g\|_{\mathcal{D}}$$

for all f and g in $\mathcal{D}$. Moreover, for any polynomials f and g,

$$\int_{\mathbb{D}} f(z) g'(z) \overline{h'_{\overline{\lambda}}(z)}\, dA(z) = \int_{\mathbb{D}} (\mathcal{H}_{\lambda} f)'(z) g'(\overline{z})\, dA(z). \tag{2.1}$$

Consequently,

$$\left| f(0) g(0) \overline{h_{\overline{\lambda}}(0)} \right| + \left| \int_{\mathbb{D}} (f(z) g(z))' \overline{h'_{\overline{\lambda}}(z)} dA(z) \right| \lesssim \|f\|_{\mathcal{D}} \|g\|_{\mathcal{D}}.$$

Therefore, $T_{h_{\overline{\lambda}}}$ is a bounded bilinear form on $\mathcal{D}$. It follows from Theorem A that $h_{\overline{\lambda}} \in \mathcal{X}$ and hence $|h'_{\overline{\lambda}}(z)|^2 dA(z)$ is a Carleson measure for $\mathcal{D}$. This completes the whole proof of Theorem 1.1. □

Denote by Aut($\mathbb{D}$) the Möbius group which consists of all one-to-one analytic functions that map $\mathbb{D}$ onto itself. Let $\widetilde{\mathcal{D}}$ be the space of functions f in $\mathcal{D}$ with $f(0) = 0$. For a sequence $\boldsymbol{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}$ of complex numbers, consider

$$\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}}(f)(z) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\lambda_{n+k}a_k\right)z^n, \quad z \in \mathbb{D},$$

where $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \widetilde{\mathcal{D}}$. For the bounded $\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}}$ on $\widetilde{\mathcal{D}}$, the norm of $\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}}$ is Möbius invariant in the following sense.

Proposition 2.1 *Let $\boldsymbol{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers. Suppose the Hankel type operator $\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}}$ is bounded on $\widetilde{\mathcal{D}}$. Then there exist positive constants C_1 and C_2 depending only on $\boldsymbol{\lambda}$ such that*

$$C_1\|\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}}\| \le \|\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}_{\boldsymbol{\phi}}}\| \le C_2\|\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}}\|$$

for all $\phi \in Aut(\mathbb{D})$, where $\boldsymbol{\lambda}_{\boldsymbol{\phi}} = \left\{\overline{\lambda_{\phi,n}}\right\}_{n\in\mathbb{N}}$ with

$$h_{\overline{\boldsymbol{\lambda}}} \circ \phi(z) = \sum_{n=0}^{\infty} \lambda_{\phi,n} z^n.$$

Proof By Theorem 1.1, the boundedness of $\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}}$ on $\widetilde{\mathcal{D}}$ yields

$$\||h'_{\overline{\boldsymbol{\lambda}}}|^2 dA\|_{CM(\widetilde{\mathcal{D}})} < \infty.$$

From the proof of Theorem 1.1, there are positive constants C_1 and C_2 depending only on $\boldsymbol{\lambda}$ such that

$$C_1\||h'_{\overline{\boldsymbol{\lambda}}}|^2 dA\|_{CM(\widetilde{\mathcal{D}})} \le \|\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}}\| \le C_2\||h'_{\overline{\boldsymbol{\lambda}}}|^2 dA\|_{CM(\widetilde{\mathcal{D}})}. \tag{2.2}$$

Let $\phi \in$ Aut($\mathbb{D}$). For $g \in \widetilde{\mathcal{D}}$, by the change of variables, we see

$$\int_{\mathbb{D}} |g(z)|^2 |(h_{\overline{\boldsymbol{\lambda}}} \circ \phi)'(z)|^2 dA(z) = \int_{\mathbb{D}} |g(\phi^{-1}(w))|^2 |h'_{\overline{\boldsymbol{\lambda}}}(w)|^2 dA(w).$$

Also,

$$\int_{\mathbb{D}} |g'(z)|^2 dA(z) = \int_{\mathbb{D}} |(g \circ \phi^{-1})'(w)|^2 dA(w).$$

Thus there is $C > 0$ such that

$$\int_{\mathbb{D}} |g(z)|^2 |(h_{\overline{\boldsymbol{\lambda}}} \circ \phi)'(z)|^2 dA(z) \le C \int_{\mathbb{D}} |g'(z)|^2 dA(z)$$

for all $g \in \widetilde{\mathcal{D}}$ if and only if

$$\int_{\mathbb{D}} |k(w)|^2 |h'_{\overline{\boldsymbol{\lambda}}}(w)|^2 dA(w) \le C \int_{\mathbb{D}} |k'(z)|^2 dA(z)$$

for all $k \in \widetilde{\mathcal{D}}$. This means

$$\| |h'_{\overline{\boldsymbol{\lambda}}}|^2 dA \|_{CM(\widetilde{\mathcal{D}})} = \| |(h_{\overline{\boldsymbol{\lambda}}} \circ \phi)'|^2 dA \|_{CM(\widetilde{\mathcal{D}})}.$$

Combining this with (2.2), we get the desired result. □

Remark 2.2 By a personal communication, Professor Blasco informed us that Proposition 2.1 can be stated as follows. Suppose $\phi \in \mathrm{Aut}(\mathbb{D})$ and $\boldsymbol{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers. Then $\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}} : \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$ is bounded if and only if $\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}_\phi} : \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$ is bounded and $\|\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}_\phi}\| = \|\widetilde{\mathcal{H}}_{\boldsymbol{\lambda}}\|$.

3 Bounded Hankel Type Operators $\mathcal{H}_\mu$ and Cesàro Type Operators $\mathcal{C}_\eta$ on $\mathcal{D}$

This section is devoted to an intrinsic description of the boundedness of $\mathcal{H}_\mu$ on $\mathcal{D}$. Indeed, we can characterize bounded Hankel type operators $\mathcal{H}_{\boldsymbol{\lambda}}$ on $\mathcal{D}$ induced by a decreasing sequence of positive numbers. The proof is based on our characterization of bounded Cesàro type operators on $\mathcal{D}$.

The following Hilbert's double theorem for the Dirichlet space plays a crucial role in our proofs. By using Schur's test, one can show that inequality (see the proof of Theorem 2 in [41] for more details).

Theorem B [41, p. 814] *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $\mathcal{D}$. Then there exists a positive constant C independent of f such that*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{|a_n||a_m|}{\log(n+m+1)} \le C \sum_{n=1}^{\infty} n|a_n|^2.$$

We first prove Theorem 1.3.

Proof of Theorem 1.3 Since $\|k_w\|_{\mathcal{D}} = 1$ for any $w \in \mathbb{D}$, we obtain the implication (i) ⇒ (ii).

Next, for each $t \in [0, 1)$, recall that the normalized reproducing kernel k_t for the Dirichlet space is

$$k_t(z) = \left(1 + \log\frac{1}{1-t^2}\right)^{-\frac{1}{2}} \left(1 + \sum_{n=1}^{\infty} \frac{1}{n} t^n z^n\right).$$

By direct computations,

$$\|\mathcal{C}_{\boldsymbol{\eta}} k_t\|_{\mathcal{D}}^2 = \left(1+\log\frac{1}{1-t^2}\right)^{-1}\left(|\eta_0|^2+\sum_{n=0}^{\infty}(n+1)|\eta_{n+1}|^2\left(1+\sum_{k=1}^{n+1}\frac{t^k}{k}\right)^2\right).$$

Thus for any positive integer m and $t \in [0, 1)$,

$$\begin{aligned}\|\mathcal{C}_{\boldsymbol{\eta}} k_t\|_{\mathcal{D}}^2 &\gtrsim \left(\log\frac{e}{1-t}\right)^{-1}\sum_{n=m}^{\infty}(n+1)|\eta_{n+1}|^2\left(\sum_{k=1}^{m}\frac{t^k}{k}\right)^2\\ &\gtrsim \left(\log\frac{e}{1-t}\right)^{-1} t^{2m}(\log(m+1))^2\sum_{n=m}^{\infty}(n+1)|\eta_{n+1}|^2.\end{aligned}$$

In particular, let $t = \frac{m}{m+1} \in (0, 1)$. Then

$$\frac{1}{\log(m+1)}\|\mathcal{C}_{\boldsymbol{\eta}} k_t\|_{\mathcal{D}}^2 \gtrsim \sum_{n=m}^{\infty}(n+1)|\eta_{n+1}|^2.$$

By (ii), we have $\|\mathcal{C}_{\boldsymbol{\eta}} k_t\|_{\mathcal{D}}^2 \lesssim 1$ and hence

$$\sum_{n=m}^{\infty} n|\eta_n|^2 = O\left(\frac{1}{\log(m+2)}\right),$$

which gives (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). The Widom type condition gives

$$|\eta_n|^2 \lesssim \frac{1}{(n+1)\log(n+2)}$$

for all $n \in \mathbb{N}$. For $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $\mathcal{D}$, we get

$$\begin{aligned}\left|\eta_n\sum_{k=0}^{n} a_k\right| &\le |\eta_n|\left(\sum_{k=0}^{n}\frac{1}{k+1}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{n}(k+1)|a_k|^2\right)^{\frac{1}{2}}\\ &\le |\eta_n|\,(\log(n+2))^{\frac{1}{2}}\,\|f\|_{\mathcal{D}}\\ &\lesssim \|f\|_{\mathcal{D}}\end{aligned}$$

for all nonnegative integers n. Thus $\mathcal{C}_{\boldsymbol{\eta}}(f)$ is well defined in $H(\mathbb{D})$.

Because of (iii),

$$\sum_{n=0}^{\infty}(n+1)|\eta_{n+1}|^2 < \infty. \tag{3.1}$$

Let $g(z)=\sum_{k=0}^{\infty} a_k z^k$ be in the Dirichlet space $\mathcal{D}$. Then

$$\|\mathcal{C}_{\eta} g\|_{\mathcal{D}}^2 \leq |\eta_0 a_0|^2 + 2\sum_{n=0}^{\infty}(n+1)|\eta_{n+1}|^2|a_0|^2 + 2\sum_{n=0}^{\infty}(n+1)|\eta_{n+1}|^2\left(\sum_{k=1}^{n+1}|a_k|\right)^2.$$

Using (3.1), (iii), and Theorem B, we deduce

$$\begin{aligned}
\|\mathcal{C}_{\eta} g\|_{\mathcal{D}}^2 &\lesssim \|g\|_{\mathcal{D}}^2 + \sum_{n=0}^{\infty}(n+1)|\eta_{n+1}|^2\left(\sum_{k=1}^{\infty}|a_k|\chi_{\{k\leq n+1\}}(k)\right)\left(\sum_{j=1}^{\infty}|a_j|\chi_{\{j\leq n+1\}}(j)\right)\\
&\approx \|g\|_{\mathcal{D}}^2 + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|a_k||a_j|\left(\sum_{n=\max\{k-1,j-1\}}^{\infty}(n+1)|\eta_{n+1}|^2\right)\\
&\lesssim \|g\|_{\mathcal{D}}^2 + \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{|a_k||a_j|}{\log(k+j+1)}\\
&\lesssim \|g\|_{\mathcal{D}}^2,
\end{aligned}$$

where $\chi_{\{k\leq n+1\}}(k)$ is the characteristic function on $\{k\leq n+1\}\cap\mathbb{N}^+$. Here $\mathbb{N}^+$ is the set of positive integers. Hence $\mathcal{C}_{\eta}$ is bounded on $\mathcal{D}$. The proof of Theorem 1.3 is complete. □

The proof of Theorem 1.2 follows from the next more general result.

Theorem 3.1 *Suppose $\boldsymbol{\lambda}=\{\lambda_n\}_{n=0}^{\infty}$ is a decreasing sequence of positive real numbers. Then the following conditions are equivalent.*

(i) *The Hankel type operator $\mathcal{H}_{\boldsymbol{\lambda}}$ is bounded on $\mathcal{D}$.*
(ii) *The reproducing kernel thesis holds; that is,*

$$\sup_{t\in[0,1)}\|\mathcal{H}_{\boldsymbol{\lambda}} k_t\|_{\mathcal{D}}<\infty,$$

where k_t is the normalized reproducing kernel of $\mathcal{D}$ at t in $[0,1)$.
(iii) *The Widom type condition is true; that is,*

$$\sum_{n=m}^{\infty} n\lambda_n^2 = O\left(\frac{1}{\log(m+2)}\right).$$

Proof If $\mathcal{H}_{\boldsymbol{\lambda}}$ is bounded on $\mathcal{D}$, it is clear that (ii) holds.

(ii) $\Rightarrow$ (iii). Since $\{\lambda_n\}$ is a decreasing sequence of positive numbers,

$$\begin{aligned}
\|\mathcal{H}_{\boldsymbol{\lambda}} k_t\|_{\mathcal{D}}^2 &\geq \left(1+\log\frac{1}{1-t^2}\right)^{-1}\sum_{n=1}^{\infty}(n+1)\left(\sum_{k=1}^{\infty}\lambda_{n+k+1}\frac{t^k}{k}\right)^2\\
&\geq \left(1+\log\frac{1}{1-t^2}\right)^{-1}\sum_{n=m}^{\infty}(n+1)\left(\sum_{k=1}^{n}\lambda_{n+k+1}\frac{t^k}{k}\right)^2\\
&\geq \left(1+\log\frac{1}{1-t^2}\right)^{-1}\sum_{n=m}^{\infty}(n+1)\lambda_{2n+1}^2\left(\sum_{k=1}^{m}\frac{t^k}{k}\right)^2\\
&\gtrsim \left(\log\frac{e}{1-t}\right)^{-1} t^{2m}(\log(m+1))^2\sum_{n=m}^{\infty}(n+1)\lambda_{2n+1}^2
\end{aligned}$$

for all $t \in [0, 1)$ and all positive integers m.

For any fixed integer m, taking $t = \frac{m}{m+1}$ in the above, we get

$$\sum_{n=m}^{\infty}(2n+1)\lambda_{2n+1}^2 \lesssim \frac{1}{\log(m+1)}$$

and

$$\sum_{n=m}^{\infty}(2n+2)\lambda_{2n+2}^2 \lesssim \frac{1}{\log(m+1)}.$$

Then the desired result holds.

(iii) $\Rightarrow$ (i). It follows from (iii) that

$$\lambda_k^2 \lesssim \frac{1}{k\log(k+1)} \tag{3.2}$$

for all positive integers k. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ belong to $\mathcal{D}$. By (3.2) and the monotonicity of $\{\lambda_k\}$,

$$\begin{aligned}
\left|\sum_{k=0}^{\infty}\lambda_{n+k}a_k\right| &\leq \lambda_n|a_0| + \sum_{k=1}^{\infty}\lambda_{n+k}|a_k|\\
&\lesssim \|f\|_{\mathcal{D}} + \|f\|_{\mathcal{D}}\left(\sum_{k=1}^{\infty}\frac{\lambda_{n+k}^2}{k+1}\right)^{\frac{1}{2}}\\
&\lesssim \|f\|_{\mathcal{D}} + \|f\|_{\mathcal{D}}\left(\sum_{k=1}^{\infty}\frac{1}{(k+1)^2\log(k+1)}\right)^{\frac{1}{2}}\\
&\lesssim \|f\|_{\mathcal{D}}
\end{aligned}$$

for all nonnegative integers n. Then $\mathcal{H}_{\boldsymbol{\lambda}}(f)$ is analytic on the unit disk $\mathbb{D}$ for any f in $\mathcal{D}$.

Observe that by condition (iii), we have

$$\sum_{n=0}^{\infty}(n+1)\lambda_{n+1}^2 < \infty. \tag{3.3}$$

Then the Hölder inequality gives $\sum_{k=0}^{\infty}\lambda_k|a_k| \lesssim \|f\|_{\mathcal{D}}$. Consequently,

$$\begin{aligned}\|\mathcal{H}_{\boldsymbol{\lambda}} f\|_{\mathcal{D}}^2 &\le \left(\sum_{k=0}^{\infty}\lambda_k|a_k|\right)^2 + \sum_{n=0}^{\infty}(n+1)\left(\sum_{k=0}^{\infty}\lambda_{n+k+1}|a_k|\right)^2 \\ &\lesssim \|f\|_{\mathcal{D}}^2 + \sum_{n=0}^{\infty}(n+1)\left(\sum_{k=0}^{n+1}\lambda_{n+k+1}|a_k|\right)^2 \\ &\quad + \sum_{n=0}^{\infty}(n+1)\left(\sum_{k=n+1}^{\infty}\lambda_{n+k+1}|a_k|\right)^2. \end{aligned} \tag{3.4}$$

By Theorem 1.3 and the monotonicity of the sequence $\boldsymbol{\lambda}$, it is true that

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)\left(\sum_{k=0}^{n+1}\lambda_{n+k+1}|a_k|\right)^2 &\le \sum_{n=0}^{\infty}(n+1)\lambda_{n+1}^2\left(\sum_{k=0}^{n+1}|a_k|\right)^2 \\ &\le \|\mathcal{C}_{\boldsymbol{\lambda}} f_2\|_{\mathcal{D}}^2 \lesssim \|f\|_{\mathcal{D}}^2, \end{aligned} \tag{3.5}$$

where $f_2(z) = \sum_{k=0}^{\infty}|a_k|z^k$ with the same Dirichlet norm of f.

Note that the monotonicity of the sequence $\boldsymbol{\lambda}$ again, the Hölder inequality, and formula (3.3). Then

$$\begin{aligned}\sum_{n=0}^{\infty}(n+1)\left(\sum_{k=n+1}^{\infty}\lambda_{n+k+1}|a_k|\right)^2 &\le \sum_{n=0}^{\infty}(n+1)\left(\sum_{k=n+1}^{\infty}\lambda_k|a_k|\right)^2 \\ &\lesssim \|f\|_{\mathcal{D}}^2\sum_{n=0}^{\infty}(n+1)\left(\sum_{k=n+1}^{\infty}\frac{\lambda_k^2}{k}\right) \\ &\approx \|f\|_{\mathcal{D}}^2\sum_{k=1}^{\infty}\frac{\lambda_k^2}{k}\sum_{n=0}^{k-1}(n+1) \\ &\approx \|f\|_{\mathcal{D}}^2\sum_{k=1}^{\infty}k\lambda_k^2 \lesssim \|f\|_{\mathcal{D}}^2. \end{aligned} \tag{3.6}$$

Joining (3.4), (3.5), and (3.6), we get the boundedness of $\mathcal{H}_{\boldsymbol{\lambda}}$ on $\mathcal{D}$. The proof is complete. □

4 Compact Hankel and Cesàro Type Operators on $\mathcal{D}$

In this section, we give corresponding results about the compactness of Hankel and Cesàro type operators on $\mathcal{D}$.

The result below is the compact version of Theorem 1.1.

Theorem 4.1 *Suppose $\boldsymbol{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers. Then the Hankel type operator $\mathcal{H}_{\boldsymbol{\lambda}}$ is compact on the Dirichlet space $\mathcal{D}$ if and only if $h_{\overline{\boldsymbol{\lambda}}}$ is analytic on $\mathbb{D}$ and the measure $|h'_{\overline{\boldsymbol{\lambda}}}(z)|^2 dA(z)$ is a vanishing Carleson measure for the Dirichlet space $\mathcal{D}$.*

Proof Suppose $h_{\overline{\boldsymbol{\lambda}}} \in \mathcal{X}_0$. Then the identity map

$$I_d : \mathcal{D} \to L^2(\mathbb{D}, |h'_{\overline{\boldsymbol{\lambda}}}|^2 dA)$$

is compact. Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence in $\mathcal{D}$ such that $\{f_k\}_{k=1}^{\infty}$ tends to 0 uniformly in compact subsets of $\mathbb{D}$ as $k \to \infty$. From the proof of Theorem 1.1, we obtain

$$|\langle \mathcal{H}_{\boldsymbol{\lambda}} f_k, g\rangle_{\mathcal{D}}| \lesssim |\lambda_0||f_k(0)||g(0)| + \|g\|_{\mathcal{D}} \left(\int_{\mathbb{D}} \left| \frac{f_k(z) - f_k(0)}{z} \right|^2 |h'_{\overline{\boldsymbol{\lambda}}}(z)|^2 dA(z) \right)^{\frac{1}{2}} + \|g\|_{\mathcal{D}} \left(\int_{\mathbb{D}} |f_k(z)|^2 |h'_{\overline{\boldsymbol{\lambda}}}(z)|^2 dA(z) \right)^{\frac{1}{2}}$$

for all $g \in \mathcal{D}$. Hence, for any $\varepsilon > 0$, there is an integer k_0 such that

$$|\langle \mathcal{H}_{\boldsymbol{\lambda}} f_k, g\rangle_{\mathcal{D}}| \lesssim \|g\|_{\mathcal{D}}\, \varepsilon$$

for all $k > k_0$ and all $g \in \mathcal{D}$. Then $\|\mathcal{H}_{\boldsymbol{\lambda}} f_k\|_{\mathcal{D}} \to 0$ as $k \to \infty$. Thus $\mathcal{H}_{\boldsymbol{\lambda}}$ is compact on $\mathcal{D}$.

Conversely, suppose $\mathcal{H}_{\boldsymbol{\lambda}}$ is compact on $\mathcal{D}$. For a bounded sequence $\{(f_n, g_n)\}_{n=1}^{\infty} \subseteq \mathcal{D} \times \mathcal{D}$, both $\{f_n\}$ and $\{g_n\}$ are bounded sequences in $\mathcal{D}$. Then both $\{\mathcal{H}_{\boldsymbol{\lambda}}(f_n)\}$ and $\{\mathcal{H}_{\boldsymbol{\lambda}}(g_n)\}$ have convergent subsequences. Because of (2.1), $T_{h_{\overline{\boldsymbol{\lambda}}}}$ extends to a compact bilinear form on $\mathcal{D}$. Then Theorem A yields $h_{\overline{\boldsymbol{\lambda}}} \in \mathcal{X}_0$. The proof is complete. □

For the compactness of $\mathcal{C}_{\boldsymbol{\eta}}$ on $\mathcal{D}$, we also have the following conclusion.

Theorem 4.2 *Suppose $\boldsymbol{\eta} = \{\eta_n\}_{n=0}^{\infty}$ is a sequence of complex numbers. Then the following conditions are equivalent.*

(i) *The Cesàro type operator $\mathcal{C}_{\boldsymbol{\eta}}$ is compact on $\mathcal{D}$.*
(ii) *The reproducing kernel thesis holds; that is,*

$$\lim_{t\to 1^-} \|\mathcal{C}_{\boldsymbol{\eta}} k_t\|_{\mathcal{D}} = 0,$$

where k_t is the normalized reproducing kernel of $\mathcal{D}$ at t in $[0, 1)$.

(iii) *The Widom type condition is true; that is,*

$$\sum_{n=m}^{\infty} n|\eta_n|^2 = o\left(\frac{1}{\log(m+2)}\right).$$

Proof (i) $\Rightarrow$ (ii). Note that $\|k_t\|_{\mathcal{D}} = 1$ for each $t \in [0, 1)$ and k_t tends to zero uniformly in compact subsets of $\mathbb{D}$ as $t \to 1^-$. Then (ii) holds.

(ii) $\Rightarrow$ (iii). Checking the proof of Theorem 1.3, we see

$$\log(m+1) \sum_{n=m}^{\infty} (n+1)|\eta_{n+1}|^2 \lesssim \|\mathcal{C}_{\boldsymbol{\eta}} k_{\frac{m}{m+1}}\|_{\mathcal{D}}^2 \tag{4.1}$$

for all positive integers m. Taking $m \to \infty$ in (4.1), we obtain (iii).

(iii) $\Rightarrow$ (i). Let m be a positive integer. For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{D}$, consider

$$\mathcal{C}_{\boldsymbol{\eta}}^{(m)}(f)(z) = \sum_{n=0}^{m} \left(\eta_n \sum_{k=0}^{n} a_k\right) z^n, \quad z \in \mathbb{D}.$$

Then $\mathcal{C}_{\boldsymbol{\eta}}^{(m)}$ is a finite rank operator. Thus, $\mathcal{C}_{\boldsymbol{\eta}}^{(m)}$ is compact on $\mathcal{D}$. Because of (iii), for every $\epsilon > 0$, there exists a positive integer N such that

$$\sum_{n=m}^{\infty} (n+1)|\eta_{n+1}|^2 < \frac{\epsilon}{\log(m+2)}$$

for $m > N$. Using the proof of Theorem 1.3, we have

$$\begin{aligned}
\sum_{n=m}^{\infty} (n+1)|\eta_{n+1}|^2 \left(\sum_{k=1}^{n+1} |a_k|\right)^2 &\lesssim \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |a_k||a_j| \sum_{n=\max\{k-1,j-1,m\}}^{\infty} (n+1)|\eta_{n+1}|^2 \\
&\lesssim \epsilon \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{|a_k||a_j|}{\log(k+j+m)} \\
&\lesssim \epsilon \|f\|_{\mathcal{D}}^2
\end{aligned}$$

for all $m > N$. Thus, for $m > N$,

$$\begin{aligned}
\|(\mathcal{C}_{\boldsymbol{\eta}} - \mathcal{C}_{\boldsymbol{\eta}}^{(m)})(f)\|_{\mathcal{D}}^2 &\lesssim |a_0|^2 \sum_{n=m}^{\infty} (n+1)|\eta_{n+1}|^2 + \sum_{n=m}^{\infty} (n+1)|\eta_{n+1}|^2 \left(\sum_{k=1}^{n+1} |a_k|\right)^2 \\
&\lesssim \epsilon \|f\|_{\mathcal{D}}^2.
\end{aligned}$$

In other words, $\|\mathcal{C}_{\boldsymbol{\eta}} - \mathcal{C}_{\boldsymbol{\eta}}^{(m)}\| \to 0$ as $m \to \infty$. Hence $\mathcal{C}_{\boldsymbol{\eta}}$ is compact on $\mathcal{D}$. The proof is finished. □

The compact result corresponding to Theorem 3.1 also holds.

Theorem 4.3 *Suppose $\boldsymbol{\lambda} = \{\lambda_n\}_{n=0}^{\infty}$ is a decreasing sequence of positive real numbers. Then the following conditions are equivalent.*

(i) *The Hankel type operator $\mathcal{H}_{\boldsymbol{\lambda}}$ is compact on $\mathcal{D}$.*
(ii) *The reproducing kernel thesis holds; that is,*

$$\lim_{t\to 1^-} \|\mathcal{H}_{\boldsymbol{\lambda}} k_t\|_{\mathcal{D}} = 0,$$

where k_t is the normalized reproducing kernel of $\mathcal{D}$ at t in $[0, 1)$.
(iii) *The Widom type condition is true; that is,*

$$\sum_{n=m}^{\infty} n\lambda_n^2 = o\left(\frac{1}{\log(m+2)}\right).$$

Proof By the proof of Theorem 3.1, the arguments here are similar to that of Theorem 4.2. We omit it. □

5 Random Hankel Type Operators on $\mathcal{D}$ and a Result Related to Rudin's $\Lambda(p)$ Sets

In this section, we prove Theorem 1.4 and Corollary 1.5. Many known real random variables sequences $(X_n)_{n\in\mathbb{N}}$ satisfy the conditions in Theorem 1.4. First of all, all bounded mean zero random variables are included in Thereom 1.4. The typical example is the Bernoulli random variables sequence $(X_n)_{n\in\mathbb{N}}$. In other words, $(X_n)_{n\in\mathbb{N}}$ is independent such that $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$. In addition, for the classical independent normal distribution sequence $X_n = N(0, 1)$, we have $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^4] = 3$ (cf. [32, p. 2]). Therefore, Theorem 1.4 can be applied to the corresponding random Gaussian analytic Dirichlet functions.

For a complex number sequence $\boldsymbol{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}$ and a sequence of i.i.d. real random variables $(X_n)_{n\in\mathbb{N}}$ with $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^4] < \infty$, if $h_{\boldsymbol{\lambda}} \in \mathcal{D}$, then $\sum_{n=0}^{\infty} |\lambda_n|^2 < \infty$. Moreover, by i.i.d. and $\mathbb{E}[X_n^2] \le \sqrt{\mathbb{E}[X_n^4]}$, we have

$$\mathbb{E}\left[\sum_{n=0}^{\infty} |\lambda_n X_n|^2\right] = \sum_{n=0}^{\infty} |\lambda_n|^2 \mathbb{E}[|X_n|^2] < \infty$$

and almost surely

$$h_{\boldsymbol{\lambda},\omega}(z) = \sum_{n=0}^{\infty} X_n(\omega)\lambda_n z^n$$

is analytic on the unit disk.

For our proof, we need the following elementary inequalities which may be well known. However, we cannot locate literature. For the sake of completeness, we have included a brief proof.

Lemma 5.1 *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. real random variables with $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^4] = 1$. Then for all finite complex numbers $a_1, \ldots, a_n$, we have*

$$\mathbb{E}\left[\left|\sum_{i=1}^{n} a_i X_i\right|^4\right] \leq 2\left[\sum_{i=1}^{n} |a_i|^2\right]^2.$$

Proof For any fixed natural number $n \geq 1$, let $A_n = \sum_{i=1}^{n} a_i X_i$. It is easy to see

$$|A_n|^2 = \sum_{i=1}^{n} |a_i|^2 X_i^2 + 2\sum_{i<j} Re(a_i \bar{a}_j) X_i X_j.$$

Note that $(X_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. real random variables with $\mathbb{E}[X_n] = 0$ for each n. Then

$$\begin{aligned}\mathbb{E}\big[|A_n|^4\big] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} |a_i|^2 X_i^2\right)^2\right] + 4\mathbb{E}\left[\left(\sum_{i<j} Re(a_i\bar{a}_j) X_i X_j\right)^2\right] \\ &= \sum_{i,j} |a_i|^2 |a_j|^2 \mathbb{E}[X_i^2 X_j^2] + 4\sum_{i<j} (Re(a_i\bar{a}_j))^2 \mathbb{E}[X_i^2 X_j^2] \\ &\leq \sum_{i,j} |a_i|^2 |a_j|^2 \mathbb{E}[X_i^2 X_j^2] + 4\sum_{i<j} |a_i\bar{a}_j|^2 \mathbb{E}[X_i^2 X_j^2].\end{aligned}$$

Observe that $\left(\mathbb{E}[X_i^2 X_j^2]\right)^2 \leq \mathbb{E}[X_i^4]\mathbb{E}[X_j^4] = 1$, hence

$$\mathbb{E}\left[\left|\sum_{i=1}^{n} a_i X_i\right|^4\right] \leq 2\left[\sum_{i=1}^{n} |a_i|^2\right]^2,$$

and the proof is completed. □

Let $f \in H(\mathbb{D})$. For $0 < p < \infty$ and $0 \leq r < 1$, let

$$M_p(f, r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}$$

and

$$M_\infty(f, r) = \max_{\theta\in[0,2\pi]} |f(re^{i\theta})|.$$

Denote by H^p the Hardy space consisting of those functions f in $H(\mathbb{D})$ with

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(f,r) < \infty.$$

It is well known (cf. [20, p. 3]) that $\mathcal{D}$ is a subset of H^p for any $p > 0$.

From a result of Brown and Shields [13, p. 300], if an analytic function f satisfies $M_p(f',r) \in L^2([0,1],dr)$ with $2 < p \le \infty$, then $|f'(z)|^2 dA(z)$ is a Carleson measure for $\mathcal{D}$. We strengthen this conclusion as follows.

Lemma 5.2 *Let $2 < p \le \infty$. Suppose ϕ is an analytic function on the open unit disk $\mathbb{D}$ with $M_p(\phi',r) \in L^2([0,1],dr)$. Then $|\phi'(z)|^2 dA(z)$ is a vanishing Carleson measure for the Dirichlet space.*

Proof Let $2 < p < \infty$. Suppose $\{f_m\}_{m=1}^{\infty}$ is a sequence in $\mathcal{D}$ such that $\sup_m \|f_m\|_{\mathcal{D}} < \infty$ and functions f_m tend to zero uniformly in compact subsets of $\mathbb{D}$ as $m \to \infty$. Due to $M_p(\phi',r) \in L^2([0,1],dr)$, for any $\epsilon > 0$, there exists a δ in $(0,1)$ such that

$$\int_\delta^1 \left(\int_0^{2\pi} |\phi'(re^{i\theta})|^p d\theta \right)^{\frac{2}{p}} dr < \epsilon.$$

Since $p > 2$, using the Hölder inequality with indices $\frac{p}{2}$ and $\frac{p}{p-2}$, we get for any $r \in (0,1)$

$$\begin{aligned}\int_0^{2\pi} |f_m(re^{i\theta})|^2 |\phi'(re^{i\theta})|^2 \frac{d\theta}{2\pi} &\le \left(\int_0^{2\pi} |\phi'(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{2}{p}} \\ &\quad \times \left(\int_0^{2\pi} |f_m(re^{i\theta})|^{\frac{2p}{p-2}} \frac{d\theta}{2\pi} \right)^{\frac{p-2}{p}} \\ &\le \left(\int_0^{2\pi} |\phi'(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{\frac{2}{p}} \|f_m\|^2_{H^{\frac{2p}{p-2}}}.\end{aligned}$$

Therefore, for the above $\delta \in (0,1)$,

$$\begin{aligned}\int_{\{z\in\mathbb{D}:|z|>\delta\}} |f_m(z)|^2 |\phi'(z)|^2 dA(z) &= \int_\delta^1 \left(\frac{1}{\pi} \int_0^{2\pi} |f_m(re^{i\theta})|^2 |\phi'(re^{i\theta})|^2 d\theta \right) dr \\ &\le 2(2\pi)^{-2/p} \|f_m\|^2_{H^{\frac{2p}{p-2}}} \\ &\quad \times \int_\delta^1 \left(\int_0^{2\pi} |\phi'(re^{i\theta})|^p d\theta \right)^{\frac{2}{p}} dr \\ &\lesssim \epsilon \|f_m\|^2_{H^{\frac{2p}{p-2}}} \lesssim \epsilon \|f_m\|^2_{\mathcal{D}} \lesssim \epsilon.\end{aligned}$$

Since functions f_m tend to zero uniformly in $\{z \in \mathbb{D} : |z| \leq \delta\}$, there is a positive integer N such that

$$\int_{\{z\in\mathbb{D}:|z|\leq\delta\}} |f_m(z)\phi'(z)|^2 dA(z) \leq \epsilon$$

for all $m > N$. Consequently,

$$\lim_{m\to\infty} \int_{\mathbb{D}} |f_m(z)\phi'(z)|^2 dA(z) = 0.$$

Thus $|\phi'(z)|^2 dA(z)$ is a vanishing Carleson measure for $\mathcal{D}$. The proof of the case of $p = \infty$ is similar. We omit it. □

Note that if $p = 2$ and $\phi \in H(\mathbb{D})$, then $M_p(\phi', r) \in L^2([0, 1], dr)$ if and only if $\phi \in \mathcal{D}$. In general, the measure $|\phi'(z)|^2 dA(z)$ with $\phi \in \mathcal{D}$ is not a vanishing Carleson measure for the Dirichlet space.

Remark 5.3 For $\phi \in H(\mathbb{D})$, if $|\phi'(z)|^2 dA(z)$ is a vanishing Carleson measure for $\mathcal{D}$, then of course $|\phi'(z)|^2 dA(z)$ is a Carleson measure for $\mathcal{D}$. By a personal communication, Professor Blasco has a condition weaker than that in Lemma 5.2 to show that $|\phi'(z)|^2 dA(z)$ is a Carleson measure for $\mathcal{D}$.

Now, we are ready to prove Theorem 1.4.

Proof If $\mathbb{E}[X_n^4] = 0$, then $X_n = 0$ almost surely. Hence without loss of generality, we can assume that $\mathbb{E}[X_n^4] = 1$. Recall that

$$h_{\overline{\boldsymbol{\lambda}},\omega}(z) = \sum_{n=0}^{\infty} X_n(\omega)\overline{\lambda_n} z^n.$$

By Theorem 1.1, we shall show that almost surely

$$|h'_{\overline{\boldsymbol{\lambda}},\omega}(z)|^2 dA(z)$$

is a vanishing Carleson measure for $\mathcal{D}$. By Lemma 5.2, it is sufficient to show that

$$\mathbb{E}\left[\int_0^1 \left(\int_0^{2\pi} |h'_{\overline{\boldsymbol{\lambda}},\omega}(re^{i\theta})|^4 \frac{d\theta}{2\pi}\right)^{1/2} dr\right] < \infty.$$

It follows from Fubini's theorem that

$$\mathbb{E}\left[\int_0^1 \left(\int_0^{2\pi} |h'_{\overline{\boldsymbol{\lambda}},\omega}(re^{i\theta})|^4 \frac{d\theta}{2\pi}\right)^{1/2} dr\right]$$
$$= \int_0^1 \int_{\Omega} \left(\int_0^{2\pi} |h'_{\overline{\boldsymbol{\lambda}},\omega}(re^{i\theta})|^4 \frac{d\theta}{2\pi}\right)^{1/2} d\mathbb{P}(w) dr.$$

Applying the Hölder inequality with respect to the probability measure $d\mathbb{P}$, we have

$$\int_0^1 \int_\Omega \left(\int_0^{2\pi} |h'_{\overline{\boldsymbol{\lambda}},\omega}(re^{i\theta})|^4 \frac{d\theta}{2\pi}\right)^{1/2} d\mathbb{P}(w)dr \\ \leq \int_0^1 \left(\int_\Omega \int_0^{2\pi} |h'_{\overline{\boldsymbol{\lambda}},\omega}(re^{i\theta})|^4 \frac{d\theta}{2\pi} d\mathbb{P}(w)\right)^{1/2} dr.$$

Consequently,

$$\int_\Omega \int_0^{2\pi} |h'_{\overline{\boldsymbol{\lambda}},\omega}(re^{i\theta})|^4 \frac{d\theta}{2\pi} d\mathbb{P}(w) = \int_0^{2\pi} \int_\Omega |h'_{\overline{\boldsymbol{\lambda}},\omega}(re^{i\theta})|^4 d\mathbb{P}(w) \frac{d\theta}{2\pi}.$$

Recall that

$$h'_{\overline{\boldsymbol{\lambda}},\omega}(re^{i\theta}) = \sum_{n=1}^{\infty} \overline{\lambda_n} n r^{n-1} e^{i(n-1)\theta} X_n(\omega)$$

and Lemma 5.1, we get

$$\int_0^1 \left(\int_\Omega \left|\sum_{n=1}^{\infty} \overline{\lambda_n} n r^{n-1} e^{i(n-1)\theta} X_n\right|^4 d\mathbb{P}(w)\right)^{\frac{1}{2}} dr \leq 3 \sum_{n=1}^{\infty} \int_0^1 |\overline{\lambda_n} n r^{n-1}|^2 dr.$$

Therefore, there is a positive constant C such that

$$\mathbb{E}\left[\int_0^1 \left(\int_0^{2\pi} |h'_{\boldsymbol{\lambda},\omega}(re^{i\theta})|^4 \frac{d\theta}{2\pi}\right)^{1/2} dr\right] \leq C \sum_{n=1}^{\infty} n|\lambda_n|^2 < \infty.$$

This completes the whole proof. □

To prove Corollary 1.5, we start with a simple observation.

Lemma 5.4 *Suppose $p > 2$, E is a Rudin's $\Lambda(p)$ set and $\boldsymbol{\lambda} = \{\lambda_n : n \in E\}$ is a sequence of complex numbers. If $\sum_{n\in E} n|\lambda_n|^2 < \infty$, then $|h'_{\overline{\boldsymbol{\lambda}}}(z)|^2 dA(z)$ is a vanishing Carleson measure for the Dirichlet space, where $h_{\overline{\boldsymbol{\lambda}}}(z) = \sum_{n\in E} \overline{\lambda_n} z^n$.*

Proof Let $p > 2$. For any $0 < r < 1$, observe that

$$h'_{\overline{\boldsymbol{\lambda}}}(re^{i\theta}) = e^{-i\theta} \sum_{n\in E} \overline{\lambda_n} n r^{n-1} e^{in\theta},$$

we have

$$\|h'_{\overline{\boldsymbol{\lambda}}}(re^{i\theta})\|_{L^p([0,2\pi],d\theta)} = \left\|\sum_{n\in E} \overline{\lambda_n} n r^{n-1} e^{in\theta}\right\|_{L^p([0,2\pi],d\theta)}.$$

Since E is a Rudin's $\Lambda(p)$, there exists a positive constant C such that

$$\begin{aligned}\left\|\sum_{n\in E}\overline{\lambda_n}nr^{n-1}e^{in\theta}\right\|^2_{L^p([0,2\pi],d\theta)} &\le C\left\|\sum_{n\in E}\overline{\lambda_n}nr^{n-1}e^{in\theta}\right\|^2_{L^2([0,2\pi],d\theta)}\\ &= C\sum_{n\in E}|\lambda_n|^2n^2r^{2(n-1)}.\end{aligned}$$

Recall that $\sum_{n\in E} n|\lambda_n|^2 < \infty$, therefore,

$$\begin{aligned}\int_0^1\left\|\sum_{n\in E}\overline{\lambda_n}nr^{n-1}e^{in\theta}\right\|^2_{L^p([0,2\pi],d\theta)} dr &\le C\int_0^1\sum_{n\in E}|\lambda_n|^2n^2r^{2(n-1)}dr\\ &< +\infty.\end{aligned}$$

The lemma then follows from Lemma 5.2. □

Proof of Corollary 1.5 Suppose $\mathcal{H}_{\boldsymbol{\lambda}}$ is bounded on $\mathcal{D}$. By Theorem 1.1, $h_{\overline{\boldsymbol{\lambda}}} \in \mathcal{D}$. By the definition of Dirichlet norm, the condition (iii) holds. (i) $\Rightarrow$ (ii) is clear. The implication (iii) $\Rightarrow$ (i) follows from Lemma 5.4 and Theorem 4.1. The proof is complete. □

6 Final Remarks

In this section, we give some remarks about some functions in $\mathcal{X}$ and the action of Hankel matrices on the Bergman space A^2.

Our results, as we stated in Corollary 1.6, in this paper yield a complete characterization of functions in $\mathcal{X}$ with decreasing Taylor's sequences of positive numbers. More precisely, suppose $\boldsymbol{\lambda} = \{\lambda_n\}_{n=0}^{\infty}$ is a decreasing sequence of positive real numbers. From Theorems 1.1 and 3.1, the measure $|h'_{\boldsymbol{\lambda}}(z)|^2dA(z)$ is a Carleson measure for $\mathcal{D}$ (i.e. $h_{\boldsymbol{\lambda}} \in \mathcal{X}$) if and only if

$$\sum_{n=m}^{\infty} n\lambda_n^2 = O\left(\frac{1}{\log(m+2)}\right).$$

By Theorems 4.1 and 4.3, the measure $|h'_{\boldsymbol{\lambda}}(z)|^2dA(z)$ is a vanishing Carleson measure for $\mathcal{D}$ (i.e. $h_{\boldsymbol{\lambda}} \in \mathcal{X}_0$) if and only if

$$\sum_{n=m}^{\infty} n\lambda_n^2 = o\left(\frac{1}{\log(m+2)}\right).$$

Let $p > 2$, for an analytic function f with non-zero Fourier coefficients belonging to a Rudin's $\Lambda(p)$ set E, we write

$$f(z) = \sum_{n \in E} a_n z^n,$$

by Corollary 1.5, the following statements are equivalent:

(a) $f \in \mathcal{X}$;
(b) $f \in \mathcal{X}_0$;
(c) $\sum_{n \in E}(n+1)|a_n|^2 < \infty$.

Suppose $\boldsymbol{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}$ is a sequence of complex number. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. real random variables with $\mathbb{E}[X_n] = 0$ and $\mathbb{E}[X_n^4] < \infty$. If $h_{\boldsymbol{\lambda}} \in \mathcal{D}$, Theorems 1.4 and 4.1 yield that

$$\mathbb{P}(h_{\overline{\boldsymbol{\lambda}},\omega} \in \mathcal{X}) = \mathbb{P}(h_{\overline{\boldsymbol{\lambda}},\omega} \in \mathcal{X}_0) = 1,$$

where $h_{\overline{\boldsymbol{\lambda}},\omega}(z) = \sum_{n=0}^{\infty} X_n(\omega)\overline{\lambda_n} z^n$.

Next we consider Hankel type operators $\mathcal{H}_{\boldsymbol{\lambda}}$ on the Bergman space A^2 which is also a Hilbert space of analytic functions on $\mathbb{D}$ and it is equipped with the inner product

$$\langle f, g\rangle_{A^2} = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z).$$

From [16, p. 349], $(A^2)^* \cong \mathcal{D}$ and $\mathcal{D}^* \cong A^2$ under the pairing

$$\langle f, g\rangle = \sum_{k=0}^{\infty} a_k b_k, \tag{6.1}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$.

For a sequence $\boldsymbol{\lambda} = \{\lambda_n\}_{n\in\mathbb{N}}$ of complex numbers, it is easy to see

$$\langle \mathcal{H}_{\boldsymbol{\lambda}} f, g\rangle = \langle f, \mathcal{H}_{\boldsymbol{\lambda}} g\rangle.$$

Consequently, $\mathcal{H}_{\boldsymbol{\lambda}}$ is bounded (resp. compact) on the Dirichlet space $\mathcal{D}$ if and only if $\mathcal{H}_{\boldsymbol{\lambda}}$ is bounded (resp. compact) on A^2. If we replace the pairing (6.1) by the Cauchy pairing

$$(f, g) = \sum_{k=0}^{\infty} a_k \overline{b_k}.$$

Then

$$(\mathcal{H}_{\boldsymbol{\lambda}} f, g) = (f, \mathcal{H}_{\overline{\boldsymbol{\lambda}}} g).$$

Hence we also get that $\mathcal{H}_\lambda$ is bounded (resp. compact) on $\mathcal{D}$ if and only if $\mathcal{H}_{\overline{\lambda}}$ is bounded (resp. compact) on A^2.

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