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Maximal Integral Inequalities and Hausdorff–Young

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Abstract
We discuss the Hausdorff–Young inequality in the context of maximal integral estimates, including the case of Hermite and Laguerre expansions. We establish a maximal inequality for integral operators with bounded kernel on $\mathbb{R}$, which in particular allows for the pointwise evaluation of these operators, including the Fourier transform, for functions in appropriate Lorentz and Orlicz spaces. In the case of the Hermite expansions we prove a refined Hausdorff–Young inequality, further sharpened by considering the maximal Hermite coefficients in place of the Hermite coefficients when estimating the appropriate Lorentz and Orlicz norms. We also consider the refined companion Hausdorff–Young inequality and Hardy–Littlewood type inequalities for the Hermite expansions. Similar results are proved for the Laguerre expansions.

Keywords Hausdorff–Young inequality · Hermite expansions · Laguerre expansions

Mathematics Subject Classification 33C45 · 42B35 · 46E30 · 47A30

1 Introduction

Given an integrable function f defined on $\mathbb{R}$, let $\widehat{f}$ denote its Fourier transform, i.e.,

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-ixy}\, dy, \quad x \in \mathbb{R}. \tag{1.1}$$

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Zygmund discusses the Fourier transform on $\mathbb{R}$, in particular the definition of $\widehat{f}$ when f is merely locally integrable, in [32, Vol. 2, Chap. XVI]. When f is in $L^2(\mathbb{R})$, $\widehat{f}$ is defined as an $L^2(\mathbb{R})$ limit in (1.1) above, and satisfies the Parseval–Plancherel formula $\|\widehat{f}\|_2 = \|f\|_2$. And, when f is in $L^p(\mathbb{R})$ for $1 < p < 2$, the integral in Eq. (1.1) has meaning as an $L^{p'}(\mathbb{R})$ limit where p' denotes the conjugate index to p, and $\widehat{f}$ satisfies the Hausdorff–Young inequality $\|\widehat{f}\|_{p'} \lesssim \|f\|_p$, [4], [32, Vol. 2, Theorem 2.3, p. 101].

Zygmund further establishes that the maximal Fourier transform given by

$$\widehat{f}^*(x) = \sup_{\alpha,\beta>0} \frac{1}{\sqrt{2\pi}} \Big| \int_{-\alpha}^{\beta} f(y) e^{-ixy}\, dy \Big|$$

maps $L^p(\mathbb{R})$ continuously into $L^{p'}(\mathbb{R})$, where $1 < p < 2$ and p' is the conjugate to p, and, consequently, for f in $L^p(\mathbb{R})$ it follows that

$$\widehat{f}(x) = \lim_{\alpha\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f(y) e^{-ixy}\, dy \quad \text{a.e.,}$$

[31], [32, Vol. 2, Theorem 3.14, p. 257].

Zygmund's maximal theorem is the underlying principle of this note, which concerns the Hausdorff–Young inequality in the context of maximal integral estimates, including the case of Hermite and Laguerre expansions.

We prove in Theorem 3.1 a maximal inequality for integral operators with bounded kernel on $\mathbb{R}$ that are of type (p_0, q_0) for $1 < p_0, q_0 < \infty$. Theorem 3.1 rests on elementary principles and, since by the Parseval–Plancherel formula the Fourier transform is of type $(2, 2)$, it yields Zygmund's maximal theorem for $1 < p < 3/2$.

The full range of Zygmund's result follows from the Christ–Kiselev maximal inequality [13]. This principle, originally proved in the context of linear operators on L^p spaces, has been considered for some Lorentz spaces and Orlicz classes [12], and extended to quasi–Banach function lattices, with applications to Lorentz spaces, Wiener amalgams, and maximal Fourier operators among others [23]. We complement these results by proving in Theorem 4.1 the Christ–Kiselev maximal inequality in the framework of Orlicz spaces; Corollary 4.2 then gives Zygmund's maximal theorem for the Fourier transform acting on Orlicz spaces in $\mathbb{R}^n$. An alternative approach to Zygmund's result for the $L^p(\mathbb{R})$ spaces using Carleson's theorem is considered in [1, p. 165].

Now, the type (p, p') of the Hausdorff–Young inequality for $1 < p < 2$ extends to general expansions in terms of orthogonal functions, but more can be argued. We will examine orthogonal expansions in terms of the Hermite functions, $\mathcal{H}_m(x)$, and the Laguerre functions, $\mathcal{L}_m^\alpha(x)$, for $-1/2 < \alpha < -1/3$. Our approach rests on a remarkable estimate for the Hermite functions established by Hille [18, p. 436], [28, p. 240], to wit,

$$\big|\mathcal{H}_m(x)\big| \lesssim m^{-1/12}. \tag{1.2}$$

Hille notes that Eq. (1.2) is the best possible estimate but that in applications he will only use the weaker formula $|\mathcal{H}_m(x)| \lesssim 1$. On the other hand, we will use Eq. (1.2) to obtain an improved, or *refined*, Hausdorff–Young estimate for orthogonal expansions. We refer to these estimates as refined because they are of type (p, q) with $q < p'$.

In the case of the Hermite expansions we establish in Theorem 5.1 that for f in $L^p(\mathbb{R})$, the sequence $\{c_m\}$ of its Hermite coefficients is in the sequence space ℓ^q, provided that p, q verify

$$1 < p < 2, \quad \text{and,} \quad \frac{5}{6}\frac{1}{p} + \frac{1}{q} = \frac{11}{12},$$

and $\|\{c_m\}\|_{\ell^q} \lesssim_p \|f\|_p$. Moreover, the conclusion remains valid for the sequence of the maximal Hermite coefficients of f. We complete the result with the corresponding statements for Lorentz and Orlicz spaces. For the Lorentz spaces the sequence of Hermite and maximal Hermite coefficients of a function in $L(p, s)$ is in the Lorentz sequence space $\ell^{q,s}$ for p, q as above and $1 \le s \le \infty$, with the corresponding norm estimate. And, for the Orlicz spaces, with A, B Young functions satisfying appropriate growth conditions, if f is in the Orlicz space $L^A(\mathbb{R})$, its Hermite and maximal Hermite coefficients are in the Orlicz sequence space ℓ^B provided that A, B verify

$$B^{-1}(t) = t^{11/12} A^{-1}(t^{-5/6}), \quad t > 0,$$

and the corresponding norm inequalities hold.

A companion result to the Hausdorff–Young inequality addresses under what conditions $\{c_m\}$ is the sequence of Fourier coefficients of a function f in the Hausdorff–Young range [4], [32, Vol. 2, Theorem 2.3, p 101]. In a refined formulation Theorem 5.2 establishes that for the Hermite expansions, if $12/11 < p < 2$, and q is such that

$$\frac{1}{p} + \frac{5}{6}\frac{1}{q} = \frac{11}{12},$$

then, given $\{c_m\}$ in ℓ^p, there is f in $L^q(\mathbb{R})$ such that the c_m are the Hermite coefficients of $f(x)$, $\|f\|_q \lesssim_p \|\{c_m\}\|_{\ell^p}$, and

$$f(x) = \lim_M \sum_{m=0}^{M} c_m \mathcal{H}_m(x) \quad \text{a.e.}$$

As in the case of Theorems 5.1, 5.2 holds for Lorentz and Orlicz spaces as well.

In addition to the Hausdorff–Young inequality and its companion inequality in $L^p(\mathbb{R})$, Ditzian discusses Hardy–Littlewood type inequalities in the context of expansions by orthogonal polynomials with respect to a family of Freud–type weights [14]; we prove a sharp version of the Hardy-Littlewood inequality for Hermite expansions in Theorem 5.3.

As for the Laguerre case, we rely on the fact established in [7, p. 277] that for $\alpha > -1/2$, the Laguerre functions satisfy the estimate

$$\left|\mathcal{L}_m^\alpha(x)\right| \lesssim_\alpha m^{\alpha/2+1/4-1/12}\, x^{\alpha/2}, \quad x > 0,$$

which stems from Eq. (1.2) above.

We then consider the refined Hausdorff–Young inequality, Theorem 5.4, and its companion inequality, Theorem 5.5, for the Laguerre expansions with $-1/2 < \alpha < -1/3$. And, to complete the picture we prove a Hausdorff–Young analogue of type (p, p') in Theorem 5.6, and a complement of this result in Theorem 5.7, along the lines of Ditzian's results for the Hermite expansions [14, Corollaries 4.1, 4.2, p. 586].

The paper consists of six sections, the second of which presents some background material. The next three sections cover the topics described above, to wit, integral operators, the Christ–Kiselev maximal inequality for Orlicz spaces, and a refined version of the Hausdorff–Young inequality for Hermite and Laguerre orthogonal expansions and related estimates including the Hardy–Littlewood inequality for Hermite expansions and a Hausdorff–Young analogue for the Laguerre expansions. In the closing section we discuss discrete maximal inequalities, which allow for the consideration of Fourier series rather than integrals, a version of Zygmund's result for the Orlicz type class of measurable functions f defined on $\mathbb{R}^n$, L_Ψ, [5], which constitute the building blocks of the Lorentz spaces [9], and we describe briefly the Hausdorff-Young inequality in the context of Banach spaces of distributions of Wiener type introduced by Feichtinger [15, 16].

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2 Preliminaries

Let (X, μ) be σ–finite measure space, where μ has no atoms and $\mu(X) = \infty$. Given a measurable function f defined on X, let $m(f, \lambda)$ denote the *distribution function* of f,

$$m(f, \lambda) = \mu\big(\{x \in X : |f(x)| > \lambda\}\big), \quad \lambda > 0.$$

Then, $m(f, \lambda)$ is nonincreasing and right continuous, and the *nonincreasing rearrangement* f^* of f defined for $t > 0$ by

$$f^*(t) = \inf\{\lambda > 0 : m(f, \lambda) \le t\}, \quad \inf \emptyset = 0,$$

is informally its inverse. f^* is nonincreasing and right continuous and, at its points of continuity t, $f^*(t) = \lambda$ is equivalent to $m(f, \lambda) = t$.

The *Lorentz space* $L^{p,q}(X) = L(p, q)$, $0 < p < \infty$, $0 < q \le \infty$, consists of those measurable functions f with finite quasinorm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = \Big(\frac{q}{p}\int_0^\infty \big(t^{1/p} f^*(t)\big)^q \,\frac{dt}{t}\Big)^{1/q}, \quad 0 < q < \infty,$$

and,

$$\|f\|_{p,\infty} = \sup_{t>0} \big(t^{1/p} f^*(t)\big) = \sup_{\lambda>0} \lambda\, m(f,\lambda)^{1/p}, \quad q = \infty.$$

Note that, in particular, $L(p, p) = L^p(X)$, and $L(p,\infty)$ is weak-$L^p(X)$.

The Lorentz spaces are monotone with respect to the second index, that is, if $0 < q < q_1 \le \infty$, then $L(p, q) \subseteq L(p, q_1)$ and

$$\|f\|_{p,q_1} \lesssim \|f\|_{p,q}.$$

Given a sequence $c = \{c_m\}$, let $\{c^*_m\}$ denote the sequence obtained by ordering $\{|c_m|\}$ in a nonincreasing fashion. The *Lorentz sequence space* $\ell^{p,q}$, $1 \le p < \infty$, $1 \le q \le \infty$, consists of those sequences $c = \{c_m\}$ with finite quasinorm $\|c\|_{\ell^{p,q}}$ given by

$$\|c\|_{\ell^{p,q}} = \Big(\sum_{m=1}^{\infty} \big(m^{1/p} c^*_m\big)^q \,\frac{1}{m}\Big)^{1/q}, \quad 1 \le q < \infty,$$

and,

$$\|c\|_{\ell^{p,\infty}} = \sup_{m\ge1} m^{1/p} c^*_m, \quad q = \infty.$$

As for the Orlicz spaces, the letters A, B are reserved for *Young functions*, i.e., for functions $A(t)$ defined for $t \ge 0$ that are zero at zero, increasing, and convex, or, more generally, $A(t)/t$ increasing to ∞ as $t \to \infty$. The *Orlicz space* $L^A(X)$ consists of those measurable functions f (modulo equality μ-a.e.) such that $\int_X A\big(|f(x)|/M\big)\, d\mu < \infty$ for some M, normed by

$$\|f\|_A = \inf\Big\{\lambda > 0 : \int_X A\Big(\frac{|f(x)|}{\lambda}\Big)\, d\mu \le 1\Big\}.$$

A couple of useful observations. First, if $A(t)/t^p$ decreases and

$$\int_X A\big(|f(x)|\big)\, d\mu = \varepsilon < 1,$$

then

$$\int_X A\Big(\frac{1}{\varepsilon^{1/p}}\,\big|f(x)\big|\Big)\, d\mu \le \int_X \frac{1}{\varepsilon} A\big(\big|f(x)\big|\big)\, d\mu = 1,$$

and, consequently,

$$\|f\|_A \le \varepsilon^{1/p}. \tag{2.1}$$

On the other hand, if $B(t)/t^q$ increases, for $\varepsilon < 1$ it follows that,

$$\int_X B\big(|f(x)|\big)\, d\mu \le \varepsilon^q \int_X B\Big(\frac{|f(x)|}{\varepsilon}\Big)\, d\mu. \tag{2.2}$$

The *Orlicz sequence space* ℓ^A consists of those sequence $c = \{c_m\}$ such that for some M,

$$\sum_m A\big(|c_m|/M\big) < \infty,$$

normed by

$$\|c\|_{\ell^A} = \inf\Big\{\lambda > 0 : \sum_m A\Big(\frac{|c_m|}{\lambda}\Big) \le 1\Big\}.$$

Finally, a mapping T of a class of functions f on (X, μ) into a class of functions on (Y, ν) is said to be *sublinear* provided that,

(i) If T is defined for f_0, f_1, then T is defined for $f_0 + f_1$, and

$$\big|T(f_0 + f_1)(x)\big| \le \big|T(f_0)(x)\big| + \big|T(f_1)(x)\big|.$$

(ii) $\big|T(kf)(x)\big| = |k|\,\big|T(f)(x)\big|$, for any scalar k.

A sublinear operation T such that

$$\|T(\chi_E)\|_q \lesssim \|\chi_E\|_{p,1},$$

where χ_E denotes the characteristic function of a measurable set E of finite measure, is said to be of *restricted type* (p, q). By [27, Theorem 3.13, p. 195] T then maps the Lorentz space $L(p, 1)$ continuously into $L^q(X)$, i.e.,

$$\|T(f)\|_q \lesssim \|f\|_{p,1}, \quad f \in L(p, 1). \tag{2.3}$$

A sublinear operator T defined for $f \in L^A(X)$ and taking values $T(f)$ in $L^B(Y)$ is said to be *bounded* if there is a constant $K > 0$ such that

$$\int_Y B\Big(\frac{|Tf(y)|}{K}\Big)\, d\nu \le 1, \quad \text{whenever} \quad \int_X A\big(|f(x)|\big)\, d\mu \le 1.$$

The smallest K above is called the *norm* of T, is denoted by $\|T\|$, and the operator is said to be of *type* (A, B). These operators satisfy $\|T(f)\|_B \lesssim \|f\|_A$. When $A(t) = t^p$ and $B(t) = t^q$, we say that T is of *type* (p, q).

Similar considerations apply to bounded sublinear operators T with domain, or range, a sequence space.

3 Maximal Integral Inequality

We begin by proving the maximal inequality for integral operators referred to in the Introduction, namely:

Theorem 3.1 *Suppose that the operator T has an integral representation with bounded kernel k on $\mathbb{R}$, i.e.,*

$$Tf(x) = \int_{-\infty}^{\infty} k(x, y) f(y)\, dy$$

whenever f is integrable, and consider the maximal operator T^* associated to T, to wit, for $\alpha, \beta > 0$, with $\chi_{\alpha,\beta}$ the characteristic function of the interval $(-\alpha, \beta)$, put

$$T^* f(x) = \sup_{\alpha,\beta>0} \big| T(f\chi_{\alpha,\beta}) \big|.$$

Then, if T is of type (p_0, q_0) for $1 < p_0, q_0 < \infty$, with p_0' the conjugate index to p_0, p' that to p, $\gamma = q_0/p_0'$, and

$$1 < r = \frac{q_0 + 1}{(q_0/p_0) + 1} < p_0, \tag{3.1}$$

T^* maps the Lorentz space $L(p, s)$ continuously into the Lorentz space $L(q, s)$, where $1 \le s \le \infty$, and p and q verify,

$$1 < p < r, \quad \text{and,} \quad q = \gamma\, p'. \tag{3.2}$$

Moreover, if p satisfies Eq. (3.2) and $1 \le s < \infty$, for $f \in L(p, s)$ we have

$$Tf(x) = \lim_{\alpha\to\infty} \int_{-\alpha}^{\alpha} k(x, y) f(y)\, dy \quad \text{a.e.} \tag{3.3}$$

Furthermore, T^* is of type (p, q) whenever Eq. (3.2) holds, and if A, B are Young functions such that $B(t)/t^q$ increases and $B(t)/t^m$ decreases for some $m > q$, T^* is continuous from the Orlicz space $L^A(\mathbb{R})$ into the Orlicz space $L^B(\mathbb{R})$ provided that A, B verify

$$B^{-1}(t) = t^{1/\gamma} A^{-1}\big(t^{-1/\gamma}\big), \quad t > 0.$$

And, when T^* is of type (A, B), Eq. (3.3) holds for $f \in L^A(\mathbb{R})$.

Proof First note that for f in some $L^p(\mathbb{R})$, $1 \le p \le \infty$, $f\chi_{\alpha,\beta}$ is integrable, and so,

$$T(f\chi_{\alpha,\beta})(x) = \int_{-\alpha}^{\beta} k(x,y)f(y)\,dy.$$

Now, since for these f's,

$$\begin{aligned} T^*f(x) &= \sup_{\alpha,\beta>0}\Big|\int_{-\alpha}^{\beta} k(x,y)f(y)\,dy\Big| \\ &\le \sup_{\alpha>0}\Big|\int_{-\alpha}^{0} k(x,y)f(y)\,dy\Big| + \sup_{\beta>0}\Big|\int_{0}^{\beta} k(x,y)f(y)\,dy\Big|, \end{aligned}$$

it suffices to consider functions f supported on either side of the origin. Both terms are dealt with in the same fashion, and we assume that f vanishes for $y \le 0$.

We claim that with r as in Eq. (3.1), T^* is of restricted type (r, q_0+1). To see this, let E be a measurable set of finite Lebesgue measure and note that, since χ_E is integrable and bounded, we have the identity,

$$\begin{aligned} &\Big(\int_0^{\beta} k(x,y)\chi_E(y)\,dy\Big)^{q_0+1} \\ &\quad = \int_0^{\beta} \frac{d}{dy}\Big(\int_0^{y} k(x,t)\chi_E(t)\,dt\Big)^{q_0+1} dy \\ &\quad\quad = (q_0+1)\int_0^{\beta}\Big(\int_0^{y} k(x,t)\chi_E(t)\,dt\Big)^{q_0} k(x,y)\chi_E(y)\,dy, \end{aligned}$$

where $\int_0^y k(x,t)\chi_E(t)\,dt = T\big(\chi_{[0,y]}\,\chi_E\big)(x)$.

Hence,

$$\begin{aligned} &\Big|\int_0^{\beta} k(x,y)\chi_E(y)\,dy\Big|^{q_0+1} \\ &\quad\quad \le (q_0+1)\int_0^{\infty} |k(x,y)|\,\chi_E(y)\,\Big|T\big(\chi_{[0,y]}\,\chi_E\big)(x)\Big|^{q_0}\,dy \end{aligned}$$

and, consequently,

$$T^*(\chi_E)(x)^{q_0+1} \le (q_0+1)\int_0^{\infty} |k(x,y)|\chi_E(y)\,\Big|T\big(\chi_{[0,y]}\,\chi_E\big)(x)\Big|^{q_0}\,dy.$$

Thus, integrating, with K a bound for k, it follows that

$$\begin{aligned} &\int_{-\infty}^{\infty} T^*(\chi_E)(x)^{q_0+1}\,dx \\ &\quad\quad \le (q_0+1)\,K\int_0^{\infty}\chi_E(y)\int_{-\infty}^{\infty}\Big|T\big(\chi_{[0,y]}\,\chi_E\big)(x)\Big|^{q_0}\,dx\,dy \end{aligned}$$

$$\le (q_0+1)\,K \int_0^\infty \chi_E(y)\|T\|^{q_0}\,\|\chi_E\|_{p_0}^{q_0}\,dy$$
$$= (q_0+1)\,K\,\|T\|^{q_0}\,|E|^{(q_0/p_0)+1}.$$

Therefore, $\|T^*(\chi_E)\|_{q_0+1} \le c\,\|\chi_E\|_{r,1}$, with $c = \big((q_0+1)\,K\,\|T\|^{q_0}\big)^{1/(q_0+1)}$, and T^* is of restricted type (r, q_0+1). Hence, by Eq. (2.3) above, for $f \in L(r,1)$ we have

$$\|T^*(f)\|_{q_0+1} \lesssim \|f\|_{r,1}.$$

Moreover, since for $f \in L^1(\mathbb{R})$,

$$T^*(f)(x) \le K \int_{-\infty}^{\infty} |f(y)|\,dy = K\,\|f\|_1,$$

T^* is of type $(1,\infty)$, and, consequently, interpolating, by [6, Corollary to Theorem 10, p. 293], with $0<\theta<1$, and

$$\frac{1}{p} = (1-\theta) + \frac{\theta}{r}, \quad \text{and,} \quad \frac{1}{q} = \frac{\theta}{q_0+1},$$

it follows that

$$\|T^*(f)\|_{q,s} \lesssim_{p_0,q_0,K,p,s} \|f\|_{p,s}, \quad 1 \le s \le \infty. \tag{3.4}$$

Now, eliminating θ yields

$$\frac{q_0+1}{q} = \Big(1-\frac{1}{p}\Big)\Big/\Big(1-\frac{1}{r}\Big),$$

and replacing r by its value given in Eq. (3.1) above it readily follows that $q = \gamma\,p'$, and the claim involving the Lorentz spaces has been established.

The existence of the pointwise limit Eq. (3.3) follows along the lines the comments after Corollary 4.2 below. If $f \in L(p,s)$ with $p>1$ and $s<\infty$, with χ_R the characteristic function of $B(0,R)$, since

$$m(f\,\chi_R, \lambda) \lesssim_f \min\big(R, \lambda^{-p}\big),$$

it readily follows that for the dense family of compactly supported $g \in L(p,s)$, $g\chi_R$ is integrable and equal to g for all sufficiently large R, and so,

$$Tg(x) = \lim_{R\to\infty} \int_{B(0,R)} k(x,y)\,g(y)\,dy, \quad x \in \mathbb{R}.$$

Moreover, since for $f \in L(p,s)$, for each λ, $m(f - f\chi_R, \lambda)$ decreases to 0 as $R \to \infty$, it readily follows that

$$\|f - f\chi_R\|_{p,s} \to 0 \quad \text{as } R \to \infty,$$

and since T is linear and continuous on $L(p,s)$, for $f \in L(p,s)$ we have

$$\|T(f) - T(f\chi_R)\|_{q,s} \lesssim_p \|f - f\chi_R\|_{p,s} \to 0 \quad \text{as } R \to \infty.$$

Hence, we conclude that for $f \in L(p,s)$ we have

$$Tf(x) = \lim_{R\to\infty} \int_{B(0,R)} k(x,y)\, f(y)\, dy \quad \text{a.e.},$$

and (3.3) has been established.

Now, on account of the monotonicity of the Lorentz norms with respect to the second index, since for p, q verifying Eq. (3.2) we have $p < r < q_0 + 1 < q$, setting $s = q$ in Eq. (3.4), it follows that

$$\begin{aligned}\|T^*(f)\|_q \lesssim \|T^*(f)\|_{q,q} &\lesssim_{p_0,q_0,K,p} \|f\|_{p,q} \\ &\lesssim_{p_0,q_0,K,p} \|f\|_{p,p} \lesssim_{p_0,q_0,K,p} \|f\|_p\,,\end{aligned}$$

and T^* is of type (p,q).

Furthermore, since the assumptions to apply the interpolation theorem for operators of types (p,q) and $(1,\infty)$ are satisfied [29, Theorem 2.11, p. 187], the conclusion for Orlicz spaces obtains.

Finally, if A, B are Young functions such that T^* is of type (A,B), the pointwise convergence in this case follows mutatis mutandis the Lorentz spaces case, and we have finished. □

Theorem 3.1 can be recast in two different contexts. On the one hand, $\mathbb{R}$ can be replaced by a closed bounded interval I in the line, and on the other hand by $\mathbb{R}^n$, $n > 1$, [8].

4 Orlicz Spaces Maximal Inequalities

We proceed now to prove the Orlicz spaces version of the Christ–Kiselev maximal inequality. The proof rests on the notion of filtrations [13], [18, Theorem 2.11.1, p. 169].

Let (X,μ) be a σ-finite measure space with no atoms. A *filtration* is a family of measurable sets A_α such that

(i) If $\beta < \alpha$, then $A_\beta \subset A_\alpha$.
(ii) $\lim_{\varepsilon\to 0^+} \mu\big(A_{\alpha+\varepsilon}\setminus A_\alpha\big) = \lim_{\varepsilon\to 0^+} \mu\big(A_\alpha\setminus A_{\alpha-\varepsilon}\big) = 0\,.$
(iii) $\mu\Big(\bigcap_\alpha A_\alpha\Big) = 0, \quad \bigcup_\alpha A_\alpha = X.$

Examples of filtrations are $A_\alpha = (-\infty, \alpha]$ in $\mathbb{R}$, and $A_\alpha = B(0,\alpha) \subset \mathbb{R}^n$, provided that $\mu\big(\{x \in \mathbb{R}^n : |x| = \alpha\}\big) = 0$ for all $\alpha \geq 0$.

We then have the Orlicz spaces maximal inequality, that is:

Theorem 4.1 *Suppose that T is a bounded sublinear mapping from $L^A(X)$ into $L^B(Y)$ with norm $\|T\|$, where A, B are Young functions such that $A(t)/t^p$ decreases and $B(t)/t^q$ increases, where $1 \leq p < q < \infty$. Further, assume that $B(t)/t^{q_1}$ decreases for some $q_1 > q$.*

Let T^* denote the maximal operator associated to T and the filtration A_α, i.e.,

$$T^* f(x) = \sup_\alpha |T(f\chi_\alpha)(x)|.$$

Then T^* is of type (A, B), and $\|T^*\| \leq \|T\| \sum_{m=1}^\infty 2^{m(1-q/p)(1/q_1)}$.

Proof Assume first that $\|f\|_A = 1$, and so $\displaystyle\int_X A\big(|f(x)|\big)\,d\mu = 1$. Let

$$G(\alpha) = \int_X A\big(\,|f\chi_{A_\alpha}(x)|\,\big)\,d\mu, \quad \alpha > 0.$$

From the properties of the filtrations it follows that $G(\alpha)$ is continuous, and

$$\lim_{\alpha\to 0} G(\alpha) = 0, \quad \text{and,} \quad \lim_{\alpha\to\infty} G(\alpha) = \int_X A(|f(x)|)\,d\mu = 1.$$

Also, that $G(\alpha)$ is monotone in α, and assumes values from 0 to 1, and so, for $m = 1, 2, \ldots$ and $j = 1, 2, \ldots, 2^m - 1$, the relation

$$G(\alpha_j^m) = \frac{j}{2^m},$$

determines the points α_j^m, with α_j^m the smallest such solution on an interval of constancy of G.

For $m = 1, 2, \ldots$ $j = 1, \ldots, 2^m$, where $\chi_{\alpha_{2^m}^m} = 1$ and $\chi_{\alpha_0^m} = 0$, let

$$f_j^m = f\big(\chi_{\alpha_j^m} - \chi_{\alpha_{j-1}^m}\big).$$

Then $\int_X A\big(|f_j^m(x)|\big)\,d\mu = 2^{-m}$ for $m = 1, 2, \ldots$, and since $A(t)/t^p$ decreases, by Eq. (2.1) we have

$$\|f_j^m\|_A \leq 2^{-m/p}. \tag{4.1}$$

As in [25, Theorem 2.11.1, p. 169], for $\alpha \in [0, 1]$, consider the expression for $G(\alpha)$ given by

$$G(\alpha) = \sum_{\ell=1}^\infty \frac{k_\ell}{2^\ell}, \quad k_\ell = 0, 1,$$

and define

$$j_m(\alpha) = 2^m \sum_{\ell=1}^{m} \frac{k_\ell}{2^\ell}.$$

Then, by the continuity of G, as functions in $L^A(X)$ (up to sets of μ measure 0), it follows that

$$f\chi_\alpha = \sum_{\substack{m=1 \\ k_m=1}} f^m_{j_m(\alpha)},$$

and, consequently,

$$T^*(f)(y) \le \sum_{m=1}^{\infty} \Big(\sup_{1\le j\le 2^m} \big|T(f_j^m)(y)\big| \Big).$$

Moreover, since $\max_{1\le\ell\le L} |a_\ell| \le B^{-1}\big(\sum_{\ell=1}^{L} B\big(|a_\ell|\big)\big)$, we have

$$T^*(f)(y) \le \|T\| \sum_{m=1}^{\infty} B^{-1}\Big(\sum_{j=1}^{2^m} B\Big(\frac{\big|T(f_j^m)(y)\big|}{\|T\|}\Big)\Big),$$

and so, by Minkowski's integral inequality,

$$\|T^*(f)\|_B \le \|T\| \sum_{m=1}^{\infty} \Big\| B^{-1}\Big(\sum_{j=1}^{2^m} B\Big(\frac{\big|T(f_j^m)\big|}{\|T\|}\Big)\Big)\Big\|_B. \tag{4.2}$$

Let

$$\Phi_m(y) = B^{-1}\Big(\sum_{j=1}^{2^m} B\Big(\frac{\big|T(f_j^m)(y)\big|}{\|T\|}\Big)\Big), \quad m = 1, 2, \ldots$$

To estimate the contribution of each Φ_m, fix m, observe that

$$\int_Y B\big(\Phi_m(y)\big)\, d\nu = \sum_{j=1}^{2^m} \int_Y B\Big(\frac{\big|T(f_j^m)(y)\big|}{\|T\|}\Big)\, d\nu,$$

and consider each summand above separately.

Now, we can write

$$\int_Y B\Big(\frac{\big|T(f_j^m)(y)\big|}{\|T\|}\Big)\, d\nu = \int_Y B\Big(\frac{\|f_j^m\|_A \big|T(f_j^m)(y)\big|}{\|T\|\|f_j^m\|_A}\Big)\, d\nu,$$

and, since $B(t)/t^q$ increases and $\|f_j^m\|_A < 1$, by Eq. (2.2) it follows that,

$$\int_Y B\Big(\frac{|T(f_j^m)(y)|}{\|T\|}\Big)\,d\nu \le \|f_j^m\|_A^q \int_Y B\Big(\frac{|T(f_j^m)(y)|}{\|T\|\,\|f_j^m\|_A}\Big)\,d\nu. \tag{4.3}$$

And, since T is of type (A, B), the integral above is ≤ 1, and so, by Eqs. (4.1, 4.3),

$$\int_Y B\Big(\frac{|T(f_j^m)(y)|}{\|T\|}\Big)\,d\nu \le \|f_j^m\|_A^q \le 2^{-mq/p}.$$

Therefore, the sum in question is bounded by

$$\int_Y B\big(\Phi_m(y)\big)\,d\nu \le \sum_{j=1}^{2^m}\big(\|f_j^m\|_A\big)^{q/p} \le (2^{-m})^{q/p}\sum_{j=1}^{2^m} 1 = 2^{m\left(1-\frac{q}{p}\right)} < 1.$$

Finally, since $B(t)/t^{q_1}$ decreases, by Eq. (2.1),

$$\|\Phi_m\|_B \le 2^{m\left(1-\frac{q}{p}\right)(1/q_1)},$$

and, consequently, summing over m from Eq. (4.2) it follows that

$$\|T^*(f)\|_B \le \|T\| \sum_{m=1}^{\infty} 2^{m\left(1-\frac{q}{p}\right)(1/q_1)} < \infty,$$

which is the desired conclusion in this case.

The estimate for arbitrary $f \in L^A(X)$ follows upon replacing f by $f/\|f\|_A$ above, and the proof is finished. □

Now, the Orlicz spaces maximal inequality yields the pointwise convergence of the operators in question, namely:

Corollary 4.2 *Suppose that the conditions for Theorem 4.1 hold. In addition, suppose that for g in a dense subset of $L^A(X)$, $\lim_{\alpha\to\infty} T_\alpha(g)(y) = T(g)(y)$ exists ν–a.e., and that for all $f \in L^A(X)$, $\lim_{\alpha\to\infty}\|T_\alpha(f) - T(f)\|_B = 0$. Then, for $f \in L^A(X)$,*

$$\lim_{\alpha\to\infty} T_\alpha(f)(y) = T(f)(y) \quad \nu\text{–a.e.}$$

Proof For $f \in L^A(X)$ put

$$F(y) = \limsup_\alpha T_\alpha(f)(y) - \liminf_\alpha T_\alpha(f)(y),$$

and note that for g in the dense set where the convergence holds ν–a.e. we have

$$F(y) \le 2\,\sup_\alpha \big|T_\alpha(f-g)(y)\big| \quad \nu - - \text{a.e.}$$

Hence, by the Orlicz spaces maximal inequality,

$$\|F\|_B \le 2\,\|T^*(f-g)\|_B \le 2\,\|T^*\|\,\|f-g\|_A,$$

which, by the density of the g's, can be made arbitrarily small. Thus $\|F\|_B = 0$, and so $F(y) = 0$ ν-a.e., and, consequently,

$$\lim_{\alpha\to\infty} T_\alpha(f)(y) \text{ exists } \nu - \text{ a.e.} \tag{4.4}$$

Now, since by assumption $\lim_{\alpha\to\infty} \left\|T_\alpha(f) - T(f)\right\|_B = 0$, there is a sequence $\{\alpha_m\}$ tending to ∞ such that $\lim_{\alpha_m\to\infty} T_{\alpha_n}(f)(y) = T(f)(y)$ ν-a.e., which combined with Eq. (4.4) gives that $\lim_{\alpha\to\infty} T_\alpha(f)(y) = T(f)(y)$ $\nu -$ a.e., and the proof is finished. □

Zygmund's result concerning the Fourier transform on $\mathbb{R}^n$ in the context of Orlicz spaces follows from Corollary 4.2. Indeed, by [19], the Fourier transform is bounded from $L^A(\mathbb{R}^n)$ into $L^B(\mathbb{R}^n)$ whenever $B(t)/t^m$ decreases for some $m > 2$ and the Young functions A, B verify

$$B^{-1}(t) = t\,A^{-1}(1/t), \quad t > 0.$$

Note that this implies that, if $A(t)/t^p$ decreases and $q = p'$, then

$$\frac{B^{-1}(t)}{t^{1/q}} = \frac{A^{-1}(1/t)}{1/t^{1/p}}$$

also decreases, and so $B(t)/t^q$ increases. Thus, A and B satisfy the assumptions of Theorem 4.1 with $1 < p < q = p' < \infty$ there. Also, functions in $L^1(\mathbb{R}^n) \cap L^A(\mathbb{R}^n)$, for which there is a.e. convergence, are dense in $L^A(\mathbb{R}^n)$. For the filtrations we take $A_\alpha = B(0,\alpha)$, $\alpha > 0$. Then, for every $f \in L^A(\mathbb{R}^n)$, with $\widehat{f}$ denoting the function in $L^B(\mathbb{R}^n)$ determined by Hausdorff–Young, we have

$$\widehat{f}(x) = \lim_{\alpha\to\infty} \frac{1}{(2\pi)^{n/2}} \int_{B(0,\alpha)} e^{-i<x,y>} f(y)\,dy \quad \text{a.e.}$$

It is also possible to produce, as J. E. Littlewood used to say, a proof of the pointwise convergence of the Fourier integral for functions in the Lorentz and Orlicz spaces for a "mathematician in a hurry." Indeed, by the Christ–Kiselev maximal inequality, the n–dimensional Zygmund maximal Fourier transform is of type (p, p') for $1 < p < 2$, and, therefore, by interpolation it maps $L(p,s)$ continuously into $L(p',s)$ for $1 \le s \le \infty$, and is of type (A, B) whenever A, B satisfy the conditions given after Corollary 4.2. This observation applies to other contexts as well.

5 Hausdorff–Young Inequalities

In this section we prove a refined version of the Hausdorff–Young inequality, and related estimates, for Hermite and Laguerre expansions.

Hermite Expansions

Szegö discusses the Hermite and Laguerre polynomials in Chapter V of [28]. Earlier, Hille had considered the *Hermite polynomials*, $H_m(x)$, and discussed some remarkable formulas and estimates [18, 30]. In particular, Hille considered the generating formula

$$\sum_{m=0}^{\infty} H_m(x)\frac{u^m}{m!} = e^{2xu - u^2},$$

and defined the *Hermite functions*, $\mathcal{H}_m(x)$, by

$$\mathcal{H}_m(x) = \frac{1}{(m!)^{1/2}}\frac{1}{2^{m/2}} H_m(x)\, e^{-x^2/2}.$$

The Hermite functions constitute an ONS in $\mathbb{R}$ with respect to the Lebesgue measure there.

We prove now the refined Hausdorff–Young inequality for the Hermite expansion stated in the Introduction:

Theorem 5.1 *Suppose the Hermite expansion of the function f is given by $f(x) \sim \sum_{m=0}^{\infty} c_m \mathcal{H}_m(x)$, where*

$$c_m = \int_{-\infty}^{\infty} \mathcal{H}_m(x)\, f(x)\, dx, \quad m = 0, 1, 2, \ldots,$$

and let T be the mapping that assigns to f the sequence $\{c_m\}$ of its Hermite coefficients. Then, T maps the Lorentz space $L(p, s)$ continuously into the Lorentz sequence space $\ell^{q,s}$, $1 \le s \le \infty$, provided that p, q verify

$$1 < p < 2, \quad \textit{and}, \quad \frac{5}{6}\frac{1}{p} + \frac{1}{q} = \frac{11}{12}. \tag{5.1}$$

In particular, T is of type (p, q) whenever Eq. (5.1) holds.

Furthermore, if A, B are Young's functions such that $B(t)/t^2$ increases, $B(t)/t^{12}$ decreases, and $\int_t^{\infty} \left(B(s)/s^{12}\right) ds/s \lesssim B(t)/t^{12}$, T maps $L^A(\mathbb{R})$ continuously into the Orlicz sequence space ℓ^B provided that A, B verify

$$B^{-1}(t) = t^{11/12} A^{-1}(t^{-5/6}), \quad t > 0.$$

And, if the maximal Hermite coefficients C_m are given by

$$C_m = \sup_{\alpha>0} \Big| \int_{-\alpha}^{\alpha} \mathcal{H}_m(x)\, f(x)\, dx \Big|, \quad m = 0, 1, 2, \ldots,$$

all norm inequalities above hold with C_m in place of c_m there.

Proof Let μ denote the atomic measure concentrated on the integer atoms $m = 0, 1, 2, \ldots$, taking the value $\mu(m) = 1$ on each such atom. Given $\lambda > 0$, let $\mathcal{I}_\lambda = \{m : |c_m| > \lambda\}$; we are interested in estimating $\mu\big(\mathcal{I}_\lambda\big)$. Now, if $0 \neq m \in \mathcal{I}_\lambda$, on account of Eq. (1.2) we have

$$\lambda < |c_m| \lesssim \|f\|_1\, m^{-\frac{1}{12}},$$

and, consequently, for such m we have

$$m \lesssim \Big(\frac{\|f\|_1}{\lambda}\Big)^{12}.$$

Hence it readily follows that

$$\lambda^{12}\, \mu\big(\{m \neq 0 : |c_m| > \lambda\}\big) \lesssim \|f\|_1^{12}, \tag{5.2}$$

which gives the desired estimate for $\mu(\mathcal{I}_\lambda)$ when $0 \notin \mathcal{I}_\lambda$.

Now, if $0 \in \mathcal{I}_\lambda$, since $\mathcal{H}_0(x) = 1$ we get that $\lambda < |c_0| \leq \|f\|_1$, and so

$$\lambda^{12}\, \mu(0) = \lambda^{12} \leq \|f\|_1^{12},$$

which combined with Eq. (5.2) above gives that, also in this case, $\lambda^{12}\, \mu\big(\mathcal{I}_\lambda\big) \lesssim \|f\|_1^{12}$.

Thus,

$$\|\{c_m\}\|_{\ell^{12,\infty}} = \sup_{\lambda>0} \lambda\, \mu\big(\{m : |c_m| > \lambda\}\big)^{1/12} \lesssim \|f\|_1, \tag{5.3}$$

and T is continuous from $L(1,1) = L^1(\mathbb{R})$ into the Lorentz sequence space $\ell^{12,\infty}$.

Also, since T is of type $(2,2)$ and the Lorentz norms are monotone with respect to the second index, we have

$$\|\{c_m\}\|_{\ell^{2,\infty}} \lesssim \|\{c_m\}\|_{\ell^2} \lesssim \|f\|_2 \lesssim \|f\|_{2,1},$$

and, thus, interpolating, by [6, Corollary to Theorem 10, p. 293] it follows that T maps the Lorentz space $L(p,s)$ continuously into the Lorentz sequence space $\ell^{q,s}$, $1 \leq s \leq \infty$, where, for $0 < \theta < 1$,

$$\frac{1}{p} = \theta + \frac{1-\theta}{2}, \quad \frac{1}{q} = \frac{\theta}{12} + \frac{1-\theta}{2}.$$

Now, simple algebraic manipulations allow us to eliminate θ giving Eq. (5.1), and, provided that Eq. (5.1) holds, we get that

$$\|\{c_m\}\|_{\ell^{q,s}} \lesssim_{p,s} \|f\|_{p,s}, \quad 1 \le s \le \infty. \tag{5.4}$$

Moreover, on account of the monotonicity of the Lorentz norms with respect to the second index, since for p, q verifying Eq. (5.1) we have $p < 2 < q$, setting $s = q$ in Eq. (5.4), it follows that

$$\|\{c_m\}\|_{\ell^q} \lesssim \|\{c_m\}\|_{\ell^{q,q}} \lesssim_p \|f\|_{p,q} \lesssim_p \|f\|_{p.p} \lesssim_p \|f\|_p, \tag{5.5}$$

and T is of type (p, q).

The Orlicz spaces estimate follows now by interpolation [29, Theorem 2.8, p. 184].

To proceed with the maximal estimates, we transfer the results from the atomic measure to the Lebesgue measure on $\mathbb{R}$ by means of a technique introduced in [6], and conclude that Eq. (5.5) holds with $\{C_m\}$ in place of $\{c_m\}$ there.

More precisely, let

$$\mathcal{H}(u, x) = \mathcal{H}_m(x), \quad m \le u < m + 1, \quad m = 0, 1, 2, \dots,$$

and from

$$c_m = \int_{-\infty}^{\infty} \mathcal{H}_m(x)\, f(x)\, dx, \quad m = 0, 1, 2, \dots,$$

pass to

$$C(f)(u) = \int_{-\infty}^{\infty} \mathcal{H}(u, x)\, f(x)\, dx, \quad u \in \mathbb{R}^+.$$

Let p, q satisfy the relation Eq. (5.1) above. Note that

$$\begin{aligned}\|C(f)\|_q^q &= \int_0^{\infty} \Big| \int_{-\infty}^{\infty} \mathcal{H}(u, x) f(x)\, dx \Big|^q du \\ &= \sum_{m=0}^{\infty} \int_m^{m+1} \Big| \int_{-\infty}^{\infty} \mathcal{H}_m(x) f(x)\, dx \Big|^q du = \sum_{m=0}^{\infty} |c_m|^q,\end{aligned}$$

and, consequently, by Eq. (5.5),

$$\|C(f)\|_q = \|\{c_m\}\|_{\ell^q} \lesssim_p \|f\|_p.$$

Now, if $\chi_\alpha = \chi_{[-\alpha,\alpha]}$, the conditions of the Christ–Kiselev maximal inequality, or Theorem 4.1, are satisfied, and so, with

$$C^*(f)(u) = \sup_\alpha \big| C(f\chi_\alpha)(u) \big| = \sup_\alpha \Big| \int_{-\alpha}^{\alpha} \mathcal{H}(u, x) f(x)\, dx \Big|,$$

it follows that $\|C^*(f)\|_q \lesssim \|f\|_p$. Again, as above,

$$\|C^*(f)\|_q^q = \sum_{m=0}^{\infty} \int_m^{m+1} \Big(\sup_\alpha \Big| \int_{-\alpha}^{\alpha} \mathcal{H}_m(x) f(x)\, dx \Big| \Big)^q du = \sum_{m=0}^{\infty} C_m^q,$$

and, consequently,

$$\|\{C_m\}\|_{\ell^q} \lesssim_p \|f\|_p, \tag{5.6}$$

and Eq. (5.5) holds with $\{C_m\}$ in place of $\{c_m\}$ there.

Let now S be the sublinear mapping that assigns to f the sequence $\{C_m\}$ of its maximal Hermite coefficients. Then Eq. (5.6) holds for those p, q that verify Eq. (5.1) above. The estimates for $\{C_m\}$ in the Lorentz and Orlicz spaces follow now by interpolation; in the case of Lorentz spaces we use [6, Corollaty to Theorem 10, p. 293], and for the Orlicz spaces we essentially repeat the argument for the $\{c_m\}$. The proof is thus finished. □

We prove next the (refined) companion result to the Hausdorff–Young inequality for Hermite expansions, namely:

Theorem 5.2 *Let* $12/11 < p < 2$, *and suppose that* q *is such that*

$$\frac{1}{p} + \frac{5}{6}\frac{1}{q} = \frac{11}{12}. \tag{5.7}$$

Then, given $\{c_m\}$ *in the Lorentz sequence space* $\ell^{p,s}$, *there is* f *in the Lorentz space* $L(q, s)$, $1 \le s \le \infty$, *such that* $f(x) \sim \sum_{m=0}^{\infty} c_m \mathcal{H}_m(x)$, *and*

$$\|f\|_{q,s} \lesssim_{p,s} \|\{c_m\}\|_{\ell^{p,s}}.$$

In particular, if τ *denotes the mapping that assigns* f *to the sequence* $\{c_m\}$, τ *is of type* (p, q) *whenever Eq. (5.7) holds.*

Moreover, if A, B *are Young's functions such that* $B(t)/t^2$ *increases, and for some* $r > 2$, $B(t)/t^r$ *decreases and* $\int_t^\infty \big(B(s)/s^r\big)\, ds/s \lesssim B(t)/t^r$, *then* τ *maps the Orlicz sequence space* ℓ^A *continuously into the Orlicz space* $L^B(\mathbb{R})$, *provided that* A, B *verify*

$$B^{-1}(t) = t^{11/10} A^{-1}(t^{-6/5}), \quad t > 0.$$

Furthermore, the maximal operator τ^* *associated to* τ *is of type* (A, B), *and for* $f = \tau(\{c_m\})$ *we have*

$$f(x) = \lim_M \sum_{m=0}^{M} c_m \mathcal{H}_m(x) \quad a.e.$$

Proof Let $b(x) = \{\mathcal{H}_m(x)\}$. Then, by Eq. (1.2), as in Eq. (5.3) it follows that $b(x) \in \ell^{12,\infty}$ uniformly in x, and so, for a sequence $\{c_m\}$ in $\ell^{12/11,1}$, we have

$$\Big|\sum_m c_m\, \mathcal{H}_m(x)\Big| \lesssim \|\{c_m\}\|_{\ell^{12/11,1}}, \quad \text{uniformly in } x.$$

Hence, if $f(x) \sim \sum_{m=0}^{\infty} c_m \mathcal{H}_m(x)$, then $f \in L^{\infty}(\mathbb{R})$, and

$$\|f\|_{\infty,\infty} = \|f\|_{\infty} \lesssim \|\{c_m\}\|_{\ell^{12/11,1}}.$$

And, by a now familiar argument, τ is of type $(2, 2)$ and we have $\|f\|_{2,\infty} \lesssim \|c_m\|_{\ell^{2,1}}$, and so, interpolating, by [6, Corollary to Theorem 10, p. 293] it follows that τ maps the Lorentz sequence space $\ell^{p,s}$ continuously into the Lorentz space $L(q, s)$, $1 \le s \le \infty$, where, for $0 < \theta < 1$,

$$\frac{1}{p} = \frac{11}{12}\theta + \frac{1-\theta}{2}, \quad \frac{1}{q} = \frac{1-\theta}{2}.$$

Now, eliminating θ gives Eq. (5.7), and, provided that Eq. (5.7) holds, we get that

$$\|f\|_{q,s} \lesssim_{p,s} \|\{c_m\}\|_{\ell^{p,s}}, \quad 1 \le s \le \infty.$$

And, since $p < q$, setting $s = q$ gives that τ is of type (p, q), provided that Eq. (5.7) holds.

The result for the Orlicz spaces follows now by interpolation [21, Theorem 2.8, p. 184], but we can say more. Referring to the Orlicz spaces discrete maximal inequality, Theorem 6.1, to be proved in the next section, let

$$\tau^*\big(\{c_m\}\big) = \sup_M \Big|\sum_{m=0}^{M} c_m \mathcal{H}_m(x)\Big|.$$

Then, by Theorem 6.1 it follows that τ^* maps ℓ^A continuously into $L^B(\mathbb{R})$ whenever τ is of type (A, B).

Let now $f_M = \sum_{m=1}^{M} c_m \mathcal{H}_m(x)$, and observe that by the linearity and boundedness of τ, with $c_{M_1}^{M_2}$ denoting the sequence with terms c_m for $M_1 + 1 \le m \le M_2$ and 0 otherwise, we have

$$\|f_{M_2} - f_{M_1}\|_B \lesssim_A \|c_{M_1}^{M_2}\|_{\ell^A} \to 0 \quad \text{as } M_1, M_2 \to \infty,$$

and, consequently, $\{f_M\}$ is Cauchy in $L^B(\mathbb{R})$. If we denote the limit of this sequence by f, then $f(x) \sim \sum_{m=0}^{\infty} c_m\, \mathcal{H}_m(x)$, $\|f\|_B \lesssim_A \|\{c_m\}\|_{\ell^A}$, and

$$\lim_M \Big\| f - \sum_{m=0}^{M} c_m \mathcal{H}_m \Big\|_B = 0.$$

Also, for a dense subset of ℓ^A, namely, those sequences with finitely many nonzero terms, $\sum_{m=0}^{\infty} c_m \mathcal{H}_m(x)$ is actually a finite sum, and so,

$$\lim_{M\to\infty} \sum_{m=0}^{M} c_m \mathcal{H}_m(x) = \sum_{m=0}^{\infty} c_m \mathcal{H}_m(x), \quad \text{all } x \in \mathbb{R}.$$

Hence, since these conditions are met, by the Orlicz spaces pointwise convergence result (Corollary 4.2, or an argument similar to that of Theorem 6.2) we have,

$$f(x) = \lim_{M} \sum_{m=0}^{M} c_m \mathcal{H}_m(x) \quad \text{a.e.},$$

and the proof is finished. □

A refined Hausdorff-Young inequality and companion result for n–dimensional Hermite expansions is given in [10].

Ditzian completes the picture with the consideration of Hardy–Littlewood type inequalities in a context that includes the Hermite expansions as well as type (p, p') Hausdorff–Young analogues [14].

We consider Hardy–Littlewood type estimates next. The key ingredient here is the sharp Hardy–Littlewood estimate for functions f in the real Hardy space $H^1(\mathbb{R})$, [21, Theorem 1.1, p. 270], namely, if $f(x) \sim \sum_{m=0}^{\infty} c_m \mathcal{H}_m(x)$ denotes the Hermite expansion of f in $H^1(\mathbb{R})$, then

$$\sum_{m=0}^{\infty} |c_m|\,(1+m)^{-\frac{3}{4}} \lesssim \|f\|_{H^1}. \tag{5.8}$$

This result is sharp because if we replace the $H^1(\mathbb{R})$ norm by the $L^1(\mathbb{R})$ norm on the right-hand side above, Eq. (5.8) holds with $3/4$ replaced by $3/4+\varepsilon$, $\varepsilon > 0$, on the left-hand side, and there is an $L^1(\mathbb{R})$ function f such that the expression on the left–hand side of Eq. (5.8) is infinite [20].

We then have the Hardy–Littlewood inequality for Hermite expansions:

Theorem 5.3 *Let $1 < p < 2$, and let μ denote the measure on the integers with mass $\mu(m) = (1+m)^{-\frac{3}{2}}$ for $n = 0, 1, 2, \ldots$ Given a function $f \in L^p(\mathbb{R})$ with Hermite series expansion $f(x) \sim \sum_{m=0}^{\infty} c_m \mathcal{H}_m(x)$, let T be the mapping that assigns to f the sequence $\{C_m\}$ given by*

$$C_m = c_m\,(1+m)^{\frac{3}{4}}, \quad m = 0, 1, 2 \ldots$$

Then, T maps the Lorentz space $L(p, s)$ into the Lorentz sequence space $\ell_\mu^{p,s}$, $1 \le s \le \infty$, and, in particular,

$$\sum_{m=0}^{\infty} |c_m|^p (1+m)^{(p-2)\frac{3}{4}} \lesssim_p \|f\|_p^p. \tag{5.9}$$

Furthermore, if $q > 2$ and $\{c_m\}$ is a sequence that satisfies

$$\sum_{m=0}^{\infty} |c_m|^q \left(1+m\right)^{(q-2)\frac{3}{4}} < \infty, \tag{5.10}$$

there is a function $f \in L^q(\mathbb{R})$ such that $c_m = c_m(f)$, the Hermite coefficients of f, and

$$\|f\|_q^q \lesssim_q \sum_{m=0}^{\infty} |c_m|^q \left(1+m\right)^{(q-2)\frac{3}{4}} . \tag{5.11}$$

Proof We begin by proving Eq. (5.9). First note that, since

$$\|\{C_m\}\|_{\ell^2_\mu}^2 = \sum_{m=0}^{\infty} |c_m|^2 (1+m)^{\frac{3}{4}\,2}\ (1+m)^{-\frac{3}{2}} \lesssim \|f\|_2^2,$$

T is continuous from $L^2(\mathbb{R})$ into ℓ^2_μ.

Also, by Eq. (5.8),

$$\sum_{m=0}^{\infty} |c_m|\,(1+m)^{\frac{3}{4}}\,(1+m)^{-\frac{3}{2}} = \sum_{m=0}^{\infty} |c_m|\,(1+m)^{-\frac{3}{4}} \lesssim \|f\|_{H^1},$$

and T is continuous from the real Hardy space $H^1(\mathbb{R})$ into ℓ^1_μ.

Then, for $1 < p < 2$, with

$$\frac{1}{p} = (1-\theta) + \frac{\theta}{2}, \quad 0 < \theta < 1,$$

by [24, Eq. (2), p. 401] it follows that T maps the Lorentz space $L(p,s)$ into the Lorentz sequence space $\ell^{p,s}_\mu$, $1 \le s \le \infty$, and, in particular, when $s = p$, $L^p(\mathbb{R})$ into ℓ^p_μ, and so, Eq. (5.9) holds.

Next, let $\{c_m\}$ be a sequence that satisfies Eq. (5.10) above for some $q > 2$, and let $f_M(x) = \sum_{m=0}^{M} c_m \mathcal{H}_m(x)$, $M = 1, 2, \ldots$ We claim that $f_M \in L^q(\mathbb{R})$, all M, with an appropriate bound. To see this, let $p = q'$ denote the conjugate to q, and for $g \in L^p(\mathbb{R})$ with $\|g\|_p \le 1$, let $g(x) \sim \sum_{m=0}^{\infty} B_m \mathcal{H}_m(x)$. Note that since

$$\left(1-\frac{2}{q}\right)\frac{3}{4} + \left(1-\frac{2}{p}\right)\frac{3}{4} = 0,$$

it follows that for all M,

$$\int_{\mathbb{R}} f_M(x)\, g(x)\, dx = \sum_{m=0}^{M} c_m B_m = \sum_{m=0}^{M} c_m\, (1+m)^{(1-\frac{2}{q})\frac{3}{4}}\, B_m\, (1+m)^{(1-\frac{2}{p})\frac{3}{4}},$$

and, therefore,

$$\Big| \int_{\mathbb{R}} f_M(x)\, g(x)\, dx \Big| \\ \leq \Big(\sum_{m=0}^{M} |c_m|^q \, (1+m)^{(q-2)\frac{3}{4}} \Big)^{1/q} \Big(\sum_{m=0}^{M} |B_m|^p (1+m)^{(p-2)\frac{3}{4}} \Big)^{1/p}.$$

Now, Eq. (5.9) implies that

$$\Big(\sum_{m=0}^{M} |B_m|^p (1+m)^{(p-2)\frac{3}{4}} \Big)^{1/p} \lesssim_p \|g\|_p \lesssim 1,$$

and so,

$$\|f_M\|_q \lesssim_q \Big(\sum_{m=0}^{M} |c_m|^q (1+m)^{(q-2)\frac{3}{4}} \Big)^{1/q},$$

and $f_M \in L^q(\mathbb{R})$, all M.

A similar argument gives that for $M_1 < M_2$,

$$\|f_{M_2} - f_{M_1}\|_q \lesssim_q \Big(\sum_{m=M_1+1}^{M_2} |c_m|^q \, (1+m)^{(q-2)\frac{3}{4}} \Big)^{1/q},$$

and, consequently, $\{f_M\}$ is Cauchy in $L^q(\mathbb{R})$. If we denote the limit of this sequence by f, then $f(x) \sim \sum_{m=0}^{\infty} c_m \, \mathcal{H}_m(x)$,

$$\|f\|_q \lesssim_q \Big(\sum_{m=0}^{\infty} |c_m|^q (1+m)^{(q-2)\frac{3}{4}} \Big)^{1/q},$$

and Eq. (5.11) has been established. □

Laguerre Expansions

We will now consider the Laguerre expansions. The *Laguerre polynomials*, $L_m^\alpha(x)$, were introduced by Szegö, [28, p. 96] and are defined by

$$\sum_{m=1}^{\infty} L_m^\alpha(x) u^m = (1-u)^{1-\alpha} e^{-ux/(1-u)}, \quad \alpha > -1,$$

and the *Laguerre functions*, $\mathcal{L}_m^\alpha(x)$, by

$$\mathcal{L}_m^\alpha(x) = \Big(\frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} \Big)^{1/2} L_m^\alpha(x)\, e^{-x/2} x^{\alpha/2}.$$

The functions $\mathcal{L}_m^\alpha(x)$ are orthonormal on $[0,\infty)$, and for $\alpha > -1/2$ satisfy the following estimate established in [7, p. 277],

$$\left|\mathcal{L}_m^\alpha(x)\right| \lesssim_\alpha m^{\frac{\alpha}{2}+\frac{1}{4}-\frac{1}{12}} x^{\alpha/2}, \tag{5.12}$$

which stems from Eq. (1.2) above. Note that if in addition $\alpha < -1/3$, then

$$\frac{\alpha}{2}+\frac{1}{4}-\frac{1}{12} = \frac{(3\alpha+1)}{6} < 0.$$

We then have a refined Hausdorff–Young inequality for the Laguerre expansion:

Theorem 5.4 *With $-1/2 < \alpha < -1/3$, let the Laguerre expansion of f be given by $f(x) \sim \sum_{m=1}^\infty c_m^\alpha \mathcal{L}_m^\alpha(x)$, where*

$$c_m^\alpha = \int_0^\infty \mathcal{L}_m^\alpha(x)\, f(x)\, dx, \quad m = 1, 2, \ldots,$$

and let T be the mapping that assigns to a function f the sequence $\{c_m^\alpha\}$ of its Laguerre coefficients. Then, T maps the Lorentz space $L(p,s)$ continuously into the Lorentz sequence space $\ell^{q,s}$, $1 \le s \le \infty$, provided that

$$\frac{2}{2+\alpha} < p < 2, \quad \textit{and}, \quad \left(\frac{4+3\alpha}{3}\right)\frac{1}{p} + (1+\alpha)\frac{1}{q} = \frac{7+6\alpha}{6}. \tag{5.13}$$

In particular, whenever p, q verify Eq. (5.13),

$$\|T(f)\|_{\ell^q} = \left(\sum_{m=1}^\infty |c_m^\alpha|^q\right)^{1/q} \lesssim_{\alpha,p} \|f\|_p. \tag{5.14}$$

Moreover, if the maximal Laguerre coefficients are defined by

$$C_m^\alpha = \sup_{\beta>0}\left|\int_0^\beta \mathcal{L}_m^\alpha(x)\, f(x)\, dx\right|,$$

then Eq. (5.14) above holds with C_m^α in place of c_m^α there.

Proof Let

$$\frac{1}{\gamma} = -\frac{(3\alpha+1)}{6}, \quad 12 < \gamma < \infty, \tag{5.15}$$

and note that by Eq. (5.12) it follows that for $m = 1, 2, \ldots$,

$$|c_m^\alpha| \lesssim_\alpha m^{-1/\gamma} \int_0^\infty x^{\alpha/2} |f(x)|\, dx.$$

Now, since $\alpha < 0$, the Hardy-Littlewood inequality [2, p. 44] yields

$$\int_0^\infty x^{\alpha/2} |f(x)]\, dx \lesssim_\alpha \int_0^\infty t^{\alpha/2} f^*(t)\, dt \\ = \|f\|_{2/(\alpha+2),1},$$

and, therefore,

$$\left|c_m^\alpha\right| \lesssim_\alpha m^{-1/\gamma}\, \|f\|_{2/2+\alpha,1},$$

and, as in Eq. (5.3) it follows that

$$\|\{c_m^\alpha\}\|_{\ell^{\gamma,\infty}} \lesssim_\alpha \|f\|_{(2/(2+\alpha)),1},$$

i.e., T is continuous from $L(2/(2+\alpha), 1)$ into $\ell^{\gamma,\infty}$.

This estimate, together with the type $(2, 2)$ of T that yields $\|\{c_m^\alpha\}\|_{\ell^{2,\infty}} \lesssim \|f\|_{2,1}$, constitute the right frame for the application of [6, Corollary to Theorem 10, p. 293], and, consequently, interpolating we have

$$\|c_m^\alpha\|_{\ell^{q,s}} \lesssim_{\alpha,p,s} \|f\|_{p,s}, \quad 1 \le s \le \infty, \tag{5.16}$$

where, with $0 < \theta < 1$,

$$\frac{1}{p} = \theta\,\frac{2+\alpha}{2} + \frac{1-\theta}{2}, \quad \text{and,} \quad \frac{1}{q} = \theta\,\frac{1}{\gamma} + \frac{1-\theta}{2}.$$

Hence, replacing γ by its value gives,

$$\frac{1}{p} + \frac{1}{q} = 1 - \frac{\theta}{6},$$

where, by some algebraic manipulations,

$$1 - \frac{\theta}{6} = 1 + \frac{1}{6\,(1+\alpha)}\Big(1 - \frac{2}{p}\Big).$$

It then readily follows that

$$\Big(\frac{4+3\alpha}{3}\Big)\,\frac{1}{p} + (1+\alpha)\,\frac{1}{q} = \frac{7+6\alpha}{6},$$

and Eq. (5.13) has been established.

Moreover, since $p < 2 < q$, setting $s = q$ in Eq. (5.16), yields

$$\|\{c_m^\alpha\}\|_{\ell^q} \lesssim \|\{c_m^\alpha\}\|_{\ell^{q,q}} \lesssim_{\alpha,p} \|f\|_{p,q} \lesssim_{\alpha,p} \|f\|_{p,p} \lesssim_{\alpha,p} \|f\|_p,$$

and Eq. (5.14) holds.

From this point on the proof for the maximal coefficients proceeds as that in the Hermite case, and is left to the reader. □

The continuity properties of T above on the Orlicz spaces follow along the lines established for the Hermite expansion using Eq. (5.14) now, and the indices depend on α. Rather than dealing with the resulting cumbersome expressions, we state the underlying principle to obtain them [29, p. 181]. Suppose that a sublinear mapping T is of types, weak–types, or mixed types (p_0, q_0) and (p_1, q_1), with $p_) \neq p_1$, and let the equation of the straight line passing through the points $(1/p_0, 1/q_0)$, $(1/p_1, 1/q_1)$ be given by $y = \varepsilon x + \gamma$. Then, under appropriate growth conditions on the Young's functions A, B, the mapping T is of type (A, B) provided that

$$B^{-1}(t) = t^{\gamma} A^{-1}(t^{\varepsilon}).$$

As for the refined Hausdorff–Young companion inequality for Laguerre expansions we have:

Theorem 5.5 *With $-1/2 < \alpha < -1/3$, suppose that p, q are such that*

$$\frac{6}{7+3\alpha} < p < 2, \quad \text{and} \quad (1+\alpha)\,\frac{1}{p} + \Big(\frac{4+3\alpha}{3}\Big)\,\frac{1}{q} = \frac{7+6\alpha}{6}. \tag{5.17}$$

Then, given $\{c_m\}$ in the Lorentz sequence space $\ell^{p,s}$, there is f in the Lorentz space $L(q,s)$, $1 \le s \le \infty$, such that $f(x) \sim \sum_{m=1}^{\infty} c_m\, \mathcal{L}_m^{\alpha}(x)$, and

$$\|f\|_{q,s} \lesssim_{\alpha,p,s} \|\{c_m\}\|_{\ell^{p,s}}.$$

In particular, if τ denotes the mapping that assigns f to the sequence $\{c_m\}$, τ is of type (p,q) whenever (5.17) holds. Furthermore, in that case the maximal operator τ^* associated to τ is of type (p,q), and for $f = \tau(\{c_m\})$ we have

$$f(x) = \lim_M \sum_{m=1}^{M} c_m\, \mathcal{L}_m^{\alpha}(x) \quad a.e.$$

Proof With γ as in Eq. (5.15), let η, $1 < \eta < 12/11$, denote its conjugate index, i.e.,

$$\frac{1}{\eta} = 1 - \frac{1}{\gamma} = \frac{(3\alpha+7)}{6}.$$

Now, if $b(x) = \{\mathcal{L}_m^{\alpha}(x)\}$, and since as in Eq. (5.3) it follows that the sequence $\{m^{-\gamma}\}$ is in $\ell^{\gamma,\infty}$, by Eq. (5.12) it follows that

$$\Big|\sum_{m=1}^{\infty} c_m\, \mathcal{L}_m^{\alpha}(x)\Big| \lesssim_{\alpha} \|c\|_{\ell^{\eta,1}}\, x^{\alpha/2},$$

and, consequently, if $f \sim \sum_{m=1}^{\infty} c_m \mathcal{L}_m^{\alpha}(x)$, then $f \in L(2/|\alpha|, \infty)$ and

$$\|f\|_{2/|\alpha|,\infty} \lesssim_{\alpha} \|c\|_{\ell^{\eta,1}},$$

and τ is a continuous mapping from the Lorentz sequence space $\ell^{\eta,1}$ into the Lorentz space $L(2/|\alpha|, \infty)$.

Hence, by the type (2, 2) of τ, by a familiar argument it follows that $\|f\|_{2,\infty} \lesssim \|c_m\|_{\ell^{2,1}}$, and so, interpolating, by [6, Corollary to Theorem 10, p. 293] it follows that τ maps the Lorentz sequence space $\ell^{p,s}$ continuously into the Lorentz space $L(q, s)$, $1 \le s \le \infty$, where, for $0 < \theta < 1$,

$$\frac{1}{p} = \frac{3\alpha + 7}{6}\theta + \frac{1-\theta}{2}, \quad \text{and,} \quad \frac{1}{q} = \frac{-\alpha}{2}\theta + \frac{1-\theta}{2}.$$

Now, it readily follows that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\theta}{6},$$

and since

$$1 + \frac{\theta}{6} = \frac{7\ + 6\alpha}{(6 + 6\alpha)} - \frac{1}{q}\frac{1}{3}\frac{1}{1+\alpha},$$

from above we get that

$$(1+\alpha)\,\frac{1}{p} + \left(\frac{4 + 3\alpha}{3}\right)\frac{1}{q} = \frac{7 + 6\alpha}{6},$$

and Eq. (5.17) holds.

From this point on the proof proceeds as that in the Hermite case, and is left to the reader. □

We prove next the (p, p') Hausdorff–Young analogue inequality for Laguerre expansions anticipated in the Introduction:

Theorem 5.6 *Let $-1/2 < \alpha < -1/3$, $1 < p < 2$, and $q = p'$, the conjugate to p. Then, with $d\nu_p(x) = x^{(\alpha/2)\left(\frac{2}{p}-1\right)}\,dx$, given a function $f \in L^p_{\nu_p}(\mathbb{R}^+)$ with Laguerre series expansion $f(x) \sim \sum_{m=1}^{\infty} c_m^{\alpha} \mathcal{L}_m^{\alpha}(x)$, it follows that*

$$\left(\sum_{m=1}^{\infty} |c_m^{\alpha}|^q \, m^{((3\alpha+1)/6)\left(1-\frac{2}{q}\right)} \right)^{1/q} \lesssim_{\alpha,p} \left(\int_0^{\infty} |f(x)|^p \, x^{(\alpha/2)\left(\frac{2}{p}-1\right)}\,dx \right)^{1/p}. \tag{5.18}$$

Furthermore, given a sequence $c = \{c_m\}$ such that

$$\sum_{m=1}^{\infty} |c_m|^p \, m^{((3\alpha+1)/6)\left(\frac{2}{p}-1\right)} < \infty,$$

there is a function $f(x) \sim \sum_{m=1}^{\infty} c_m^\alpha \mathcal{L}_m^\alpha(x)$ with $c_m = c_m^\alpha$, the Laguerre coefficients of f, and

$$\left(\int_0^\infty |f(x)|^q \, x^{(\alpha/2)\left(1-\frac{2}{q}\right)} \, dx \right)^{1/q} \lesssim_{\alpha,p} \left(\sum_{m=1}^{\infty} |c_m|^p \, m^{((3\alpha+1)/6)\left(\frac{2}{p}-1\right)} \right)^{1/p}. \tag{5.19}$$

Proof Let γ be defined as in Eq. (5.15). Let T be the mapping that assigns to a function $f(x) \sim \sum_{m=1}^{\infty} c_m^\alpha \mathcal{L}_m^\alpha(x)$, the sequence $\{c_m^\alpha\}$ of its Laguerre coefficients. Then, with $d\nu(x) = x^{\alpha/2}\, dx$, if $f \in L_\nu^1(\mathbb{R}^+)$, by Eq. (5.12),

$$|c_m^\alpha| \, m^{1/\gamma} \lesssim_\alpha \int_0^\infty |f(x)| \, x^{\alpha/2} \, dx, \quad n = 1, 2, \ldots,$$

and $\{c_m^\alpha \, m^{1/\gamma}\} \in \ell^\infty$. Now, with $\mu(m) = m^{-1/\gamma}$, $m = 1, 2, \ldots$, the linear functional

$$L(c) = \sum_{m=1}^{\infty} c_m^\alpha \, c_m, \quad c = \{c_m\} \in \ell_\mu^1,$$

satisfies

$$\left| L(c) \right| = \left| \sum_{m=1}^{\infty} c_m^\alpha \, m^{1/\gamma} \, c_m \frac{1}{m^{1/\gamma}} \right| \le \|\{c_m^\alpha \, m^{1/\gamma}\}\|_{\ell^\infty} \, \|\{c_m\}\|_{\ell_\mu^1},$$

and is thus bounded on ℓ_μ^1 with norm $\le \|\{c_m^\alpha \, m^{1/\gamma}\}\|_{\ell^\infty}$. Hence, since ℓ_μ^1 is σ–finite, its dual is ℓ_μ^∞, and, therefore, $\{c_m^\alpha\} \in \ell_\mu^\infty$, and

$$\|\{c_m^\alpha\}\|_{\ell_\mu^\infty} \le \|\{c_m^\alpha \, m^{1/\gamma}\}\|_{\ell^\infty},$$

and so, T maps $L_\nu^1(\mathbb{R}^+)$ continuously into ℓ_μ^∞.

Since T is also a bounded mapping from $L^2(\mathbb{R}^+)$ into ℓ^2, we can apply the Stein–Weiss complex interpolation theorem with change of measure [26, Theorem 2.11, p. 164], or the real interpolation theorem with change of measure [22, Theorem 4, p. 773], and get, for $0 < \theta < 1$, with

$$\frac{1}{p} = \theta + \frac{(1-\theta)}{2}, \quad \frac{1}{q} = \frac{(1-\theta)}{2},$$

and

$$\mu_\theta(m) = m^{-(1/\gamma)(\theta)}, m = 1, 2, \ldots, \quad d\nu_\theta(x) = x^{(\alpha/2)\,(\theta)}\,dx,$$

that T maps $L^p_{\nu_\theta}(\mathbb{R}^+)$ continuously into $\ell^q_{\mu_\theta}$, and, consequently,

$$\Big(\sum_{m=1}^{\infty} |c_m^\alpha|^q\,\mu_\theta(m)\Big)^{1/q} \lesssim_{\alpha,p} \Big(\int_0^\infty |f(x)|^p\,d\nu_\theta(x)\Big)^{1/p}.$$

Now, since

$$\theta = \Big(\frac{2}{p} - 1\Big) = \Big(1 - \frac{2}{q}\Big),$$

it readily follows that the measure in the statement is $d\nu_p(x) = x^{(\alpha/2)\left(\frac{2}{p}-1\right)}dx$, and, consequently,

$$\Big(\sum_{m=1}^{\infty} |c_m^\alpha|^q\, m^{(-1/\gamma)\left(1-\frac{2}{q}\right)}\Big)^{1/q} \lesssim_{\alpha,p} \Big(\int_0^\infty |f(x)|^p\, x^{(\alpha/2)\left(\frac{2}{p}-1\right)}\,dx\Big)^{1/p},$$

and Eq. (5.18) follows upon substituting γ by its value.

Next, let τ be the mapping that assigns to the sequence $c = \{c_m\}$ the function $\tau(c) = f(x)$, where $f(x) \sim \sum_{m=1}^\infty c_m \mathcal{L}_m^\alpha(x)$. Then, by Eq. (5.12),

$$x^{-\alpha/2}|f(x)| \lesssim_\alpha \sum_{m=1}^{\infty} |c_m|\, m^{-1/\gamma},$$

and so, with $\mu(m) = m^{-1/\gamma}, m = 1, 2, \ldots, x^{-\alpha/2} f(x) \in L^\infty(\mathbb{R}^+)$ whenever $c \in \ell^1_\mu$. Moreover, since $x^{-\alpha/2}$ is locally integrable, ν is σ–finite, and as above it follows that $f \in L^\infty_\nu(\mathbb{R}^+)$ and

$$\|f(x)\|_{L^\infty_\nu(\mathbb{R}^+)} \le \|f(x)\, x^{-\alpha/2}\|_{L^\infty(\mathbb{R}^+)}.$$

Thus, $\|f\|_{L^\infty_\nu(\mathbb{R}^+)} \le \|c\|_{\ell^1_\mu}$, and τ is continuous from ℓ^1_μ into $L^\infty_\nu(\mathbb{R}^+)$.

And, since τ is bounded from ℓ^2 into $L^2(\mathbb{R}^+)$, interpolating with change of measure [22, Theorem 4, p. 773], [26, Theorem 2.11, p. 164], we get that for $0 < \theta < 1$, with

$$\frac{1}{p} = \theta + \frac{(1-\theta)}{2}, \quad \frac{1}{q} = \frac{(1-\theta)}{2},$$

and

$$\mu_\theta(m) = m^{-(1/\gamma)\theta}, \; m = 1, 2, \ldots, \quad d\nu_\theta(x) = x^{(\alpha/2)\,\theta}\,dx,$$

τ maps $\ell^p_{\mu_\theta}$ into $L^q_{\nu_\theta}(\mathbb{R}^+)$, i.e.,

$$\Big(\int_0^\infty |f(x)|^q \, d\nu_\theta(x) \Big)^{1/q} \lesssim_{\alpha,p} \Big(\sum_{m=1}^\infty |c_m|^p \mu_\theta(m) \Big)^{1/p}.$$

And, since

$$\theta = \Big(\frac{2}{p} - 1\Big) = \Big(1 - \frac{2}{q}\Big),$$

Equation (5.19) follows readily upon substituting γ by its value, and the proof is finished. □

Because the analogue Hausdorff–Young inequality involves weighted norms, it is of interest to identify those functions f whose Laguerre coefficients are in ℓ^q, for some $q > 2$, and explore the dual result. In this direction we have:

Theorem 5.7 *With $-1/2 < \alpha < -1/3$, let $1 < p < 2$, and suppose that p, q satisfy the relation*

$$\Big(\frac{4+3\alpha}{3}\Big)\frac{1}{p} + \frac{1}{q} = \frac{7+3\alpha}{6}. \tag{5.20}$$

Then, with $d\nu_p(x) = x^{\alpha(1/p-1/2)}\,dx$, if $f \in L^p_{\nu_p}(\mathbb{R}^+)$ has the Laguerre series expansion $f(x) \sim \sum_{m=1}^\infty c^\alpha_m \mathcal{L}^\alpha_m(x)$, we have

$$\big\| \{c^\alpha_m\} \big\|_{\ell^q} \lesssim_{\alpha,p} \Big(\int_0^\infty |f(x)|^p \, x^{\alpha\left(\frac{1}{p}-\frac{1}{2}\right)} dx \Big)^{1/p}. \tag{5.21}$$

Furthermore, let $1 < p < 2$, and suppose that now p, q verify the relation

$$\frac{1}{p} + \Big(\frac{4+3\alpha}{3}\Big)\frac{1}{q} = \frac{7+3\alpha}{6}. \tag{5.22}$$

Then, if $c = \{c_m\} \in \ell^p$, there is a function f such that $c_m = c^\alpha_m(f)$ are the Laguerre coefficients of f, and

$$\Big(\int_0^\infty |f(x)|^q x^{\alpha\left(\frac{1}{2}-\frac{1}{q}\right)(1-q)} \, dx \Big)^{1/q} \lesssim_{\alpha,p} \|\{c_m\}\|_{\ell^p}. \tag{5.23}$$

Proof Let γ, $12 < \gamma < \infty$, be given by Eq. (5.15), and let T be the mapping that assigns to a function f the sequence $\{c^\alpha_m\}$ of its Laguerre coefficients. Now, if $f \in L^1_{\nu_1}(\mathbb{R}^+)$ where $d\nu_1(x) = x^{\alpha/2}\,dx$, by Eq. (5.12), as above it readily follows that

$$\big|c^\alpha_m\big|\, m^{1/\gamma} \lesssim_\alpha \int_0^\infty |f(x)|\, x^{\alpha/2}\, dx, \quad m = 1, 2, \ldots,$$

and so, an argument similar to that leading to Eq. (5.3) yields that T is bounded from $L^1_{\nu_1}(\mathbb{R}^+)$ into $\ell^{\gamma,\infty}$. This combined with the fact that T maps $L^2(\mathbb{R}^+)$ boundedly into ℓ^2 allows for real interpolation with change of measure, and by [3, Sect. 5.5, p. 119], T is continuous from $L^p_{\nu_\theta}(\mathbb{R}^+)$ into ℓ^q, where, with $0 < \theta < 1$,

$$\frac{1}{p} = \theta + \frac{(1-\theta)}{2} = \frac{(1+\theta)}{2}, \quad \frac{1}{q} = \frac{\theta}{\gamma} + \frac{(1-\theta)}{2},$$

and $d\nu_\theta(x) = x^{(\alpha/2)\theta}\,dx$. Moreover, since from the above relation it follows that $\theta = (2/p) - 1$, we have

$$d\nu_\theta(x) = x^{\left(\alpha/2\right)\left(\frac{2}{p}-1\right)}\,dx.$$

As for q, simple algebraic manipulations yield

$$\left(1 - \frac{2}{\gamma}\right)\frac{1}{p} + \frac{1}{q} = \left(1 - \frac{1}{\gamma}\right), \tag{5.24}$$

and so, Eq. (5.20) follows replacing γ by its value. For these values of p, q we have Eq. (5.21), and the proof of the first assertion is finished.

Next, from Eq. (5.22) it follows that $1 < q' < 2$, and that

$$\left(1 - \frac{2}{\gamma}\right)\frac{1}{q'} + \frac{1}{p'} = \left(1 - \frac{1}{\gamma}\right).$$

Thus, q', p' verify Eqs. (5.24, 5.21) holds for q', p', and, with

$$d\nu(x) = x^{\alpha\left(\frac{1}{q'}-\frac{1}{2}\right)}\,dx,$$

if $g \in L^{q'}_\nu(\mathbb{R}^+)$ has a Laguerre series expansion $g(x) \sim \sum_{m=1}^\infty d_m^\alpha\,\mathcal{L}_m^\alpha(x)$, then,

$$\|\{d_m^\alpha\}\|_{\ell^{p'}} \lesssim_{\alpha,q'} \left(\int_0^\infty |g(x)|^{q'}\, x^{\alpha\left(\frac{1}{q'}-\frac{1}{2}\right)}\,dx\right)^{1/q'}.$$

Now, for $f_M(x) = \sum_{m=1}^M c_m\mathcal{L}_m^\alpha(x)$ and $g(x)$ as above we have

$$\int_0^\infty f_M(x)g(x)\,dx = \sum_{m=1}^M c_m\,d_m^\alpha,$$

and so,

$$\Big| \int_0^\infty f_M(x)\, g(x)\, dx \Big| \le \Big(\sum_{m=1}^M |c_m|^p \Big)^{1/p} \|\{d_m^\alpha\}\|_{\ell^{p'}}$$

$$\lesssim_{\alpha,q'} \Big(\sum_{m=1}^M |c_m|^p \Big)^{1/p} \Big(\int_0^\infty |g(x)|^{q'}\, x^{\alpha\left(\frac{1}{q'}-\frac{1}{2}\right)}\, dx \Big)^{1/q'}.$$

As for the integral on the left–hand side above, observe that for our range of values of α, with $\beta = \alpha\big(\frac{1}{q'} - \frac{1}{2}\big)$, x^β is locally integrable, and with $d\nu(x) = x^\beta\, dx$, it equals

$$\int_0^\infty f_M(x)\, g(x)\, dx = \int_0^\infty f_M(x)\, x^{-\beta} g(x)\, d\nu(x),$$

and taking the sup over those functions g with $\|g\|_{L^{q'}_\nu} \le 1$, it follows that

$$\Big(\int_0^\infty |f_M(x)|^q x^{-\beta q}\, d\nu(x) \Big)^{1/q} = \Big(\int_0^\infty |f_M(x)|^q x^{\beta\,(1-q)}\, dx \Big)^{1/q}$$

$$\lesssim_{\alpha,p} \Big(\sum_{m=1}^M |c_m|^p \Big)^{1/p}.$$

A similar argument shows that for $M_1 < M_2$,

$$\| f_{M_2} - f_{M_1} \|_q \lesssim_{\alpha,p} \Big(\sum_{m=M_1+1}^{M_2} |c_m|^p \Big)^{1/p},$$

and, consequently, $\{f_M\}$ is Cauchy in $L^q_\nu(\mathbb{R})$. If we denote the limit of this sequence by f, then $f(x) \sim \sum_{m=1}^\infty c_m\, \mathcal{L}_m^\alpha(x)$, and

$$\Big(\int_0^\infty |f(x)|^q x^{\beta\,(1-q)}\, dx \Big)^{1/q} \lesssim_{\alpha,p} \Big(\sum_{m=1}^\infty |c_m|^p \Big)^{1/p}.$$

Finally, since also $\beta = \alpha\big(\frac{1}{2} - \frac{1}{q}\big)$, Eq. (5.23) holds, and the proof is finished. □

6 Concluding Remarks

Interpolation arguments are at the core of our results. Since the hybrid Orlicz–Lorentz spaces enjoy interpolation properties similar to those of both the Lorentz and Orlicz spaces [29], the interested reader may prove results for those spaces adapting the ideas above.

Also, note that the maximal inequalities extend to the discrete case, i.e, when $L^A(X)$ is replaced by the sequence space ℓ^A, as is the case for ℓ^p spaces [18, Theorem 2.11.3, p. 171], and the Lorentz spaces [23]. The results are similar and also follow by interpolation, or else by transferring estimates from the atomic measure to the Lebesgue measure on $\mathbb{R}$, as was done in the proof of the Hausdorff–Young inequality for Hermite and Laguerre expansions. These results can then be applied to Fourier series rather than integrals.

More precisely, consider the following setting. Let $S : \ell^A \to L^A(\mathbb{R})$ be the mapping that assigns to a sequence $z = \{z_m\}$ in ℓ^A the function $S(z)$ on $\mathbb{R}$ defined piecewise by

$$S(z)(x) = z_m, \quad m \le x < m+1,$$

and let an inverse $\tilde{S} : L^A(\mathbb{R}) \to \ell^A$ be defined by $z = \tilde{S}(f)$ where

$$z_m = \tilde{S}(f)_m = \int_m^{m+1} f(x)\,dx, \quad \text{all } m.$$

Note that $\|S(z)\|_A = \|z\|_{\ell^A}$, and that, by Jensen's inequality, $\|\tilde{S}(f)\|_{\ell^A} \le \|f\|_A$. Also, $\tilde{S}S = I$. We then have the Orlicz spaces discrete maximal inequality:

Theorem 6.1 *Suppose the Young functions A, B satisfy the conditions of Theorem 3.1. Let $\tau : \ell^A \to L^B(\mathbb{R})$ be a bounded linear mapping, and let χ_m denote the characteristic function of $(-\infty, m]$. Consider the maximal operator*

$$\tau^*(z)(x) = \sup_m \left|\tau(\chi_m z)(x)\right|.$$

Then, we have $\|\tau^*(z)\|_B \lesssim \|z\|_{\ell^A}$.

Proof Let $T = \tau\tilde{S}$; clearly, $T : L^A(\mathbb{R}) \to L^B(\mathbb{R})$ boundedly, and so, by the Orlicz spaces maximal inequality, $\|T^*(f)\|_B \lesssim \|f\|_A$. Now, since $S\chi_m = \chi_m S$ and $\tilde{S}S = I$, it follows that $\tau(z) = TS(z)$ and

$$\tau^*(z)(x) \le T^*(S(z))(x).$$

The conclusion follows readily from this. □

Theorem 6.1 yields Menshov's theorem for Orlicz spaces along the lines of the proof for the ℓ^p spaces [18, Theorem 2.11.5, p. 172]. Indeed, we have:

Theorem 6.2 *Suppose that the Young function A satisfies $A(t)/t^2$ decreasing, $A(1) = 1$, and let $\{\varphi_m\}$ be an ONS in some L^2. Given a sequence $\{c_m\}$ define*

$$c^*(x) = \sup_n \left|\sum_{m=1}^{n} c_m \varphi_m(x)\right|.$$

Then we have, $\| c^* \|_2 \lesssim \| c \|_{\ell^A}$. *Moreover, if for* $c \in \ell^2$ *we interpret* $\sum_{m=1}^{\infty} c_m \varphi_m$ *as an* L^2 *sum, then for* $c \in \ell^A$ *we have*

$$\lim_{n\to\infty} \sum_{m=1}^{n} c_m \varphi_k(m) = \Big(\sum_{m=1}^{\infty} c_m \varphi_m \Big)(x) \ a.e.$$

Proof It is readily seen that, if $c = \{c_m\} \in \ell^A$, then $c \in \ell^2$, and $\| c \|_{\ell^2} \le \| c \|_{\ell^A}$, and so $\sum_{m=1}^{\infty} c_m \varphi_m(x)$ defines an element in L^2, which is precisely $\tau(c)$, such that

$$\|\tau(c)\|_2 \lesssim \| c \|_{\ell^2} \lesssim \| c \|_{\ell^A}.$$

The norm conclusion follows readily from Theorem 6.1. As for the limit, it holds for finite sequences, and since these sequences are dense in ℓ^A, essentially Corollary 4.2 gives the conclusion for all sequences. □

We also point out that a refined Hausdorff–Young inequality holds for expansions by orthogonal polynomials with respect to a class of Freud–type weights on $\mathbb{R}$, thus completing Ditzian's results [14]. The proof follows along the lines to that of Theorem 5.1, and, therefore, it can be extended to include the corresponding results for the maximal coefficients, n–dimensional expansion, and Lorentz and Orlicz space estimates [11].

We consider next an observation concerning the Orlicz classes, L_Ψ, that constitute the building blocks of the Lorentz spaces $L(p,r)$ on $\mathbb{R}^n$ for $0 < r < p$, $p > 1$. They are defined as follows.

Given a nondecreasing function $\Psi \ge 0$ defined on $(0,\infty)$ such that

$$\int_0^\infty \frac{t^{q-1}}{\Psi(t)^{q/p}}\, dt < \infty, \tag{6.1}$$

where $0 < r < p$, $p > 1$, and $1/p + 1/q = 1/r$, the Orlicz type class L_Ψ consists of those measurable functions f defined on $\mathbb{R}^n$ such that the nonincreasing rearrangement f^* of $|f|$ satisfies

$$\int_0^\infty \Psi\big(f^*(t)\big)\, dt < \infty.$$

Then, for this range of values, $L(p,r) = \bigcup_\Psi L_\Psi$, where the Ψ's satisfy Eq. (5.1). In particular, for any such Ψ and $f \in L_\Psi$ we have [9],

$$\|f\|_{p,r} \lesssim_{p,\Psi} \Big(\int_0^\infty \Psi\big(f^*(t)\big)\, dt \Big)^{1/p}. \tag{6.2}$$

The following version of Zygmund's maximal theorem holds for the classes L_Ψ:

Theorem 6.3 *Let* T *be a linear mapping of types* $(1,\infty)$ *and* $(2,2)$*, and let* T^* *denote the maximal operator associated to* T *as in the maximal inequality theorem. For*

$1 < p < 2$, $0 < r < p$, and $1/p + 1/q = 1/r$, consider the Orlicz class L_Ψ where Ψ satisfies Eq. (6.1). Then, with p' the conjugate to p, for $f \in L_\Psi$ we have

$$\|T^*(f)\|_{p',r} \lesssim_{p,\Psi} \Big(\int_0^\infty \Psi\big(f^*(t)\big)\,dt\Big)^{1/p}. \tag{6.3}$$

Proof Since T satisfies the conditions of the maximal inequality with $A(t) = t^p$, $B(t) = t^{p'}$ there, $1 < p < 2$, interpolating it readily follows that

$$\|T^*(f)\|_{p',s} \lesssim_{p,s} \|f\|_{p,s} \quad 1 < p < 2,\ 1 \le s \le \infty.$$

Hence, letting $s = r$ above, Eq. (6.3) follows from Eq. (6.2), and the proof is finished. □

We close this note with a remark about the Banach spaces of distributions of Wiener's type [15, 16], or the Wiener amalgam spaces as they are referred to nowadays [17], in the context of the Hausdorff–Young inequality. These spaces, which were introduced by Feichtinger, allow for the consideration of the fact that the local properties of f are reflected in the global properties of $\widehat{f}$, and vice versa [15, Prop. 3.3]; these results are, in various directions, best possible [15, Remark 3.1]. In view that all essential elements, including the Christ–Kiselev inequality, are in place [23], it is a tantalizing challenge to consider a similar result for more general expansions in terms of orthogonal functions, including the ones considered here.

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