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Matrix Spherical Functions on Finite Groups

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Abstract

In this paper we focus our attention on matrix or operator-valued spherical functions associated to finite groups (G, K), where K is a subgroup of G. We introduce the notion of matrix-valued spherical functions on G associated to any K-type $\delta \in \hat{K}$ by means of solutions of certain associated integral equations. The main properties of spherical functions are established from their characterization as eigenfunctions of right convolution multiplication by functions in $A[G]^K$, the algebra of K-central functions in the group algebra $A[G]$. The irreducible representations of $A[G]^K$ are closely related to the irreducible spherical functions on G. This allows us to study and compute spherical functions via the representations of this algebra.

Keywords Matrix-valued spherical functions · Finite groups · Finite Gelfand pairs

Mathematics Subject Classification 43A90 · 43A65 · 20C05

1 Introduction

The theory of scalar-valued spherical functions, or zonal spherical functions, goes back to the classical papers of É. Cartan and H. Weyl. They showed that spherical harmonics arise naturally from the study of functions on G/K, where G is the special orthogonal group in Euclidean n-space and K consists of those transformations in G which leave a given vector invariant. This study is carried out using methods of group representations. However, to develop a theory applicable to larger classes of

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"special functions", it is necessary to consider other families of pairs (G, K). The first general results were obtained in 1950 by Gelfand [11], who considered zonal spherical functions on a Riemannian symmetric pair (G, K). Shortly thereafter, the fundamental papers of Godement [12] and Harish-Chandra [15, 16] on spherical trace functions appeared.

In the context of finite groups, Travis in [21], considers trace spherical functions based on the paper of Godement. Brender in [2] also deals with these complex-valued spherical functions and computes those corresponding to the pair of symmetric groups $(\mathfrak{S}_n, \mathfrak{S}_{n-1})$.

Stanton, in the survey paper [18], studies scalar values spherical functions for Chevalley groups over the finite fields $\mathrm{G}F(q)$. It is well known that zonal spherical functions on rank one compact symmetric spaces lead to classical orthogonal polynomials. Instead, Stanton considers a finite group G of automorphisms of a finite metric space X and assumes that the metric is integer-valued and that X is a two-point homogeneous G-space. Under these general hypotheses, the spherical functions are certain sets of discrete orthogonal polynomials, given by basic hypergeometric series, or q-series. We refer to the book by Terras [19] for a comprehensive introduction to zonal spherical functions on finite groups.

More recently, in [6] Ceccherini-Sliberstein, Scarabotti, and Tolli consider finite Gelfand pairs and study spherical functions associated to them. These functions are also complex-valued and they are associated to the trivial representation of K. Later in Chapter 13 of [5] they develop a theory of complex-valued spherical functions associated with a multiplicity-free induced one-dimensional representation.

Following Godement's work, in [20] and later in [10] the emphasis was put on to work directly with the spherical functions associated to an irreducible representation of a locally compact or a Lie group G rather than with their traces, giving an intrinsic definition of them. Over time, this point of view has proven to be very fruitful in developing research on matrix special functions, matrix orthogonal polynomials, time and band limiting problems, and matrix differential operators, among others.

The first example of these connections was the seminal paper [13] on spherical matrix functions associated to the complex projective plane $\mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$, where a rich connection with matrix orthogonal polynomial was established. See also [14]. After this, many papers on spherical functions and matrix orthogonal polynomials have appeared involving different Lie groups and Gelfand pairs (G, K).

The present paper is the starting point for the study of matrix valued spherical functions on finite groups. We focus on *matrix or operator-valued spherical functions* associated to finite pairs (G, K). These functions arise by considering any irreducible representation δ of K. The particular case when δ is the trivial representation corresponds to classical spherical functions mentioned before.

The definition of a spherical function is based on an interesting functional equation that replaces the multiplicative property of a representation of a group G.

Let G be a finite group and let K be a subgroup of G. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ, i.e. the dimension of any representation in the class δ, and $\chi_\delta = d(\delta)\xi_\delta$.

We shall denote by V a finite dimensional vector space over the field $\mathbb{C}$ of complex numbers and by $\mathrm{End}(V)$ the space of all linear transformations of V into V.

We define a matrix valued spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ of type $\delta \in \hat{K}$ as a solution of the functional equation

$$\Phi(x)\Phi(y) = \frac{1}{|K|}\sum_{k\in K} \chi_\delta(k^{-1})\Phi(xky), \tag{1}$$

for all $x, y \in G$, where $\chi_\delta = d(\delta)\xi_\delta$, $d(\delta)$ and ξ_δ are respectively the dimension and the character of δ, (Definition 3.1).

The algebra $A[G]^K$, of K-central functions in the group algebra $A[G]$, is one of the main character in the theory of matrix valued spherical functions of finite groups. It plays a similar role as the subalgebra $D(G)^K$ of right invariant differential operators under K in the algebra of all left invariant differential operators on a connected Lie group G, with K a compact subgroup.

The main properties of matrix valued spherical functions Φ are obtained from their characterization as eigenfunctions of the convolution operators $\Phi \mapsto \Phi * f$, for all $f \in A[G]^K$. More precisely,

$$(\Phi * f)(g) = \Phi(g)(\Phi * f)(e), \qquad \text{for all } g \in G,\ f \in A[G]^K. \tag{2}$$

The equivalence between Eqs. (1) and (2), given in Theorem 3.6, is original and highly nontrivial.

The irreducible representations of the algebra $A[G]^K$ are closely related to the irreducible spherical functions on G. This allows us to study and compute spherical functions through the representations of this algebra. First of all, we prove that $A[G]^K$ is a direct sum of certain subalgebras,

$$A[G]^K = \bigoplus_{\delta\in\hat{K}} A_\delta^K[G],$$

where $A_\delta^K[G] = A_\delta[G] \cap A[G]^K$ and $A_\delta[G] = \{f \in A[G] : \overline{\chi}_\delta * f = f * \overline{\chi}_\delta = |K| f\}$.

Given a spherical function $\Phi : G \to \mathrm{End}(V)$ of type $\delta \in \hat{K}$, the map

$$f \mapsto \Phi(f) = \sum_{g\in G} f(g)\Phi(g) = (\Phi * \check{f})(e)$$

is a representation of the algebra $A_\delta[G]$, where $\check{f}(g) = f(g^{-1})$. Moreover, we prove that all irreducible representations of this algebra are obtained in this way from an irreducible spherical function Φ of type δ. See Theorem 4.4. We also prove that any irreducible representation of the algebra $A_\delta^K[G]$ can be obtain from an irreducible spherical function of type δ. (Theorem 4.13).

Irreducible spherical functions of a pair (G, K) also arise in a natural way upon considering irreducible representations of G. Let (V, ρ) be an irreducible representation of G in a vector space V and let P_δ be the projection of V onto the isotypical component of type δ. For $g \in G$,

$$\Phi(g) = P_\delta \rho(g) P_\delta$$

is an irreducible spherical function of G of type δ. Another main result in the paper is that all irreducible spherical functions can be obtained in this way, the proof is demanding and given in Theorem 7.1.

In a forthcoming paper, we describe all spherical functions of the pairs of symmetric groups $(G, K) = (\mathfrak{S}_n, \mathfrak{S}_{n-m} \times \mathfrak{S}_m)$ for $m = 1, 2$. It is known that $A[G]^K = A[K]^K[x]$ respectively $A[G]^K = A[K]^K[x, y]$. An important and new consequence is an interesting presentation of this algebra by generators and relations.

This paper is organized as follows. In Sect. 2 we briefly recall some basic facts on the representation theory of finite groups. In Sect. 3 we give the precise definition and the generalities of spherical functions. We prove different characterizations of such functions: one as eigenfunctions of certain convolution operators (Theorem 3.6) and another as functions canonically associated to pairs $(\rho, \pi) \in \hat{G} \times \hat{K}$ where π is a subrepresentation of ρ, (Theorem 3.13). An alternative definition of spherical function is also given in Definition 3.12.

In Sect. 4 we study the algebras $A_\delta[G]$, $A_\delta^K[G]$ and $A[G]^K$ and we characterize their irreducible representations in terms of irreducible spherical functions on G. Furthermore, we prove that $A_\delta^K[G]$ is a semisimple algebra with identity: it is a direct sum of complex matrix algebras.

We consider the particular case when $A_\delta^K[G]$ is a commutative algebra and we prove that this happens precisely when all spherical functions of type δ are of height one.

In Sect. 5 we study the relation between spherical functions and Gelfand pairs. We say that (G, K) is a *Gelfand pair* if the algebra $A[G]^{K \times K}$ of bi-K-invariant functions on G is commutative, while it is a *strong Gelfand pair* if $A[G]^K$ is commutative. We prove that for a strong Gelfand pair (G, K), the set of all irreducible spherical functions of (G, K) are in a one to one correspondence with the set of all K-conjugacy classes in G. Finally in Sect. 6 we compute the spherical Plancherel measure of (G, K), generalizing the classical Plancherel identity of a finite group, and we give the inverse spherical transform.

The classic theory of spherical functions associated with Gelfand pairs has numerous connections with classical special functions. We look forward to obtain soon concrete examples of discrete matrix orthogonal polynomials from these matrix valued spherical functions.

2 Background on Representation Theory of Finite Groups

Let G be a finite group and let $\hat{G}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of G.

The *group algebra* $A[G]$ of a finite group, is the associative algebra of all complex-valued functions on G, with the convolution product

$$(f_1 * f_2)(g) = \sum_{x \in G} f_1(gx^{-1}) f_2(x) = \sum_{x \in G} f_1(x) f_2(x^{-1}g), \quad \text{for all } f_1, f_2 \in A[G]. \tag{3}$$

The algebra $A[G]$ bears the left and right regular representations L and R of G defined by $(L_g f)(x) = f(g^{-1}x)$ and $(R_g f)(x) = f(xg)$. If δ_g denotes the delta function at g, it is not difficult to see that $L_g(f) = \delta_g * f$ and $R_g(f) = f * \delta_{g^{-1}}$, for all $f \in A[G]$, and $g \in G$. We also have in $A[G]$ a left and right invariant inner product given by $\langle f_1, f_2 \rangle = \sum_{g \in G} f_1(g)\overline{f_2(g)}$. Moreover $A[G]$ is a $G \times G$-module with the representation $L \otimes R$, given by

$$\big((L \otimes R)(x, y) f\big)(g) = \big(L_x R_y f\big)(g) = f(x^{-1}gy). \tag{4}$$

Let ρ be a unitary irreducible finite dimensional representation of G on V_ρ. Given $v, w \in V_\rho$, the function $\rho_{v,w}(g) = \langle \rho(g)w, v \rangle$ for all $g \in G$, is the *matrix coefficient* $\rho_{v,w}$ of the representation ρ. The map $w \mapsto \rho_{v,w}$ is an injective intertwining operator from V_ρ to $(A[G], R)$ and also the map $v \mapsto \rho_{v,w}$ is an injective intertwining operator from V_ρ to $(A[G], L)$.

We denote with E_ρ the complex subspace of $A[G]$ generated by all matrix coefficients of ρ. By Schur's orthogonality relations we have that E_ρ and E_σ are orthogonal subspaces when ρ and σ are not equivalent and

$$\langle \rho_{v,w}, \rho_{v',w'} \rangle = \sum_{x \in G} \langle \rho(x)v, w \rangle \overline{\langle \rho(x)v', w' \rangle} = \frac{|G|}{d(\rho)} \langle v, v' \rangle \langle w', w \rangle,$$
$$\text{for all } v, w, v', w' \in V_\rho,$$

where $d(\delta)$ is the dimension of δ. Let ξ_ρ be the character of $\rho \in \hat{G}$ and $\chi_\rho = d(\rho)\xi_\rho$. For $\sigma, \rho \in \hat{G}$, it is easy to prove that

$$\chi_\rho * \chi_\sigma = \begin{cases} |G|\chi_\rho & \text{if } \rho = \sigma, \\ 0 & \text{in other cases.} \end{cases} \tag{5}$$

Let (V_ρ, ρ) be a unitary finite dimensional representation of G and let K be a subgroup of G. Thus $V_\rho = \sum_{\delta \in \hat{K}} m_\delta V_\delta$. The sum of all submodules of type δ appearing in the decomposition of V_ρ is called the δ-isotypic component of V_ρ and m_δ is the multiplicity of V_δ in V_ρ. The linear operator

$$P_\delta = \frac{1}{|K|} \sum_{k \in K} \chi_\delta(k^{-1}) \rho(k) \tag{6}$$

is the orthogonal projection of V_ρ onto the δ-isotypic component $V_{(\delta)}$.

If (V_ρ, ρ) is an irreducible representation of G, then the *contragradient representation* $(V_{\rho'}, \rho')$ is defined in $V_{\rho'} = V'_\rho$ the dual vector space of V_ρ by

$$(\rho'(g)\lambda)(v) = \lambda(\rho(g^{-1})v), \quad \text{for all} \quad g \in G, \ \lambda \in V'_\rho, \ v \in V_\rho.$$

Given $\lambda \in V'_\rho$, $w \in V_\rho$ we consider $\rho_{\lambda,w} \in A[G]$, given by $\rho_{\lambda,w}(g) = \lambda(\rho(g)w)$. We know that there exists a unique $v \in V_\rho$ such that $\lambda(w) = \langle w, v\rangle$ for all $w \in V_\rho$. Thus, $\rho_{\lambda,w} = \rho_{v,w}$ is the matrix coefficient associated to the elements $v, w \in V_\rho$.

Since $L_x R_y \rho_{\lambda,w} = \rho_{\rho'(x)\lambda,\rho(y)w}$ we have that the linear map $\lambda \otimes w \mapsto \rho_{\lambda,w}$ is a $G \times G$-morphism. Moreover, $V'_\rho \otimes V_\rho$ is $G \times G$-irreducible, because it is the tensor product of two irreducible G-modules. Therefore

$$V'_\rho \otimes V_\rho \longrightarrow A[G] \qquad \lambda \otimes w \mapsto \rho_{\lambda,w} \tag{7}$$

is an injective $G \times G$-morphism and $V'_\rho \otimes V_\rho$ can be identified with E_ρ, the space of ρ-matrix coefficients in $A[G]$. These $G \times G$-modules are orthogonal to each other.

Theorem 2.1 (Peter–Weyl theorem) *If G is a finite group, then*

$$A[G] = \bigoplus_{\rho \in \hat{G}} V'_\rho \otimes V_\rho$$

where the sum on the right-hand side is an orthogonal direct sum of irreducible $G \times G$-modules. Moreover $L \otimes R = \bigoplus_{\rho \in \hat{G}} \rho' \otimes \rho$.

Let us observe that the orthogonal projection P of $A[G]$ onto $V'_\rho \otimes V_\rho$ is given by

$$Pf = \frac{1}{|G|}\chi_\rho * f = \frac{1}{|G|} f * \chi_\rho, \tag{8}$$

and $\frac{1}{|G|}\chi_\rho$ is the identity of the two-sided ideal $V'_\rho \otimes V_\rho$ of $A[G]$. Therefore $A[G]$ is a semisimple algebra since $V'_\rho \otimes V_\rho$ is a matrix algebra.

A function $f \in A[G]$ is *central* if $f(xy) = f(yx)$ for all $x, y \in G$. In other words, if it is constant on each conjugacy class of G. Let $A[G]^G$ be the space of all central functions on G. Therefore $A[G]^G$ is the center of the group algebra $A[G]$. Let $\mathcal{C}(G)$ be the set of all conjugacy classes of G.

Corollary 2.2 *If ρ is an irreducible representation of G and let ξ_ρ be its character. Then $\{\xi_\rho : \rho \in \hat{G}\}$ is a basis of $A[G]^G$. In particular $|\hat{G}| = |\mathcal{C}(G)|$.*

Now we introduce the subalgebra of all K-central functions f in $A[G]$, that is $f(kxk^{-1}) = f(x)$ for all $x \in G$ and $k \in K$,

$$A[G]^K = \{f \in A[G] : \delta_k * f = f * \delta_k, \ \text{for all } k \in K\}. \tag{9}$$

This subalgebra of $A[G]$ will play a crucial role in the theory of spherical functions of finite groups. For the benefit of the reader we remind the following notations: If

$\sigma \in \hat{G}$ and $\pi \in \hat{K}$ let σ' (resp. π') be the contragredient representation of σ (resp. π), and let $(V'_\sigma)_{(\pi')}$ (resp. $(V_\sigma)_{(\pi)}$) be the π' (resp. π) -isotypic component of (V'_σ) (resp. $V\sigma$).

Proposition 2.3 *We have*

$$A[G]^K = \bigoplus_{\sigma\in\hat{G},\pi\in\hat{K}} \left((V'_\sigma)_{(\pi')} \otimes (V_\sigma)_{(\pi)}\right)^K$$

where the summands on the right are two-sided ideals. In particular the algebra $A[G]^K$ is semisimple.

Proof We have that $A[G] = \bigoplus_{\sigma\in\hat{G}} V'_\sigma \otimes V_\sigma \simeq \bigoplus_{\sigma\in\hat{G}} \operatorname{End}(V_\sigma)$. Then

$$A[G]^K = \bigoplus_{\sigma\in\hat{G}} \operatorname{End}_K(V_\sigma) = \bigoplus_{\sigma\in\hat{G}} \bigoplus_{\pi\in\hat{K}} \operatorname{End}_K((V_\sigma)_{(\pi)}) = \bigoplus_{\sigma,\pi} \left((V'_\sigma)_{(\pi')} \otimes (V_\sigma)_{(\pi)}\right)^K.$$

Let $f \in (V'_\sigma)_{(\pi')} \otimes (V_\sigma)_{(\pi)}$ and $h \in (V'_\rho)_{(\delta')} \otimes (V_\rho)_{(\delta)}$, with $\sigma, \rho \in \hat{G}$. Then we get $f * h = 0$, for $\sigma \neq \rho$. In the case $\sigma = \rho$ and $\pi, \delta \in \hat{K}$, $\pi \neq \delta$ we can take $\{v_r\}$ and $\{w_i\}$ orthonormal bases of $(V_\sigma)_{(\pi)}$ and $(V_\rho)_{(\delta)}$, respectively. Let $\{\lambda_r\}$ and $\{\mu_i\}$ be the dual basis of $(V'_\sigma)_{(\pi')}$ and $(V'_\rho)_{(\delta')}$, respectively. We compute

$$\begin{aligned} ((\lambda_r \otimes v_s) * (\mu_i \otimes w_j))(g) &= \sum_{x\in G} \lambda_r(\sigma(gx)v_s)\mu_i(\sigma(x^{-1})w_j) \\ &= \sum_{x\in G} \langle \rho((gx))v_s, v_r\rangle \langle \rho(x^{-1})w_j, w_i\rangle = \sum_{x\in G} \langle \sigma(x)v_s, \sigma(g^{-1})v_r\rangle \overline{\langle \sigma(x)w_i, w_j\rangle} = 0, \end{aligned}$$

by Schur orthogonality relations. Hence $(V'_\sigma)_{(\pi')} \otimes (V_\sigma)_{(\pi)}$ is an ideal in $A[G]$. Therefore $\left((V'_\sigma)_{(\pi')} \otimes (V_\sigma)_{(\pi)}\right)^K$ is an ideal in $A[G]^K$, which is isomorphic to the matrix algebra $\operatorname{End}_K((V_\sigma)_{(\pi)})$. This completes the proof. □

If σ is an irreducible finite dimensional representation of G we extend it to a function σ of $A[G]$ into $\operatorname{End}(V_\sigma)$ by

$$\sigma(f) = \sum_{g\in G} f(g)\sigma(g), \qquad f \in A[G].$$

Proposition 2.4 *If $\sigma \in \hat{G}$, then the linear map $f \mapsto \sigma(f)$ is an irreducible representation of $A[G]$. Conversely, if L is an irreducible representation of $A[G]$, then L extends some $\sigma \in \hat{G}$. Therefore there is a bijective correspondence between the irreducible representations of G and those of its group algebra.*

Moreover, the set of all irreducible representations of $A[G]$ separates points. In other words, if $f \in A[G]$ and $\sigma(f) = 0$ for all $\sigma \in \hat{G}$, then $f = 0$.

Proof The first assertion follows at once upon observing that $\sigma(\delta_g) = \sigma(g)$. To prove the second one let $0 \neq f \in A[G]$. From the Peter-Weyl theorem, we can write $f = \sum_{\rho \in \hat{G}} f_\rho$ with $f_\rho \in V'_\rho \otimes V_\rho$. By hypothesis $f_{\sigma'} \neq 0$ for some $\sigma \in \hat{G}$. If $\{v_i\}$ is an orthonormal basis of $V_{\sigma'}$, then $f_{\sigma'}(g) = \sum_{i,j} a_{i,j} \langle \sigma'(g)v_i, v_j \rangle = \sum_{i,j} a_{i,j} \overline{\langle \sigma(g)v_i, v_j \rangle}$. From the orthogonality relations we get

$$\begin{aligned}\langle \sigma(f)v_s, v_r \rangle &= \sum_{g \in G} f_{\sigma'}(g) \langle \sigma(g)v_s, v_r \rangle = \sum_{i,j} a_{i,j} \sum_{g \in G} \overline{\langle \sigma(g)v_i, v_j \rangle} \langle \sigma(g)v_s, v_r \rangle \\ &= \frac{|G|}{d(\sigma)} a_{s,r}.\end{aligned}$$

Therefore $\sigma(f) \neq 0$, and this completes the proof. □

Let (V_ρ, ρ) be an irreducible representation of G then (V'_ρ, ρ'), the contragradient representation of ρ is also an irreducible representation of G. From Proposition 2.4 we have that $\rho' : A[G] \to \mathrm{End}(V'_\rho)$ is an algebra homorphism. Therefore it is a $(G \times G)$-morphism. Recall that in $A[G]$ the action of G is given by $L(g)f = \delta_g * f$ and $R(g)f = f * \delta_{g^{-1}}$ and $\mathrm{End}(V_\rho)$ becomes a $G \times G$-module by defining $(g_1, g_2) \cdot T = \rho(g_1) T \rho(g_2^{-1})$.

Proposition 2.5 *The linear map $\rho' : V'_\rho \otimes V_\rho \to \mathrm{End}(V'_\rho)$ is a $G \times G$-algebra isomorphism.*

Proof We already know that $\rho' : A[G] \to \mathrm{End}(V'_\rho)$ is an algebra homomorphism and that $V'_\rho \otimes V_\rho$ is a two-sided ideal in $A[G]$ with $\frac{1}{|G|}\chi_\rho$ as an identity. Now we have that $\rho'\big(\frac{1}{|G|}\chi_\rho\big) = I$ is the identity of V'_ρ, because for $\lambda \in V'_\rho$ and $v \in V_\rho$ we get

$$\rho'\big(\tfrac{1}{|G|}\chi_\rho\big)(\lambda)(v) = \lambda\Big(\tfrac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \rho(g^{-1}) v\Big) = \lambda(v).$$

Therefore $\rho' : V'_\rho \otimes V_\rho \to \mathrm{End}(V'_\rho)$ is a nonzero $G \times G$-morphism between irreducible modules, thus it is an isomorphism. □

3 Spherical Functions

A *zonal spherical function* φ on G is a complex valued function which satisfies $\varphi(e) = 1$ and

$$\varphi(x)\varphi(y) = \frac{1}{|K|} \sum_{k \in K} \varphi(xky) \qquad x, y \in G. \tag{10}$$

A fruitful generalization of the above concept is given in the following definition.

Definition 3.1 A spherical function Φ on G of type $\delta \in \hat{K}$ is a function $\Phi : G \to \mathrm{End}(V)$ such that

(i) $\Phi(e) = I$ ($I = I_V : V \to V$ the identity transformation of V),
(ii) $\Phi(x)\Phi(y) = \frac{1}{|K|}\sum_{k\in K}\chi_\delta(k^{-1})\Phi(xky)$, for all $x, y \in G$.

As an immediate consequence of the definition of a spherical function, we have the following result.

Proposition 3.2 *If* $\Phi : G \to \mathrm{End}(V)$ *is a spherical function of type* δ *then:*

(i) $\Phi(k_1 g k_2) = \Phi(k_1)\Phi(g)\Phi(k_2)$, *for all* $k_1, k_2 \in K$, $g \in G$,
(ii) $k \mapsto \Phi(k)$ *is a representation of* K *such that any irreducible subrepresentation belongs to* δ.

Proof Let $k_1 \in K$ and $g \in G$. From the definition we have

$$\Phi(k_1x) = \Phi(e)\Phi(k_1x) = \frac{1}{|K|}\sum_{k\in K}\chi_\delta(k^{-1})\Phi(kk_1x) = \frac{1}{|K|}\sum_{k\in K}\chi_\delta(k^{-1})\Phi(k_1kx)$$
$$= \Phi(k_1)\Phi(x).$$

In the same way we prove that $\Phi(xk_2) = \Phi(x)\Phi(k_2)$.

Now we observe that from part i) and $\Phi(e) = I$, we have $\Phi(k_1k_2) = \Phi(k_1)\Phi(k_2)$, therefore $k \mapsto \Phi(k)$ is a representation of K. By definition we get

$$I = \Phi(e)\Phi(e) = \frac{1}{|K|}\sum_{k\in K}\chi_\delta(k^{-1})\Phi(k),$$

and by (6) the right-hand side is the orthogonal projection of V onto the isotypical component of type δ under the representation $k \mapsto \Phi(k)$. Thus, we have that $I = P_\delta$, and therefore all irreducible subrepresentations of $k \mapsto \Phi(k)$ are of type δ. □

Remark 3.3 Concerning the definition let us point out that the spherical function Φ determines its type univocally (Proposition 3.2) and the number of times that δ occurs in the representation $k \mapsto \Phi(k)$ is called the *height* of Φ.

When K is a subgroup contained in the center of G a spherical function Φ on G is nothing but a representation of G. In fact we have for all $x, y \in G$:

$$\Phi(x)\Phi(y) = \frac{1}{|K|}\sum_{k\in K}\chi_\delta(k^{-1})\Phi(xky) = \frac{1}{|K|}\sum_{k\in K}\chi_\delta(k^{-1})\Phi(k)\Phi(xy) = \Phi(xy).$$

Therefore when $K = \{e\}$ the spherical functions on G are precisely the finite dimensional representations of G and when G is abelian or $G = K$, the spherical functions are the finite dimensional representations of G which satisfy that all irreducible subrepresentations are equivalent to each other.

A spherical function $\Phi : G \to \mathrm{End}(V)$ is said to be *irreducible* if the only subspaces of V invariant under the set of linear transformations $\{\Phi(g) : g \in G\}$ are 0 and V.

Proposition 3.4 *Any spherical function Φ of type δ is the direct sum of irreducible spherical functions of type δ.*

Proof Let $(\cdot,\cdot)$ be an inner product on V and define $\langle u, v\rangle = \sum_{g\in G}(\Phi(g)u, \Phi(g)v)$. Then it is easy to see that $\langle\cdot,\cdot\rangle$ is an inner product such that

$$\langle \Phi(x)u, v\rangle = \langle u, \Phi(x^{-1})v\rangle \qquad \text{for all } x \in G. \tag{11}$$

If $U < V$ is an invariant subspace under $\Phi(g)$ for all $g \in G$ and U' is the orthogonal complement of U with respect to the inner product $\langle\cdot,\cdot\rangle$, then it follows from (11) that U' is also invariant. We complete the proof by induction on dim V. □

Let φ be a complex-valued solution of equation (10). If φ is not identically zero then $\varphi(e) = 1$. (cf. [17], Proposition 2.2, p. 400). This result generalizes in the following way.

Proposition 3.5 *Let Φ be an* End(V)*-valued nonzero solution of equation ii) in Definition 3.1. If Φ is irreducible, then $\Phi(e) = I$.*

Proof See Proposition 1.3 in [20]. For $v \in V$, the vector space W_v spanned by $\{\Phi(g)v : g \in G\}$ is $\Phi(G)$-invariant, therefore W_v is either 0 or V. Hence, we can choose $v \in V$ such that $W_v = V$. We also have,

$$\begin{aligned}\Phi(x)\Phi(e)\Phi(y) &= \frac{1}{|K|^2}\sum_{k\in K}\Big(\sum_{k_1\in K}\chi_\delta(k^{-1})\chi_\delta(k_1^{-1}k)\Big)\Phi(xk_1y)\\ &= \frac{1}{|K|}\sum_{k\in K}\chi_\delta(k_1^{-1})\Phi(xk_1y) = \Phi(x)\Phi(y),\end{aligned}$$

where we have used that $\chi_\delta * \chi_\delta = |K|\chi_\delta$. Thus $\Phi(x)\Phi(e) = \Phi(x)$ and in particular $\Phi(e)$ is a non-zero projection. On the other hand, if $\Phi(e)\Phi(y) \neq \Phi(y)$ for some $y \in G$, then there exists $v \in V$ such that $(\Phi(e)\Phi(y) - \Phi(y))v \neq 0$. Hence, by irreducibility, the linear space $\{w \in V : \Phi(x)w = 0 \text{ for all } x \in G\} = V$, which is a contradiction. Therefore $\Phi(e)\Phi(y) = \Phi(y)$ for all $y \in G$ and $\Phi(e)$ is a projection that commutes with $\Phi(x)$ for all $x \in G$. Once again by irreducibility $\Phi(e) = I$. This completes the proof of the proposition. □

The matrix valued spherical functions associated to a connected semisimple Lie group G and K a compact subgroup can be characterized as eigenfunctions of the subalgebra $D(G)^K$ of right invariant differential operators under K in the algebra of all left invariant differential operators on G. In the case of finite groups, we can obtain a similar result: the spherical functions are eigenfunctions of operators defined from the subalgebra $A[G]^K$ of K-central functions on $A[G]$.

Given $f \in A[G]$ we denote by D_f the right multiplication by $\check{f}$ on $A[G]$, where $\check{f}$ is defined by $\check{f}(g) = f(g^{-1})$. The map $D : A[G] \to \mathrm{End}(A[G])$ given by $D(f) = D_f$ is a representation of the algebra $A[G]$ on the vector space $A[G]$. For a function

$\Phi : G \to \mathrm{End}(V)$ we extend this definition by $D_f(\Phi) = \Phi * \check{f}$, i.e.

$$[D_f\Phi](g) = (\Phi * \check{f})(g) = \sum_{x\in G} f(g^{-1}x)\Phi(x).$$

For any $f \in A[G]^K$, the operator D_f is left invariant under G and right invariant under K, i.e. $L_g D_f = D_f L_g$ and $D_f R_k = R_k D_f$ for $g \in G, k \in K$.

The main goal of the rest of this section is to prove the following characterization of a spherical function on G of K-type δ.

Theorem 3.6 *A function $\Phi : G \to \mathrm{End}(V)$ is a spherical function of type δ if and only if*

(i) $\Phi(e) = I$,
(ii) $\Phi(k_1 g k_2) = \Phi(k_1)\Phi(g)\Phi(k_2)$ *for all* $k_1, k_2 \in K$, $g \in G$,
(iii) $[D_f\Phi](g) = \Phi(g)\,[D_f\Phi](e)$ *for all* $f \in A[G]^K$,
(iv) *the restriction* $\pi = \Phi_{|K}$ *as a representation of* K *is equivalent to a direct sum of copies of* δ.

We start by proving the following proposition, which completes the proof that a spherical function $\Phi : G \to \mathrm{End}(V)$ of type $\delta \in \hat{K}$ satisfies conditions (i) - (iv) in Theorem 3.6.

Proposition 3.7 *If $\Phi : G \to \mathrm{End}(V)$ is a spherical function then*

$$[D_f\Phi](g) = \Phi(g)\,[D_f\Phi](e),$$

for all $f \in A[G]^K$, $g \in G$.

Proof By using that $[D_f\Phi](e) = (\Phi * \check{f})(e)$ we have that for $f \in A[G]^K$

$$\Phi(g)[D_f\Phi](e) = \Phi(g)\sum_{y\in G}\Phi(y)f(y) = \frac{1}{|K|}\sum_{y\in G}\sum_{k\in K}\chi_\delta(k^{-1})\Phi(gy)f(k^{-1}y)$$

since f is a K-central function

$$= \frac{1}{|K|}\sum_{y\in G}\sum_{k\in K}\chi_\delta(k^{-1})\Phi(gyk)f(y) = \sum_{y\in G}\Phi(gy)f(y) = (\Phi * \check{f})(g) = [(D_f\Phi](g).$$

This concludes the proof of the proposition. □

To give a proof of the converse of Theorem 3.6, we will show that for certain functions Φ the condition (iii) in this theorem is equivalent to identity (ii) in Definition 3.1. See Theorem 3.11 below. For this purpose, we introduce a function Ψ closely related with Φ, which in fact could be taken as an alternative way to handle the concept of spherical function.

Let (V, π) be a finite dimensional representation of K that it is a multiple of $\delta \in \hat{K}$. We consider the following vector spaces

$$\mathcal{A} = \{\Phi : G \to \mathrm{End}(V) \; : \; \Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)\},$$
$$\mathcal{B} = \{\Psi : G \to \mathrm{End}_K(V) : \Psi \text{ is } K\text{-central and } \chi_\delta * \Psi = |K| \, \Psi\}.$$

Proposition 3.8 *For $\Phi \in \mathcal{A}$ and $\Psi \in \mathcal{B}$, we define the linear maps T and S by*

$$(T\Phi)(g) = \frac{1}{|K|} \sum_{k \in K} \pi(k)\Phi(g)\pi(k^{-1}) \quad \textit{and} \quad (S\Psi)(g) = \frac{d(\delta)^2}{|K|} \sum_{k \in K} \pi(k)\Psi(k^{-1}g).$$

Then T is an isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ and S is the inverse of T.

Proof If $\Phi \in \mathcal{A}$ we will see that $T\Phi \in \mathcal{B}$: It is clear that $(T\Phi)(g) \in \mathrm{End}_K(V)$ for all $g \in G$, and that $T\Phi$ is a K-central function. Furthermore $T\Phi$ satisfies $\chi_\delta * (T\Phi) = |K| T\Phi$:

$$\begin{aligned}
(\chi_\delta * (T\Phi))(g) &= \frac{1}{|K|} \sum_{k, k_1 \in K} \chi_\delta(k)\pi(k_1)\Phi(k^{-1}g)\pi(k_1^{-1}) \\
&= \sum_{k_1 \in K} \pi(k_1)\Big(\frac{1}{|K|} \sum_{k \in K} \chi_\delta(k)\pi(k^{-1})\Big)\Phi(g)\pi(k_1^{-1}) \\
&= \sum_{k_1 \in K} \pi(k_1)\Phi(g)\pi(k_1^{-1}) = |K|(T\Phi)(g),
\end{aligned}$$

since $\frac{1}{|K|} \sum_{k \in K} \chi_\delta(k)\pi(k^{-1}) = I$. On the other hand, if $\Psi \in \mathcal{B}$ then $S\Psi \in \mathcal{A}$. In fact,

$$\begin{aligned}
(S\Psi)(k_1 g k_2) &= \frac{d(\delta)^2}{|K|} \sum_{k \in K} \pi(k)\Psi(k^{-1}k_1 g k_2) = \frac{d(\delta)^2}{|K|} \sum_{k \in K} \pi(k)\Psi(k_2 k^{-1} k_1 g) \\
&= \frac{d(\delta)^2}{|K|} \sum_{k \in K} \pi(k_1 k k_2)\Psi(k^{-1}g) \\
&= \frac{d(\delta)^2}{|K|} \pi(k_1) \sum_{k \in K} \pi(k)\Psi(k^{-1}g)\pi(k_2) = \pi(k_1)(S\Psi)(g)\pi(k_2),
\end{aligned}$$

we have used that, by hypothesis, $\pi(k)\Psi(g) = \Psi(g)\pi(k)$ for all $g \in G$ and $k \in K$.

To see that S is a right inverse of T we take $\Psi \in \mathcal{B}$ and we observe that

$$\begin{aligned}
(TS\Psi)(g) &= \frac{d(\delta)^2}{|K|^2} \sum_{k_1 \in K} \sum_{k_2 \in K} \pi(k_1)\pi(k_2)\Psi(k_2^{-1}g)\pi(k_1^{-1}) \\
&= \frac{d(\delta)^2}{|K|^2} \sum_{k_2 \in K} \sum_{k_1 \in K} \pi(k_1 k_2 k_1^{-1})\Psi(k_2^{-1}g).
\end{aligned}$$

Now we note

$$\sum_{k_1 \in K} \pi(k_1 k_2 k_1^{-1}) = \frac{|K|\chi_\delta(k_2)}{d(\delta)^2} I. \tag{12}$$

In fact, by Schur's Lemma $\sum_{k_1 \in K} \delta(k_1 k_2 k_1^{-1}) = c(k_2) I_\delta$ for some $c(k_2) \in \mathbb{C}$. By taking trace in both sides we obtain, $|K|\xi_\delta(k_2) = c(k_2) d(\delta)$. Since π is a direct sum of copies of δ, (12) follows. Therefore

$$(TS\Psi)(g) = \frac{1}{|K|} \sum_{k_2 \in K} \chi_\delta(k_2) \Psi(k_2^{-1} g) = \frac{1}{|K|} (\chi_\delta * \Psi)(g) = \Psi(g).$$

This proves that S is a right inverse of T. To see that S is a left inverse of T we take $\Phi \in \mathcal{A}$, then we get

$$\begin{aligned}(ST\Phi)(g) &= \frac{d(\delta)^2}{|K|^2} \sum_{k_1 \in K} \sum_{k_2 \in K} \pi(k_1)\pi(k_2)\Phi(k_1^{-1} g)\pi(k_2^{-1}) \\ &= \frac{d(\delta)^2}{|K|^2} \sum_{k_1 \in K} \sum_{k_2 \in K} \pi(k_1)\pi(k_2)\pi(k_1^{-1})\Phi(g)\pi(k_2^{-1}) \\ &= \frac{1}{|K|} \sum_{k_2 \in K} \chi_\delta(k_2)\Phi(g)\pi(k_2^{-1}) = \Phi(g).\end{aligned}$$

This completes the proof of the proposition. □

Remark 3.9 If $(\mathbb{C}, 1)$ is the trivial one dimensional representation of K, then $\mathcal{A} = \mathcal{B} = A[G]^{K\times K}$ and $T = S = I$. In fact, that $\mathcal{A} = A[G]^{K\times K}$ is obvious. Moreover if Ψ is a K-central function such that $\chi_1 * \Psi = |K|\Psi$, then $(\chi_1 * \Psi)(g) = \sum_{k\in K} \chi_1(k^{-1})\Psi(kg) = \sum_{k\in K} \Psi(kg) = |K|\Psi(g)$. Therefore Ψ is K-left invariant, and since Ψ is K-central it is also K-right invariant. Conversely, $A[G]^{K\times K} \leq \mathcal{B}$ is obvious. Then $T = S = I$ is a straightforward consequence of the definitions.

In particular, if ϕ is a zonal spherical function, then $\phi = T\phi = \psi$.

Lemma 3.10 *Let $\Psi : G \to \mathrm{End}(V)$ be K-central. If Ψ satisfies $[D_f \Psi](e) = 0$ for all $f \in A[G]^K$, then $\Psi = 0$.*

Proof If $f \in A[G]$, let $f^\circ(g) = \frac{1}{|K|} \sum_{k\in K} f(kgk^{-1})$. Then $f \mapsto f^\circ$ is the K-projection of $A[G]$ onto $A[G]^K$. If $f \in A[G]$ we have $[D_f \Psi](e) = (\Psi * \check{f})(e) = \sum_{g\in G} \Psi(g) f(g)$. Therefore

$$\begin{aligned}[D_f \Psi](e) &= \frac{1}{|K|} \sum_{k_1 \in K} \sum_{g \in G} \Psi(k_1 g k_1^{-1}) f(k_1 g k_1^{-1}) = \frac{1}{|K|} \sum_{g \in G} \sum_{k_1 \in K} \Psi(g) f(k_1 g k_1^{-1}) \\ &= \sum_{g \in G} \Psi(g) f^\circ(g) = [D_{f^\circ} \Psi](e) = 0 \end{aligned} \tag{13}$$

for all $f \in A[G]$, by hypothesis. In particular, by taking $f = \delta_g$ it follows $\Psi(g) = 0$ for any $g \in G$. □

Theorem 3.11 *Let $\Phi \in \mathcal{A}$ and $\Psi = T\Phi$. Then the following conditions are equivalent:*

(i) Ψ *satisfies the functional equation*

$$\Psi(x)\Psi(y) = \frac{1}{|K|}\sum_{k\in K}\Psi(kxk^{-1}y), \qquad \text{for all } x, y \in G.$$

(ii) Φ *satisfies the functional equation*

$$\Phi(x)\Phi(y) = \frac{1}{|K|}\sum_{k\in K}\chi_\delta(k^{-1})\Phi(xky), \quad \text{for all } x, y \in G.$$

(iii) *for all $f \in A[G]^K$*

$$[D_f\Phi](x) = \Phi(x)[D_f\Phi](e), \qquad \text{for all } x \in G.$$

(iv) *for all $f \in A[G]^K$*

$$[D_f\Psi](x) = \Psi(x)[D_f\Psi](e), \qquad \text{for all } x \in G.$$

Proof (i) ⇒ (ii). By assumption $\Phi = S\Psi$, then

$$\begin{aligned}
\Phi(x)\Phi(y) &= \frac{d(\delta)^4}{|K|^2}\sum_{k_1,k_2\in K}\pi(k_1)\Psi(k_1^{-1}x)\pi(k_2)\Psi(k_2^{-1}y)\\
&= \frac{d(\delta)^4}{|K|^3}\sum_{k_1,k_2\in K}\pi(k_1k_2)\sum_{k\in K}\Psi(kk_1^{-1}xk^{-1}k_2^{-1}y)\\
&= \frac{d(\delta)^4}{|K|^3}\sum_{k,k_1,k_2\in K}\pi(k_1k_2)\sum_{k\in K}\Psi(k_2^{-1}kk_1^{-1}xk^{-1}y)\\
&= \frac{d(\delta)^2}{|K|^2}\sum_{k,k_1\in K}\pi(k_1)\Phi(kk_1^{-1}xk^{-1}y)\\
&= \frac{d(\delta)^2}{|K|^2}\sum_{k,k_1\in K}\pi(k_1kk_1^{-1})\Phi(xk^{-1}y)\\
&= \frac{1}{|K|}\sum_{k\in K}\chi_\delta(k)\Phi(xk^{-1}y),
\end{aligned}$$

in the last equality, we have used (12).

(ii) ⇒ (iii). This was proved in Proposition 3.7.

(ii) $\Rightarrow$ (iv). For $f \in A[G]^K$, we have

$$
\begin{aligned}
[D_f\Psi](e) &= \sum_{y\in G}\Psi(y)f(y) = \frac{1}{|K|}\sum_{y\in G}\sum_{k\in K}\Phi(kyk^{-1})f(y) \\
&= \frac{1}{|K|}\sum_{y\in G}\sum_{k\in K}\Phi(y)f(y) = [D_f\Phi](e),
\end{aligned}
$$

and

$$
\begin{aligned}
\pi(k)[D_f\Phi](e)\pi(k^{-1}) &= \pi(k)\sum_{y\in G}\Phi(y)f(y)\pi(k^{-1}) \\
&= \sum_{y\in G}\Phi(kyk^{-1})f(y) = \sum_{y\in G}\Phi(y)f(y) = [D_f\Phi](e).
\end{aligned}
$$

We also have,

$$
\begin{aligned}
[D_f\Psi](x) &= \sum_{y\in G}\Psi(xy)f(y) = \frac{1}{|K|}\sum_{y\in G}\sum_{k\in K}\pi(k)\Phi(xy)f(y)\pi(k^{-1}) \\
&= \frac{1}{|K|}\sum_{k\in K}\pi(k)[D_f\Phi](x)\pi(k^{-1}) \\
&= \frac{1}{|K|}\sum_{k\in K}\pi(k)\Phi(x)[D_f\Phi](e)\pi(k^{-1}) \\
&= \Psi(x)[D_f\Phi](e) = \Psi(x)[D_f\Psi](e).
\end{aligned}
$$

(iv) $\Rightarrow$ (i). We observe that for any $x \in G$ the function

$$
\Psi(x)\Psi(y) - \frac{1}{|K|}\sum_{k\in K}\Psi(kxk^{-1}y)
$$

is K-central as a function of y. If we apply the operator D_f to it and evaluate at the identity $e \in G$ we obtain

$$
\Psi(x)[D_f\Psi](e) - \frac{1}{|K|}\sum_{k\in K}\Psi(kxk^{-1})[D_f\Psi](e) = 0.
$$

Therefore, by applying Lemma 3.10, (i) follows. □

Now we are in condition to complete the proof of Theorem 3.6.

Proof of Theorem 3.6 If $\Phi : G \to \mathrm{End}(V)$ is a spherical function of type $\delta \in \hat{K}$ then (i) holds by definition, (ii) and (iv) follow from Proposition 3.2 and (iii) follows from Proposition 3.7.

For the converse, if a function $\Phi : G \to \mathrm{End}(V)$ satisfies (i) - (iv) then Φ belongs to the vector space $\mathcal{A}$, with $\pi = \Phi_{|K}$ and satisfies (iii) in Theorem 3.11, which is equivalent to the functional equation defining a spherical function of type δ. □

If $\Phi : G \to \mathrm{End}(V)$ is a spherical function of type δ and height p, then the function

$$\Psi = T\Phi : G \longrightarrow \mathrm{End}_K(V) \simeq \mathrm{Mat}_{p\times p}(\mathbb{C})$$

should be considered as the other face of the same coin: □

Proposition 3.12 *The function* $\Psi : G \longrightarrow \mathrm{Mat}_{p\times p}(\mathbb{C})$ *satisfies*

(i) $\Psi(e) = I$,
(ii) $\chi_\delta * \Psi = |K|\Psi$,
(iii) $\Psi(x)\Psi(y) = \frac{1}{|K|}\sum_{k\in K}\Psi(kxk^{-1}y)$, *for all* $x, y \in G$.

Spherical functions of a pair (G, K) arise in a natural way upon considering representations of G. If (V, ρ) is a representation of G in a vector space V that contains the K-type δ, we recall that

$$P_\delta = \frac{1}{|K|}\sum_{k\in K}\chi_\delta(k^{-1})\rho(k)$$

is the K-projection of V onto $V_{(\delta)}$, the isotypical component of type δ. (See (6)).

Theorem 3.13 *Let* (V, ρ) *be a finite dimensional irreducible representation of G that contains the* K*-type* δ. *Then* $\Phi(g) = P_\delta\rho(g)P_\delta$, $g \in G$, *is an irreducible spherical function of G of type* δ. *Conversely, any irreducible spherical function of the pair* (G, K) *is of this form.*

Proof In fact, if $v \in V_{(\delta)}$ we have

$$\Phi(x)\Phi(y)v = P_\delta\rho(x)P_\delta\rho(y)a = \frac{1}{|K|}\sum_{k\in K}\chi_\delta(k^{-1})P_\delta\rho(x)\rho(k)\rho(y)v$$
$$= \Big(\frac{1}{|K|}\sum_{k\in K}\chi_\delta(k^{-1})\Phi(xky)\Big)v.$$

To prove that Φ is irreducible let W be a nonzero $\Phi(G)$-invariant subspace of $V_{(\delta)}$ and let Q be a $\Phi(G)$-projection of $V_{(\delta)}$ onto W. Then

$$0 = P_\delta\rho(g)QP_\delta - QP_\delta\rho(g)QP_\delta = (I-Q)P_\delta\rho(g)QP_\delta$$

(I= identity transformation of $V_{(\delta)}$). Since the linear span $\langle\rho(g)a : a \in W\rangle = V$, it follows that $I = Q$ which completes the proof of the first part of the proposition.

The last part of the statement is proved in the Appendix at the end of the paper because we need more sophisticated tools which will be developed in the next sections. □

4 Spherical Functions and Representations of Algebras

We extend to $A[G]$ any function $\Phi : G \to \mathrm{End}(V)$ by defining

$$\Phi : A[G] \to \mathrm{End}(V), \qquad \Phi(f) = \sum_{g\in G} f(g)\Phi(g).$$

Observe that $\Phi(f) = (\Phi * \check{f})(e)$ and $\Phi(\delta_g) = \Phi(g)$, for all $g \in G$.

Definition 4.1 Let $\delta \in \hat{K}$, and $\chi_\delta = \mathrm{d}(\delta)\xi_\delta$ where ξ_δ is the character of δ. We introduce the following subalgebra of $A[G]$

$$A_\delta[G] = \{f \in A[G] : \overline{\chi}_\delta * f = f * \overline{\chi}_\delta = |K|\, f\}.$$

Observe that $\frac{1}{|K|}\overline{\chi}_\delta$ is an identity of $A_\delta[G]$.

Lemma 4.2 *The map $P : f \mapsto \frac{1}{|K|^2}\overline{\chi}_\delta * f * \overline{\chi}_\delta$ is a linear projection of $A[G]$ onto $A_\delta[G]$.*

Proof We get $\overline{\chi}_\delta * \overline{\chi}_\delta = |K|\, \overline{\chi}_\delta$. Therefore for all $f \in A[G]$ we have $P^2(f) = P(f)$ and $P(f) \in A_\delta[G]$. Moreover if $f \in A_\delta[G]$ then $\overline{\chi}_\delta * f * \overline{\chi}_\delta = |K|^2 f$. Thus $f = P(f)$ and the proof is completed. □

Proposition 4.3 *Let $\Phi : G \to \mathrm{End}(V)$ be a function such that $\chi_\delta * \Phi = \Phi * \chi_\delta = |K|\, \Phi$. Then Φ satisfies the functional equation*

$$\Phi(x)\Phi(y) = \frac{1}{|K|}\sum_{k\in K} \chi_\delta(k^{-1})\Phi(xky),$$

if and only if Φ is a representation of $A_\delta[G]$.

Proof Let $f \in A[G]$, then $\Phi(f) = \sum_{g\in G} f(g)\Phi(g) = (\Phi * \check{f})(e)$. Therefore

$$\begin{aligned} \Phi(\overline{\chi}_\delta * f * \overline{\chi}_\delta) &= (\Phi * (\overline{\chi}_\delta * f * \overline{\chi}_\delta)^\vee)(e) = (\Phi * (\chi_\delta * \check{f} * \chi_\delta))(e) \\ &= |K|\, (\Phi * \check{f} * \chi_\delta)(e) \\ &= |K|\, (\chi_\delta * \Phi * \check{f})(e) = |K|^2\, (\Phi * \check{f})(e) = |K|^2\, \Phi(f), \end{aligned} \tag{14}$$

where we have used that $(f * h)^\vee = \check{h} * \check{f}$, $\overline{\chi}_\delta = \check{\chi}_\delta$ and $(f * h)(e) = (h * f)(e)$.

Now, by using (14) we obtain

$$\begin{aligned}
&\Phi\big((\overline{\chi}_\delta * f * \overline{\chi}_\delta) * (\overline{\chi}_\delta * h * \overline{\chi}_\delta)\big)\\
&\qquad = |K|\,\Phi((\overline{\chi}_\delta * f * \overline{\chi}_\delta * h * \overline{\chi}_\delta) = |K|^3\,\Phi(f * \overline{\chi}_\delta * h)\\
&\qquad = |K|^3 \sum_{y\in G}(f * \overline{\chi}_\delta * h)(y)\Phi(y) = |K|^3 \sum_{y\in G}\sum_{x\in G}(f * \overline{\chi}_\delta)(x)h(x^{-1}y)\Phi(y)\\
&\qquad = |K|^3 \sum_{y\in G}\sum_{x\in G}\sum_{k\in K} f(xk^{-1})\overline{\chi}_\delta(k)h(y)\Phi(xy)\\
&\qquad = |K|^4 \sum_{y\in G}\sum_{x\in G} f(x)h(y)\Big(\sum_{k\in K}\frac{1}{|K|}\,\chi_\delta(k^{-1})\Phi(xky)\Big).
\end{aligned}$$

On the other hand, by (14) we have

$$\begin{aligned}
&\Phi(\overline{\chi}_\delta * f * \overline{\chi}_\delta)\,\hat{\Phi}(\overline{\chi}_\delta * h * \overline{\chi}_\delta)\\
&\quad = |K|^4\Phi(f)\hat{\Phi}(h) = |K|^4 \sum_{x\in G}\sum_{y\in G} f(x)h(y)\Phi(x)\Phi(y).
\end{aligned}$$

Now the proposition follows immediately. □

We are in a position to state a very important result that establishes a close connection between spherical functions of type δ and representations of the algebra $A_\delta[G]$.

Theorem 4.4 *If Φ is an irreducible spherical function of type $\delta \in \hat{K}$, then the linear map*

$$\Phi : f \mapsto \sum_{g\in G} f(g)\Phi(g)$$

is an irreducible representation of $A_\delta[G]$. Conversely, if L is an irreducible representation of $A_\delta[G]$, then L is the representation Φ defined by an irreducible spherical function Φ of G of type δ.

Proof Let $\Phi : G \to \mathrm{End}(V)$ be an irreducible spherical function of type δ. Then

$$(\Phi * \chi_\delta)(g) = \sum_{k\in K}\Phi(gk)\chi_\delta(k^{-1}) = \Phi(g)\sum_{k\in K}\Phi(k)\chi_\delta(k^{-1}) = |K|\,\Phi(g), \tag{15}$$

because $\Phi_{|K}$ is a direct sum of representations of K all in the class δ and then $\frac{1}{|K|}\sum_{k\in K}\chi_\delta(k^{-1})\Phi(k) = I$ by (6). Similarly, we get that $\chi_\delta * \Phi = |K|\Phi$.

Now, by Proposition 4.3, we have that $\Phi : A_\delta[G] \to \mathrm{End}(V)$ is a representation of $A_\delta[G]$. Since $\Phi(\delta_g) = \Phi(g)$ the irreducibility of a spherical function is equivalent to the irreducibility of the representation Φ of $A_\delta[G]$.

Conversely, let $L : A_\delta[G] \to \mathrm{End}(V)$ be an irreducible representation of $A_\delta[G]$. Let Θ be the $\mathrm{End}(V)$-valued function on G defined by

$$\Theta(g) = \frac{1}{|K|^2} L(\overline{\chi}_\delta * \delta_g * \overline{\chi}_\delta).$$

If $f \in A[G]$, then $f = \sum_g f(g)\delta_g$. Thus by linearity, we have

$$L(\overline{\chi}_\delta * f * \overline{\chi}_\delta) = \sum_{g\in G} f(g)L(\overline{\chi}_\delta * \delta_g * \overline{\chi}_\delta) = |K|^2 \sum_{g\in G} f(g)\Theta(g) = |K|^2\,\Theta(f).$$

Therefore if $f \in A_\delta[G]$, then $\Theta(f) = \frac{1}{|K|^2} L(\overline{\chi}_\delta * f * \overline{\chi}_\delta) = L(f)$ is a representation of $A_\delta[G]$.

Let $\Phi = \frac{1}{|K|^2}\chi_\delta * \Theta * \chi_\delta$. Thus $\chi_\delta * \Phi = \Phi * \chi_\delta = |K|\,\Phi$. We also have, for all $f \in A_\delta[G]$

$$\begin{aligned}\Phi(f) &= (\Phi * \check{f})(e) = \frac{1}{|K|^2}\big(\chi_\delta * \Theta * \chi_\delta * \check{f}\big)(e) = \frac{1}{|K|^2}(\Theta * \chi_\delta * \check{f} * \chi_\delta)(e)\\ &= \frac{1}{|K|^2}\big(\Theta * (\overline{\chi}_\delta * f * \overline{\chi}_\delta)^\vee\big)(e) = (\Theta * \check{f})(e) = \Theta(f) = L(f).\end{aligned}$$

Therefore Φ is an irreducible representation of the algebra $A_\delta[G]$. By Proposition 4.3 and Proposition 3.5, we have that Φ is an irreducible spherical function of type δ such that $\Phi(f) = L(f)$ for all $f \in A_\delta[G]$. □

Corollary 4.5 *The irreducible representations of $A_\delta[G]$ separate points.*

Proof Let $0 \neq f \in A_\delta[G]$. From Proposition 2.4 there exists $\rho \in \hat{G}$ such that $\rho(f) \neq 0$. By hypothesis $f = \frac{1}{|K|^2}\overline{\chi}_\delta * f * \overline{\chi}_\delta$. Therefore

$$\rho(f) = \frac{1}{|K|^2}\rho(\overline{\chi}_\delta)\rho(f)\hat{\rho}(\overline{\chi}_\delta) = \Phi(f) \neq 0,$$

where Φ is the spherical function of type δ associated to ρ. □

We say that the spherical functions $\Phi : G \to \mathrm{End}(V)$ and $\Phi_1 : G \to \mathrm{End}(V_1)$ are *equivalent* if there exists a linear isomorphism $T : V \to V_1$ such that $\Phi_1(g)T = T\Phi_1(g)$, for all $g \in G$.

Proposition 4.6 *The irreducible spherical functions $\Phi : G \to \mathrm{End}(V)$ and $\Phi_1 : G \to \mathrm{End}(V_1)$ of type δ are equivalent, if and only if the corresponding representations $\Phi : A_\delta[G] \to \mathrm{End}(V)$ and $\Phi_1 : A_\delta[G] \to \mathrm{End}(V_1)$ are equivalent.*

Proof Let T be an isomorphism of V onto V_1 such that $\Phi_1(f) = T\Phi(f)T^{-1}$ for all $f \in A_\delta[G]$. Then, using (14), we have

$$\Phi_1(f) = \frac{1}{|K|^2}\Phi_1(\overline{\chi}_\delta * f * \overline{\chi}_\delta) = \frac{1}{|K|^2}T\Phi(\overline{\chi}_\delta * f * \overline{\chi}_\delta)T^{-1} = T\Phi(f)T^{-1}$$

for any $f \in A[G]$. Therefore $\Phi_1(g) = T\Phi(g)T^{-1}$ for all $g \in G$. The converse assertion is obvious. □

As a corollary of Theorem 4.4 and Proposition 4.6 we obtain the following result.

Proposition 4.7 *The irreducible spherical functions Φ and Φ_1 are equivalent if and only if* $\operatorname{tr}\Phi(g) = \operatorname{tr}\Phi_1(g)$ *for all $g \in G$.*

Proof It is obvious that if Φ and Φ_1 are equivalent they have the same trace. Conversely, since in particular $\operatorname{tr}\Phi(k) = \operatorname{tr}\Phi_1(k)$ for all $k \in K$, Φ and Φ_1 are of the same K-type δ. Moreover, $\operatorname{tr}\Phi(g) = \operatorname{tr}\Phi_1(g)$ for all $g \in G$, implies that $\operatorname{tr}\Phi(f) = \operatorname{tr}\Phi_1(f)$ for all $f \in A_\delta[G]$. Since Φ and Φ_1 are two irreducible finite dimensional representations of an associative algebra over $\mathbb{C}$ having the same trace they are equivalent. Hence, by Proposition 4.6 the spherical functions Φ and Φ_1 are equivalent. □

Recall that $A[G]^K$ denotes the subalgebra of $A[G]$ of all K-central functions. Let us define

$$A_\delta^K[G] = A[G]^K \cap A_\delta[G] \quad \text{and} \quad A_\delta[K] = A[K] \cap A_\delta[G]. \tag{16}$$

Hence $A_\delta^K[G]$ and $A_\delta[K]$ are subalgebras of $A_\delta[G]$ with identity $\frac{1}{|K|}\overline{\chi}_\delta$.

We also observe that

$$A_\delta[K] = A[K] * \overline{\chi}_\delta. \tag{17}$$

In fact if $f \in A[K] \cap A_\delta[G]$, then $f = \frac{1}{|K|} f * \overline{\chi}_\delta \in A[K] * \overline{\chi}_\delta = A_\delta[K]$. Conversely, for any $f \in A[K]$ we have $f * \overline{\chi}_\delta = \overline{\chi}_\delta * f$ because $\bar{\chi}_\delta$ is a K-central function. Then $f * \overline{\chi}_\delta \in A_\delta[G] \cap A[K]$. In particular, we obtain that $A_\delta[K]$ is a two-sided ideal in $A[K]$.

If $\delta = 1$ is the trivial representation of K, then $A_1[G] = A[G]^{K\times K} = A_1^K[G]$. In fact, $(\overline{\chi}_1 * f)(g) = |K| f(g)$ (respectively $f * \overline{\chi}_1 = |K| f$) is equivalent to f being K-left invariant (resp. K-right invariant). On the other hand, $A[G]^{K\times K} \leq A_1^K[G] \leq A_1[G] = A[G]^{K\times K}$.

Lemma 4.8 *Let $\delta \in \hat{K}$, if $V = V_\delta \oplus \cdots \oplus V_\delta$ as K-modules and $\Delta(T) = T \oplus \cdots \oplus T$ for $T \in \operatorname{End}(V_\delta)$, then the linear map $p : \operatorname{End}_K(V) \otimes \operatorname{End}(V_\delta) \to \operatorname{End}(V)$ defined by $p(S \otimes T) = S\Delta(T) = \Delta(T)S$ is a surjective isomorphism of algebras.*

Proof Let $\{v_i\}$ be a basis of V_δ and $w_i^j = (0, \ldots, 0, v_i, 0, \ldots, 0)$ the element v_i in the j^{th}-position. Then $\{w_i^j\}$ is a basis of V. Given an ordered pair (j, k) let $S_{j,k} \in \operatorname{End}(V)$ be defined by $S_{j,k}(w_i^j) = w_i^k$ and $S_{j,k}(w_i^r) = 0$ for all i, if $r \neq j$. Then $S_{j,k} \in \operatorname{End}_K(V)$ maps the j^{th}-summand onto the k^{th}-summand and all the other summands to zero. Besides let $T_{i,r} \in \mathrm{End}(V_\delta)$ be defined by $T_{i,r}(v_s) = \delta_i^s v_r$. Then

$$\begin{aligned}
&(T_{i,r} \oplus \cdots \oplus T_{i,r})S_{j,k}(w_i^j) = w_r^k,\\
&(T_{i,r} \oplus \cdots \oplus T_{i,r})S_{j,k}(w_{i'}^{j'}) = 0 \text{ if } (i', j') \neq (i, j),\\
&S_{j,k}(T_{i,r} \oplus \cdots \oplus T_{i,r})(w_i^j) = w_r^k,\\
&S_{j,k}(T_{i,r} \oplus \cdots \oplus T_{i,r})(w_{i'}^{j'}) = 0 \text{ if } (i', j') \neq (i, j).
\end{aligned}$$

Then the linear map $p : \mathrm{End}_K(V) \otimes \mathrm{End}(V_\delta) \to \mathrm{End}(V)$ is onto, and that $(T \oplus \cdots \oplus T)S = S(T \oplus \cdots \oplus T)$.

On the other hand, if h is the multiplicity of V_δ in V, then

$$\dim\big(\mathrm{End}_K(V) \otimes \mathrm{End}(V_\delta)\big) = h^2 d(\delta)^2 = (hd(\delta))^2 = \dim\big(\mathrm{End}(V)\big).$$

This completes the proof of the lemma. □

Proposition 4.9 *The map $m : A_\delta^K[G] \otimes A_\delta[K] \longrightarrow A_\delta[G]$, given by $f \otimes h \mapsto f * h$ is an algebra isomorphism, i.e.*

$$A_\delta^K[G] \otimes A_\delta[K] \simeq A_\delta[G].$$

Proof The linear map m is a homomorphism of algebras because $f * h = h * f$ for all $f \in A_\delta^K[G]$, $h \in A_\delta[K]$.

By the Peter-Weyl theorem, we have $A[G] = \bigoplus_{\rho \in \hat{G}} V'_\rho \otimes V_\rho$. We know that $Pf = \frac{1}{|K|^2}\overline{\chi}_\delta * f * \overline{\chi}_\delta$ is a projection of $A[G]$ onto $A_\delta[G]$.

Given (ρ, V_ρ) a representation of G, (ρ', V'_ρ) denotes the contragredient representation of ρ. We also denote $P_{\rho',\delta}$ the K-projection of V'_ρ onto the K-isotypic component $(V'_\rho)_{(\delta)}$.

For $w \in V_\rho$ and $\lambda \in V'_\rho$ we have

$$\frac{1}{|K|}(\overline{\chi}_\delta * (\lambda \otimes w))(g) = \frac{1}{|K|}\sum_{k \in K} \chi_\delta(k^{-1})(\rho'(k)\lambda)(\rho(g)w) = (P_{\rho',\delta}\lambda \otimes w)(g),$$

and similarly we get

$$\frac{1}{|K|}((\lambda \otimes w) * \overline{\chi}_\delta)(g) = \frac{1}{|K|}\sum_{k \in K}(\lambda(\rho(k)w)(g)\chi_{\delta'}(k^{-1}) = (\lambda \otimes P_{\rho,\delta'}w)(g).$$

Then

$$\frac{1}{|K|}\overline{\chi}_\delta * (\lambda \otimes w) = P_{\rho',\delta}\lambda \otimes w \quad \text{and} \quad \frac{1}{|K|}(\lambda \otimes w) * \overline{\chi}_\delta = \lambda \otimes P_{\rho,\delta'}w.$$

Therefore, $P(\lambda \otimes w) = P_{\rho',\delta}\lambda \otimes P_{\rho,\delta'}w$, $P(V'_\rho \otimes V_\rho) = (V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}$ and

$$A_\delta[G] = \bigoplus_{\rho\in\hat{G}} (V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}. \tag{18}$$

Also

$$A_\delta^K[G] = \bigoplus_{\rho\in\hat{G}} \big((V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}\big)^K \tag{19}$$

is a direct sum of two-sided ideals. On the other hand, from (17) we have $A_\delta[K] = A[K] * \overline{\chi}_\delta = \overline{\chi}_\delta * A[K]$. Then the homomorphism $m : A_\delta^K[G] \otimes A_\delta[K] \longrightarrow A_\delta[G]$ is the direct sum of the homomorphisms

$$m_\rho : \big((V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}\big)^K \otimes A_\delta[K] \longrightarrow (V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')},$$

$\rho \in \hat{G}$, defined by $m_\rho(f \otimes a) = f * a$. Let $p_\rho : \mathrm{End}_K\big((V_\rho)_{(\delta)}\big) \otimes \mathrm{End}(V_\delta) \longrightarrow \mathrm{End}\big((V_\rho)_{(\delta)}\big)$ be defined by $p_\rho(S \otimes T) = S\Delta(T)$, see Lemma 4.8.

We use Proposition 2.5, by changing ρ by ρ' and taking into account that $\chi_{\rho'} = \overline{\chi}_\rho$, to obtain that $\rho : V_\rho \otimes V'_\rho \to \mathrm{End}(V_\rho)$ is an algebra isomorphism, and that $\rho\big((V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}\big) \subseteq \mathrm{End}((V_\rho)_{(\delta)})$, because

$$\rho\left(\frac{1}{|K|^2}\overline{\chi}_\delta * f * \overline{\chi}_\delta\right) = \frac{1}{|K|^2}\rho(\overline{\chi}_\delta)\rho(f)\rho(\overline{\chi}_\delta) = P_{\rho,\delta}\rho(f)P_{\rho,\delta},$$

since $\frac{1}{|K|}\rho(\overline{\chi}_\delta) = P_{\rho,\delta}$. Moreover $\dim\big((V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}\big) = \dim\big(\mathrm{End}((V_\rho)_{(\delta)})\big)$, therefore

$$\rho : (V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')} \longrightarrow \mathrm{End}((V_\rho)_{(\delta)}) \tag{20}$$

is an algebra isomorphism. By the same reasoning, $\delta : A_\delta[K] \to \mathrm{End}(V_\delta)$ is an algebra isomorphism.

Now we prove that the following diagram of algebras is commutative,

$$\begin{array}{ccc} \big((V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}\big)^K \otimes A_\delta[K] & \xrightarrow{m_\rho} & (V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')} \\ {\scriptstyle \rho\otimes\delta}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathrm{End}_K\big((V_\rho)_{(\delta)}\big) \otimes \mathrm{End}(V_\delta) & \xrightarrow{p_\rho} & \mathrm{End}\big((V_\rho)_{(\delta)}\big), \end{array}$$

where the vertical arrows are isomorphisms of algebras. We get

$$\rho(a) = \sum_{k\in K} a(k)\rho(k) = \sum_{k\in K} a(k)\Delta(\delta(k)) = \Delta(\delta(a)). \tag{21}$$

Hence, on one hand, we have

$$\rho(m_\rho(f \otimes a)) = \rho(f * a) = \rho(f)\rho(a) = \rho(f)\Delta(\delta(a)),$$

and on the other we get

$$p_\rho((\rho \otimes \delta)(f \otimes a)) = p_\rho(\rho(f) \otimes \delta(a)) = \rho(f)\Delta(\delta(a)).$$

This proves that the diagram is commutative.

By Lemma 4.8, $p_\rho : \mathrm{End}_K\big((V_\rho)_{(\delta)}\big) \otimes \mathrm{End}(V_\delta) \to \mathrm{End}\big((V_\rho)_{(\delta)}\big)$ is an isomorphism, hence

$$m_\rho : \big((V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}\big)^K \otimes A_\delta[K] \to (V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}$$

is a surjective isomorphism for all $\rho \in \hat{G}$. Therefore $m : A_\delta^K[G] \otimes A_\delta[K] \to A_\delta[G]$ is a surjective isomorphism of algebras, completing the proof of the proposition. □

Remark 4.10 For $\delta = 1$, since $A[K]$ can be viewed as a subalgebra of $A[G]$, it follows that $A_1[K] \leq A_1[G] = A[G]^{K\times K}$. Therefore $A_1[K] \leq A[K]^{K\times K} = \mathbb{C}$ (the constant functions). Hence the proof of $A_\delta^K[G] \otimes A_\delta[K] \simeq A_\delta[G]$ is trivial when $\delta = 1$, in fact it reduces to: $A_1^K[G] \otimes A_1[K] \simeq A_1^K[G] \otimes \mathbb{C} \simeq A_1[G]$.

Corollary 4.11 *For $\delta \in \hat{K}$, $A_\delta^K[G] = \bigoplus_{\rho\in\hat{G}} \mathrm{End}_K((V_\rho)_{(\delta)})$. In particular $A_\delta^K[G]$ is a semisimple algebra.*

Proof From (19) and (20) we obtain that

$$A_\delta^K[G] = \bigoplus_{\rho\in\hat{G}} \big((V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}\big)^K \simeq \bigoplus_{\rho\in\hat{G}} \big(\mathrm{End}((V_\rho)_{(\delta)})\big)^K \simeq \bigoplus_{\rho\in\hat{G}} \mathrm{End}_K((V_\rho)_{(\delta)}).$$

Then $A_\delta^K[G]$ is a direct sum of matrix algebras, which are simple. □

Given $\Phi : G \to \mathrm{End}(V)$ a spherical function of type δ of (G, K), we defined in Sect. 3 a function $\Psi : G \to \mathrm{End}_K(V)$, closely related to Φ, by

$$\Psi(g) = T\Phi(g) = \frac{1}{|K|}\sum_{k\in K} \Phi(kgk^{-1}).$$

Now, we will see that these functions Ψ are representations of the algebra $A_\delta^K[G]$.

Proposition 4.12 *Let $\Phi : G \to \mathrm{End}(V)$ be a spherical function of type δ and let $\Psi = T\Phi$. Then its extension $\Psi : A_\delta^K[G] \to \mathrm{End}_K(V)$ is a representation of $A_\delta^K[G]$. Moreover, if Φ is irreducible, then Ψ is irreducible.*

Proof If $f \in A_\delta^K[G]$ and $\pi(k) = \Phi(k)$ for $k \in K$, then

$$\begin{aligned}\Psi(f) &= \sum_{g\in G} f(g)\Psi(g) = \frac{1}{|K|}\sum_{g\in G}\sum_{k\in K} f(g)\pi(k)\Phi(g)\pi(k^{-1})\\ &= \frac{1}{|K|}\sum_{g\in G}\sum_{k\in K} f(g)\Phi(kgk^{-1})\\ &= \frac{1}{|K|}\sum_{g\in G}\sum_{k\in K} f(k^{-1}gk)\Phi(g) = \sum_{g\in G} f(g)\Phi(g) = \Phi(f)\end{aligned}$$

and

$$\Phi(f) = \sum_{g\in G} f(kgk^{-1})\Phi(g) = \sum_{g\in G} f(g)\Phi(k^{-1}gk) = \pi(k^{-1})\Phi(f)\pi(k) \in \operatorname{End}_K(V).$$

Since Φ is a representation of $A_\delta[G]$ and $\operatorname{End}_K(V)$ is a matrix algebra because $V = V_\delta \oplus \cdots \oplus V_\delta$, the first assertion follows.

Now we want to prove that if Φ is irreducible, then Ψ is an irreducible representation of $A_\delta^K[G]$. By Burnside theorem, it is enough to prove that $\Psi : A_\delta^K[G] \to \operatorname{End}_K(V)$ is surjective.

Given $M \in \operatorname{End}_K(V)$ let $f \in A_\delta[G]$ be such that $M = \Phi(f)$. Recall that if $f \in A_\delta[G]$, then $f^\circ \in A_\delta^K[G]$ where $f^\circ(g) = \frac{1}{|K|}\sum_{k\in K} f(kgk^{-1})$. Now we observe that

$$\begin{aligned}\Psi(f^\circ) = \Phi(f^\circ) &= \frac{1}{|K|}\sum_{g\in G}\sum_{k\in K} f(g)\Phi(kgk^{-1})\\ &= \frac{1}{|K|}\sum_{k\in K}\pi(k)\Phi(f)\pi(k^{-1}) = M,\end{aligned}$$

since $\Phi(f) = M \in \operatorname{End}_K(V)$. Hence $M = \Psi(f^\circ)$ with $f^\circ \in A_\delta^K[G]$ and therefore Ψ is an irreducible representation of $A_\delta^K[G]$. This completes the proof of the lemma. □

We observe that in the proof of the previous proposition we obtained

$$\Phi(f) = \Psi(f), \qquad \text{for all } f \in A_\delta^K[G]. \tag{22}$$

With the notation of Proposition 4.9, we have that the following diagram is commutative:

$$\begin{array}{ccc} A_\delta^K[G] \otimes A_\delta[K] & \xrightarrow{\ m\ } & A_\delta[G] \\ {\scriptstyle \Psi\otimes\delta}\big\downarrow & & \big\downarrow{\scriptstyle \Phi} \\ \operatorname{End}_K(V) \otimes \operatorname{End}(V_\delta) & \xrightarrow{\ p\ } & \operatorname{End}(V) \end{array}. \tag{23}$$

In fact, if $f \in A_\delta^K[G]$ and $a \in A_\delta[K]$ then

$$\begin{aligned}\Phi(f * a) &= \sum_{g\in G} \Phi(g)(f * a)(g) = \sum_{g\in G}\sum_{k\in K} \Phi(gk^{-1}) f(g) a(k^{-1}) \\ &= \sum_{g\in G} \Phi(g) f(g) \sum_{k\in K} a(k)\pi(k) = \Phi(f)\pi(a).\end{aligned}$$

From (21) we obtain $\pi(a) = \sum_{k\in K} a(k)\pi(k) = \sum_{k\in K} a(k)\Delta(\delta(k)) = \Delta(\delta(a))$. Thus,

$$\Phi(f * a) = \Phi(f)\Delta(\delta(a)) = \Psi(f)\Delta(\delta(a)) = p\big(\Psi(f) \otimes \delta(a)\big).$$

We also have that Ψ, δ, and Φ are irreducible representations of algebras.

Next, we characterize the irreducible representations of the algebra $A_\delta^K[G]$.

Theorem 4.13 *The irreducible representations of the algebra $A_\delta^K[G]$ are, up to equivalence, the extensions of the functions $\Psi = T\Phi$, for irreducible spherical functions Φ of type δ. Moreover, if $f \in A_\delta^K[G]$ and $\Psi(f) = 0$ for all Ψ, then $f = 0$.*

Proof From Proposition 4.12 we have that Ψ is an irreducible representation of $A_\delta^K[G]$ for $\Psi = T\Phi$ and $\Phi : G \to \mathrm{End}(V)$ an irreducible spherical function of type δ.

Let M be a finite dimensional irreducible representation of $A_\delta^K[G]$, without lost of generality we may assume that M is a matrix representation $M : A_\delta^K[G] \to M(p, \mathbb{C})$. Let us consider the K-module $V = V_\delta \oplus \cdots \oplus V_\delta$, sum of p-copies of V_δ. Then, by Schur's Lemma, $M(p, \mathbb{C}))$ can be identify with $\mathrm{End}_K(V)$.

With the previous notation we define $L = p(M \otimes \delta)m^{-1}$ (see diagram (23)). Since δ is an irreducible representation of $A[K]$ we obtain that $M \otimes \delta$ is an irreducible representation of $A_\delta^K[G] \otimes A_\delta[K]$. Thus L is an irreducible representation of $A_\delta[G]$. Therefore, from Theorem 4.4 we know that $L = \Phi$ for an irreducible spherical function of type δ, and it is easy to check that $\Psi = M$.

To prove the last assertion let $0 \neq f \in A_\delta^K[G]$. From (19) we know that $A_\delta^K[G] = \bigoplus_{\rho\in\hat{G}} \big((V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}\big)^K$ and write $f = \sum_{\rho\in\hat{G}} f_\rho$ with $f_\rho \in \big((V'_\rho)_{(\delta)} \otimes (V_\rho)_{(\delta')}\big)^K$. By hypothesis $f_{\sigma'} \neq 0$ for some $\sigma \in \hat{G}$. Let Ψ be the spherical function of type δ' associated to σ. If $\{v_i\}$ is an orthonormal basis of $\big(V_\sigma\big)_{(\delta')}$, then $f_{\sigma'}(g) = \sum_{i,j} a_{i,j}\langle\sigma'(g)v_i, v_j\rangle = \sum_{i,j} a_{i,j}\overline{\langle\sigma(g)v_i, v_j\rangle}$. By recalling that $\Psi(f) = \Phi(f)$, from the orthogonality relations we get

$$\begin{aligned}\langle\Psi(f)v_s, v_r\rangle &= \sum_{g\in G,\rho\in\hat{G}} f_\rho(g)\langle\Phi(g)v_s, v_r\rangle = \sum_{g\in G,\rho\in\hat{G}} f_\rho(g)\langle\sigma(g)v_s, v_r\rangle \\ &= \sum_{i,j} a_{i,j} \sum_{g\in G} \overline{\langle\sigma(g)v_i, v_j\rangle}\langle\sigma(g)v_s, v_r\rangle = \frac{|G|}{d(\delta)} a_{s,r}.\end{aligned}$$

Therefore $\Psi(f) \neq 0$. This completes the proof of the proposition. □

Proposition 4.14 *For any (G, K) the following isomorphism of algebras holds*

$$A[G]^K = \bigoplus_{\delta \in \hat{K}} A_\delta^K[G]. \tag{24}$$

The corresponding projection $Q_\delta : A[G]^K \to A_\delta^K[G]$ is given by $Q_\delta(f) = \frac{1}{|K|} f * \bar{\chi}_\delta$ and it is an algebra homomorphism.

Proof The first statement follows from Theorem 2.3 and (19). We have that $Q_\delta(f) = \frac{1}{|K|} f * \bar{\chi}_\delta$ is the projection because $\overline{\chi}_\delta * \overline{\chi}_\delta = |K| \overline{\chi}_\delta$ and $\overline{\chi}_\delta * \overline{\chi}_\sigma = 0$ if $\delta \neq \sigma$. We also have that $f * \bar{\chi}_\delta = \bar{\chi}_\delta * f$, for all K-central functions f. Thus

$$Q_\delta(f * h) = \frac{1}{|K|}(f * h) * \bar{\chi}_\delta = \frac{1}{|K|^2} f * \bar{\chi}_\delta * h * \bar{\chi}_\delta = Q_\delta(f) * Q_\delta(h),$$

and this completes the proof of the proposition. □

Theorem 4.15 *For any (G, K), let $Q_\delta : A[G]^K \to A_\delta^K[G]$ be the projection onto $A_\delta^K[G]$. The irreducible representations L of $A[G]^K$ are precisely those of the form $L = \Phi \circ Q_\delta$, where Φ is the extension of an irreducible spherical function Φ of type δ.*

Proof The proof follows at once from Theorem 4.13, since $\Psi(f) = \Phi(f)$ for all $f \in A_\delta^K[G]$. □

Proposition 4.16 *The following properties are equivalent:*

(i) *$A_\delta^K[G]$ is commutative.*
(ii) *Every irreducible spherical function of type δ is of height one.*
(iii) *$A_\delta^K[G]$ is the center of $A_\delta[G]$.*

Let us recall that the height of a spherical function $\Phi : G \to \mathrm{End}(V)$ of type $\delta \in \hat{K}$ is the multiplicity of δ in the representation $k \mapsto \Phi(k)$. Note that height one means that V is an irreducible K-module and $\dim \mathrm{End}_K(V) = 1$.

Proof (i)$\Longleftrightarrow$(ii). Let $\Phi : G \to \mathrm{End}(V)$ be an irreducible spherical function of type δ. Thus $\Psi : A_\delta^K[G] \to \mathrm{End}_K(V)$ is an irreducible representation of the $A_\delta^K[G]$.

If $A_\delta^K[G]$ is a commutative algebra then every finite dimensional irreducible representation of $A_\delta^K[G]$ is one dimensional and this implies that Φ is of height one.

On the other hand, if every irreducible spherical function of type δ is of height one, then all irreducible representations of $A_\delta^K[G]$ are one dimensional, and they separate points of $A_\delta^K[G]$. Therefore (i) holds.

(ii) $\Longrightarrow$ (iii). Let $f \in A_\delta^K[G]$ and $h \in A_\delta[G]$. For any irreducible spherical function Φ of type δ and height one we have that $\Phi(f) = \Psi(f)$ is a scalar. Then we have

$$\Phi(f * h) = \Phi(f)\Phi(h) = \Phi(h)\Phi(f) = \Phi(h * f).$$

Therefore $f * h = h * f$, i.e. $A_\delta^K[G]$ is contained in the center of $A_\delta[G]$.

For $f \in A[G]$ the projection of $A[G]$ onto $A^K[G]$ is given by $f \mapsto f^\circ$, with $f^\circ(g) = \frac{1}{|K|}\sum_{k\in K} f(kgk^{-1})$.

Let f be in the center of $A_\delta[G]$, then $\Phi(f)$ is a scalar for every irreducible spherical function Φ of type δ. Hence

$$\Phi(f^\circ) = \frac{1}{|K|}\sum_{g\in G}\sum_{k\in K}\Phi(g) f(kgk^{-1}) = \frac{1}{|K|}\sum_{k\in K}\Phi(k^{-1})\Phi(f)\Phi(k) = \Phi(f),$$

which proves that $f^\circ = f$. Therefore $f \in A_\delta^K[G]$.

(iii) $\Longrightarrow$ (i) is trivial and this completes the proof of the proposition. □

Theorem 4.17 *If $A_\delta^K[G]$ is commutative, then the irreducible spherical functions on G of type δ are in a one to one correspondence with the one dimensional representations β of $A[G]^K$ such that $\beta(\bar{\chi}_\delta) \neq 0$ and $\beta(\bar{\chi}_\sigma) = 0$ for all $\delta \neq \sigma \in \hat{K}$.*

Proof We have seen that there is a one to one correspondence between irreducible spherical functions on G of type δ and the irreducible representations L of the algebra $A_\delta^K[G]$, which in this case are one dimensional.

Hence, given L an irreducible representation of $A_\delta^K[G]$, we have that $\beta = L \circ Q_\delta$: $A[G]^K \to \mathbb{C}$ is a one dimensional representation of $A[G]^K$ that satisfies $\beta(\bar{\chi}_\sigma) = 0$, for all $\sigma \neq \delta$ and $\beta(\bar{\chi}_\delta) = |K|$, because $\frac{1}{|K|}\bar{\chi}_\delta$ is the identity of $A_\delta^K[G]$.

Conversely, any one dimensional representation of $A[G]^K$ is of the form $\beta = L \circ Q_{\delta'}$, for some one dimensional representation L of $A_{\delta'}^K[G]$. If $\delta' \neq \delta$ we get $|K| = \beta(\bar{\chi}_\delta) = L(Q_{\delta'}(\bar{\chi}_\delta)) = 0$, which is a contradiction and this completes the proof. □

5 Spherical Functions and Strong Gelfand Pairs

Let K be a subgroup of a finite group G. We say that (G, K) is a *Gelfand pair* if the algebra $A[G]^{K\times K}$ of bi-K-invariant functions on G is commutative, while it is a *strong Gelfand pair* if the algebra of the K-central functions $A[G]^K$ is commutative.

A representation (V_ρ, ρ) of a group G is *multiplicity-free* if all irreducible subrepresentations are pairwise nonequivalent. A subgroup K of G is a *multiplicity-free subgroup* of G if for every $\rho \in \hat{G}$ the restriction $\mathrm{Res}_K^G \rho$ is multiplicity free.

Now we give a useful characterization of finite Gelfand pairs.

Theorem 5.1 *The following conditions are equivalent:*

(i) *(G, K) is a Gelfand pair, i.e, the algebra $A[G]^{K\times K}$ is commutative.*
(ii) *The G-module $A[G/K]$ is multiplicity free.*
(iii) *For any $\rho \in \hat{G}$, $\mathrm{Res}_K^G \rho$ contains the trivial representation of K at most once.*

Proof The equivalence between (i) and (ii) is contained in [4], Theorem 4.4.2.

Let (V, ρ) be an irreducible representation of G. The trivial representation of K is contained in ρ as many times as the dimension of the K-invariants vectors in V,

$$V^K = \{v \in V : \rho(k)v = v, \text{ for all } k \in K\}.$$

In [4], Theorem 4.6.2, it is proved that (G, K) is a Gelfand pair if and only if $\dim V^K \leq 1$, for all irreducible representations (V, ρ) of G. ☐

Theorem 5.2 *The following conditions are equivalent:*

(i) *(G, K) is a strong Gelfand pair.*
(ii) *$(G \times K, \tilde{K})$ is a Gelfand pair, where $\tilde{K} = \{(\sigma, \sigma) : \sigma \in K\}$.*
(iii) *K is a multiplicity free subgroup of G.*

Proof We start by proving that (ii) and (iii) are equivalent. Let us consider $A[G]$ as a $G \times K$-module with the action

$$((g, k)f)(x) = f(g^{-1}xk), \quad \text{for all} \quad g, x \in G,\ k \in K,\ f \in A[G].$$

The decomposition of $A[G]$ into irreducible $G \times K$-modules follows from the Peter-Weyl theorem (Theorem 2.1):

$$A[G] = \bigoplus_{\rho \in \hat{G}} V'_\rho \otimes V_\rho = \bigoplus_{\rho \in \hat{G}} \bigoplus_{\sigma \in \hat{K}} \langle \mathrm{Res}^G_K \rho, \sigma \rangle V'_\rho \otimes V_\sigma = \bigoplus_{\rho \in \hat{G}} \bigoplus_{\sigma \in \hat{K}} \mathrm{End}(V_\sigma, V_\rho).$$

Therefore

$$A[G]^K = \bigoplus_{\rho \in \hat{G}} \bigoplus_{\sigma \in \hat{K}} \mathrm{End}_K(V_\sigma, V_\rho).$$

Moreover, this equality is indeed an isomorphism of algebras. Now it is obvious that $A[G]^K$ is commutative if and only if $\langle \rho, \sigma \rangle \leq 1$ for all $\rho \in \hat{G}$ and all $\sigma \in \hat{K}$.

To prove that (i) is equivalent to (ii) we will show that the algebras $A[G]^K$ and $A[G \times K]^{\tilde{K} \times \tilde{K}}$ are isomorphic; recall that by definition $(G \times K, \tilde{K})$ is a Gelfand pair if $A[G \times K]^{\tilde{K} \times \tilde{K}}$ is commutative.

We start by observing that the $\tilde{K} \times \tilde{K}$-orbit of $(g, k) \in G \times K$ is the set $\{(k_1 g k_2, k_1 k k_2) : k_1, k_2 \in K\}$. Hence, any $\tilde{K} \times \tilde{K}$-orbit in $G \times K$ is the $\tilde{K} \times \tilde{K}$-orbit of an element of the form (g, e) with $g \in G$. Then, we can define a map from the set $\tilde{K} \backslash (G \times K) / \tilde{K}$ of $\tilde{K} \times \tilde{K}$-orbits in $G \times K$ into the set $C(K, G)$ of conjugacy classes of K in G by setting $\gamma : (\tilde{K} \times \tilde{K}) \cdot (g, e) \mapsto K \cdot g$. It is not difficult to see that γ is a bijection. This bijection lifts to the following map

$$\Gamma : A[G \times K]^{\tilde{K} \times \tilde{K}} \to A[G]^K, \quad (\Gamma f)(g) = |K| f(g, e).$$

It is immediate to see that Γ is a linear map into $A[G]^K$. Moreover, it is easy to check that the characteristic function of the $\tilde{K} \times \tilde{K}$-orbit of (g, e) maps to the characteristic function of the K-conjugacy class of g. Therefore, Γ is a linear isomorphism of $A[G \times K]^{\tilde{K} \times \tilde{K}}$ onto $A[G]^K$. We are only left to show that Γ is multiplicative. Let

$f_1, f_2 \in A[G \times K]^{\tilde{K} \times \tilde{K}}$. Then

$$\begin{aligned}(\Gamma(f_1 * f_2))(g) &= |K|(f_1 * f_2)(g, e) \\ &= |K| \sum_{(x,k)\in G\times K} f_1((g, e)(x, k)) f_2(x^{-1}, k^{-1}) \\ &= |K| \sum_{x\in G} \sum_{k\in K} f_1(gxk^{-1}, e) f_2(kx^{-1}, e) \\ &= |K|^2 \sum_{y\in G} f_1(gy, e) f_2(y^{-1}, e) = ((\Gamma f_1) * (\Gamma f_1))(g).\end{aligned}$$

The theorem is thus proved. □

Theorem 5.3 *Let (G, K) be a strong Gelfand pair, $\mathcal{F}$ the set of all equivalence classes of irreducible spherical functions of (G, K) and $\mathcal{C}(K, G)$ the set of all K-conjugacy classes in G. Then*

$$|\mathcal{F}| = |\mathcal{C}(K, G)|.$$

Proof The characteristic functions of K-conjugacy classes in G form a basis of $A[G]^K$. Then $\dim A^K[G] = |\mathcal{C}(K, G)|$. The set $\mathcal{F}$ is the (disjoint) union of the sets $\mathcal{F}(\delta)$, of all equivalence classes of irreducible spherical functions of type δ, which are in a one to one correspondence with the equivalence classes of irreducible representations of $A^K_\delta[G]$, by Propositions 4.13 and 4.6. From Corollary 4.11 we know that

$$A^K_\delta[G] = \bigoplus_{\rho\in\hat{G}} \mathrm{End}_K((V_\rho)_{(\delta)}).$$

Since K is a multiplicity-free subgroup of G, the representation δ appears in ρ at most once and, by Schur's lemma, we get $\mathrm{End}_K((V_\rho)_{(\delta)} = \mathbb{C}$ if and only if $\langle\rho, \delta\rangle = 1$. Thus $\dim A^K_\delta[G] = |\{\rho \in \hat{G} : \langle\rho, \delta\rangle \geq 1\}|$ because a simple algebra has only one equivalence class of irreducible representations. Therefore

$$|\mathcal{C}(K, G)| = \dim A[G]^K = \sum_{\delta\in\hat{K}} \dim A^K_\delta[G] = \sum_{\delta\in\hat{K}} |\mathcal{F}(\delta)| = |\mathcal{F}|. \tag{25}$$

The theorem is proved. □

Proposition 5.4 *If (G, K) is a strong Gelfand pair, then the set of representations of $A[G]^K$ which are extensions of $\Phi \in \mathcal{F}$ is a basis of the dual of the complex linear space $A[G]^K$.*

Proof For $\Phi \in \mathcal{F}$, we have that Φ is a one dimensional representation of the algebra $A[G]^K$ by Proposition 4.15. Taking also into account Theorem 5.3 we have

$$|\{\Phi : A[G]^K \to \mathbb{C} : \Phi \in \mathcal{F}\}| = |\mathcal{F}| = \dim A[G]^K.$$

Thus we only have to prove that the set $\{\Phi : A[G]^K \to \mathbb{C} : \Phi \in \mathcal{F}\}$ is linearly independent.

Let $\{\Phi : A[G]^K \to \mathbb{C} : \Phi \in \mathcal{F}\} = \{\Phi_1, \Phi_2, \dots, \Phi_m\}$ and take $h \in A[G]^K$ such that $\Phi_i(h) \neq \Phi_j(h)$ for all $i \neq j$ (choose h in the complement of the union of the hyperplanes $\ker(\Phi_i - \Phi_j)$ with $i \neq j$). If $\sum_{1 \le j \le m} a_j \Phi_j = 0$, then for $0 \le i \le m-1$

$$\sum_{1 \le j \le m} a_j \Phi_j(h^i) = \sum_{1 \le j \le m} a_j \Phi_j(h)^i = 0$$

is a system of m linear equations in m unknowns a_j's. The coefficient matrix is

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \Phi_1(h) & \Phi_2(h) & \cdots & \Phi_m(h) \\ \Phi_1(h)^2 & \Phi_2(h)^2 & \cdots & \Phi_m(h)^2 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \Phi_1(h)^{m-1} & \Phi_2(h)^{m-1} & \cdots & \Phi_1(h)^{m-1} \end{pmatrix},$$

which is a nonsingular Vandermonde matrix. Therefore $a_j = 0$ for all $1 \le j \le m$. This completes the proof of the proposition. □

5.1 The Pair $(G, K) = (\mathfrak{S}_n, \mathfrak{S}_{n-m} \times \mathfrak{S}_m)$

Let $G = \mathfrak{S}_n$ denote the group of all permutations of the set $\{1, \dots, n\}$ and $K = \mathfrak{S}_{n-m} \times \mathfrak{S}_m$, where $\mathfrak{S}_{n-m}$ and $\mathfrak{S}_m$ are, respectively, the subgroups of G of all permutations of $\{1, \dots, n-m\}$ and of $\{n-m+1, \dots, n\}$, for $1 \le m \le n-m$. Let (1) denote the identity permutation and (i, j) be the transposition of the elements i and j.

The group G acts transitively on the set X of all subsets of $n-m$ elements of $\{1, \dots, n\}$. The isotropy subgroup at $O_0 = \{1, \dots, n-m\} \in X$ is K. Thus $X = G/K$ is the set of all subsets of $n-m$ elements of $\{1, \dots, n\}$. The K-orbits in X are the subsets of all sets with the same number of elements as in O_0. Thus X is the disjoint union of its K-orbits, $X = K \cdot O_0 \cup \cdots \cup K \cdot O_m$, where

$$O_r = \{1, \dots, n-m-r\} \cup \{n-m+1, \dots, n-m+r\}, \qquad 1 \le r \le m.$$

Let us also introduce the subset $A = \{x_0, \dots, x_m\}$ of G, where the finite sequence $\{x_i\}$ is defined inductively by:

$$x_0 = (1), \quad \text{and} \quad x_i = (n-m-i+1, n-m+i)x_{i-1}, \quad \text{for } 1 \le i \le m.$$

Observe that A is a commuting set of permutations and $a^2 = 1$, for all $a \in A$. We get the following relation between the K-orbits $O_r = x_r O_0$.

A very important property of this set A is that

$$G = KAK. \tag{26}$$

In fact, if $p : G \longrightarrow X$ denotes the projection map then for any $g \in G$ there exists $0 \leq j \leq m$ such that $p(g) \in K \cdot O_j$. Since $O_j = x_j O_0$ we get $p(g) \in Kx_j \cdot O_0$ and thus $g \in Kx_j K$.

Lemma 5.5 (Gelfand's Lemma, cf. Section 4.3 in [3]) *Let G is a finite group and let K be a subgroup. Suppose there exists an automorphism τ of G such that $g^{-1} \in K\tau(g)K$ for all $g \in G$. Then (G, K) is a Gelfand pair.*

Proof We observe that if $f \in A[G]^{K\times K}$ we have that $f(\tau(g)) = f(g^{-1})$ for all $g \in G$. Then, for $f_1, f_2 \in A[G]^{K\times K}$ and $g \in G$ we get

$$\begin{aligned}(f_1 * f_2)(\tau(g)) &= \sum_{h\in G} f_1(\tau(gh) f_2(\tau(h^{-1})) = \sum_{h\in G} f_1((gh)^{-1}) f_2(h) \\ &= \sum_{h\in G} f_2(h) f_1(h^{-1}g^{-1}) = (f_2 * f_1)(g^{-1}) = (f_2 * f_1)(\tau(g)).\end{aligned}$$

Therefore $A[G]^{K\times K}$ is commutative. □

Proposition 5.6 *For $1 \leq m \leq n - m$ we have*

i) $(\mathfrak{S}_n, \mathfrak{S}_{n-m} \times \mathfrak{S}_m)$ *and* $(\mathfrak{S}_n, \mathfrak{S}_m \times \mathfrak{S}_{n-m})$ *are Gelfand pairs, for all m.*
ii) $(\mathfrak{S}_n, \mathfrak{S}_{n-m} \times \mathfrak{S}_m)$ *and* $(\mathfrak{S}_n, \mathfrak{S}_m \times \mathfrak{S}_{n-m})$ *are strong Gelfand pairs if and only if $m = 1$ or $m = 2$.*

Proof From (26), if $g = k_1 a k_2 \in G$ with $k_1, k_2 \in K$, $a \in A$, then

$$g^{-1} = k_2^{-1} a^{-1} k_1^{-1} = (k_1 k_2)^{-1} k_1 a k_2 (k_1 k_2)^{-1} \in KgK.$$

Hence, by Lemma 5.5 we have that (G, K) is a Gelfand pair.

On the other hand, it is known that the irreducible representations of $\mathfrak{S}_n$ are in a one to one correspondence with the Young diagrams of n elements. In terms of this parameterization if V_ν denotes the irreducible $\mathfrak{S}_n$-module associated to the Young diagram ν, Pieri's formula says that $\mathrm{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} V_\nu = \sum_\lambda V_\lambda$, the sum over all diagrams obtained from ν by removing one box (see [9] p. 58, also Section 3.5.3 in [4]). As a consequence of this branching rule, it is not difficult to obtain that $K = \mathfrak{S}_{n-m} \times \mathfrak{S}_m$, with $1 \leq m \leq n - m$, is a multiplicity-free subgroup of $G = \mathfrak{S}_n$ if and only if $m = 1$ or $m = 2$. Hence, from Theorem 5.2 we have that $(\mathfrak{S}_n, \mathfrak{S}_{n-m} \times \mathfrak{S}_m)$ is a strong Gelfand pair. See also [1], Theorem 1.2 for a classification of all strong Gelfand pairs $(\mathfrak{S}_n, K)$. □

6 Spherical Transform

It is known that when G is a finite group the following Plancherel identity holds,

$$\sum_{g\in G} |f(g)|^2 = \frac{1}{|G|} \sum_{\rho\in\hat{G}} d(\rho)\,\mathrm{tr}(\rho(f)\rho(f)^*) \tag{27}$$

for all $f \in A[G]$. In other words, the Plancherel measure is $\mu(\rho) = d(\rho)/|G|$. This result can be generalized to matrix valued spherical functions as follows.

Proposition 6.1 *Let $\mathcal{F} = \{\Phi = \Phi_{\rho,\delta} : \rho \in \hat{G}, \delta \in \hat{K}, V_\rho \geq V_\delta\}$ be the space of all equivalence classes of irreducible spherical functions of the pair (G, K). Then*

$$\sum_{g\in G} |f(g)|^2 = \frac{1}{|G|} \sum_{\Phi=\Phi_{\rho,\delta}\in\mathcal{F}} d(\rho)\,\mathrm{tr}(\Phi(f)\Phi(f)^*) \tag{28}$$

for all $f \in A[G]^K$. (The $*$ stands for the adjoint operator defined by an inner product such that $\Phi^* = \check{\Phi}$).

Proof Since $\Phi(g)^* = \Phi(g^{-1})$, we get that $\Phi(f)^* = \Phi(\check{\bar{f}})$. For any $f \in A[G]^K$ we have that $\check{\bar{f}} \in A[G]^K$ and $\Phi(f)\Phi(f)^* = \Phi(f)\Phi(\overline{\check{f}}) = \Phi(f * \check{\bar{f}})$. By Proposition 2.3 we have

$$A[G]^K = \bigoplus_{\sigma\in\hat{G},\pi\in\hat{K}} \big((V'_\sigma)_{(\pi')} \otimes (V_\sigma)_{(\pi)}\big)^K,$$

where the summands on the right-hand side are two-sided ideals. For $f \in A[G]^K$, we let $f = \sum_{\sigma,\pi} f_{\sigma,\pi}$ with $f_{\sigma,\pi} \in \big((V'_\sigma)_{(\pi')} \otimes (V_\sigma)_{(\pi)}\big)^K$ and we get

$$f * \check{\bar{f}} = \sum_{\pi<\sigma} f_{\sigma,\pi} * \check{\bar{f}}_{\sigma,\pi}.$$

Therefore

$$\mathrm{tr}(\Phi(f)\Phi(f)^*) = \mathrm{tr}(\Phi(f * \check{\bar{f}}) = \sum_{\pi<\sigma} \mathrm{tr}(\Phi(f_{\sigma,\pi} * \check{\bar{f}}_{\sigma,\pi})).$$

On the other hand, since $\{f_{\sigma,\pi} : \pi \in \hat{K}, \sigma \in \hat{G}\}$ is an orthogonal set of functions we have $|f|^2 = \sum_{\sigma,\pi} |f_{\sigma,\pi}|^2$.

We may assume that $f \in \big((V'_\sigma)_{(\pi')} \otimes (V_\sigma)_{(\pi)}\big)^K$ for some $\sigma \in \hat{G}, \pi \in \hat{K}, \pi < \sigma$. Let $\{v_r\}$ be an orthonormal basis of $(V_\sigma)_{(\pi)}$ and $\{\lambda_r\}$ the dual basis of $(V'_\sigma)_{(\pi')}$. The functions $\lambda_r \otimes v_s$ in $A[G]$ are the matrix coefficients of the representation σ

$$c^\sigma_{r,s}(g) = (\lambda_r \otimes v_s)(g) = \langle\sigma(g)v_s, v_r\rangle. \tag{29}$$

It is easy to verify that $c^{\sigma}_{r,s} = \overline{\check{c^{\sigma}_{s,r}}}$ and $c^{\sigma'}_{r,s} = \overline{c^{\sigma}_{r,s}}$, where σ' is the contragradient representation of σ. By using the Schur orthogonality relations we also get $c^{\sigma}_{i,j} * c^{\sigma}_{r,s} = \frac{|G|}{d(\sigma)}\delta_{j,r}\, c^{\sigma}_{i,s}$.

Let $f = \sum_{r,s} a_{r,s}c^{\sigma}_{r,s} \in ((V'_{\sigma})_{(\pi')} \otimes (V_{\sigma})_{(\pi)})^K$. Then $\check{\bar{f}} = \sum_{r,s} \bar{a}_{r,s}c^{\sigma}_{s,r}$ and $f * \check{\bar{f}} = \frac{|G|}{d(\sigma)} \sum_{r,s,p} a_{rs}\bar{a}_{ps}c^{\sigma}_{r,p}$. Hence

$$\Phi(f * \check{\bar{f}}) = \frac{|G|}{d(\sigma)} \sum_{r,s,p} a_{rs}\bar{a}_{ps} \sum_{g\in G} c^{\sigma}_{r,p}(g)\Phi(g).$$

If $\Phi = \Phi_{\rho,\delta}$, with $\rho \in \hat{G}, \delta \in \hat{K}$ and $V_\rho > V_\delta$, then $\Phi(g)u_i = \sum_j \langle \rho(g)u_i, u_j\rangle u_j$, where $\{u_i\}$ is an orthonormal basis of $(V_\rho)_{(\delta)}$ (cf. Theorem 3.13). Thus,

$$\Phi(f * \check{\bar{f}})u_i = \frac{|G|}{d(\sigma)} \sum_{r,s,p} a_{rs}\bar{a}_{ps} \sum_{j} \Big(\sum_{g\in G} \overline{c^{\sigma'}_{r,p}(g)} c^{\rho}_{j,i}(g) \Big) u_j.$$

By the Schur orthogonality relations, for all $(\rho, \delta) \neq (\sigma', \pi')$, we get $\Phi(f * \check{\bar{f}}) = 0$. For $(\rho, \delta) = (\sigma', \pi')$ we may assume that $u_i = v_i$ for all i. Then

$$\mathrm{tr}(\Phi(f * \check{\bar{f}})) = \frac{|G|}{d(\sigma)} \sum_{i,r,s,p} a_{r,s}\bar{a}_{p,s} \sum_{g\in G} c^{\sigma}_{r,p}(g)\overline{c^{\sigma}_{i,i}}(g) = \frac{|G|^2}{d(\sigma)^2} \sum_{r,s} a_{r,s}\bar{a}_{r,s}$$

By the Schur orthogonality relations, we get $\sum_{g\in G} |f(g)|^2 = \frac{|G|}{d(\sigma)} \sum_{r,s} |a_{r,s}|^2$. Thus

$$\mathrm{tr}(\Phi(f)\Phi(f)^*) = \mathrm{tr}(\Phi(f * \check{\bar{f}})) = \frac{|G|}{d(\sigma)} \sum_{g\in G} |f(g)|^2,$$

for any $f \in ((V'_{\sigma})_{(\pi')} \otimes (V_{\sigma})_{(\pi)})^K$. This completes the proof of the proposition. □

Proposition 6.2 *(Inverse spherical transform) Any $f \in A[G]^K$ can be recovered from its spherical transform $\hat{f}(\Phi) = \Phi(f)$ by*

$$f(g) = \frac{1}{|G|} \sum_{\Phi=\Phi_{\rho,\delta}\in\mathcal{F}} d(\rho)\,\mathrm{tr}\left(\Phi(g^{-1})\Phi(f)\right).$$

Proof Let $f \in \left(V'_{\sigma})_{(\pi')} \otimes (V_{\sigma})_{(\pi)}\right)^K$, and $\Phi = \Phi_{\rho,\delta}$, with $\sigma, \rho \in \hat{G}, \pi, \delta \in \hat{K}, \pi < \sigma$ and $\delta < \rho$. With the same notation as in Proposition 6.1 we have

$$f(g) = \sum_{r,s} a_{r,s}c^{\sigma}_{r,s}(g) = \sum_{r,s} a_{r,s}\langle \sigma(g)v_s, v_r\rangle.$$

We also have $\Phi(g)u_i = \sum_j \langle \rho(g)u_i, u_j\rangle u_j$, where $\{u_i\}$ is an orthonormal basis of $(V_\rho)_{(\delta)}$. Then $\Phi(f)v_i = \sum_{j,r,s} a_{r,s} v_j \sum_{g\in G} \langle \sigma(g)v_s, v_r\rangle\langle \rho(g)u_i, u_j\rangle$. Since $\langle \rho(g)u_i, u_j\rangle = c^{\sigma}_{j,i} = \overline{c^{\sigma'}}_{j,i}$, by the Schur orthogonality relations it follows that $\Phi(f) = 0$ for all $(\pi, \sigma) \neq (\delta', \rho')$. In the case $(\pi, \sigma) = (\delta', \rho')$, we may assume $u_s = v_s$ for all s and we obtain $\Phi(f)u_i = \frac{|G|}{d(\rho)} \sum_j a_{i,j} v_j$. Now we compute

$$\Phi(g^{-1})\Phi(f)u_i = \frac{|G|}{d(\rho)} \sum_{j,k} a_{i,j} \langle \rho(g^{-1})u_j, u_k\rangle u_k,$$

and

$$\operatorname{tr}\left(\Phi(g^{-1})\Phi(f)\right) = \frac{|G|}{d(\rho)} \sum_{i,j} a_{i,j} \langle \rho(g^{-1})u_j, u_i\rangle = \frac{|G|}{d(\rho)} \sum_{i,j} a_{i,j} c^{\rho'}_{i,j}(g) = \frac{|G|}{d(\rho)} f(g).$$

The general case, for any $f = \sum_{\pi<\sigma} f_{\sigma,\pi} \in \bigoplus_{\sigma,\pi} \left(V'_\sigma)_{(\pi')} \otimes (V_\sigma)_{(\pi)}\right)^K$, follows easily by linearity. ◻

In the particular case when $K = \{e\}$ is the trivial subgroup of G, the irreducible spherical functions of G are exactly the irreducible representations of G, i.e. $\Phi_{\rho,1} = \rho$. Hence we get the classical Fourier's inversion formula, as a corollary of Proposition 6.2.

Corollary 6.3 (Inverse Fourier transform) *Any $f \in A[G]$ can be recovered from its Fourier transform $\hat{f}(\rho) = \rho(f)$ by*

$$f(g) = \frac{1}{|G|} \sum_{\rho\in\hat{G}} d(\rho) \operatorname{tr}\left(\rho(g^{-1})\hat{f}(\rho)\right).$$

7 Appendix: Proof of Theorem 3.13

Theorem 7.1 *Let (V, ρ) be a finite dimensional irreducible representation of G that contains the K-type δ. Then $\Phi(g) = P_\delta\rho(g)P_\delta$, $g \in G$, is an irreducible spherical function of G of type δ. Conversely, any irreducible spherical function of the pair (G, K) is of this form.*

Proof The first part was proved in Theorem 3.13. Now let $\Phi : G \to \operatorname{End}(V)$ be an irreducible spherical function of type δ and let L be a maximal left ideal in $\operatorname{End}(V)$. If

$$I = \{f \in A_\delta[G] : \text{such that } \Phi(f) \in L\},$$

then I is a maximal left ideal in $A_\delta[G]$. In fact, $\Phi : A_\delta[G] \to \operatorname{End}(V)$ is a surjective homomorphism, see Theorem 4.4. Hence $I = \Phi^{-1}(L)$ is a left ideal in $A_\delta[G]$.

Moreover $A_\delta[G]/I$ and $\mathrm{End}(V)/L$ are left $A_\delta[G]$-modules by defining

$$h \cdot (f + I) = h * f + I, \quad h \cdot (T + L) = \Phi(h)T + L$$

for all $h, \ f \in A_\delta[G]$ and $T \in \mathrm{End}(V)$. Then the linear map $\mu : A_\delta[G]/I \to \mathrm{End}(V)/L$ defined by $\mu(f + I) = \Phi(f) + L$ is an $A_\delta[G]$-morphism.

Considering $\mathrm{End}(V)$ as left $\mathrm{End}(V)$-module it is known that $\mathrm{End}(V)$ is semisimple. Therefore there exists a left ideal L_1 such that $\mathrm{End}(V) = L \oplus L_1$. Moreover $L_1 \simeq V$ as left modules. Hence $\mathrm{End}(V)/L \simeq L_1 \simeq V$. Thus by considering V as a left $A_\delta[G]$-module by defining $f \cdot v = \Phi(f)v$ we get $A_\delta[G]/I \simeq \mathrm{End}(V)/L \simeq V$ which implies that I is a maximal left ideal of $A_\delta[G]$ since V is an irreducible $A_\delta[G]$-module.

Let

$$J = \{f \in A[G] : \bar{\chi}_\delta * h * f * \bar{\chi}_\delta \in I \text{ for every } h \in A[G]\}.$$

We will prove now that

- J is a regular maximal left ideal in $A[G]$.
- $I = J \cap A_\delta[G]$.
- $f * \bar{\chi}_\delta \equiv |K| f \mod (J)$, for all $f \in A[G]$.

The fact that J is a left ideal is obvious; that J is regular means that there exists $u \in A[G]$ such that u is a right identity of $A[G] \mod (J)$. For $f, h \in A[G]$, we have

$$\bar{\chi}_\delta * h * (f * \tfrac{1}{|K|}\bar{\chi}_\delta - f) * \bar{\chi}_\delta = 0.$$

Hence $f * \bar{\chi}_\delta \equiv |K| f \mod (J)$ and $u = \frac{1}{|K|}\bar{\chi}_\delta$ satisfies that $f * u - f \in J$, i.e. u is a right identity of $A[G] \mod (J)$.

To see that $I \subset J \cap A_\delta[G]$ let $f \in I$ and $h \in A[G]$. Thus $\bar{\chi}_\delta * h * f * \bar{\chi}_\delta = \bar{\chi}_\delta * h * \bar{\chi}_\delta * f \in I$, because $\bar{\chi}_\delta * h * \bar{\chi}_\delta \in A_\delta[G]$. Hence $f \in J \cap A_\delta[G]$.
By the maximality of I, it is enough to prove that $J \cap A_\delta[G]$ is a nontrivial ideal in $A_\delta[G]$. But we have that $\frac{1}{|K|}\bar{\chi}_\delta \notin J$, otherwise would have that

$$\bar{\chi}_\delta * h * \bar{\chi}_\delta = \bar{\chi}_\delta * h * \tfrac{1}{|K|}\bar{\chi}_\delta * \bar{\chi}_\delta \in I, \quad \text{for all } h \in A[G]$$

and hence $A_\delta[G] = I$.

In order to prove that J is maximal let N be a left ideal, $J \subseteq N \subsetneq A[G]$. If $N \cap A_\delta[G] = A_\delta[G]$ then $u = \frac{1}{K}\bar{\chi}_\delta \in N$. Since $f * u - f \in J \subseteq N$, for all $f \in A[G]$ we get that $f \in N$, that is $N = A[G]$. Therefore, by the maximality of I we see that $N \cap A_\delta[G] = I$. If $f \in N$, $f * \bar{\chi}_\delta - |K| f \in J \subseteq N$, hence $f * \bar{\chi}_\delta \in N$. Therefore

$$\bar{\chi}_\delta * h * f * \bar{\chi}_\delta = \bar{\chi}_\delta * h * f * u * \bar{\chi}_\delta \in N \cap A_\delta[G] = I \quad \text{for all } h \in A[G].$$

Thus $f \in J$ which proves that $N = J$.

The left regular representation on $A[G]$ induces a natural representation ρ of G on the space $E = A[G]/J$, by $\rho(g)(f + J) = \delta_g * f + J$, because J is a left ideal in $A[G]$.

It is easy to see that the extension of U to $A[G]$ is given by $\rho(f)(h+J) = f*h+J$. Since J is maximal, ρ is irreducible.

Let $E_{(\delta)}$ be the isotypical component of type δ in E and let $P_\delta : E \longrightarrow E$ be the orthogonal projection onto $E_{(\delta)}$. Thus

$$P_\delta(f+J) = \tfrac{1}{|K|}\sum_{k\in K}\bar{\chi}_\delta(k)\rho(k)(f+J) = \tfrac{1}{|K|}\rho(\bar{\chi}_\delta)(f+J) = \tfrac{1}{|K|}\bar{\chi}_\delta * f + J.$$

Hence $\alpha : f \mapsto f+J$ is a mapping of $A_\delta[G]$ into $E_{(\delta)}$. Since $f*\bar{\chi}_\delta \equiv |K| f \pmod{J}$ for all $f \in A[G]$ we have $P_\delta(f+J) = P_\delta\big(\frac{1}{|K|^2}\bar{\chi}_\delta * f * \bar{\chi}_\delta + J\big)$ which proves that α is onto. In this way α gives rise to the isomorphism of $A_\delta[G]$-modules $E_{(\delta)} \simeq A_\delta[G]/I$, because $I = J \cap A_\delta[G]$.

The spherical function $\Phi_1 : G \to \mathrm{End}(E_{(\delta)})$ of type δ associated to the irreducible representation ρ of G is given by $\Phi_1(g) = P_\delta\rho(g)P_\delta$. To see that Φ_1 is equivalent to Φ it is sufficient to show that the representations $\Phi : A_\delta[G] \to \mathrm{End}(V)$ and $\Phi_1 : A_\delta[G] \to \mathrm{End}(E_{(\delta)})$ are equivalent (Proposition 4.6).

We observe that if $\Phi(f) = 0$, $f \in A_\delta[G]$, then $\Phi_1(f) = 0$. In fact, if $\Phi(f) = 0$ we have that $\Phi(f*h) = 0$ for all $h \in A_\delta[G]$, which implies that $f*h \in I \le J$. Thus $\Phi_1(f)(h+J) = f*h+J = 0$.

Therefore, the well defined linear map $\Phi(f) \mapsto \Phi_1(f)$ is an algebra isomorphism of $\mathrm{End}(V)$ onto $\mathrm{End}(E_{(\delta)})$, because $E_{(\delta)} \simeq A_\delta[G]/I \simeq V$. Hence, there exists a linear isomorphism $T : V \to E_{(\delta)}$ such that $\Phi_1(f) = T\Phi(f)T^{-1}$, since any automorphism of the associative algebra $\mathrm{End}(V)$ is inner (See [7], Theorem 3.26). This completes the proof of the theorem. □

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