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The Density Theorem for Operator-Valued Frames via Square-Integrable Representations of Locally Compact Groups

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Abstract

In this paper, we first prove a density theorem for operator-valued frames via square-integrable representations restricted to closed subgroups of locally compact groups, which is a natural extension of the density theorem in classical Gabor analysis. More precisely, it is proved that for such an operator-valued frame, the index subgroup is co-compact if and only if the generator is a Hilbert–Schmidt operator. Then we present some applications of this density theorem, and in particular establish necessary and sufficient conditions for the existence of such operator-valued frames with Hilbert–Schmidt generators. We also introduce the concept of wavelet transform for Hilbert–Schmidt operators, and use it to prove that if the representation space is infinite-dimensional, then the system indexed by the entire group is Bessel system but not a frame for the space of all Hilbert–Schmidt operators on the representation space.

Keywords Density theorem · Operator-valued frame · Wavelet transform · Locally compact group · Hilbert–Schmidt operator

Mathematics Subject Classification 42C15 · 42C40 · 43A70 · 46B15 · 47B10

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1 Introduction

Gabor analysis is devoted to expanding complex signals as linear combinations of basic signals derived from a single window function by varying it in time and frequency over specific lattices. Classical harmonic analysis on locally compact abelian groups plays a crucial role in modern time–frequency analysis. Let G be a second countable, locally compact, abelian group, let $\widehat{G}$ be the Pontryagin dual group of G, and let π be the Weyl-Heisenberg representation of $G \times \widehat{G}$ [5]. Jakobsen and Lemvig [12] showed the following density theorem for Gabor frames:

Theorem 1.1 *Let Δ be a closed subgroup of $G \times \widehat{G}$ and let $g \in L^2(G)$. If the system of the form*

$$\pi(\Delta)g := \{\pi(\nu)g\}_{\nu \in \Delta} \tag{1.1}$$

is a Gabor frame for $L^2(G)$ with bounds $0 < \alpha \leq \beta < \infty$, then the following statements hold:

(i) *The index subgroup Δ is co-compact.*
(ii) $\alpha \mathrm{vol}(\Delta) \leq \|g\|^2 \leq \beta \mathrm{vol}(\Delta)$*, where the size of Δ is given by*

$$\mathrm{vol}(\Delta) := \mu_{(G\times\widehat{G})/\Delta}((G \times \widehat{G})/\Delta).$$

The more general setup is as in the following proposition, in which the essential orthogonality relation for the matrix coefficient functions of irreducible square-integrable projective unitary representations is the foundation for our deduction and this proposition will get repeated use in this paper.

Proposition 1.2 [16] *Let π be a σ-projective irreducible unitary representation of a unimodular locally compact group G on $\mathcal{H}_\pi$. Then the following are equivalent:*

(i) *There exist nonzero vectors $\xi, \eta \in \mathcal{H}_\pi$ such that $\int_G |\langle \xi, \pi(x)\eta\rangle|^2 dx < \infty$.*
(ii) *For every $\xi, \eta \in \mathcal{H}_\pi$, we have that $\int_G |\langle \xi, \pi(x)\eta\rangle|^2 dx < \infty$.*
(iii) *π is a sub-representation of the σ-twisted left regular representation of G.*

If any of the above assumptions holds, then there exists a positive number $\mathrm{d}_\pi > 0$, called the *formal dimension* of π, such that

$$\int_G \langle \xi, \pi(x)\eta\rangle \langle \pi(x)\eta', \xi'\rangle dx = \frac{\langle \xi, \xi'\rangle\langle \eta', \eta\rangle}{\mathrm{d}_\pi}$$

for all $\xi, \xi', \eta, \eta' \in \mathcal{H}_\pi$.

Representations satisfying the above equivalent conditions in Proposition 1.2 are called *square-integrable*. The formal dimension d_π is related to the Haar measure on G. In some concrete settings, we can explicitly compute it; see [17, Section 9].

Throughout this paper, we will suppose that $(\pi, \mathcal{H}_\pi)$ is a square-integrable representation of a unimodular locally compact group G with formal dimension $\mathrm{d}_\pi > 0$ and Λ is a closed subgroup of G. We point out that these groups may not be abelian and discrete. The research of spanning properties of Λ-indexed systems of the form

$$\pi(\Lambda)\eta := \{\pi(\lambda)\eta\}_{\lambda\in\Lambda}, \tag{1.2}$$

where $\eta \in \mathcal{H}_\pi$, is fundamental in some aspects of applied and computational harmonic analysis. It includes Gabor analysis, wavelet analysis and so on [1, 4, 5, 8, 12, 17]. Systems with some special structure is well worth being investigated because they have a strong influence on practical applications. The system of the form (1.1) is a special case of (1.2). Under suitable assumptions on G and π, many fundamental results, known as density theorems, give basic obstructions to the spanning properties of such systems which is closely related to density of Λ in G.

Inspired by operator-valued frames [13] and frames of the form (1.2), in this paper we mainly concentrate on the study of Λ-indexed operator-valued systems of the form

$$A\pi(\Lambda) := \{A\pi(\lambda)\}_{\lambda\in\Lambda}, \tag{1.3}$$

where A is a bounded linear operator on $\mathcal{H}_\pi$. Based on Theorem 1.1, we consider the following density problem for operator-valued frames of the form (1.3):

Problem 1.3 *Let $(\pi, \mathcal{H}_\pi)$ be a square-integrable representation of G with formal dimension $\mathrm{d}_\pi > 0$, let Λ be a closed subgroup of G and let A be a bound linear operator on $\mathcal{H}_\pi$. Suppose that $A\pi(\Lambda)$ is an operator-valued frame on $\mathcal{H}_\pi$ with bounds $0 < \alpha \le \beta < \infty$. Under what condition is the index subgroup Λ co-compact? In this case, what can we say about the density of Λ? Do we have*

$$\alpha \mathrm{vol}(\Lambda) \le \frac{\|A\|^2}{\mathrm{d}_\pi} \le \beta \mathrm{vol}(\Lambda)$$

for some appropriate norm $||A||$ of the generator A?

To the best of our knowledge, the problem in this general setting has not been considered yet in the literature. The main purpose of the paper is to address the answers to this problem. We remark the investigation of density theorems is essential in Gabor analysis; see [10] for a history. If G, Λ, π and A are replaced by $G \times \widehat{G}$, a closed subgroup of $G \times \widehat{G}$, the Weyl-Heisenberg representation and a rank-one operator on $L^2(G)$, respectively, then this problem reduces to Theorem 1.1.

This paper is organized as follows: In Sect. 2, we collect some concepts, notations and properties on Hilbert–Schmidt operators on Hilbert spaces, harmonic analysis on locally compact groups and frame theory. In Sect. 3, we prove a density theorem that says that for an operator-valued frame $A\pi(\Lambda)$ with bounds $0 < \alpha \le \beta < \infty$, the index subgroup Λ is co-compact if and only if the generator A is a Hilbert–Schmidt operator on $\mathcal{H}_\pi$, which is a main theorem in this paper. In this case, we obtain the inequalities

$$\alpha \mathrm{vol}(\Lambda) \leq \frac{\|A\|_2^2}{\mathrm{d}_\pi} \leq \beta \mathrm{vol}(\Lambda),$$

where $\|A\|_2$ is the Hilbert–Schmidt norm of A. In addition, the generator of an operator-valued Bessel system indexed by a closed co-compact subgroup is necessarily a Hilbert–Schmidt operator. Moreover, some applications of this theorem are given in this section. Especially, we establish necessary and sufficient conditions for the existence of a Hilbert–Schmidt operator A on $\mathcal{H}_\pi$ such that $A\pi(\Lambda)$ is an operator-valued frame. Finally, in Sect. 4, the concept of the wavelet transform for Hilbert–Schmidt operators is introduced to prove that the system (1.3) indexed by the entire group must be Bessel, but not a frame for the space of all Hilbert–Schmidt operators on $\mathcal{H}_\pi$.

2 Preliminaries

2.1 Hilbert–Schmidt Operators on Hilbert Spaces

Let $\{\xi_i\}_{i\in\mathbb{I}}$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$, where $\mathbb{I}$ is a finite or countable set. Denote by $\mathcal{B}(\mathcal{H})$ the space of all bound linear operators on $\mathcal{H}$. Let $A \in \mathcal{B}(\mathcal{H})$. Then the operator A is called a *Hilbert–Schmidt operator* on $\mathcal{H}$ if

$$\|A\|_2 := \left(\sum_i \|A\xi_i\|^2\right)^{\frac{1}{2}} < \infty.$$

Also, we have that $\|A^*\|_2 = \|A\|_2$, where A^* is the adjoint operator of A. We denote the class of all Hilbert–Schmidt operators on $\mathcal{H}$ by $\mathcal{S}_2(\mathcal{H})$.

It is well known that $\mathcal{S}_2(\mathcal{H})$ is a Hilbert space with the inner product

$$\langle A, B\rangle_2 := \mathrm{tr}(B^*A)$$

for all $A, B \in \mathcal{S}_2(\mathcal{H})$, where $\mathrm{tr}(\cdot)$ is the usual trace function. Moreover, $\mathcal{S}_2(\mathcal{H})$ is a two-side ideal containing the set $\mathcal{F}(\mathcal{H})$ of all finite-rank operators in $\mathcal{B}(\mathcal{H})$. In addition, $\|A\| \leq \|A\|_2$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ hold for all $A, B \in \mathcal{S}_2(\mathcal{H})$. Further, $\|AB\|_2 \leq \|A\|\|B\|_2$ holds for all $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{S}_2(\mathcal{H})$.

Fix $\eta, \xi \in \mathcal{H}$. We can define the operator $\eta \otimes \xi$ on $\mathcal{H}$ as follows:

$$(\eta \otimes \xi)\rho = \langle \rho, \xi\rangle \eta$$

for all $\rho \in \mathcal{H}$. If $\eta, \eta_1, \xi, \xi_1 \in \mathcal{H}$ and $A, B \in \mathcal{B}(\mathcal{H})$, then the following equalities hold:

$$(\eta \otimes \eta_1)(\xi \otimes \xi_1) = \langle \xi, \eta_1\rangle (\eta \otimes \xi_1),$$
$$(\eta \otimes \xi)^* = \xi \otimes \eta,$$
$$A(\eta \otimes \xi)B = (A\eta) \otimes (B^*\xi).$$

We can refer to [14] for more information about Hilbert–Schmidt operators.

2.2 Harmonic Analysis on Locally Compact Groups

In this subsection, we collect some elementary notions and properties about locally compact groups.

A *2-cocycle* (or a *multiplier*) on G is a function $\sigma : G \times G \to \mathbb{T}$ satisfying

$$\sigma(x, y)\sigma(xy, z) = \sigma(x, yz)\sigma(y, z)$$

for all $x, y, z \in G$ and $\sigma(e, e) = 1$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and e is the identity of G. Denote by $Z^2(G, \mathbb{T})$ the set of all 2-cocycles on G. Fix $x \in G$. We say that the element x is σ-*regular* if $\sigma(x, y) = \sigma(y, x)$ whenever y commutes with x. If x is σ-regular, then every element in the conjugacy class $C_x := \{y^{-1}xy : y \in G\}$ of x is also σ-regular. So it makes sense to consider σ-regular conjugacy classes. We say that the pair (G, σ) satisfies *Kleppner's condition* if the only finite σ-regular conjugacy class is the trivial one. By a σ-*projective unitary representation* of G, we mean a strongly continuous mapping $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ into the group of all unitary operators on a separable Hilbert space $\mathcal{H}_\pi$ satisfying

$$\pi(x)\pi(y) = \sigma(x, y)\pi(xy)$$

for all $x, y \in G$. The Hilbert space $\mathcal{H}_\pi$ is called the *representation space* of π. We will always write the pair $(\pi, \mathcal{H}_\pi)$ instead of $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$.

Let Λ be a closed subgroup of G. Then Λ and G/Λ are also locally compact groups. There exists a non-zero positive regular Borel measure μ_G on G which is left-translation invariant. That is, $\mu_G(xB) = \mu_G(B)$ holds for any element $x \in G$ and any $B \in \mathcal{B}_G$, where $\mathcal{B}_G$ denotes the σ-algebra of Borel sets of G. This measure μ_G is called a *left Haar measure* on G [9]. Moreover, the integral over G is left-translation invariant in the following sense that

$$\int_G f\big(y^{-1}x\big)d\mu_G(x) = \int_G f(x)d\mu_G(x)$$

for all $f \in C_c(G)$ and all $y \in G$, where $C_c(G)$ denotes the set of all continuous functions on G having compact support [6]. From this measure μ_G, we can define the space $L^p(G)$, $1 \leq p < \infty$ to be the space of equivalence classes of measurable functions f on G, modulo equality a.e., with the property that

$$\int_G |f(x)|^p d\mu_G(x) < \infty.$$

This Haar measure on G is unique up to multiplication with a positive constant. More precisely, there is a continuous homomorphism $\Delta_G : G \to (0, \infty)$ with respect to multiplication on $(0, \infty)$, called the *modular function*, such that

$$\int_G f(xy) d\mu_G(x) = \Delta_G\left(y^{-1}\right) \int_G f(x) d\mu_G(x)$$

and

$$\int_G f\left(x^{-1}\right) d\mu_G(x) = \int_G f(x) \Delta_G\left(x^{-1}\right) d\mu_G(x)$$

for any $f \in L^1(G, \mu_G)$ and any $y \in G$. This modular function is independent of the choice of Haar measures on G. If $\Delta_G \equiv 1$, then G is called *unimodular*. There are many unimodular locally compact groups. For example, $\mathbb{R}^d$, discrete groups, the general linear group $\mathrm{GL}(d, \mathbb{R})$ of all $d \times d$ invertible matrices with real entries and connected nilpotent Lie groups. The subgroup Λ also has a modular function Δ_Λ and a left Haar measure μ_Λ, whose scale we also fix.

The relation between the Haar measures on the three groups can be given as follows: Once two out of three Haar measures have been chosen on G, Λ and G/Λ, the last one can be chosen so that Weil's formula:

$$\int_G f(x) d\mu_G(x) = \int_{G/\Lambda} \int_\Lambda f(xy) d\mu_\Lambda(y) d\mu_{G/\Lambda}(x\Lambda)$$

for all $f \in L^1(G)$; see [11, Section 3]. The subgroup Λ is called *co-compact* if the quotient group G/Λ is compact under the quotient topology. The *volume* of Λ is defined by

$$\mathrm{vol}(\Lambda) := \mu_{G/\Lambda}(G/\Lambda).$$

Note that this volume $\mathrm{vol}(\Lambda)$ depend on the Haar measure μ_G. We also know that Λ is co-compact if and only if $\mathrm{vol}(\Lambda) < \infty$. The subgroup Λ is called a *uniform lattice* if Λ is a discrete co-compact subgroup of G.

A classical example of square-integrable representations is the Weyl-Heisenberg representation. Let G be a second countable, locally compact, abelian group. For any $\nu = (\lambda, \gamma) \in G \times \widehat{G}$. Let $\pi(\nu)$ denote the time–frequency shift operator $E_\gamma T_\lambda$, where the translation operator $T_\lambda, \quad \lambda \in G$ as follows:

$$T_\lambda : L^2(G) \to L^2(G), \quad (T_\lambda f)(x) = f\left(x\lambda^{-1}\right)$$

and the modulation operator $E_\gamma, \quad \gamma \in \widehat{G}$ as follows:

$$E_\gamma : L^2(G) \to L^2(G), \quad (E_\gamma f)(x) = \gamma(x) f(x)$$

for all $f \in L^2(G)$ and all $x \in G$. It is easy to verify that each $\pi(\nu)$ is a unitary operator on $L^2(G)$. We use the essential equality commutator relation

$$\gamma(\lambda) T_\lambda E_\gamma = E_\gamma T_\lambda$$

to obtain the following useful identities:

$$\pi(\nu_1)\pi(\nu_2) = \overline{\gamma_2(\lambda_1)}\pi(\nu_1\nu_2)$$

for all $\nu_i = (\lambda_i, \gamma_i), i = 1, 2$. So the *Weyl-Heisenberg representation* π [5] is a projective unitary representation with the 2-cocycle $\sigma \in Z^2(G \times \widehat{G}, \mathbb{T})$ given by

$$\sigma(\nu_1, \nu_2) = \overline{\gamma_2(\lambda_1)}, \quad \nu_i = (\lambda_i, \gamma_i), i = 1, 2.$$

By [2, Theorem 1.3] and [12, Lemma 4.2], this representation is square-integrable irreducible and $\mathrm{d}_\pi = 1$. Moreover, $(G \times \widehat{G}, \sigma)$ satisfies Kleppner's condition [5, Section 6.2].

For more information about harmonic analysis on locally compact groups, we refer the readers to the books [3, 9].

2.3 Frames

In this subsection, we recall some basic concepts about frames.

Definition 2.1 (Frames [12]) Let $\mathcal{H}$ be a complex Hilbert space and let (Ω, Σ, μ) be a measure space. A family of $\{f_k\}_{k\in\Omega}$ is a *frame* for $\mathcal{H}$ with respect to (Ω, Σ, μ) with bounds $0 < \alpha \le \beta < \infty$ if the following statements hold:

(i) The mapping $\Omega \to \mathcal{H}, k \mapsto f_k$ is weakly measurable, i.e., for all $f \in \mathcal{H}$, the mapping $\Omega \to \mathbb{C}, k \mapsto \langle f, f_k\rangle$ is measurable.
(ii) The inequalities

$$\alpha\|f\|^2 \le \int_\Omega |\langle f, f_k\rangle|^2 d\mu(k) \le \beta\|f\|^2$$

hold for all $f \in \mathcal{H}$.

Let $(\pi, \mathcal{H}_\pi)$ be a square-integrable representation of a unimodular locally compact group G and Λ is a closed subgroup of G. We say that the pair (π, Λ) admits a frame if there exists $\eta \in \mathcal{H}_\pi$ such that $\pi(\Lambda)\eta$ is a frame for $\mathcal{H}_\pi$. That is, there exist constants $0 < \alpha \le \beta < \infty$ such that

$$\alpha\|\xi\|^2 \le \int_\Lambda |\langle \xi, \pi(\lambda)\eta\rangle|^2 d\lambda \le \beta\|\xi\|^2$$

for all $\xi \in \mathcal{H}_\pi$. In this case, this vector η is called the *generator* of the system $\pi(\Lambda)\eta$.

Motivated by the concept of operator-valued frames in Hilbert spaces proposed by Kaftal et al. in [13], we introduce the notion of operator-valued frames with special structure.

Definition 2.2 (Operator-valued frames with special structure) Let $(\pi, \mathcal{H}_\pi)$ be a square-integrable representation of G with formal dimension $\mathrm{d}_\pi > 0$, let Λ be a closed subgroup of G and let A be a bounded linear operator on $\mathcal{H}_\pi$. The system $A\pi(\Lambda)$ is called an *operator-valued frame* on $\mathcal{H}_\pi$ with bounds $0 < \alpha \leq \beta < \infty$ if this system satisfies

$$\alpha\|\xi\|^2 \leq \int_\Lambda \|A\pi(\lambda)\xi\|^2 d\lambda \leq \beta\|\xi\|^2$$

for all $\xi \in \mathcal{H}_\pi$.

If one can choose $\alpha = \beta$ in the above inequalities, then $A\pi(\Lambda)$ is called *tight*. The system $A\pi(\Lambda)$ is called *Parseval* if $\alpha = \beta = 1$. If $A\pi(\Lambda)$ satisfies the upper bound inequality, then it is called *Bessel*. In this case, this operator A is called the *generator* of the system $A\pi(\Lambda)$.

We say that (π, Λ) admits an operator-valued frame on $\mathcal{H}_\pi$ if there exists an operator $A \in \mathcal{B}(\mathcal{H}_\pi)$ such that $A\pi(\Lambda)$ is an operator-valued frame. In particular, if $A \in \mathcal{F}(\mathcal{H}_\pi)$, then we say that (π, Λ) admits a *FR-operator-valued frame*. if $A \in \mathcal{S}_2(\mathcal{H}_\pi)$, then we say that (π, Λ) admits a *HS-operator-valued frame*.

3 A Density Theorem and Its Applications

In this section, we first provide a density theorem for operator-valued frames of the form (1.3), which is a main theorem in this paper. From this, we also give a partial answer to Problem 1.3 in the introduction. This theorem suggests that when we deal with such operator-valued frames, the co-compactness of the index subgroup is closely related to the special property of the generator of this system. Hence if we want to obtain properties of such operator-valued frames indexed by closed co-compact subgroups, then this special property of the generators brings the great benefit to our proof and deduction. So we will investigate properties of such operator-valued frames indexed by closed co-compact subgroups. Moreover, applications of this density theorem are listed later.

3.1 A Density Theorem for Operator-Valued Frames

For the proof of our main theorem, we will use famous Weil's identity in harmonic analysis on locally compact groups and the orthogonality relation for square-integrable representations.

Theorem 3.1 *Let $(\pi, \mathcal{H}_\pi)$ be a square-integrable representation of G with formal dimension $\mathrm{d}_\pi > 0$, let Λ be a closed subgroup of G and let A be a bounded linear* *operator on $\mathcal{H}_\pi$. If $A\pi(\Lambda)$ is an operator-valued frame with bounds $0 < \alpha \le \beta < \infty$, then the following conditions are equivalent:*

(i) *The generator A is a Hilbert–Schmidt operator on $\mathcal{H}_\pi$, i.e., $A \in \mathcal{S}_2(\mathcal{H}_\pi)$.*
(ii) *The index subgroup Λ is co-compact, i.e., $\mathrm{vol}(\Lambda) < \infty$.*

If any of the above conditions holds, then

$$\alpha \mathrm{vol}(\Lambda) \le \frac{\|A\|_2^2}{\mathrm{d}_\pi} \le \beta \mathrm{vol}(\Lambda).$$

In particular, the generator of an operator-valued Bessel system indexed by a closed co-compact subgroup is necessarily a Hilbert–Schmidt operator.

Proof Suppose that $A\pi(\Lambda)$ is an operator-valued frame with bounds $0 < \alpha \le \beta < \infty$. Since $\mathcal{H}_\pi$ is separable, we can assume that $\{\xi_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}_\pi$, where $\mathbb{I}$ is a finite or countable set.

(i)⇒(ii): If the generator A is a Hilbert–Schmidt operator on $\mathcal{H}_\pi$, then $\|A\|_2 < \infty$ and so $\|A\|_2^2 = \sum_i \|A^*\xi_i\|^2$. Fix $\xi \in \mathcal{H}_\pi$ and $i \in \mathbb{I}$. By Proposition 1.2, we see that

$$\int_G |\langle \xi, \pi(x)A^*\xi_i\rangle|^2 dx = \frac{\|A^*\xi_i\|^2\|\xi\|^2}{\mathrm{d}_\pi} < \infty,$$

which implies that the function $x \mapsto |\langle \xi, \pi(x)A^*\xi_i\rangle|^2$ lies in $L^1(G)$. By Weil's identity, we obtain that

$$\int_G |\langle \xi, \pi(x)A^*\xi_i\rangle|^2 dx = \int_{G/\Lambda}\int_\Lambda |\langle \xi, \pi(x\lambda)A^*\xi_i\rangle|^2 d\lambda d(x\Lambda).$$

We also have that

$$\int_{G/\Lambda}\int_\Lambda |\langle \xi, \pi(x\lambda)A^*\xi_i\rangle|^2 d\lambda d(x\Lambda) = \int_{G/\Lambda}\int_\Lambda |\langle \pi(\lambda^{-1})\pi(x)^*\xi, A^*\xi_i\rangle|^2 d\lambda d(x\Lambda).$$

Since Λ is a unimodular locally compact group, we get that

$$\int_\Lambda |\langle \pi(\lambda^{-1})\pi(x)^*\xi, A^*\xi_i\rangle|^2 d\lambda = \int_\Lambda |\langle \pi(\lambda)\pi(x)^*\xi, A^*\xi_i\rangle|^2 d\lambda$$

for all $x \in G$. By Tonelli's Theorem, we know that

$$\sum_i \int_{G/\Lambda}\int_\Lambda |\langle A\pi(\lambda)\pi(x)^*\xi, \xi_i\rangle|^2 d\lambda d(x\Lambda) = \int_{G/\Lambda}\int_\Lambda \|A\pi(\lambda)\pi(x)^*\xi\|^2 d\lambda d(x\Lambda).$$

Combining the above formulas, we conclude that

$$\begin{aligned}\frac{\|A\|_2^2\|\xi\|^2}{\mathrm{d}_\pi} &= \int\limits_{G/\Lambda}\int\limits_{\Lambda} \|A\pi(\lambda)\pi(x)^*\xi\|^2 d\lambda d(x\Lambda) \\ &\geq \int\limits_{G/\Lambda} \alpha\|\pi(x)^*\xi\|^2 d(x\Lambda) = \int\limits_{G/\Lambda} \alpha\|\xi\|^2 d(x\Lambda) \\ &= \alpha\|\xi\|^2 \int\limits_{G/\Lambda} d(x\Lambda) = \alpha\|\xi\|^2 \mathrm{vol}(\Lambda),\end{aligned}$$

where we use the condition that the system $A\pi(\Lambda)$ satisfies the lower frame bound inequality. Thus

$$\mathrm{vol}(\Lambda) \leq \frac{\|A\|_2^2}{\alpha \mathrm{d}_\pi} < \infty,$$

which shows that Λ is co-compact.

(ii)⇒(i): If the index subgroup Λ is co-compact, then $\mathrm{vol}(\Lambda) < \infty$. Fix $\xi \in \mathcal{H}_\pi$. Similar to the above proof, using the upper frame bound inequality for the system $A\pi(\Lambda)$ yields that

$$\begin{aligned}\sum_i \frac{\|A^*\xi_i\|^2\|\xi\|^2}{\mathrm{d}_\pi} &= \int\limits_{G/\Lambda}\int\limits_{\Lambda} \|A\pi(\lambda)\pi(x)^*\xi\|^2 d\lambda d(x\Lambda) \\ &\leq \beta\|\xi\|^2 \mathrm{vol}(\Lambda).\end{aligned}$$

Hence

$$\sum_i \|A^*\xi_i\|^2 \leq \beta \mathrm{d}_\pi \mathrm{vol}(\Lambda) < \infty.$$

That is, $A^* \in \mathcal{S}_2(\mathcal{H}_\pi)$ and $\|A^*\|_2^2 \leq \beta \mathrm{d}_\pi \mathrm{vol}(\Lambda)$. Thus $A \in \mathcal{S}_2(\mathcal{H}_\pi)$ and

$$\frac{\|A\|_2^2}{\mathrm{d}_\pi} \leq \beta \mathrm{vol}(\Lambda).$$

This completes the proof. □

Remark 3.2 By Theorem 3.1, the answer to Problem 1.3 (i) is sure if and only if A is necessarily a Hilbert–Schmidt operator on $\mathcal{H}_\pi$. Since $\|A\| \leq \|A\|_2$, we see that

$$\frac{\|A\|^2}{\mathrm{d}_\pi} \leq \beta \mathrm{vol}(\Lambda),$$

which is the right hand inequality in Problem 1.3 (ii). However, we do not obtain the left hand inequality.

As a special case, we also obtain the classical density theorem for systems of the form (1.2) via square-integrable representations restricted to closed subgroups. The following density theorem generalizes [12, Theorem 5.1] and [17, Proposition 7.2].

Theorem 3.3 *Let Λ be a closed subgroup of G and let $\eta \in \mathcal{H}_\pi$. If $\pi(\Lambda)\eta$ is a frame for $\mathcal{H}_\pi$ with bounds $0 < \alpha \le \beta < \infty$, then the following statements hold:*

(i) *The index subgroup Λ is co-compact, i.e.,* $\mathrm{vol}(\Lambda) < \infty$.
(ii) $\alpha \mathrm{vol}(\Lambda) \le \frac{\|\eta\|^2}{\mathrm{d}}_\pi \le \beta \mathrm{vol}(\Lambda)$.

Proof We assume that $\pi(\Lambda)\eta$ is a frame for $\mathcal{H}_\pi$ with bounds $0 < \alpha \le \beta < \infty$. Fix η_0 in $\mathcal{H}_\pi$ with $\|\eta_0\| = 1$. Put

$$A := \eta_0 \otimes \eta.$$

Then $A \in \mathcal{S}_2(\mathcal{H}_\pi)$ and $\|A\|_2 = \|\eta\|$. For any $\xi \in \mathcal{H}_\pi$, we compute that

$$\int_\Lambda \|A\pi(\lambda)\xi\|^2 d\lambda = \int_\Lambda |\langle \xi, \pi(\lambda)\eta \rangle|^2 d\lambda.$$

Thus $A\pi(\Lambda)$ is a HS-operator-valued frame with same bounds $0 < \alpha \le \beta < \infty$. Now this theorem follows from Theorem 3.1. □

Based on operator-valued frames [13] and ordinary Gabor frames of the form (1.1), we introduce the notation of operator-valued Gabor frames on locally compact abelian groups. Let Δ be a closed subgroup of the phase space $G \times \widehat{G}$, let A be a bounded linear operator on $L^2(G)$ and let π be the Weyl-Heisenberg representation of $G \times \widehat{G}$, where G is a second countable, locally compact, abelian group and $\widehat{G}$ is the Pontryagin dual group of G. We say that the system $A\pi(\Delta)$ is an *operator-valued Gabor frame* on $L^2(G)$ with bounds $0 < \alpha \le \beta < \infty$ if it satisfies

$$\alpha \|f\|^2 \le \int_\Delta \|A\pi(\nu) f\|^2 d\nu \le \beta \|f\|^2$$

for all $f \in L^2(G)$. If $A\pi(\Delta)$ satisfies the upper bound inequality, then it is called *Bessel*. In this case, this operator A is called the *generator* of the system $A\pi(\Delta)$. Since this representation π is a square-integrable representation with formal dimension $\mathrm{d}_\pi = 1$, we can obtain the following density theorem for operator-valued Gabor frames, which is a generalization of [12, Theorem 5.1].

Theorem 3.4 *If $A\pi(\Delta)$ is an operator-valued Gabor frame with bounds $0 < \alpha \le \beta < \infty$, then the following conditions are equivalent:*

(i) *The generator A is a Hilbert–Schmidt operator on $L^2(G)$, i.e., $A \in \mathcal{S}_2(L^2(G))$.*

(ii) *The index subgroup* Δ *is co-compact, i.e.,* $\text{vol}(\Delta) < \infty$.

If any of the above conditions holds, then

$$\alpha \text{vol}(\Delta) \leq \|A\|_2^2 \leq \beta \text{vol}(\Delta).$$

In particular, the generator of an operator-valued Bessel system indexed by a closed co-compact subgroup of the phase space is necessarily a Hilbert–Schmidt operator.

3.2 Analysis Operators and Synthesis Operators

One can define $L^2(\Lambda, \mathcal{H}_\pi)$ to be the space of equivalence classes of strongly measurable functions $\Psi : \Lambda \to \mathcal{H}_\pi$ (in the sense that the scalar-valued function $\lambda \mapsto \|\Psi(\lambda)\|$ is measurable), modulo equality a.e., with the property that

$$\int_\Lambda \|\Psi(\lambda)\|^2 d\lambda < \infty.$$

Then it becomes a Hilbert space with the inner product

$$\langle \Psi_1, \Psi_2 \rangle = \int_\Lambda \langle \Psi_1(\lambda), \Psi_2(\lambda) \rangle d\lambda$$

for all $\Psi_1, \Psi_2 \in L^2(\Lambda, \mathcal{H}_\pi)$. Let $A\pi(\Lambda)$ be an operator-valued Bessel system on $\mathcal{H}_\pi$. We can define the operator Θ_A associated to $A\pi(\Lambda)$ as follows:

$$\Theta_A : \mathcal{H}_\pi \to L^2(\Lambda, \mathcal{H}_\pi), \quad \Theta_A \xi(\lambda) = A\pi(\lambda)\xi$$

for all $\xi \in \mathcal{H}_\pi$ and all $\lambda \in \Lambda$. This operator Θ_A is called the *analysis operator* of the system $A\pi(\Lambda)$. It is easy to show that $A\pi(\Lambda)$ is a Bessel system with a bound $\beta > 0$ if and only if Θ_A is a bounded linear operator with a bound $\beta^{\frac{1}{2}}$. The adjoint operator of Θ_A,

$$\Theta_A^* : L^2(\Lambda, \mathcal{H}_\pi) \to \mathcal{H}_\pi, \quad \Psi \mapsto \Theta_A^* \Psi$$

satisfying

$$\langle \Theta_A^* \Psi, \xi \rangle = \int_\Lambda \langle \pi(\lambda)^* A^* \Psi(\lambda), \xi \rangle d\lambda$$

for all $\xi \in \mathcal{H}_\pi$, is called the *synthesis operator* of $A\pi(\Lambda)$. If $A\pi(\Lambda)$ and $B\pi(\Lambda)$ are two operator-valued Bessel systems on $\mathcal{H}_\pi$, where $A, B \in \mathcal{B}(\mathcal{H}_\pi)$, then one can define the (mixed) frame operator for them in the weak sense by

$$\langle S_{A,B}\xi, \eta\rangle = \int_\Lambda \langle A\pi(\lambda)\xi, B\pi(\lambda)\eta\rangle d\lambda$$

for all $\xi, \eta \in \mathcal{H}_\pi$. Then $S_{A,B} = \Theta_B^*\Theta_A$. If their generators are same, i.e., $A = B$, then we write the frame operator S. We say that operator-valued Bessel systems $A\pi(\Lambda)$ and $B\pi(\Lambda)$ are *dual* if they satisfy

$$\int_\Lambda \langle A\pi(x)\xi, B\pi(x)\eta\rangle dx = \langle \xi, \eta\rangle$$

for all $\xi, \eta \in \mathcal{H}_\pi$. That is, the frame operator for them satisfies the equality $S_{A,B} = I$. We also say that operator-valued Bessel systems $A\pi(\Lambda)$ and $B\pi(\Lambda)$ are *HS-dual* if they satisfy $A, B \in \mathcal{S}_2(\mathcal{H}_\pi)$ and $S_{A,B} = I$.

Lemma 3.5 *Let Λ be a unimodular closed subgroup of G and let $A, B \in \mathcal{B}(\mathcal{H}_\pi)$. If $A\pi(\Lambda)$ and $B\pi(\Lambda)$ are two operator-valued Bessel systems on $\mathcal{H}_\pi$, then*

(i) $S_{A,B}\pi(\lambda) = \pi(\lambda)S_{A,B}, \quad for \quad all \quad \lambda \in \Lambda.$
(ii) *If $A\pi(\Lambda)$ is an operator-valued frame, then*

$$S^{-1}\pi(\lambda) = \pi(\lambda)S^{-1}, \quad S^{-\frac{1}{2}}\pi(\lambda) = \pi(\lambda)S^{-\frac{1}{2}}$$

for all $\lambda \in \Lambda$.

Proof (i) Fix $\kappa \in \Lambda$. For any $\xi, \eta \in \mathcal{H}_\pi$, we can get

$$\begin{aligned}
\langle S_{A,B}\pi(\kappa)\xi, \eta\rangle &= \int_\Lambda \langle A\pi(\lambda)\pi(\kappa)\xi, B\pi(\lambda)\eta\rangle d\lambda \\
&= \int_\Lambda \langle A\pi(\iota\kappa^{-1})\pi(\kappa)\xi, B\pi(\iota\kappa^{-1})\pi(\kappa)\pi(\kappa)^*\eta\rangle d\iota \\
&= \int_\Lambda \langle A\pi(\iota)\xi, B\pi(\iota)\pi(\kappa)^*\eta\rangle d\iota \\
&= \langle S_{A,B}\xi, \pi(\kappa)^*\eta\rangle \\
&= \langle \pi(\kappa)S_{A,B}\xi, \eta\rangle,
\end{aligned}$$

where we use the fact that the Haar measure on Λ is transition invariant in the second identity and π is a σ-projective unitary representation in the third identity. Hence

$$S_{A,B}\pi(\kappa) = \pi(\kappa)S_{A,B}.$$

(ii) We suppose that $A\pi(\Lambda)$ is an operator-valued frame. Then S is an invertible operator on $\mathcal{H}_\pi$. Fix $\lambda \in \Lambda$. By (i), we obtain that $S\pi(\lambda) = \pi(\lambda)S$ and so

$$S^{-\frac{1}{2}}\pi(\lambda) = \pi(\lambda)S^{-\frac{1}{2}}.$$

□

Remark 3.6 If $A\pi(\Lambda)$ is a HS-operator-valued frame indexed by a unimodular closed subgroup Λ of G, then $A \in \mathcal{S}_2(\mathcal{H}_\pi)$ and $AS^{-1}, AS^{-\frac{1}{2}} \in \mathcal{S}_2(\mathcal{H}_\pi)$ since $\mathcal{S}_2(\mathcal{H}_\pi)$ is a two-side ideal in $\mathcal{B}(\mathcal{H}_\pi)$. For any $\lambda \in \Lambda$, by Lemma 3.5, we have that $A\pi(\lambda)S^{-1} = AS^{-1}\pi(\lambda)$ and $A\pi(\lambda)S^{-\frac{1}{2}} = AS^{-\frac{1}{2}}\pi(\lambda)$. Thus $AS^{-1}\pi(\Lambda)$ is the canonical HS-dual of $A\pi(\Lambda)$. Moreover, $AS^{-\frac{1}{2}}\pi(\Lambda)$ is a Parseval HS-operator-valued frame. By Theorem 3.1, we see that

$$\left\| AS^{-\frac{1}{2}} \right\|_2^2 = \mathrm{d}_\pi \mathrm{vol}(\Lambda).$$

3.3 Applications of the Density Theorem

We study the properties of the operator-valued frames generated by square-integrable representations of the Euclidean spaces. To do this, we need to clarify the structure of the closed subgroups, co-compact subgroups and uniform lattices of the Euclidean spaces; see [3]. The following corollary is a generalized version of [12, Corollary 5.2].

Corollary 3.7 *Let $(\pi, \mathcal{H}_\pi)$ be a square-integrable representation of $\mathbb{R}^d$, and let Λ be a closed subgroup of $\mathbb{R}^d$. If (π, Λ) admits a HS-operator-valued frame, then the index subgroup Λ is of the form*

$$Q\big(\mathbb{Z}^k \times \mathbb{R}^{d-k}\big), \quad 0 \le k \le d, \quad Q \in \mathrm{GL}(d, \mathbb{R}).$$

Proof By [3, Example 21.7.2], there exist $0 \le k, l \le d$ with $0 \le k + l \le d$ such that

$$\Lambda \cong \{0\}^l \times \mathbb{Z}^k \times \mathbb{R}^{d-k-l}.$$

By Theorem 3.1, Λ is co-compact. This implies that $l = 0$ and so $\Lambda \cong \mathbb{Z}^k \times \mathbb{R}^{d-k}$. That is, $\Lambda = Q(\mathbb{Z}^k \times \mathbb{R}^{d-k})$ for some $Q \in \mathrm{GL}(d, \mathbb{R})$. □

Gabardo and Han [7] proved the following proposition which shows that for a Parseval Gabor frame indexed by full rank lattices, the norm of the window function is closely related to the determinant of the invertible matrices.

Proposition 3.8 [7] *Let $\mathcal{L} = A\mathbb{Z}^d, \mathcal{K} = B\mathbb{Z}^d$ be two full rank lattices in $\mathbb{R}^d$ and let $g \in L^2(\mathbb{R}^d)$. If $\mathscr{G}(g, \mathcal{L} \times \mathcal{K})$ is a Parseval Gabor frame for $L^2(\mathbb{R}^d)$, then*

$$\|g\|^2 = |\det(AB)|.$$

Our aim is to generalize this proposition to the more general setting.

Corollary 3.9 *Let $(\pi, \mathcal{H}_\pi)$ be a square-integrable representation of $\mathbb{R}^d$ with formal dimension $\mathrm{d}_\pi > 0$, let $A \in \mathcal{B}(\mathcal{H}_\pi)$ and let $\Lambda = P\mathbb{Z}^d$ be a full-rank lattice in $\mathbb{R}^d$. If*

$A\pi(\Lambda)$ is an operator-valued frame with bounds $0 < \alpha \leq \beta < \infty$, then $A\pi(\Lambda)$ is a HS-operator-valued frame. Moreover, we have that

$$\alpha|\mathrm{detP}| \leq \frac{\|A\|_2^2}{\mathrm{d}_\pi} \leq \beta|\mathrm{detP}|.$$

In particular, if $A\pi(\Lambda)$ is a Parseval operator-valued frame, then

$$\|A\|_2^2 = \mathrm{d}_\pi|\mathrm{detP}|.$$

Proof Since Λ is a full-rank lattice in $\mathbb{R}^d$, Λ is a closed co-compact subgroup of $\mathbb{R}^d$. Also,

$$\mathrm{vol}(\Lambda) = m(\mathbb{R}^d/(P\mathbb{Z}^d)) = |\mathrm{detP}|,$$

where m is the Lebesuge measure on the Euclidean space. The remaining part of the proof follows from Theorem 3.1. □

Now we apply our density theorem to a class of number-theoretic groups, the field $\mathbb{Q}_p$ of p-adic numbers. We know that $\mathbb{Q}_p$ is a locally compact abelian group under addition. We can refer to [15] for more information about $\mathbb{Q}_p$. The following corollary shows that for the p-adic numbers, there exists no other index subgroups corresponding to operator-valued frames. Moreover, this is a generalized version of [12, Corollary 5.2].

Corollary 3.10 *Fix a prime number p. Let $\widehat{\mathbb{Q}_p}$ be the Pontryagin dual group of $\mathbb{Q}_p$, let $(\pi, \mathcal{H}_\pi)$ be a square-integrable representation of $\mathbb{Q}_p \times \widehat{\mathbb{Q}_p}$, and let Λ be a closed subgroup of $\mathbb{Q}_p \times \widehat{\mathbb{Q}_p}$. If (π, Λ) admits a HS-operator-valued frame, then we necessarily have that*

$$\Lambda = \mathbb{Q}_p \times \widehat{\mathbb{Q}_p}.$$

Proof By Theorem 3.1, the index subgroup Λ is co-compact. Since the only co-compact subgroup of $\mathbb{Q}_p \times \widehat{\mathbb{Q}_p}$ is the entire group itself, $\Lambda = \mathbb{Q}_p \times \widehat{\mathbb{Q}_p}$. □

The following theorem shows that the existence of a HS-operator-valued frame indexed by a discrete abelian subgroup implies that the index subgroup is necessarily a uniform lattice. Furthermore, the size of the index subgroup can not be too small and the operator norm of the corresponding frame generator can not be too big.

Theorem 3.11 *Let Λ be a discrete subgroup of G equipped with the counting measure. If $A\pi(\Lambda)$ is a HS-operator-valued frame with an upper bound $\beta > 0$, then Λ is a uniform lattice in G, $\|A\|^2 \leq \beta$ and*

$$\mathrm{vol}(\Lambda) \geq \frac{\|A\|^2}{\beta \mathrm{d}_\pi}.$$

Proof By Theorem 3.1, Λ is co-compact. Since Λ is discrete, Λ is a uniform lattice. Let S be the frame operator associated with $A\pi(\Lambda)$. Then $\|S\| \leq \beta$. Noticing that $AS^{-\frac{1}{2}}\pi(\Lambda)$ is a Parseval operator-valued frame and Λ is a discrete subgroup equipped with the counting measure, we have that for each $\xi \in \mathcal{H}_\pi$,

$$\begin{aligned}\|A\xi\|^2 &\leq \sum_{\lambda\in\Lambda} \left\|AS^{-\frac{1}{2}}\pi(\lambda)S^{\frac{1}{2}}\xi\right\|^2 = \left\|S^{\frac{1}{2}}\xi\right\|^2 \\ &\leq \|S\|\|\xi\|^2 \leq \beta\|\xi\|^2\end{aligned}$$

and so $\|A\|^2 \leq \beta$. Again using Theorem 3.1, we obtain that

$$\mathrm{vol}(\Lambda) \geq \frac{\|A\|_2^2}{\beta \mathrm{d}_\pi} \geq \frac{\|A\|^2}{\beta \mathrm{d}_\pi}.$$

In particular, if $A\pi(\Lambda)$ is a Parseval HS-operator-valued frame, then $\|A\| \leq 1$ and

$$\mathrm{vol}(\Lambda) \geq \frac{\|A\|^2}{\mathrm{d}_\pi}.$$

□

For every frame $\{f_k\}_{k\in\Omega}$ for a Hilbert space $\mathcal{H}$, we know that $\{S^{-1}f_k\}_{k\in\Omega}$ is the canonical dual frame, where S is the frame operator associated with the frame $\{f_k\}_{k\in\Omega}$. The following proposition is celebrated in frame theory.

Proposition 3.12 [12] *Let $\mathcal{H}$ be a Hilbert space and let $\alpha, \beta > 0$. Then the following statements are equivalent:*

(i) *The system $\{f_k\}_{k\in\Omega}$ is a frame for $\mathcal{H}$ with bounds α and β.*
(ii) *The system $\{f_k\}_{k\in\Omega}$ is a Bessel system for $\mathcal{H}$ with a bound β and there exists another Bessel system $\{g_k\}_{k\in\Omega}$ for $\mathcal{H}$ with a bound $\frac{1}{\alpha}$ such that the equality*

$$\langle f, g\rangle = \int_\Omega \langle f, g_k\rangle\langle f_k, g\rangle d\mu(k)$$

holds for all $f, g \in \mathcal{H}$.

Based on Proposition 3.12, we get the following theorem. A necessary and sufficient condition for an operator-valued system to be an operator-valued frame is given in this theorem. Further, it will provide ideas for obtaining some duality properties about operator-valued frames with special structure.

Theorem 3.13 *Let Λ be a unimodular closed co-compact subgroup of G, let $A \in \mathcal{B}(\mathcal{H}_\pi)$ and let $0 < \alpha \leq \beta < \infty$. Then the following statements are equivalent:*

(i) *The system $A\pi(\Lambda)$ is an operator-valued frame with bounds α and β.*

(ii) *The system $A\pi(\Lambda)$ is a Bessel system with a bound β. Moreover, the pair (π, Λ) admits another HS-operator-valued Bessel system $B\pi(\Lambda)$ with a bound $\frac{1}{\alpha}$ and the equality*

$$\langle \xi, \eta \rangle = \int_{\Lambda} \langle A\pi(\lambda)\xi, B\pi(\lambda)\eta \rangle d\lambda$$

holds for all $\xi, \eta \in \mathcal{H}_\pi$.

Proof (i)⇒(ii): We suppose that S is the frame operator associated with the operator-valued frame $A\pi(\Lambda)$. Then by Theorem 3.1, $A \in \mathcal{S}_2(\mathcal{H}_\pi)$. Thus $A\pi(\Lambda)$ is a HS-operator-valued frame indexed by a unimodular subgroup Λ of G. Set

$$B := AS^{-1}.$$

By Remark 3.6, $B\pi(\Lambda)$ is the canonical HS-dual of $A\pi(\Lambda)$.

(ii)⇒(i): We assume that the statement (ii) holds. For any $\xi \in \mathcal{H}_\pi$, we have that

$$\|\xi\|^2 \leq \int_{\Lambda} |\langle A\pi(\lambda)\xi, B\pi(\lambda)\xi \rangle| d\lambda.$$

By the Cauchy–Schwarz inequality, we see that

$$\begin{aligned} \|\xi\|^2 &\leq \int_{\Lambda} \|A\pi(\lambda)\xi\| \|B\pi(\lambda)\xi\| d\lambda \\ &\leq \left(\int_{\Lambda} \|A\pi(\lambda)\xi\|^2 d\lambda \right)^{\frac{1}{2}} \left(\int_{\Lambda} \|B\pi(\lambda)\xi\|^2 d\lambda \right)^{\frac{1}{2}}. \end{aligned}$$

Since $B\pi(\Lambda)$ is Bessel with a bound $\frac{1}{\alpha}$, we obtain that

$$\int_{\Lambda} \|B\pi(\lambda)\xi\|^2 d\lambda \leq \frac{1}{\alpha} \|\xi\|^2.$$

Hence we get the lower frame bound inequality

$$\alpha \|\xi\|^2 \leq \int_{\Lambda} \|A\pi(\lambda)\xi\|^2 d\lambda.$$

Thus $A\pi(\Lambda)$ is an operator-valued frame with bounds α and β. □

Next we prove that under certain assumptions on square-integrable representations, uniform lattices-indexed HS-operator-valued frames always exist. To see this, we need the following proposition.

Proposition 3.14 [5] *Let $(\pi, \mathcal{H}_\pi)$ be a square-integrable representation of G with formal dimension $\mathrm{d}_\pi > 0$, where (G, σ) satisfies Kleppner's condition. Let Λ be a uniform lattice in G. Then (π, Λ) admits an n-multiwindow d-super frame if and only if* $\mathrm{d}_\pi \operatorname{vol}(\Lambda) \leq \frac{n}{d}$.

Theorem 3.15 *Let $(\pi, \mathcal{H}_\pi)$ be a square-integrable representation of G with formal dimension $\mathrm{d}_\pi > 0$, and let Λ be a discrete subgroup of G equipped with the counting measure. Suppose that (G, σ) satisfies Kleppner's condition. Then the following statements are equivalent:*

(i) (π, Λ) *admits a FR-operator-valued frame, i.e., there exist a finite-rank operator A such that $A\pi(\Lambda)$ is an operator-valued frame.*
(ii) (π, Λ) *admits a HS-operator-valued frame, i.e., there exist a Hilbert–Schmidt operator A such that $A\pi(\Lambda)$ is an operator-valued frame.*
(iii) Λ *is co-compact, i.e.,* $\operatorname{vol}(\Lambda) < \infty$.

In particular, there always exist a HS-operator-valued Gabor frame with the index subgroup being the uniform lattice as long as there exist a uniform lattice in the phase space.

Proof (i)⇒(ii): This is clear.

(ii)⇒(iii): Suppose that $A\pi(\Lambda)$ is a HS-operator-valued frame. By Theorem 3.1, Λ is co-compact.

(iii)⇒(i): If Λ is co-compact, then $\operatorname{vol}(\Lambda) < \infty$ and so we can choose a positive integer n satisfying

$$\mathrm{d}_\pi \operatorname{vol}(\Lambda) \leq n.$$

Since Λ is discrete, Λ is a uniform lattice in G. By Proposition 3.14, (π, Λ) admits a n-multiwindow frame $\{\pi(\lambda)\eta_i\}_{\lambda\in\Lambda, 1\leq i\leq n}$ for $\mathcal{H}_\pi$. Thus there exist constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha\|\xi\|^2 \leq \sum_i \sum_\lambda |\langle \xi, \pi(\lambda)\eta_i\rangle|^2 \leq \beta\|\xi\|^2$$

for all $\xi \in \mathcal{H}_\pi$. We can take an orthonormal set $\{\xi_i : 1 \leq i \leq n\}$ in $\mathcal{H}_\pi$. Set

$$A := \sum_i \xi_i \otimes \eta_i.$$

Then $A \in \mathcal{F}(\mathcal{H}_\pi)$. For any $\xi \in \mathcal{H}_\pi$, we compute that

$$\sum_\lambda \|A\pi(\lambda)\xi\|^2 = \sum_i \sum_\lambda |\langle \xi, \pi(\lambda)\eta_i\rangle|^2.$$

Hence $A\pi(\Lambda)$ is a FR-operator-valued frame. This completes the proof. □

4 Operator-Valued Systems Indexed by the Entire Group *G*

In this section, we study the operator-valued systems indexed by the entire group. To study their properties better, we introduce the notion of the wavelet transform for Hilbert–Schmidt operators and give some basic properties about this wavelet transform.

Fix $\eta \in \mathcal{H}_\pi$. By Proposition 1.2, the system $\pi(G)\eta$ indexed by the entire group is a tight frame for $\mathcal{H}_\pi$ with bound $\frac{\|\eta\|^2}{\mathrm{d}_\pi}$. As a generalization of this fact, the first theorem suggests that HS-operator-valued systems indexed by the entire group must be tight operator-valued frames.

Theorem 4.1 *Let $A \in \mathcal{B}(\mathcal{H}_\pi)$. Then the system $A\pi(G)$ is a tight operator-valued frame with bound $\frac{\|A\|_2^2}{\mathrm{d}_\pi}$ if and only if $A \in \mathcal{S}_2(\mathcal{H}_\pi)$.*

Proof For the proof of the necessary part, we use Theorem 3.1. Conversely, we suppose that $A \in \mathcal{S}_2(\mathcal{H}_\pi)$ and $\{\xi_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}_\pi$, where $\mathbb{I}$ is a finite or countable set. Then $\|A\|_2 < \infty$. For any $\eta \in \mathcal{H}_\pi$ and any $x \in G$, we have that

$$\|A\pi(x)\eta\|^2 = \sum_i |\langle A^*\xi_i, \pi(x)\eta\rangle|^2.$$

By Proposition 1.2, we obtain that

$$\int_G |\langle A^*\xi_i, \pi(x)\eta\rangle|^2 dx = \frac{\|A^*\xi_i\|^2\|\eta\|^2}{\mathrm{d}_\pi}$$

for all $i \in \mathbb{I}$. Hence we conclude that

$$\int_G \|A\pi(x)\eta\|^2 dx = \frac{\|A\|_2^2}{\mathrm{d}_\pi}\|\eta\|^2.$$

Thus $A\pi(G)$ is a tight HS-operator-valued frame with bound $\frac{\|A\|_2^2}{\mathrm{d}_\pi}$. □

Definition 4.2 (Wavelet transform [5]) Let $(\pi, \mathcal{H}_\pi)$ be a σ-projective unitary representation of locally compact group G. Given $\eta, \xi \in \mathcal{H}_\pi$. The function $\phi_{\eta,\xi} : G \to \mathbb{C}$ is given by

$$\phi_{\eta,\xi}(x) = \langle \xi, \pi(x)\eta\rangle$$

for all $x \in G$. This function $\phi_{\eta,\xi}$ is called the *wavelet transform of η with respect to ξ*.

Based on Definition 4.2, we introduce the wavelet transform for Hilbert–Schmidt operators. This wavelet transform is closely related to square-integrable representations of locally compact groups.

Definition 4.3 (Wavelet transform for Hilbert–Schmidt operators) Let $(\pi, \mathcal{H}_\pi)$ be a square-integrable representation of G with formal dimension $\mathrm{d}_\pi > 0$, and let $A, B \in \mathcal{S}_2(\mathcal{H}_\pi)$. The function $\Phi_{A,B} : G \to \mathbb{C}$ is given by

$$\Phi_{A,B}(x) = \langle B, A\pi(x)\rangle_2$$

for all $x \in G$. This function $\Phi_{A,B}$ is called the *wavelet transform of A with respect to B*.

We remark that the notion of the wavelet transform for Hilbert–Schmidt operators is a generalization of the classical wavelet transform. Indeed, if $\rho, \eta, \xi \in \mathcal{H}_\pi$ and $\|\rho\| = 1$, then the wavelet transform of $\rho \otimes \xi$ with respect to $\rho \otimes \eta$ is precisely the wavelet transform of η with respect to ξ. To see this, for any $x \in G$, we have that

$$\Phi_{\rho\otimes\xi,\rho\otimes\eta}(x) = \langle \rho \otimes \eta, (\rho \otimes \xi)\pi(x)\rangle_2 = \langle \pi(x)^*\xi, \eta\rangle = \phi_{\eta,\xi}(x).$$

Next we collect some basic properties about the wavelet transform for Hilbert–Schmidt operators as follows.

Proposition 4.4 *Let $A, B, C, D \in \mathcal{S}_2(\mathcal{H}_\pi)$. Then*

(i) $\Phi_{AB,C} = \Phi_{B,A^*C}$ *and* $\Phi_{A,BC} = \Phi_{B^*A,C}$. *Moreover, this wavelet transform is conjugate-linear in the first variable and linear in the second variable in the sense that*

$$\Phi_{\alpha A+B,C} = \overline{\alpha}\Phi_{A,C} + \Phi_{B,C}, \quad \Phi_{A,\beta C+D} = \beta\Phi_{A,C} + \Phi_{A,D}$$

hold for all $\alpha, \beta \in \mathbb{C}$, where $\overline{\alpha}$ is the complex conjugate of α.

(ii) *For any $x, y \in G$, we have that*

$$\Phi_{A,B\pi(x)}(y) = \overline{\sigma(xy^{-1}, y)\Phi_{B,A}(xy^{-1})}.$$

(iii) *The function $\Phi_{A,B}$ is a bounded function on G and*

$$\|\Phi_{A,B}\|_\infty \le \|A\|_2\|B\|_2,$$

where $\|\Phi_{A,B}\|_\infty$ is the sup-norm of $\Phi_{A,B}$.

(iv) *The function $\Phi_{A,B}$ belongs to $L^2(G)$ and*

$$\|\Phi_{A,B}\|^2 = \frac{\|A^*B\|_2^2}{\mathrm{d}_\pi}.$$

More generally, we obtain a generalized Moyal identity:

$$\langle \Phi_{A,B}, \Phi_{C,D}\rangle = \frac{\langle A^*B, C^*D\rangle_2}{\mathrm{d}_\pi}.$$

Proof (i) It is immediate from Definition 4.3.

(ii) For any $x, y \in G$, we see that

$$\Phi_{A,B\pi(x)}(y) = \mathrm{tr}(B\pi(x)\pi(y)^* A^*).$$

Since π is a σ-projective unitary representation, we have that

$$\pi(x)\pi(y)^* = \overline{\sigma(xy^{-1}, y)}\pi(xy^{-1}).$$

Thus

$$\begin{aligned}\Phi_{A,B\pi(x)}(y) &= \overline{\sigma(xy^{-1}, y)}\mathrm{tr}(B\pi(xy^{-1})A^*)\\ &= \overline{\sigma(xy^{-1}, y)}\langle B\pi(xy^{-1}), A\rangle_2\\ &= \overline{\sigma(xy^{-1}, y)\Phi_{B,A}(xy^{-1})}.\end{aligned}$$

(iii) By the Cauchy–Schwarz inequality, we obtain that

$$|\Phi_{A,B}(x)|^2 \le \|B\|_2^2\|A\pi(x)\|_2^2 = \|A\|_2^2\|B\|_2^2 < \infty$$

for all $x \in G$. So $\Phi_{A,B}$ is a bounded function on G and

$$\|\Phi_{A,B}\|_\infty \le \|A\|_2\|B\|_2.$$

(iv) By Definition 4.3, we know that

$$\int_G \Phi_{A,B}(x)\overline{\Phi_{C,D}(x)}dx = \int_G \langle B, A\pi(x)\rangle_2\langle C\pi(x), D\rangle_2 dx.$$

We assume that $\{\xi_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}_\pi$, where $\mathbb{I}$ is a finite or countable set. For any $x \in G$, we see that

$$\langle B, A\pi(x)\rangle_2\langle C\pi(x), D\rangle_2 = \sum_{i,j}\langle B\xi_i, A\pi(x)\xi_i\rangle\langle C\pi(x)\xi_j, D\xi_j\rangle.$$

By Proposition 1.2, we have that

$$\int_G \langle A^*B\xi_i, \pi(x)\xi_i\rangle\langle\pi(x)\xi_j, C^*D\xi_j\rangle dx = \frac{\langle A^*B\xi_i, C^*D\xi_j\rangle\langle\xi_j, \xi_i\rangle}{\mathrm{d}_\pi}$$

for all $i, j \in \mathbb{I}$. Thus by Tonelli's Theorem, we obtain that

$$\int_G \Phi_{A,B}(x)\overline{\Phi_{C,D}(x)}dx = \sum_{i,j}\int_G \langle B\xi_i, A\pi(x)\xi_i\rangle\langle C\pi(x)\xi_j, D\xi_j\rangle dx$$
$$= \frac{1}{\mathrm{d}_\pi}\sum_i\sum_j\langle D^*CA^*B\xi_i, \xi_j\rangle\langle\xi_j, \xi_i\rangle.$$

Since $\{\xi_j : j \in \mathbb{I}\}$ is an orthogonormal basis for $\mathcal{H}_\pi$, the equality

$$\sum_j\langle D^*CA^*B\xi_i, \xi_j\rangle\langle\xi_j, \xi_i\rangle = \langle D^*CA^*B\xi_i, \xi_i\rangle$$

holds for all $i \in \mathbb{I}$. Thus we conclude that

$$\int_G \Phi_{A,B}(x)\overline{\Phi_{C,D}(x)}dx = \frac{1}{\mathrm{d}_\pi}\langle A^*B, C^*D\rangle_2.$$

Hence

$$\int_G |\Phi_{A,B}(x)|^2 dx = \frac{\|A^*B\|_2^2}{\mathrm{d}_\pi} < \infty,$$

which implies that $\Phi_{A,B} \in L^2(G)$ and

$$\|\Phi_{A,B}\|^2 = \frac{\|A^*B\|_2^2}{\mathrm{d}_\pi}.$$

Moreover, we also obtain that

$$\langle\Phi_{A,B}, \Phi_{C,D}\rangle = \frac{\langle A^*B, C^*D\rangle_2}{\mathrm{d}_\pi}.$$

The proof is completed. □

Remark 4.5 Fix an unit vector $\rho \in \mathcal{H}_\pi$. Then for any $\xi, \xi', \eta, \eta' \in \mathcal{H}_\pi$, take

$$A = \rho\otimes\xi, B = \rho\otimes\eta, C = \rho\otimes\xi', D = \rho\otimes\eta'.$$

On the one hand, we see that

$$\langle\Phi_{\rho\otimes\xi,\rho\otimes\eta}, \Phi_{\rho\otimes\xi',\rho\otimes\eta'}\rangle = \langle\phi_{\eta,\xi}, \phi_{\eta',\xi'}\rangle = \int_G \langle\xi, \pi(x)\eta\rangle\langle\pi(x)\eta', \xi'\rangle dx.$$

On the other hand, by Proposition 4.4 (iv), we obtain that

$$\langle\Phi_{\rho\otimes\xi,\rho\otimes\eta}, \Phi_{\rho\otimes\xi',\rho\otimes\eta'}\rangle = \frac{1}{\mathrm{d}_\pi}\langle(\rho\otimes\xi)^*(\rho\otimes\eta), (\rho\otimes\xi')^*(\rho\otimes\eta')\rangle_2.$$

We compute that

$$\langle(\rho \otimes \xi)^*(\rho \otimes \eta), (\rho \otimes \xi')^*(\rho \otimes \eta')\rangle_2 = \langle \xi, \xi'\rangle\langle \eta', \eta\rangle.$$

Hence this conclusion is a generalization of the classical Moyal identity.

Theorem 4.6 *Let $A, B \in \mathcal{S}_2(\mathcal{H}_\pi)$. If they satisfy the relation*

$$\langle A, B\rangle_2 = \mathrm{d}_\pi,$$

then the systems $A\pi(G)$ and $B\pi(G)$ are HS-dual. Furthermore, we have a reproducing formula:

$$\int_G \langle \pi(x)^* B^* A\pi(x)\xi, \eta\rangle dx = \langle \xi, \eta\rangle$$

for all $\xi, \eta \in \mathcal{H}_\pi$.

Proof By Theorem 4.1, the systems $A\pi(G)$ and $B\pi(G)$ are two tight operator-valued frames. We suppose that $\{\xi_i\}_{i\in\mathbb{I}}$ is an orthonormal basis for $\mathcal{H}_\pi$, where $\mathbb{I}$ is a finite or countable set.

For any $\xi, \eta \in \mathcal{H}_\pi$ and any $x \in G$, we have that

$$\langle A\pi(x)\xi, B\pi(x)\eta\rangle = \sum_i \langle A\pi(x)\xi, \xi_i\rangle\langle \xi_i, B\pi(x)\eta\rangle.$$

Thus by Tonelli's Theorem,

$$\int_G \langle \pi(x)^* B^* A\pi(x)\xi, \eta\rangle dx = \sum_i \int_G \langle \pi(x)\xi, A^*\xi_i\rangle\langle B^*\xi_i, \pi(x)\eta\rangle dx.$$

By Proposition 1.2, we see that

$$\int_G \langle B^*\xi_i, \pi(x)\eta\rangle\langle \pi(x)\xi, A^*\xi_i\rangle dx = \frac{\langle B^*\xi_i, A^*\xi_i\rangle\langle \xi, \eta\rangle}{\mathrm{d}_\pi}.$$

We compute that

$$\langle A, B\rangle_2 = \sum_i \langle B^*\xi_i, A^*\xi_i\rangle.$$

Thus we obtain that

$$\int_G \langle \pi(x)^* B^* A\pi(x)\xi, \eta\rangle dx = \langle A, B\rangle_2 \frac{\langle \xi, \eta\rangle}{\mathrm{d}_\pi} = \langle \xi, \eta\rangle.$$

This finishes the proof. □

Theorem 4.7 *If $A, B, C \in \mathcal{S}_2(\mathcal{H}_\pi)$, then we have a reproducing formula:*

$$\mathrm{d}_\pi \int_G \langle C, A\pi(x)\rangle_2 \langle B\pi(x), D\rangle_2 dx = \langle BA^* C, D\rangle_2$$

for all $D \in \mathcal{S}_2(\mathcal{H}_\pi)$.

Proof For any $D \in \mathcal{S}_2(\mathcal{H}_\pi)$, by the generalized Moyal identity, we see that

$$\mathrm{d}_\pi \int_G \langle C, A\pi(x)\rangle_2 \langle B\pi(x), D\rangle_2 dx = \langle A^* C, B^* D\rangle_2.$$

We also have that

$$\langle A^* C, B^* D\rangle_2 = \mathrm{tr}(D^* BA^* C) = \langle BA^* C, D\rangle_2.$$

□

Set

$$\mathrm{supp}\Phi_{A,B} := \mathrm{cl}(\{x \in G : \Phi_{A,B}(x) \neq 0\}),$$

where $\mathrm{cl}(K)$ is the closure of the subset K of G. The set $\mathrm{supp}\Phi_{A,B}$ is called the *support* of the wavelet transform of A with respect to B.

Theorem 4.7 has the following straight corollary. This corollary shows that the measure of the support of the wavelet transform for Hilbert–Schmidt operators has a special lower bound.

Proposition 4.8 *Let A, B be nonzero Hilbert–Schmidt operators on $\mathcal{H}_\pi$. If the set* $\mathrm{supp}\Phi_{A,B}$ *has a strictly positive measure, then it satisfies the inequality*

$$\mu_G(\mathrm{supp}\Phi_{A,B}) \geq \frac{1}{\mathrm{d}_\pi} \frac{\|A^* B\|_2^2}{\|A\|_2^2 \|B\|_2^2}.$$

Proof We suppose that $\mathrm{supp}\Phi_{A,B}$ has a strict positive measure. Then $\|\Phi_{A,B}\|_\infty > 0$. By Theorem 4.7, we have that

$$\|A^* B\|_2^2 = \mathrm{d}_\pi \|\Phi_{A,B}\|^2$$

and

$$\|\Phi_{A,B}\|_\infty \leq \|A\|_2 \|B\|_2.$$

Since $\mathrm{supp}\Phi_{A,B} = \mathrm{cl}(\{x \in G : \Phi_{A,B}(x) \neq 0\})$, we have that

$$\begin{aligned}\|\Phi_{A,B}\|^2 &= \int_{\mathrm{supp}\Phi_{A,B}} |\Phi_{A,B}(x)|^2 dx \\ &\leq \|\Phi_{A,B}\|_\infty^2 \mu_G(\mathrm{supp}\Phi_{A,B}).\end{aligned}$$

Thus we conclude that

$$\mu_G(\mathrm{supp}\Phi_{A,B}) \geq \frac{\|\Phi_{A,B}\|^2}{\|\Phi_{A,B}\|_\infty^2} \geq \frac{1}{\mathrm{d}_\pi} \frac{\|A^* B\|_2^2}{\|A\|_2^2 \|B\|_2^2}.$$

□

5 Operator-Valued Systems in $\mathcal{S}_2(\mathcal{H}_\pi)$

Interestingly, the system of the form (1.3) indexed by the entire group is a Bessel system for $\mathcal{S}_2(\mathcal{H}_\pi)$. However, there are not such a frame in $\mathcal{S}_2(\mathcal{H}_\pi)$ under some assumptions on the representation space $\mathcal{H}_\pi$.

Theorem 5.1 *Suppose that B is a nonzero Hilbert–Schmidt operator on $\mathcal{H}_\pi$ and the representation space $\mathcal{H}_\pi$ is a separable infinite dimensional Hilbert space. Then $B\pi(G)$ indexed by the entire group is a Bessel system for $\mathcal{S}_2(\mathcal{H}_\pi)$ with bound $\frac{\|B\|^2}{\mathrm{d}_\pi}$ in the sense that there exists a positive number β such that*

$$\int_G |\langle A, B\pi(x)\rangle_2|^2 dx \leq \beta \|A\|_2^2$$

for all $A \in \mathcal{S}_2(\mathcal{H}_\pi)$. Moreover, this system $B\pi(G)$ can not be a frame for $\mathcal{S}_2(\mathcal{H}_\pi)$ in the sense of Definition 2.1. That is, there do not exist constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \|A\|_2^2 \leq \int_G |\langle A, B\pi(x)\rangle_2|^2 dx \leq \beta \|A\|_2^2$$

for all $A \in \mathcal{S}_2(\mathcal{H}_\pi)$.

Proof By Theorem 4.7, we obtain that

$$\int_G |\langle A, B\pi(x)\rangle_2|^2 dx = \frac{\|B^* A\|_2^2}{\mathrm{d}_\pi} \leq \frac{\|B\|^2}{\mathrm{d}_\pi} \|A\|_2^2$$

for all $A \in \mathcal{S}_2(\mathcal{H}_\pi)$. Since $0 < \|B\|^2 < \infty$, $B\pi(G)$ is a Bessel system with bound $\frac{\|B\|^2}{\mathrm{d}\ \pi}$.

For the proof of moreover part, we assume that there exists also a positive constant α such that

$$\int_G |\langle A, B\pi(x)\rangle_2|^2 dx \geq \alpha \|A\|_2^2$$

for all $A \in \mathcal{S}_2(\mathcal{H}_\pi)$. For any $\xi \in \mathcal{H}_\pi$, $\xi \otimes \xi \in \mathcal{S}_2(\mathcal{H}_\pi)$. On the one hand, using the above lower bound inequality for the system $B\pi(G)$ yields that

$$\alpha \|\xi \otimes \xi\|_2^2 \leq \int_G |\langle \xi \otimes \xi, B\pi(x)\rangle_2|^2 dx.$$

On the other hand, by Theorem 4.7,

$$\int_G |\langle \xi \otimes \xi, B\pi(x)\rangle_2|^2 dx = \frac{\|B^*\xi \otimes \xi\|_2^2}{\mathrm{d}_\pi}.$$

Thus the inequality

$$\|B^*\xi\|^2 \geq \alpha \mathrm{d}_\pi \|\xi\|^2$$

holds for all $\xi \in \mathcal{H}_\pi$. This implies that B is surjective.

However, since $B \in \mathcal{S}_2(\mathcal{H}_\pi)$, B is a compact operator on $\mathcal{H}_\pi$. Then B is not surjective because $\mathcal{H}_\pi$ is a separable infinite dimensional Hilbert space. We get a contradiction so far. Hence there does not exist such a positive constant. So $B\pi(G)$ is not a frame for $\mathcal{S}_2(\mathcal{H}_\pi)$. This completes the proof. □

Finally, we give the density theorem for the ordinary frame of the following form

$$B\pi(\Lambda) := \{B\pi(\lambda) : \lambda \in \Lambda\},$$

where the generator $B \in \mathcal{S}_2(\mathcal{H}_\pi)$ and the index subgroup Λ is a closed subgroup of locally compact abelian group G.

Theorem 5.2 *Let Λ be a closed subgroup of locally compact abelian group G, and let $B \in \mathcal{S}_2(\mathcal{H}_\pi)$. If $B\pi(\Lambda)$ is a frame for $S_2(\mathcal{H}_\pi)$ with bounds $0 < \alpha \leq \beta < \infty$, then the following statements hold:*

(i) *The index subgroup Λ is co-compact, i.e,* $\mathrm{vol}(\Lambda) < \infty$.
(ii) $\alpha \mathrm{vol}(\Lambda) \leq \frac{\|B\|^2}{\mathrm{d}_\pi} \leq \beta \mathrm{vol}(\Lambda)$.

Proof We suppose that $B\pi(\Lambda)$ is a frame for $S_2(\mathcal{H}_\pi)$ with bounds $0 < \alpha \leq \beta < \infty$. Then it satisfies the inequalities

$$\alpha \|A\|_2^2 \leq \int_\Lambda |\langle A, B\pi(\lambda)\rangle_2|^2 d\lambda \leq \beta \|A\|_2^2$$

for all $A \in \mathcal{S}_2(\mathcal{H}_\pi)$.

(i) By Theorem 4.7, we have that

$$\int_G |\langle A, B\pi(x)\rangle_2|^2 dx = \frac{\|B^* A\|_2^2}{\mathrm{d}_\pi} \le \frac{\|B\|^2 \|A\|_2^2}{\mathrm{d}_\pi} < \infty$$

for all $A \in \mathcal{S}_2(\mathcal{H}_\pi)$. By Weil's identity, we get that

$$\int_G |\langle A, B\pi(x)\rangle_2|^2 dx = \int_{G/\Lambda} \int_\Lambda |\langle A, B\pi(\lambda x)\rangle_2|^2 d\lambda d(x\Lambda).$$

For any $x \in G$ and any $\lambda \in \Lambda$, we compute that

$$|\langle A, B\pi(\lambda x)\rangle_2| = |\langle A\pi(x)^*, B\pi(\lambda)\rangle_2|.$$

Using the lower frame bound inequality for the system $B\pi(\Lambda)$ yields that

$$\int_\Lambda |\langle A\pi(x)^*, B\pi(\lambda)\rangle_2|^2 d\lambda \ge \alpha \|A\|_2^2.$$

Combining the above formulas, we conclude that

$$\frac{\|B\|^2 \|A\|_2^2}{\mathrm{d}_\pi} \ge \int_{G/\Lambda} \alpha \|A\|_2^2 d(x\Lambda) = \alpha \|A\|_2^2 \mathrm{vol}(\Lambda).$$

So

$$\mathrm{vol}(\Lambda) \le \frac{\|B\|^2}{\alpha \mathrm{d}_\pi} < \infty.$$

This implies that Λ is co-compact.

(ii) For every $A \in \mathcal{S}_2(\mathcal{H}_\pi)$, we also have that

$$\begin{aligned} \frac{\|B^* A\|_2^2}{\mathrm{d}_\pi} &= \int_{G/\Lambda} \int_\Lambda |\langle A\pi(x)^*, B\pi(\lambda)\rangle_2| d\lambda d(x\Lambda) \\ &\le \int_{G/\Lambda} \beta \|A\pi(x)^*\|_2^2 d(x\Lambda) \\ &= \beta \|A\|_2^2 \mathrm{vol}(\Lambda), \end{aligned}$$

where we use the the upper frame bound inequality for the system $B\pi(\Lambda)$. For any $\xi, \xi_0 \in \mathcal{H}_\pi$ with $\|\xi_0\| = 1$, we obtain that

$$\frac{\|B^*\xi \otimes \xi_0\|_2^2}{\mathrm{d}_\pi} \leq \beta \|\xi \otimes \xi_0\|_2^2 \mathrm{vol}(\Lambda).$$

That is, $\frac{\|B^*\xi\|^2}{\mathrm{d}_\pi} \leq \beta \|\xi\|^2 \mathrm{vol}(\Lambda)$ and so

$$\alpha \mathrm{vol}(\Lambda) \leq \frac{\|B\|^2}{\mathrm{d}_\pi} \leq \beta \mathrm{vol}(\Lambda).$$

This completes the proof. □

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