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Approximation by Subsequences of Matrix Transform Means of Some Two-Dimensional Rectangle Walsh–Fourier Series

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Abstract
In the present paper we discuss the rate of the approximation by the matrix transform of special partial sums of some two-dimensional rectangle (decreasing diagonal) Walsh-Fourier series in $L^p(G^2)$ space $(1 \leq p < \infty)$ and in $C(G^2)$. It implies in some special case

$$\sigma_{2^n}^{\boxslash T}(f) = \sum_{k=0}^{2^n-1} t_{k,2^n-1} S_{2^n-k,k}(f) \to f$$

norm convergence. We also show an application of our results for Lipschitz functions. At the end of the paper we show the most important result, the almost everywhere convergence theorem. We note that T summation is a common generalization of the following known summation methods Cesàro, Weierstrass, Riesz and Picar and Bessel methods.

Keywords Character system · Fourier series · Walsh-Paley system · Rate of approximation · Modulus of continuity · Norm convergence · Almost everywhere convergence · Lipschitz functions · Matrix transform

Mathematics Subject Classification 42C10

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1 Definitions and Notations

Let $\mathbb{P}$ be the set of positive natural numbers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let denote the discrete cyclic group of order 2 by $\mathbb{Z}_2$. The group operation is the modulo 2 addition. Let every subset be open. The normalized Haar measure μ on $\mathbb{Z}_2$ is given in the way that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. That is, the measure of a singleton is $1/2$. $G := \mathop{\times}\limits_{k=0}^{\infty} \mathbb{Z}_2$, G is called the Walsh group. The elements of Walsh group G are the 0, 1 sequences. That is, $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0, 1\}$ $(k \in \mathbb{N})$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure μ is the product measure and the topology is the product topology. For an other topology on the Walsh group see e.g. [9].

Dyadic intervals are defined in the usual way

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}$$

for $x \in G$, $n \in \mathbb{P}$ and denote $I_n := I_n(0)$. Intervals form a base for the neighbourhoods of G. Denote by $\mathcal{A}_n$ the σ-algebra generated by the intervals $I_n(x)$. That is, $\mathcal{A}_n := \{I_n(x) : x \in G\}$ $(n \in \mathbb{N})$.

We will use notations $\bar{I} := G \setminus I$ for any $I \subseteq G$ set, and $J_n := I_n \setminus I_{n+1}$ for any interval, where $n \in \mathbb{N}$.

Let $L^p(G)$ denote the usual Lebesgue spaces on G (with the corresponding norm $\|.\|_p$).

For the sake of brevity in notation, we agree to write L^∞ instead of C and set $\|f\|_\infty := \sup\{|f(x)| : x \in G\}$. Of course, it is clear that the space L^∞ is not the same as the space of continuous functions, i.e. it is a proper subspace of it. But since in the case of continuous functions the supremum norm and the L^∞ norm are the same, for convenience we hope the reader will be able to tolerate this simplification in notation.

Now, we introduce some concepts of Walsh-Fourier analysis. The Rademacher functions are defined as

$$r_n(x) := (-1)^{x_n} \quad (x \in G, n \in \mathbb{N}).$$

The sequence of the Walsh-Paley functions is the product system of the Rademacher functions. Namely, every natural number n can uniquely be expressed in the number system based 2, in the form

$$n = \sum_{k=0}^{\infty} n_k 2^k, \quad n_k \in \{0, 1\} \ (k \in \mathbb{N}),$$

where only a finite number of n_k's different from zero. We will use the notation $n^{(s)} := \sum_{k=s}^{\infty} n_k 2^k$, where $s \in \mathbb{N}$ and also that $n \oplus m = \sum_{k=0}^{\infty} |n_k - m_k| 2^k$ $(n, m \in \mathbb{N})$.

Let the order of $n > 0$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$. It means $2^{|n|} \leq n < 2^{|n|+1}$. The Walsh-Paley functions are $w_0(x) := 1$ and for $n \in \mathbb{P}$

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = (-1)^{\sum_{k=0}^{|n|} n_k x_k}.$$

Let $\mathcal{P}_n$ be the collection of Walsh polynomials of order less than n, that is, functions of the form

$$P(x) = \sum_{k=0}^{n-1} a_k w_k(x),$$

where $n \in \mathbb{P}$ and $\{a_k\}$ is a sequence of complex numbers, and let $\mathcal{P}_{n,m}$ be the collection of two dimensional Walsh polynomials

$$Q(x^1, x^2) = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} a_{k,j} w_k(x^1) w_j(x^2),$$

where $n, m \in \mathbb{P}$ and $\{a_{k,j}\}$ is a two dimensional sequence of complex numbers.

It is known [23] that the Walsh-Paley system $(w_n, n \in \mathbb{N})$ is the character system of $(G, +)$.

The partial modulus of continuity are defined by

$$\omega_p^1(f, \delta) := \sup_{|t|<\delta} \| f(. + t, ..) - f(., ..)\|_p ,$$
$$\omega_p^2(f, \delta) := \sup_{|t|<\delta} \| f(., .. + t) - f(., ..)\|_p ,$$

and

$$\omega_p(f, \delta) := \omega_p^1(f, \delta) + \omega_p^2(f, \delta)$$

for $f \in L^p(G^2)$, where $\delta > 0$ with the notation

$$|x| := \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} \quad \text{for all } x \in G.$$

In the case $f \in C(G^2)$ we change p by ∞. The mixed modulus of continuity is defined as follows

$$\omega_p^{1,2}(f, \delta_1, \delta_2) :=$$
$$\sup_{|t^1|\leq\delta_1, |t^2|\leq\delta_2} \left\| f(. + t^1, . + t^2) - f(. + t^1, .) - f(., . + t^2) + f(., .) \right\|_p ,$$

where $\delta_1, \delta_2 > 0$.

It is known, that

$$\omega_p^{1,2}(f, \delta, \delta) \le \omega_p(f, \delta). \tag{1.1}$$

The Lipschitz classes in $L^p(G^2)$ for each $\alpha > 0$ are defined by

$$\mathrm{Lip}(\alpha, p, G^2) := \{f \in L^p(G^2) : \omega_p(f, \delta) = O(\delta^\alpha) \text{ as } \delta \to 0\}.$$

Moreover,

$$\mathrm{Lip}(\alpha, C(G^2)) := \{f \in C(G^2) : \ |f(x+y) - f(x)| \le c|y|^\alpha, \ x, y \in G^2\},$$

where for $y = (y^1, y^2) \in G^2$ we define $|y|$ by $|y|^2 := |y^1|^2 + |y^2|^2$.

Further, for the simplicity we write $\mathrm{Lip}(\alpha, \infty, G^2) := \mathrm{Lip}(\alpha, C(G^2))$.

The (i, j)th Fourier-coefficient, the (k, l)th rectangular partial sum of the two dimensional Fourier series and the nth Dirichlet kernel is defined by

$$\hat{f}(i, j) := \int_{G^2} f(x^1, x^2) w_i(x^1) w_j(x^2) d\mu(x^1, x^2),$$

$$S_{k,l}(f)(x^1, x^2) := \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} \hat{f}(i, j) w_i(x^1) w_j(x^2),\ D_n := \sum_{k=0}^{n-1} w_k,\ \ D_0 := 0.$$

For two-dimensional variable $x = (x^1, x^2) \in G^2$ we use the notations

$$w_n^i := w_n(x^i), \quad r_n^i := r_n(x^i), \quad D_n^i := D_n(x^i), \quad K_n^i := K_n(x^i),$$

for any $n \in \mathbb{N}$, where $i \in \{1, 2\}$.

Fejér kernels are defined as the arithmetical means of Dirichlet kernels, that is,

$$K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k,$$

Marcinkiewicz kernels are defined as

$$\mathcal{K}_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k^1 D_k^2$$

and the nth decreasing diagonal mean is the following

$$\sigma_n^{\boxslash}(f) := \frac{1}{n} \sum_{k=0}^{n-1} S_{n-k,k}(f),$$

where $n \in \mathbb{P}$.

Let $T := \left(t_{i,j}\right)_{i,j=0}^{\infty}$ be a doubly infinite matrix of numbers. It is always supposed that matrix T is triangular. Let us define the nth decreasing diagonal matrix transform mean determined by the matrix T

$$\sigma_n^{\boxslash T}(f) := \sum_{k=0}^{n-1} t_{k,n-1} S_{n-k,k}(f),$$

where $\{t_{k,n-1} : 0 \leq k \leq n-1,\ k \in \mathbb{N}\}$ be a finite sequence of non-negative numbers for each $n \in \mathbb{P}$.

The nth matrix transform decreasing diagonal kernel is defined by

$$K_n^{\boxslash T} := \sum_{k=0}^{n-1} t_{k,n-1} D_{n-k}^1 D_k^2.$$

It can be seen easily seen that

$$\sigma_n^{\boxslash T}(f)(x) = \int_{G^2} f(u) K_n^{\boxslash T}(u+x) d\mu(u),$$

where $x := (x^1, x^2) \in G^2$ and $u := (u^1, u^2) \in G^2$. This equality (and its analogous versions for special means) shows us the necessity of observing kernel functions.

We introduce the notation $\Delta t_{k,n} := t_{k,n} - t_{k+1,n}$, where $k \in \{0, \ldots, n\}$, $n \in \mathbb{N}$ and $t_{n+1,n} := 0$. In Sect. 7, we give some examples of known summation methods that fall within the scope of the T summation discussed in this paper.

More material on the theory of multidimensional Fourier series can be found in Weisz's book [43].

2 Connection to Triangular Means

The triangular partial sums of the two-dimensional Walsh-Fourier series are defined as

$$S_k^{\triangle}(f)(x^1, x^2) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} \hat{f}(i,j) w_i(x^1) w_j(x^2). \tag{2.1}$$

The reason that (2.1) is called triangular partial sums is that geometrically double partial sums are defined on triangles. See Herriot [22], Weisz [43], Karagulyan and Muradian [26] and Bakhvalov [1].

The triangular (C, α) means is defined as follows

$$\sigma_n^{\triangle \alpha}(f) := \frac{1}{A_{n-1}^{\alpha}} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} S_k^{\triangle}(f),$$

where $A_n^\alpha = \frac{(1+\alpha)\dots(n+\alpha)}{n!}$ and $\alpha \neq -1, -2, \dots$.

It is easily obtained (see e.g. [16]) that

$$\sigma_n^{\triangle}(f) := \sigma_n^{\triangle 1}(f) = \sigma_n^{\boxbslash}(f), \tag{2.2}$$

so matrix transform decreasing diagonal summation is a kind of generalization of the triangular Fejér summation. This property disappears, however, for all other $\alpha \neq 1$. That is, the summation we introduce differs from the triangular summation. Matrix transformations of triangular partial sums are defined as follows:

$$\sigma_n^{\triangle T}(f) := \sum_{k=0}^{n-1} t_{k,n-1} S_k^{\triangle}(f)$$

This happens to coincide with our definition of the decreasing diagonal mean in the case of $t_{k,n-1} = 1/n$ (for all k). In other cases it does not.

Of course, it would be interesting to examine the convergence properties of the generalized triangle summation method. In this paper, we present a convergence theorem based on the comments and suggestions of one of the referees of our article.

3 Historical Notes

Matrix transform means are common generalizations of several well-known summation methods. It follows by simple consideration that the Nörlund means, the Fejér (or the $(C, 1)$) and the (C, α) means are special cases of the matrix transform summation method introduced above.

Our paper is motivated by the work of Móricz, Siddiqi [30] on the Walsh–Nörlund summation method, the result of Móricz and Rhoades [29] on the Walsh weighted mean method and work of Chripkó [10] on Jacobi-Fourier series. As special cases, Móricz and Siddiqi obtained the earlier results given by Yano [46], Jastrebova [24] and Skvortsov [36] on the rate of the approximation by Cesàro means. The approximation properties of the Walsh-Cesàro means of negative order were studied by Goginava [18], the Vilenkin case was investigated by Shavardenidze [35] and Tepnadze [37]. Common generalizations of these two results of Móricz and Siddiqi [30] and Móricz and Rhoades [29] was given by Nagy and the author [4, 5].

In 2008, Fridli, Manchanda and Siddiqi generalized the result of Móricz and Siddiqi for homogeneous Banach spaces and dyadic Hardy spaces [12]. Recently, L. Baramidze, D. Baramidze, Memić, Persson, Tephnadze and Wall presented some results with respect to this topic [2, 3, 7, 27, 28]. See [11, 40], as well. For two-dimensional results see [6, 8, 31, 32].

For the trigonometric system Herriot proved [22] the a.e. (and norm) convergence $\sigma_n^{\triangle}(f) \to f$ $(f \in L^1)$. The first result for triangular means on Walsh-Paley system is due to Goginava and Weisz [21]. They proved and each integrable function the a.e. convergence relation $\sigma_{2^n}^{\triangle}(f) \to f$ $(f \in L^1)$. Later Gát verified [16] Herriot's result with respect to the Walsh system. See also papers of Weisz [39–42, 44], Karagulyan and Muradian [26] and Bakhvalov [1]. The main difficulty is that in the trigonometric case we have a simple closed formula for the kernel functions of this triangular means and this is not the case in the Walsh situation.

We also note that it could also be interesting to investigate summation methods where the corresponding kernel function is not the weighted mean of the $D^1_{n-k}D^2_k$ two-variable functions. But some $D^1_{\alpha_1(n,k)}D^2_{\alpha_2(n,k)}$ functions. In this direction (just in the case of $t_{k,n-1} = 1/n$) one of the authors has some a.e. convergence results in the article [15] for the Walsh-Paley system. No version of these results in [15] is currently known for the trigonometric system.

4 Auxiliary Results

To prove our theorems we need the following results.

Lemma 1 (Paley's Lemma [34], p. 7.) *For $n \in \mathbb{N}$ we have*

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}$$

Lemma 2 ([34], p. 34) *For $j, n \in \mathbb{N},\ j < 2^n$ we have*

$$D_{2^n+j} = D_{2^n} + r_n D_j.$$

The next lemma is also a simple one. It can be found in several articles in the literature. See for example article [25].

Lemma 3 *For $j, n \in \mathbb{N},\ j < 2^n$ we have*

$$D_{2^n-j} = D_{2^n} - w_{2^n-1} D_j.$$

Lemma 4 ([34], p. 28.) *For $n \in \mathbb{N}$ we have*

$$D_n(x) = w_n(x) \sum_{k=0}^{\infty} n_k r_k(x) D_{2^k}(x).$$

Corollary 1 *If $x \in J_v := I_k \backslash I_{k+1}$ and $v, n \in \mathbb{N}$, then we have*

$$D_n(x) = w_{n^{(v)}}(x) \left(\sum_{k=0}^{v-1} n_k 2^k - n_v 2^v \right).$$

Proof Using Lemma 1 for $x \in J_v$ we get

$$\sum_{k=0}^{\infty} n_k (-1)^{x_k} D_{2^k}(x) = \sum_{k=0}^{v-1} n_k (-1)^0 2^k + n_v (-1)^1 2^v + \sum_{k=v+1}^{\infty} n_k (-1)^{x_k} 0$$

$$= \sum_{k=0}^{v-1} n_k 2^k - n_v 2^v.$$

On the other side,

$$w_n(x) = (-1)^{\sum_{k=0}^{|n|} n_k x_k} = (-1)^{\sum_{k=0}^{v-1} n_k 0 + \sum_{k=v}^{|n|} n_k x_k} = (-1)^{\sum_{k=v}^{|n|} n_k x_k} = w_{n^{(v)}}(x).$$

Hence Lemma 4 proves this statement. □

Lemma 5 [45] *The norm of the Fejér kernel is bounded uniformly. That is, if $n \in \mathbb{P}$, then*

$$\|K_n\|_1 \leq 2.$$

In 2018 Toledo [38] improved this result, but for our proof the knowledge of the exact supremum of $\|K_n\|_1$ is not necessary, just its boundedness.

Lemma 6 [17] *Let $\alpha_1, \ldots, \alpha_n, q \in \mathbb{R}$ and $1 < q \leq 2$. Then*

$$\frac{1}{n} \left\| \sum_{k=1}^{n} \alpha_k D_k^1 D_k^2 \right\|_1 \leq c n^{-1/q} \left(\sum_{k=1}^{n} \alpha_k^q \right)^{1/q}.$$

Corollary 2 *The norm of Marcinkiewicz kernel is bounded uniformly, namely there exists a positive constant c such that*

$$\|\mathcal{K}_n\|_1 \leq c \quad \text{for all} n \in \mathbb{N}.$$

Proof It implies from Lemma 6 immediately, choosing $\alpha_1 = \cdots = \alpha_n = 1$. □

Lemma 7 [19]

$$\int_{\bar{I}_r} \sup_{|n| \geq b} |K_n(x)| d\mu(x) \leq c \frac{b-r}{2^{b-r}},$$

for all $r \leq b \in \mathbb{N}$.

From now we will use notation

$$F(x, u) := f(x+u) - f(x).$$

Lemma 8 [33] *Let $P \in \mathcal{P}_{2^A}$, $f \in L^p(G^2)$, where $A, B \in \mathbb{P}$ and $1 \leq p \leq \infty$. Then there exists a positive constant c such that*

$$\left\| \int_{G^2} r_A(u^1) D_{2^B}(u^2) P(u^1) F(., u) d\mu(u) \right\|_p \leq c \|P\|_1 \omega_p^1(f, 2^{-A}).$$

Corollary 3 *Let $f \in L^p(G^2)$, where $A, B \in \mathbb{P}$ and $1 \le p \le \infty$. Then there exists a positive constant c such that*

$$\left\| \int_{G^2} r_A(u^i) D_{2^B}(u^{3-i}) K_j(u^i) \left(w_{2^n-1}(u^1) F(., u) \right) d\mu(u) \right\|_p \le c\omega_p^i(f, 2^{-A}),$$

for $j \le 2^A$ and $A < n$, where $i \in \{1, 2\}$.

Proof Since $|w_{2^n-1}| = 1$, and in the definition of the modulus of continuity we find the absolute value of expression $F(., u)$ (because of the L^p norm), as well as $K_j \in \mathcal{P}_{2^A}$ if $j \le 2^A$ and $\|K_j\|_1 \le c$ (see Lemma 5) we obtain this statement by Lemma 8 immediately. □

Lemma 9 [32] *Let $Q \in \mathcal{P}_{2^A, 2^A}$, $f \in L^p(G^2)$, where $A \in \mathbb{P}$ and $1 \le p \le \infty$. Then there exists a positive constant c such that*

$$\left\| \int_{G^2} r_A(u^1) r_A(u^2) Q(u) F(., u) d\mu(u) \right\|_p \le c\|Q\|_1 \omega_{1,2}^p(f, 2^{-A}, 2^{-A}).$$

Corollary 4 *Let $f \in L^p(G^2)$, where $A \in \mathbb{P}$ and $1 \le p \le \infty$. Then there exists a positive constant c such that*

$$\left\| \int_{G^2} r_A(u^1) r_A(u^2) \mathcal{K}_j(u) \left(w_{2^n-1}(u^1) F(., u) \right) d\mu(u) \right\|_p \le c\omega_{1,2}^p(f, 2^{-A}, 2^{-A}),$$

for $j \le 2^A$ and $A < n$.

Proof Similarly to Corollary 3, but it is proved by Lemma 9. □

In the next lemma, we give a decomposition of the kernels $K_{2^n}^{\boxslash T}$.

Lemma 10 *Let n be a positive integer, then we have*

$$\begin{aligned} K_{2^n}^{\boxslash T} = {} & D_{2^n}^1 \sum_{j=0}^{n-1} D_{2^j}^2 \sum_{k=0}^{2^j-1} t_{2^j+k, 2^n-1} + D_{2^n}^1 \sum_{j=0}^{n-1} r_j^2 \sum_{k=1}^{2^j-2} k K_k^2 \Delta t_{2^j+k, 2^n-1} \\ & + D_{2^n}^1 \sum_{j=0}^{n-1} r_j^2 (2^j - 1) K_{2^j-1}^2 t_{2^{j+1}-1, 2^n-1} - w_{2^n-1}^1 \sum_{j=0}^{n-1} D_{2^j}^1 D_{2^j}^2 \sum_{k=0}^{2^j-1} t_{2^j+k, 2^n-1} \\ & - w_{2^n-1}^1 \sum_{j=0}^{n-1} r_j^2 D_{2^j}^1 \sum_{k=1}^{2^j-2} k K_k^2 \Delta t_{2^j+k, 2^n-1} - w_{2^n-1}^1 \sum_{j=0}^{n-1} r_j^2 D_{2^j}^1 (2^j - 1) K_{2^j-1}^2 t_{2^{j+1}-1, 2^n-1} \\ & - w_{2^n-1}^1 \sum_{j=0}^{n-1} r_j^1 D_{2^j}^2 \sum_{k=1}^{2^j-2} k K_k^1 \Delta t_{2^j+k, 2^n-1} - w_{2^n-1}^1 \sum_{j=0}^{n-1} r_j^1 D_{2^j}^2 (2^j - 1) K_{2^j-1}^1 t_{2^{j+1}-1, 2^n-1} \end{aligned}$$

$$- w^1_{2^n-1}\sum_{j=0}^{n-1} r^1_j r^2_j \sum_{k=1}^{2^j-2} k\mathcal{K}_k \Delta t_{2^j+k,2^n-1} - w^1_{2^n-1}\sum_{j=0}^{n-1} r^1_j r^2_j (2^j-1)\mathcal{K}_{2^j-1} t_{2^{j+1}-1,2^n-1}$$

$$=: \sum_{j=1}^{10} K_{j,n}.$$

Proof We can write

$$K^{\boxslash T}_{2^n} = \sum_{j=0}^{n-1}\sum_{l=2^j}^{2^{j+1}-1} t_{l,2^n-1} D^1_{2^n-l} D^2_l .$$

Lemma 3 means for us

$$D^1_{2^n-l} = D^1_{2^n} - w^1_{2^n-1} D^1_l .$$

From Lemma 2 we get

$$\begin{aligned}
\sum_{l=2^j}^{2^{j+1}-1} t_{l,2^n-1} D^1_{2^n-l} D^2_l &= \sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1}\left(D^1_{2^n} - w^1_{2^n-1} D^1_{2^j+k}\right) D^2_{2^j+k}\\
&= D^1_{2^n} D^2_{2^j} \sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1} + D^1_{2^n} r^2_j \sum_{k=1}^{2^j-1} t_{2^j+k,2^n-1} D^2_k\\
&\quad - w^1_{2^n-1}\sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1} D^1_{2^j+k} D^2_{2^j+k}.
\end{aligned}$$

Using Abel-transform

$$\sum_{k=1}^{2^j-1} t_{2^j+k,2^n-1} D^2_k = \sum_{k=1}^{2^j-2} \Delta t_{2^j+k,2^n-1} k K^2_k + t_{2^{j+1}-1,2^n-1}(2^j-1) K^2_{2^j-1}. \tag{4.1}$$

So

$$\begin{aligned}
K^{\boxslash T}_{2^n} &= D^1_{2^n}\sum_{j=0}^{n-1} D^2_{2^j}\sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1}\\
&\quad + D^1_{2^n}\sum_{j=0}^{n-1} r^2_j\left(\sum_{k=1}^{2^j-2}\Delta t_{2^j+k,2^n-1} k K^2_k + t_{2^{j+1}-1,2^n-1}(2^j-1)K^2_{2^j-1}\right)\\
&\quad - w^1_{2^n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1} D^1_{2^j+k} D^2_{2^j+k}.
\end{aligned}$$

Let us decompose the last expression. Using Lemma 2 twice

$$
\begin{aligned}
\sum_{j=0}^{n-1}\sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1} D^1_{2^j+k} D^2_{2^j+k} &= \sum_{j=0}^{n-1} D^1_{2^j} D^2_{2^j} \sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1} \\
&+ \sum_{j=0}^{n-1} r^2_j D^1_{2^j} \sum_{k=1}^{2^j-1} t_{2^j+k,2^n-1} D^2_k \\
&+ \sum_{j=0}^{n-1} r^1_j D^2_{2^j} \sum_{k=1}^{2^j-1} t_{2^j+k,2^n-1} D^1_k \\
&+ \sum_{j=0}^{n-1} r^1_j r^2_j \sum_{k=1}^{2^j-1} t_{2^j+k,2^n-1} D^1_k D^2_k.
\end{aligned}
$$

Applying (4.1) we have for $i \in \{1,2\}$

$$
\sum_{k=1}^{2^j-1} t_{2^j+k,2^n-1} D^i_k = \sum_{k=1}^{2^j-2} \Delta t_{2^j+k,2^n-1} k K^i_k + t_{2^{j+1}-1,2^n-1}(2^j-1) K^i_{2^j-1}
$$

and

$$
\sum_{k=1}^{2^j-1} t_{2^j+k,2^n-1} D^1_k D^2_k = \sum_{k=1}^{2^j-2} \Delta t_{2^j+k,2^n-1} k \mathcal{K}_k + t_{2^{j+1}-1,2^n-1}(2^j-1)\mathcal{K}_{2^j-1}.
$$

Summarizing these it completes the proof of Lemma 10. □

Theorem 1 requires Lemma 10. But to prove the norm convergence, the following simpler decomposition suffices.

Lemma 11 *Let n be a positive integer, then we have*

$$
\begin{aligned}
K^{\boxslash T}_{2^n} &= D^1_{2^n} \sum_{l=1}^{2^n-2} \Delta t_{l,2^n-1} l K^2_l + D^1_{2^n} t_{2^n-1,2^n-1}(2^n-1) K^2_{2^n-1} \\
&\quad - w^1_{2^n-1} \sum_{l=1}^{2^n-2} \Delta t_{l,2^n-1} l \mathcal{K}_l - w^1_{2^n-1} t_{2^n-1,2^n-1}(2^n-1)\mathcal{K}_{2^n-1} \\
&=: \sum_{j=1}^{4} K'_{j,n}.
\end{aligned}
$$

Proof From Lemma 3 we get

$$\begin{aligned} K_{2^n}^{\boxslash T} &= \sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_{2^n-l} D^2_l \\ &= \sum_{l=0}^{2^n-1} t_{l,2^n-1} \left(D^1_{2^n} - w^1_{2^n-1} D^1_l \right) D^2_l \\ &= D^1_{2^n} \sum_{l=0}^{2^n-1} t_{l,2^n-1} D^2_l - w^1_{2^n-1} \sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_l D^2_l . \end{aligned}$$

Using Abel-transform, equalities

$$\sum_{l=1}^{2^n-1} t_{l,2^n-1} D^2_l = \sum_{l=1}^{2^n-2} \Delta t_{l,2^n-1} l K^2_l + t_{2^n-1,2^n-1} (2^n-1) K^2_{2^n-1}$$

and

$$\sum_{l=1}^{2^n-1} t_{l,2^n-1} D^1_l D^2_l = \sum_{l=1}^{2^n-2} \Delta t_{l,2^n-1} l \mathcal{K}_l + t_{2^n-1,2^n-1} (2^n-1) \mathcal{K}_{2^n-1} .$$

hold. By summarizing these, the proof of Lemma 11 becomes complete. □

5 The rate of the approximation in norm by subsequences of matrix transform

Theorem 1 *Let $f \in L^p(G^2)$ $(1 \le p \le \infty)$. For every $n \in \mathbb{N}$, $\{t_{k,2^n-1} : 0 \le k \le 2^n-1\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=0}^{2^n-1} t_{k,2^n-1} = 1 \tag{5.1}$$

is satisfied.

(a) *If the finite sequence $\{t_{k,2^n-1} : 0 \le k \le 2^n-1\}$ is non-decreasing for a fixed n, then*

$$\left\| \sigma_{2^n}^{\boxslash T}(f) - f \right\|_p \le c \sum_{j=0}^{n-1} 2^j t_{2^{j+1}-1,2^n-1} \omega_p \left(f, 2^{-j} \right) + c\omega^1_p \left(f, 2^{-n} \right)$$

holds, where the constant c does not depend on n.

(b) *If the finite sequence* $\{t_{k,2^n-1} : 0 \le k \le 2^n - 1\}$ *is non-increasing for a fixed n, then*

$$\left\|\sigma_{2^n}^{\boxslash T}(f) - f\right\|_p \le c \sum_{j=0}^{n-1} 2^j t_{2^j,2^n-1} \omega_p\left(f, 2^{-j}\right) + c\omega_p^1\left(f, 2^{-n}\right)$$

holds.

Remark 1 We mention, that assuming (5.1) is natural, because many well-known means satisfy it and this equality is a part of regularity conditions [47, page 74.].

Proof of Theorem 1 The proof is carried out in cases where $1 \le p < \infty$, while the proof of case $p = \infty$ is similar. Recall that by the case $p = \infty$ we mean that we are considering the space of continuous functions.

During our proofs c denotes a positive constant, which may vary at different appearances.

We use equality (5.1), the usual Minkowski inequality and Lemma 10

$$\begin{aligned}
\left\|\sigma_{2^n}^{\boxslash T}(f) - f\right\|_p &= \left(\int_{G^2} \left|\sigma_{2^n}^{\boxslash T}(f;x) - f(x)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&= \left(\int_{G^2} \left|\int_{G^2} K_{2^n}^{\boxslash T}(u) F(x,u) d\mu(u)\right|^p d\mu(x)\right)^{\frac{1}{p}} \\
&\le \sum_{j=1}^{10} \left\|\int_{G^2} K_{j,n}(u) F(.,u) d\mu(u)\right\|_p \\
&=: \sum_{j=1}^{10} I_{j,n}.
\end{aligned}$$

Using generalized Minkowski's inequality [47, vol. 1, p. 19], inequality

$$|F(x,u)| \le |f(x^1+u^1, x^2+u^2) - f(x^1+u^1, x^2)| + |f(x^1+u^1, x^2) - f(x^1, x^2)|,$$

Lemma 1 and equality (5.1) we write that

$$\begin{aligned}
I_{1,n} &\le \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1} \left\|\int_{G^2} D_{2^n}(u^1) D_{2^j}(u^2) F(.,u) d\mu(u)\right\|_p \\
&\le \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1} \int_{G^2} D_{2^n}(u^1) D_{2^j}(u^2) \left(\int_{G^2} |F(x,u)|^p d\mu(x)\right)^{\frac{1}{p}} d\mu(u) \\
&\le c\omega_p^1(f, 2^{-n}) + c \sum_{j=0}^{n-1} \omega_p^2\left(f, 2^{-j}\right) \sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1}.
\end{aligned}$$

Now, in case a)

$$I_{1,n} \le c\omega_p^1(f, 2^{-n}) + c\sum_{j=0}^{n-1} 2^j t_{2^{j+1}-1,2^n-1}\omega_p^2\left(f, 2^{-j}\right),$$

in case b)

$$I_{1,n} \le c\omega_p^1(f, 2^{-n}) + c\sum_{j=0}^{n-1} 2^j t_{2^j,2^n-1}\omega_p^2\left(f, 2^{-j}\right).$$

Recall that

$$\left|w_{2^n-1}(u^1)F(x,u)\right| = |F(x,u)|,$$

similarly to estimate of $I_{1,n}$ we get

$$\begin{aligned}
I_{4,n} &\le \sum_{j=0}^{n-1}\sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1}\left\|\int_{G^2} D_{2^j}(u^1)D_{2^j}(u^2)\left(w_{2^n-1}(u^1)F(.+u)\right)d\mu(u)\right\|_p \\
&\le \sum_{j=0}^{n-1}\sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1}\int_{G^2} D_{2^j}(u^1)D_{2^j}(u^2)\left(\int_{G^2}|F(x,u)|^p\,d\mu(x)\right)^{\frac{1}{p}} d\mu(u) \\
&\le c\sum_{j=0}^{n-1}\left(\omega_p^1\left(f, 2^{-j}\right) + \omega_p^2\left(f, 2^{-j}\right)\right)\sum_{k=0}^{2^j-1} t_{2^j+k,2^n-1}.
\end{aligned}$$

Now, in case (a)

$$I_{4,n} \le c\sum_{j=0}^{n-1} 2^j t_{2^{j+1}-1,2^n-1}\left(\omega_p^1\left(f, 2^{-j}\right) + \omega_p^2\left(f, 2^{-j}\right)\right),$$

in case b)

$$I_{4,n} \le c\sum_{j=0}^{n-1} 2^j t_{2^j,2^n-1}\left(\omega_p^1\left(f, 2^{-j}\right) + \omega_p^2\left(f, 2^{-j}\right)\right).$$

For expressions $I_{2,n}$ and $I_{3,n}$ usual Minkowski's inequality yields

$$I_{2,n} \le \sum_{j=0}^{n-1}\sum_{k=0}^{2^j-2}\left|\Delta t_{2^j+k,2^n-1}\right| k\left\|\int_{G^2} r_j(u^2)D_{2^n}(u^1)K_k(u^2)F(.,u)d\mu(u)\right\|_p,$$

$$I_{3,n} \le \sum_{j=0}^{n-1} t_{2^{j+1}-1,2^n-1}(2^j-1)\left\| \int_{G^2} r_j(u^2) D_{2^n}(u^1) K_{2^j-1}(u^2) F(.,u) d\mu(u) \right\|_p .$$

From Lemmas 5 and 8 we write

$$\begin{aligned} I_{2,n} &\le c \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-2} \left| \Delta t_{2^j+k,2^n-1} \right| k \| K_k \|_1 \omega_p^2(f, 2^{-j}) \\ &\le c \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-2} \left| \Delta t_{2^j+k,2^n-1} \right| k \omega_p^2(f, 2^{-j}), \end{aligned}$$

$$\begin{aligned} I_{3,n} &\le c \sum_{j=0}^{n-1} t_{2^{j+1}-1,2^n-1}(2^j-1) \| K_{2^j-1} \|_1 \omega_p^2(f, 2^{-n}) \\ &\le c \sum_{j=0}^{n-1} t_{2^{j+1}-1,2^n-1} 2^j \omega_p^2(f, 2^{-j}). \end{aligned}$$

At first, let us observe expression $I_{2,n}$. In case a.) we have

$$\begin{aligned} \sum_{k=0}^{2^j-2} \left| \Delta t_{2^j+k,2^n-1} \right| k &= \sum_{k=0}^{2^j-2} (t_{2^j+k+1,2^n-1} - t_{2^j+k,2^n-1}) k \\ &= (2^j-2) t_{2^{j+1}-1,2^n-1} - \sum_{k=0}^{2^j-2} t_{2^j+k,2^n-1} \\ &\le (2^j-2) t_{2^{j+1}-1,2^n-1} \end{aligned}$$

and

$$I_{2,n} \le c \sum_{j=0}^{n-1} 2^j t_{2^{j+1}-1,2^n-1} \omega_p^2 \left(f, 2^{-j} \right).$$

In case b.)

$$\begin{aligned} \sum_{k=0}^{2^j-2} \left| \Delta t_{2^j+k,2^n-1} \right| k &= \sum_{k=0}^{2^j-2} t_{2^j+k,2^n-1} - (2^j-2) t_{2^{j+1}-1,2^n-1} \\ &\le \sum_{k=0}^{2^j-2} t_{2^j+k,2^n-1} \le 2^j t_{2^j,2^n-1} \end{aligned}$$

and

$$I_{2,n} \leq c \sum_{j=0}^{n-1} 2^j t_{2^j,2^n-1} \omega_p^2 \left(f, 2^{-j}\right).$$

Now, we discuss expression $I_{3,n}$.

$$I_{3,n} \leq c \sum_{j=0}^{n-1} 2^j t_{2^{j+1}-1,2^n-1} \omega_p^2(f, 2^{-j}),$$

so we are ready in case a.). From this inequality in case b.) we obtain

$$I_{3,n} \leq c \sum_{j=0}^{n-1} 2^j t_{2^j,2^n-1} \omega_p^2(f, 2^{-j})$$

immediately.

It follows from Corollary 3 that

$$\begin{aligned} I_{5,n} \leq& \sum_{j=0}^{n-1} \sum_{k=1}^{2^j-2} \left|\Delta t_{2^j+k,2^n-1}\right| k \times \\ &\times \left\| \int_{G^2} r_j(u^2) D_{2^j}(u^1) K_k(u^2) \left(w_{2^n-1}(u^1) F(., u)\right) d\mu(u) \right\|_p \\ \leq& c \sum_{j=0}^{n-1} \sum_{k=1}^{2^j-2} \left|\Delta t_{2^j+k,2^n-1}\right| k \omega_p^1(f, 2^{-j}), \end{aligned}$$

similarly to expression $I_{2,n}$, as in the next case.

$$\begin{aligned} I_{7,n} \leq& \sum_{j=0}^{n-1} \sum_{k=1}^{2^j-2} \left|\Delta t_{2^j+k,2^n-1}\right| k \times \\ &\times \left\| \int_{G^2} r_j(u^1) D_{2^j}(u^2) K_k(u^1) \left(w_{2^n-1}(u^1) F(., u)\right) d\mu(u) \right\|_p \\ \leq& c \sum_{j=0}^{n-1} \sum_{k=1}^{2^j-2} \left|\Delta t_{2^j+k,2^n-1}\right| k \omega_p^2(f, 2^{-j}), \end{aligned}$$

and this is what we obtained estimating $I_{3,n}$. And so on

$$I_{6,n} \leq \sum_{j=1}^{n-1} (2^j - 1) t_{2^{j+1}-1,2^n-1} \times$$

$$\times \left\| \int_{G^2} r_j(u^2) D_{2^j}(u^1) K_{2^j-1}(u^2) \left(w_{2^n-1}(u^1) F(., u) \right) d\mu(u) \right\|_p$$
$$\leq c \sum_{j=1}^{n-1} 2^j t_{2^{j+1}-1,2^n-1} \omega_p^2(f, 2^{-j}).$$

It means that we got exactly the same expression, as in case $I_{2,n}$.

$$I_{8,n} \leq \sum_{j=1}^{n-1} (2^j - 1) t_{2^{j+1}-1,2^n-1} \times$$
$$\times \left\| \int_{G^2} r_j(u^1) D_{2^j}(u^2) K_{2^j-1}(u^1) \left(w_{2^n-1}(u^1) F(., u) \right) d\mu(u) \right\|_p$$
$$\leq c \sum_{j=1}^{n-1} 2^j t_{2^{j+1}-1,2^n-1} \omega_p^1(f, 2^{-j})$$

just in case $I_{6,n}$, but ω_p^1 instead of ω_p^2 in it.

At the and, let us consider Corollary 4 for $I_{9,n}$ and $I_{10,n}$.

$$I_{9,n} \leq \sum_{j=0}^{n-1} \sum_{k=1}^{2^j-2} \left| \Delta t_{2^j+k,2^n-1} \right| k \times$$
$$\times \left\| \int_{G^2} r_j(u^1) r_j(u^2) \mathcal{K}_k(u) \left(w_{2^n-1}(u^1) F(., u) \right) d\mu(u) \right\|_p$$
$$\leq c \sum_{j=0}^{n-1} \sum_{k=1}^{2^j-2} \left| \Delta t_{2^j+k,2^n-1} \right| k \omega_p^{1,2}(f, 2^{-j}, 2^{-j}),$$

similarly to $I_{2,n}$ and other expressions.

$$I_{10,n} \leq \sum_{j=1}^{n-1} (2^j - 1) t_{2^{j+1}-1,2^n-1} \times$$
$$\times \left\| \int_{G^2} r_j(u^1) D_{2^j}(u^2) \mathcal{K}_{2^j-1}(u) \left(w_{2^n-1}(u^1) F(., u) \right) d\mu(u) \right\|_p$$
$$\leq c \sum_{j=1}^{n-1} 2^j t_{2^{j+1}-1,2^n-1} \omega_p^{1,2}(f, 2^{-j}, 2^{-j}),$$

like is case $I_{3,n}$ and so on.

Summarising these facts, considering Inequality (1.1) the Theorem 1 is proved. □

6 Application of the Norm Estimation

In this section, we apply Theorem 1 for Lipschitz functions.

Theorem 2 *Let $f \in \mathrm{Lip}(\alpha, p, G^2)$ for some $\alpha > 0$ and $1 \le p \le \infty$. For matrix transform $\sigma_{2^n}^{\boxslash T}$ we suppose that the conditions in Theorem 1 are satisfied.*

(a) *Let us suppose, that equality*

$$t_{2^n-1,2^n-1} = O(2^{-n}) \tag{6.1}$$

holds. Then next estimate is true

$$\left\|\sigma_{2^n}^{\boxslash T}(f) - f\right\|_p = \begin{cases} O(2^{-n\alpha}), & \text{if } 0 < \alpha < 1, \\ O(n2^{-n}), & \text{if } \alpha = 1, \\ O(2^{-n}), & \text{if } \alpha > 1. \end{cases}$$

(b) *The equality*

$$\left\|\sigma_{2^n}^{\boxslash T}(f) - f\right\|_p = O\left(\sum_{j=0}^{n-1} t_{2^j,2^n-1} 2^{j(1-\alpha)}\right) + O(2^{-n\alpha})$$

holds.

Proof In part a), Condition (6.1) yields that

$$\begin{aligned} \left\|\sigma_{2^n}^{\boxslash T}(f) - f\right\|_p &= O\left(\sum_{j=0}^{n-1} 2^j t_{2^{j+1}-1,2^n-1} 2^{-j\alpha} + 2^{-n\alpha}\right) \\ &= O\left(t_{2^n-1,2^n-1} \sum_{j=0}^{n-1} 2^{j(1-\alpha)} + 2^{-n\alpha}\right) \\ &= O\left(2^{-n} \sum_{j=0}^{n-1} 2^{j(1-\alpha)}\right) + O(2^{-n\alpha}). \end{aligned}$$

From this form we get our statement easily.

At last, we discuss part b).

$$\begin{aligned} \left\|\sigma_{2^n}^{\boxslash T}(f) - f\right\|_p &= O\left(\sum_{j=0}^{n-1} 2^j t_{2^j,2^n-1} 2^{-j\alpha} + 2^{-n\alpha}\right) \\ &= O\left(\sum_{j=0}^{n-1} t_{2^j,n} 2^{j(1-\alpha)}\right) + O(2^{-n\alpha}). \end{aligned}$$

□

7 Boundedness of the $L^1(G^2)$ Norm of 2^nth Decreasing Diagonal Matrix Transform Kernel, Norm Convergence

Theorem 3 *For every $n \in \mathbb{N}$ let $\{t_{k,2^n-1} : 0 \le k \le 2^n - 1\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=0}^{2^n-1} t_{k,2^n-1} = O(1) \tag{7.1}$$

is satisfied.

a) If the finite sequence $\{t_{k,2^n-1} : 0 \le k \le 2^n - 1\}$ is non-decreasing for a fixed n and the condition

$$t_{2^n-1,2^n-1} = O\left(2^{-n}\right) \tag{7.2}$$

is satisfied, or

b) if the finite sequence $\{t_{k,2^n-1} : 0 \le k \le 2^n - 1\}$ is non-increasing for a fixed n, then the $L^1(G^2)$ norm of the 2^nth T decreasing diagonal kernel is bounded uniformly. Namely,

$$\left\| K_{2^n}^{\boxbslash T} \right\|_1 \le c$$

holds.

Proof of Theorem 3 We use Lemma 11 and the usual Minkowski's inequality

$$\begin{aligned} \left\| K_{2^n}^{\boxbslash T} \right\|_1 &\le \sum_{j=1}^{4} \left\| K'_{j,n} \right\|_1 \\ &=: \sum_{j=1}^{4} J_{j,n}. \end{aligned}$$

For expressions $J_{1,n}$ and $J_{2,n}$ it yields

$$J_{1,n} \le \sum_{l=1}^{2^n-2} \left|\Delta t_{l,2^n-1}\right| l \int_{G^2} \left| D_{2^n}(u^1) K_l(u^2) \right| d\mu(u),$$

$$J_{2,n} \le t_{2^n-1,2^n-1}(2^n - 1) \int_{G^2} \left| D_{2^n}(u^1) K_{2^n-1}(u^2) \right| d\mu(u).$$

From Lemmas 1 and 5 we write

$$\begin{aligned} J_{1,n} &\le c \sum_{l=1}^{2^n-2} \left|\Delta t_{l,2^n-1}\right| l, \\ J_{2,n} &\le c t_{2^n-1,2^n-1}(2^n - 1) \le c. \end{aligned}$$

Let us observe expression $J_{1,n}$. In case a.) we obtain with help of Condition (7.2)

$$\begin{aligned} J_{1,n} &\leq c\left((2^n-1)t_{2^n-1,2^n-1} - \sum_{l=1}^{2^n-1} t_{l,2^n-1}\right) \\ &\leq c2^n t_{2^n-1,2^n-1} \leq c. \end{aligned}$$

In case b.) from Condition (7.1)

$$\begin{aligned} J_{1,n} &\leq c\left(\sum_{l=1}^{2^n-1} t_{l,2^n-1} - (2^n-1)t_{2^n-1,2^n-1}\right) \\ &\leq c\sum_{l=0}^{2^n-1} t_{l,2^n-1} \leq c. \end{aligned}$$

Let us consider Corollary 2 for $J_{3,n}$ and $J_{4,n}$. We have

$$\begin{aligned} J_{3,n} &\leq \sum_{l=1}^{2^n-2} \left|\Delta t_{l,2^n-1}\right| l \int_{G^2} \left|w_{2^n-1}(u^1)\mathcal{K}_l(u)\right| d\mu(u) \\ &\leq c\sum_{l=1}^{2^n-2} \left|\Delta t_{l,2^n-1}\right| l \leq c, \end{aligned}$$

similarly to $J_{1,n}$, and

$$\begin{aligned} J_{4,n} &\leq (2^n-1)t_{2^n-1,2^n-1} \int_{G^2} \left|w_{2^n-1}(u^1)\mathcal{K}_{2^j-1}(u)\right| d\mu(u) \\ &\leq c(2^n-1)t_{2^n-1,2^n-1} \leq c, \end{aligned}$$

like in the case of $J_{2,n}$.

This completes the proof of Theorem 3. □

Theorem 4 *For every $n \in \mathbb{N}$, $\{t_{k,2^n-1} : 0 \leq k \leq 2^n-1\}$ be a finite sequence of non-negative numbers such that*

$$\sum_{k=0}^{2^n-1} t_{k,2^n-1} = 1 \tag{7.3}$$

is satisfied.

(a) *If the finite sequence $\{t_{k,2^n-1} : 0 \leq k \leq 2^n-1\}$ is non-decreasing for a fixed n and the condition*

$$t_{2^n-1,2^n-1} = O\left(2^{-n}\right) \tag{7.4}$$

is satisfied, or

(b) *if the finite sequence* $\{t_{k,2^n-1} : 0 \leq k \leq 2^n - 1\}$ *is non-increasing for a fixed n and suppose that*

$$\lim_{n\to\infty} t_{0,2^n-1} = 0. \tag{7.5}$$

Then have the $L^1(G^2)$ *and* $C(G^2)$*-norm convergence*

$$\sigma_{2^n}^{\boxbslash T}(f) \to f.$$

Remark 2 This norm convergence in $L^p(G^2)$ for $1 < p < \infty$ also holds. It is a trivial corollary of the norm convergence of the two dimensional rectangular partial sums. Of course we also need Conditions (7.3) and in case b) (7.5) (but do not need to assume (7.4) in neither case).

Remark 3 In the non-decreasing situation statement of (7.5) trivially holds.

Proof of Theorem 4 The proof is a simple consequence of Theorem 3 and the known fact, that the set of two dimensional Walsh polynomials is dense in $L^p(G^2)$ for each $1 \leq p < \infty$. Besides, for any Walsh polynomial P we have $S_{2^n-k,k}(P) = P$ for sufficiently large $n_0 \leq n$ and appropriate $2^{n_0} \leq k_0 \leq k < k_1 < 2^n$. Then, using Condition (7.5) we have

$$\begin{aligned}\sigma_{2^n}^{\boxbslash T}(P) &= \sum_{k=0}^{k_0-1} t_{k,2^n-1} S_{2^n-k,k}(P) \\ &\quad + \sum_{k=k_0}^{k_1-1} t_{k,2^n-1} P \\ &\quad + \sum_{k=k_1}^{2^n-1} t_{k,2^n-1} S_{2^n-k,k}(P) \\ &\to P\end{aligned}$$

in norm and also everywhere. □

8 Almost Everywhere Convergence

Theorem 5 *Let* $f \in L^1(G^2)$. *For every* $n \in \mathbb{N}$ *let* $\{t_{k,2^n-1} : 0 \leq k \leq 2^n - 1\}$ *be a finite sequence of non-negative numbers with properties*

$$\sum_{k=0}^{2^n-1} t_{k,2^n-1} = 1 \tag{8.1}$$

and

$$\sum_{k=0}^{2^n-1} t_{k,2^n-1}^2 = O(2^{-n}). \tag{8.2}$$

If $n \to \infty$, then

$$\sigma_{2^n}^{\boxbslash T}(f) \to f$$

almost everywhere.

In particular, we would like to emphasize that in Theorem 5 the sequence $\{t_{k,2^N-1} : 0 \le k \le 2^N - 1\}$ is not assumed to be monotone. We only assume that the sum of these non-negative numbers is 1 and the sum of their squares is $O(2^{-N})$.

We now present some examples of known summation methods that fall within the scope of the T summation discussed in this paper and in this Theorem 5. Basically, we have to check property (8.2), that is, $\sum_{k=0}^{n-1} t_{k,n-1}^2 \le c/n$ because all the other properties are trivially fulfilled. That is:

(1) The Cesàro (or (C, α)) summation. Let $\alpha > 1/2$ and $A_n^\alpha = \frac{(1+\alpha)\dots(n+\alpha)}{n!}$,

$$\sigma_n^\alpha f := \frac{1}{A_{n-1}^\alpha} \sum_{k=0}^{n-1} A_{n-1-k}^{\alpha-1} S_k f.$$

That is, let $t_{k,n} = \frac{A_{n-k-1}^{\alpha-1}}{A_{n-1}^\alpha}$. It is quite obvious that all the required properties (in this paper) hold for this summation method. Especially,

$$\sum_{k=0}^{n-1} t_{k,n-1}^2 \le \sum_{k=0}^{n-1} c_\alpha \frac{(n-k)^{2\alpha-2}}{n^\alpha} \le \frac{c_\alpha}{n^\alpha} \sum_{k=1}^{n} k^{2\alpha-2} \le c_\alpha/n.$$

(In general Cesàro summation is valid for any $\alpha \ne -1, -2, \dots,$ but property (8.2) needs restriction $\alpha > 1/2$.)

(2) The Weierstrass summation with parameter $\gamma > 1/2$. The n-th Weierstrass mean is defined as

$$\sum_{k=0}^{n} e^{-\left(\frac{k}{n+1}\right)^\gamma} \hat{f}(k) w_k.$$

That is, let $t_{k,n} = \Delta e^{-\left(\frac{k}{n+1}\right)^\gamma} = -\frac{1}{n+1} e^{-u^\gamma}(-1)\gamma u^{\gamma-1}$, where u is between $k/(n+1)$ and $(k+1)/(n+1)$. Then $t_{k,n} \le \frac{\gamma}{n+1}\left(\frac{k}{n+1}\right)^{\gamma-1}$ and this gives

$$\sum_{k=0}^{n-1} t_{k,n-1}^2 \le \frac{c}{n^{2\gamma}} \sum_{k=1}^{n-1} \frac{1}{k^{2-2\gamma}} \le \frac{c}{n^{2\gamma}} n^{2\gamma-1} = c/n.$$

(3) The Picar and Bessel summation with parameters $\alpha\gamma > 1, \gamma > 1/2$. The n-th Picar and Bessel mean is defined as

$$\sum_{k=0}^{n} \frac{1}{\left(1+\left(\frac{k}{n+1}\right)^{\gamma}\right)^{\alpha}} \hat{f}(k) w_k.$$

(Special cases $\alpha = 1$ means the Picar and $\gamma = 2$ means the Bessel summation.) Then let

$$t_{k,n} = \Delta \frac{1}{\left(1+\left(\frac{k}{n+1}\right)^{\gamma}\right)^{\alpha}} = \frac{-1}{n+1}(-\alpha)\frac{1}{(1+(u)^{\gamma})^{\alpha+1}}\gamma u^{\gamma-1},$$

where u is between $k/(n+1)$ and $(k+1)/(n+1)$. Then $t_{k,n} \le \alpha\gamma u^{\gamma-1}/(n+1)$. If $\gamma \ge 1$, then $t_{k,n} \le c/n$ and we are finished. On the other hand, in the case of $1 > \gamma > 1/2$ we have

$$\sum_{k=0}^{n-1} t_{k,n-1}^2 \le \frac{c}{n^{2\gamma}} \sum_{k=1}^{n-1} \frac{1}{k^{2-2\gamma}} \le \frac{c}{n^{2\gamma}} n^{2\gamma-1} = c/n.$$

(4) The Riesz summation with parameters $\alpha \ge 1, \gamma > 1/2$. The n-th Riesz mean is defined as

$$\sum_{k=0}^{n} \left(1-\left(\frac{k}{n+1}\right)^{\gamma}\right)^{\alpha} \hat{f}(k) w_k.$$

Then let

$$t_{k,n} = \Delta \left(1-\left(\frac{k}{n+1}\right)^{\gamma}\right)^{\alpha} = \frac{-1}{n+1}\alpha\left(1-(u)^{\gamma}\right)^{\alpha-1}(-1)\gamma u^{\gamma-1},$$

where u is between $k/(n+1)$ and $(k+1)/(n+1)$. Then we have $t_{k,n} \le \frac{c}{n+1}\frac{(n+1)^{1-\gamma}}{k^{1-\gamma}}$ and this gives

$$\sum_{k=0}^{n-1} t_{k,n-1}^2 \le \frac{c}{n^{2\gamma}} \sum_{k=1}^{n-1} \frac{1}{k^{2-2\gamma}} \le \frac{c}{n^{2\gamma}} n^{2\gamma-1} = c/n.$$

The proof of Theorem 5 is based on some lemmas below. To formulate the first one, we need to introduce a new operator and kernel function. Let $h_n(f) := f * H_n$, the definition of the kernel functions H_n is given as.

$$H_n := \left| \sum_{l=0}^{2^n-1} t_{l,2^n-1} D_l^1 D_l^2 \right|. \tag{8.3}$$

In Lemma 13 we prove the operator $h_* := \sup_n |h_n(f)|$ is quasi-local. Before this we need Lemma 12. Let

$$H_{N,a} := \left| \sum_{l=2^a}^{2^N-1} t_{l,2^N-1} D_l^1 D_l^2 \right|.$$

Now, we turn our attention to the mentioned Lemma 12.

Lemma 12 *For every $a, N \in \mathbb{N}$ let $\{t_{k,2^N-1} : 0 \le k \le 2^N - 1\}$ be a finite sequence of non-negative numbers such that Conditions (8.1) and (8.2) are fulfilled. Then we have*

$$\int_{\overline{I}_a \times G} \sup_{N>a} H_{N,a} \le c.$$

In particular, we would like to emphasize that the sequence $\{t_{k,2^N-1} : 0 \le k \le 2^N - 1\}$ is not assumed to be monotone. We only assume that the sum of these non-negative numbers is 1 and their sum of squares is $O(2^{-N})$.

Proof Set $n, N, p \in \mathbb{N}$ as $N = n + p$. We discuss the integral

$$\int_{J_{v^1} \times J_{v^2}} H_{n,a,p} := \int_{J_{v^1} \times J_{v^2}} \left| \sum_{l=2^{n-1}}^{2^n-1} t_{l,2^N-1} D_l^1 D_l^2 \right|,$$

where $v^1 < a, n, p, v^1, v^2 \in \mathbb{N}$ are fixed. We remark that $n > a$ is supposed.

Since

$$\sum_{l=2^{n-1}}^{2^n-1} t_{l,2^N-1} D_l^1 D_l^2 = \sum_{l=0}^{2^{n-1}-1} t_{2^{n-1}+l,2^N-1} D^1_{2^{n-1}+l} D^2_{2^{n-1}+l}$$

and in case of $z = (z^1, z^2) \in J_{v^1} \times J_{v^2}$ from Corollary 1 we get

$$D^1_{2^{n-1}+l}(z^1) = w_{2^{n-1}+l^{(v^1)}}(z^1) \left(\sum_{j=0}^{v^1-1} l_j 2^j - l_{v^1} 2^{v^1} \right).$$

By the help of the Cauchy–Buniakovskii inequality we have

$$\int_{J_{v^1} \times J_{v^2}} H_{n,a,p} \le \int_{J_{v^2}} 2^{-v^1/2} \left(\int_{J_{v^1}} \left(\sum_{l=0}^{2^{n-1}-1} t_{2^{n-1}+l,2^N-1} D^1_{2^{n-1}+l} D^2_{2^{n-1}+l} \right)^2 \right)^{1/2} \tag{8.4}$$

In the sequel we investigate the inner integral on the set J_{v^1}. It is,

$$\int_{J_{v^1}} \sum_{k,l=0}^{2^{n-1}-1} t_{2^{n-1}+k,2^N-1} t_{2^{n-1}+l,2^N-1} w_{2^{n-1}+k^{(v^1)}}(z^1) w_{2^{n-1}+l^{(v^1)}}(z^1)$$
$$\times \left(\sum_{j=0}^{v^1-1} k_j 2^j - k_{v^1} 2^{v^1} \right) \left(\sum_{j=0}^{v^1-1} l_j 2^j - l_{v^1} 2^{v^1} \right)$$
$$\times D^2_{2^{n-1}+k}(z^2) D^2_{2^{n-1}+l}(z^2) d\mu(z^1) \tag{8.5}$$
$$= \sum_{k,l=0}^{2^{n-1}-1} t_{2^{n-1}+k,2^N-1} t_{2^{n-1}+l,2^N-1} \left(\sum_{j=0}^{v^1-1} k_j 2^j - k_{v^1} 2^{v^1} \right) \left(\sum_{j=0}^{v^1-1} l_j 2^j - l_{v^1} 2^{v^1} \right)$$
$$\times D^2_{2^{n-1}+k}(z^2) D^2_{2^{n-1}+l}(z^2) \int_{J_{v^1}} w_{2^{n-1}+k^{(v^1)}}(z^1) w_{2^{n-1}+l^{(v^1)}}(z^1) \mu(z^1).$$

The integral $\int\limits_{J_{v^1}} w_{2^{n-1}+k^{(v^1)}}(z^1) w_{2^{n-1}+l^{(v^1)}}(z^1) \mu(z^1)$ can be different from zero only in the case when

$$k^{(v^1)} \oplus l^{(v^1)} = 0.$$

That is, $k_{v^1} = l_{v^1}, k_{v^1+1} = l_{v^1+1}, \ldots$. Use the notation: $\wedge(k,n) := \min(k,n)$. It is easy to see that

$$\left| D^2_{2^{n-1}+k}(z^2) \right| \le c 2^{\wedge(n,v^2)}.$$

That is, for the integral $\int\limits_{J_{v^1}} (\cdot)^2$ we have the following upper estimation:

$$\frac{c}{2^{v^1}} \sum_{k=0}^{2^{n-1}-1} t_{2^{n-1}+k,2^N-1} \sum_{\{l: l_j = k_j, j \ge v^1\}} t_{2^{n-1}+l,2^N-1} 2^{2v^1} 2^{2\wedge(n,v^2)}.$$

Check this sum $\sum_{k=0}^{2^n-1} t_{2^{n-1}+k,2^N-1} \sum_{\{l:l_j=k_j,j\geq v^1\}} t_{2^{n-1}+l,2^N-1}$ out. We prove that it is not bigger than $c2^{v^1}/2^N$. The obvious inequality $ab \leq a^2/2+b^2/2$ and (8.2) gives

$$\begin{aligned}
&\sum_{k=0}^{2^{n-1}-1} t_{2^{n-1}+k,2^N-1} \sum_{\{l:l_j=k_j,j\geq v^1\}} t_{2^{n-1}+l,2^N-1} \\
&= \sum_{k=0}^{2^{n-1}-1} t_{2^{n-1}+k,2^N-1} \sum_{l=0}^{2^{v^1}-1} t_{2^{n-1}+k^{(v^1)}+l,2^N-1} \\
&\leq c \sum_{k=0}^{2^{n-1}-1} \sum_{l=0}^{2^{v^1}-1} \left(t^2_{2^{n-1}+k,2^N-1} + t^2_{2^{n-1}+k^{(v^1)}+l,2^N-1}\right) \\
&\leq c2^{v^1} \sum_{k=0}^{2^n-1} t^2_{2^{n-1}+k,2^N-1} + c \sum_{k_0,\ldots,k_{v^1-1}\in\{0,1\}} \sum_{k_{v^1},\ldots,k_{n-1}\in\{0,1\}} \sum_{l=0}^{2^{v^1}-1} t^2_{2^{n-1}+k^{(v^1)}+l,2^N-1} \\
&\leq c2^{v^1} \sum_{k=0}^{2^{n-1}-1} t^2_{2^{n-1}+k,2^N-1} \leq c2^{v^1} \sum_{k=0}^{2^N-1} t^2_{k,2^N-1} \leq \frac{c2^{v^1}}{2^N}.
\end{aligned} \tag{8.6}$$

This gives an upper estimation for the integral $\int\limits_{J_{v^1}} (\cdot)^2$. More precisely, for (8.5)

$$\begin{aligned}
&\frac{c}{2^{v^1}} \sum_{k=0}^{2^{n-1}-1} t_{2^{n-1}+k,2^n-1} \sum_{\{l:l_j=k_j,j\geq v^1\}} t_{2^{n-1}+l,2^n-1} 2^{2v^1} 2^{2\wedge(n,v^2)} \\
&\leq c\frac{2^{2v^1+2\wedge(n,v^2)}}{2^N}.
\end{aligned}$$

That is,

$$\begin{aligned}
&\int_{J_{v^1}\times J_{v^2}} \left| \sum_{l=2^{n-1}}^{2^n-1} t_{l,2^N-1} D_l^1 D_l^2 \right| \\
&\leq c\int_{J_{v^2}} 2^{-v^1/2} \left[\frac{2^{2v^1+2\wedge(n,v^2)}}{2^N}\right]^{1/2} \\
&\leq c\frac{2^{v^1/2}}{2^{N/2}} \frac{2^{\wedge(n,v^2)}}{2^{v^2}}.
\end{aligned}$$

For symmetry reasons, it can be assumed that $v^1 \leq v^2$. Check the following integral (remember that it was $N = n + p$ above).

$$
\begin{aligned}
&\int_{\overline{I_a}\times G}\sum_{p=0}^{\infty}\sum_{n=a+1}^{\infty}H_{n,a,p}\\
&\sum_{v^1=0}^{a-1}\int_{J_{v^1}\times G}\sum_{p=0}^{\infty}\sum_{n=a+1}^{\infty}H_{n,a,p}\\
&\leq\sum_{p=0}^{\infty}\sum_{n=a+1}^{\infty}\sum_{v^1=0}^{a-1}\sum_{v^2=v^1}^{n-1}\int_{J_{v^1}\times J_{v^2}}\left|\sum_{l=2^{n-1}}^{2^n-1}t_{l,2^{n+p}-1}D_l^1D_l^2\right| \qquad (8.7)\\
&+\sum_{p=0}^{\infty}\sum_{n=a+1}^{\infty}\sum_{v^1=0}^{a-1}\int_{J_{v^1}\times I_n}\left|\sum_{l=2^{n-1}}^{2^n-1}t_{l,2^{n+p}-1}D_l^1D_l^2\right|\\
&=: A_1+A_2.
\end{aligned}
$$

By the above written estimations we have for term A_1:

$$
\begin{aligned}
A_1 &\leq c\sum_{p=0}^{\infty}\sum_{n=a+1}^{\infty}\sum_{v^1=0}^{a-1}\sum_{v^2=v^1}^{n-1}\frac{2^{v^1/2}}{2^{(n+p)/2}}\cdot\frac{2^{v^2}}{2^{v^2}}\\
&\leq c\sum_{p=0}^{\infty}\sum_{n=a+1}^{\infty}\sum_{v^1=0}^{a-1}(n-v^1)\frac{2^{v^1/2}}{2^{n/2+p/2}}\leq c\sum_{p=0}^{\infty}\sum_{n=a+1}^{\infty}(n-a)\frac{2^{a/2}}{2^{n/2+p/2}} \qquad (8.8)\\
&\leq\sum_{p=0}^{\infty}\frac{c}{2^{p/2}}\leq c.
\end{aligned}
$$

On the other hand, by the above written we have

$$
A_2\leq c\sum_{p=0}^{\infty}\sum_{n=a+1}^{\infty}\sum_{v^1=0}^{a-1}\frac{2^{v^1/2}}{2^{n/2+p/2}}\cdot\frac{2^n}{2^n}\leq c\sum_{p=0}^{\infty}\sum_{n=a+1}^{\infty}\frac{2^{a/2}}{2^{n/2+p/2}}\leq\sum_{p=0}^{\infty}\frac{c}{2^{p/2}}\leq c. \qquad (8.9)
$$

Since (remember that we used the notation $N=n+p$ above)

$$
\begin{aligned}
&\sup_{N>a}H_{N,a}\\
&=\sup_{N>a}\left|\sum_{l=2^a}^{2^N-1}t_{l,2^N-1}D_l^1D_l^2\right|\\
&\leq\sum_{p=0}^{\infty}\sum_{n=a+1}^{\infty}\left|\sum_{l=2^{n-1}}^{2^n-1}t_{l,2^{n+p}-1}D_l^1D_l^2\right|
\end{aligned}
$$

$$= \sum_{p=0}^{\infty} \sum_{n=a+1}^{\infty} H_{n,a,p}$$

then the inequalities (8.7), (8.8) and (8.9) complete the proof of Lemma 12. □

Lemma 13 *Suppose that the finite sequence* $\{t_{k,2^N-1} : 0 \leq k \leq 2^N - 1\}$ *satisfies properties (8.1) and (8.2). Then the operator* h_* *is quasi-local.*

Proof of Lemma 13 Because of the shift invariance, we can assume that the square Q is of the following kind: $Q = I_a \times I_a = I_a^2$ for some natural number a. Then

$$\begin{aligned}
&\int_{G^2 \setminus Q} h_*(f) \\
&\leq \int_{\overline{I_a \times I_a}} \sup_{n>a} \left| \int_{I_a \times I_a} f(x) H_{n,a}(x+u) d\mu(x) \right| d\mu(u) \\
&\quad + \int_{\overline{I_a \times I_a}} \sup_{n>a} \left| \int_{I_a \times I_a} f(x) \left| \sum_{l=0}^{2^a-1} t_{l,2^n-1} D_l(x^1+u^1)^1 D_l(x^2+u^2)^2 \right| d\mu(x) \right| d\mu(u) \\
&\quad + \int_{\overline{I_a \times I_a}} \sup_{n \leq a} \left| \int_{I_a \times I_a} f(x) H_n(x+u) d\mu(x) \right| d\mu(u) \\
&\quad =: A_1 + A_2 + A_3.
\end{aligned}$$

Since for $l \leq a$ the function $D_l^1 D_l^2$ is $\mathcal{A}_a^2$ measurable, then we have

$$A_2 = \int_{\overline{I_a \times I_a}} \sup_{n \leq a} \left| \sum_{l=0}^{2^a-1} t_{l,2^n-1} D_l(u^1)^1 D_l(u^2)^2 \right| \left| \int_{I_a \times I_a} f(x) d\mu(x) \right| d\mu(u) = 0$$

as it is given by $\int_{I_a \times I_a} f = 0$. In the same way we also have

$$A_3 = 0.$$

That is,

$$\int_{G^2 \setminus Q} h_*(f) = \int_{\overline{I_a \times I_a}} \sup_{n>a} \left| \int_{I_a \times I_a} f(x) H_{n,a}(x+u) d\mu(x) \right| d\mu(u)$$

Since we integrate on $\overline{I_a \times I_a}$ which is $(\overline{I_a} \times G) \cup (G \times \overline{I_a})$, then

$$\begin{aligned}
&\int_{\overline{I_a \times I_a}} \sup_{n>a} \left| \int_{I_a \times I_a} f(x) H_{n,a}(x+u) d\mu(x) \right| d\mu(u) \\
&\leq \int_{\overline{I_a} \times G} \sup_{n>a} \left| \int_{I_a \times I_a} f(x) H_{n,a}(x+u) d\mu(x) \right| d\mu(u) \\
&+ \int_{G \times \overline{I_a}} \sup_{n>a} \left| \int_{I_a \times I_a} f(x) H_{n,a}(x+u) d\mu(x) \right| d\mu(u) \\
&= B_1 + B_2.
\end{aligned}$$

For the estimation of B_1 we apply Lemma 12.

$$B_1 \leq \int_{I_a \times I_a} |f(x)| \int_{\overline{I_a} \times G} \sup_{n>a} H_{n,a}(u) d\mu(u) d\mu(x) \leq c \|f\|_1.$$

To discuss case term B_2, we apply the Lemma 12 in the same way, we just have to take note that the variables in the kernels $H_n, H_{n,a}$ are interchangeable. That is, $B_2 \leq c\|f\|_1$ as well. This completes the proof of Lemma 13. □

The next lemma we need:

Lemma 14 $\|H_n\|_1 \leq c$.

Proof The proof is basically a direct application of Lemma 12. More precisely, the formulas in (8.7), (8.8) and (8.9). That is,

$$\begin{aligned}
&\|H_{a+1,a,p}\|_1 \\
&\leq \int_{\overline{I_a} \times G} H_{a+1,a,p}(z) d\mu(z) \\
&+ \int_{G \times \overline{I_a}} H_{a+1,a,p}(z) d\mu(z) \\
&+ \int_{I_a \times I_a} H_{a+1,a,p}(z) d\mu(z) =: A_1 + A_2 + A_3.
\end{aligned}$$

Lemma 12 (formulas in (8.7), (8.8)) gives $A_1, A_2 \leq c/2^{p/2}$. Besides, by (8.1), (8.2) and the Cauchy-Buniakovskii inequality

$$\begin{aligned}
A_3 &= \int_{I_a \times I_a} \left| \sum_{l=2^a}^{2^{a+1}-1} t_{l,2^{a+1+p}-1} D_l^1(z^1) D_l^2(z^2) \right| d\mu(z) \\
&\leq c \sum_{l=2^a}^{2^{a+1}-1} t_{l,2^{a+1+p}-1} \leq c 2^{a/2} \left(\sum_{l=2^a}^{2^{a+1}-1} t_{l,2^{a+1+p}-1}^2 \right)^{1/2} \\
&\leq c 2^{a/2} 2^{-a/2-p/2} \leq c 2^{-p/2}.
\end{aligned}$$

That is, $\|H_{a+1,a,p}\|_1 \leq c2^{-p/2}$. Finally,

$$\begin{aligned}\|H_n\|_1 &= \left\| \sum_{l=0}^{2^n-1} t_{l,2^n-1} D_l^1 D_l^2 \right\|_1 \\ &\leq \sum_{a=0}^{n-1} \left\| \sum_{l=2^a}^{2^{a+1}-1} t_{l,2^n-1} D_l^1 D_l^2 \right\|_1 \\ &\leq c \sum_{a=0}^{n-1} 2^{-(n-a)/2} \leq c.\end{aligned}$$

This completes the proof of Lemma 14. □

The next step towards realizing the proof of Theorem 5 is based on Lemma 13 and aims to prove that operator $h_*(f) = \sup_n |f * H_n|$ is of weak type (L^1, L^1).

Lemma 15 *The operator h_* is of weak type (L^1, L^1) and of strong type (L^p, L^p) for each $1 < p \leq \infty$.*

Proof We prove that operator h_* is of strong type (L^∞, L^∞) in the first place. Basically, this property of the operator h_* is a trivial consequence of Lemma 14, the fact that the kernel functions H_n is (uniformly in n) bounded in L^1. That is, the operator h_* is of strong type (L^∞, L^∞) and since it is σ-sublinear, then by a standard argument the fact that it is quasi-local (Lemma 13) gives that it is also of weak type (L^1, L^1). Finally, the interpolation theorem of Marcinkiewicz for sublinear operators completes the proof of Lemma 15. □

Let

$$\sigma_{2^n}^{1,\boxbslash T}(f) := f * K_{2^n}^{1,\boxbslash T} = f * \left| \sum_{l=0}^{2^n-1} t_{l,2^n-1} D_{2^n-l}^1 D_k^2 \right|, \quad \sigma_*^{1,\boxbslash T}(f) := \sup_n \left| \sigma_{2^n}^{1,\boxbslash T}(f) \right|.$$

Recall that

$$\sigma_{2^n}^{\boxbslash T}(f) := f * K_{2^n}^{\boxbslash T} = f * \sum_{l=0}^{2^n-1} t_{l,2^n-1} D_{2^n-l}^1 D_k^2, \quad \sigma_*^{\boxbslash T}(f) := \sup_n \left| \sigma_{2^n}^{\boxbslash T}(f) \right|.$$

We continue to assume only that the $\{t_{l,2^n-1} : 0 \leq l \leq 2^n - 1\}$ fulfils properties (8.1) and (8.2) (no monotonicity of any kind is supposed).

Lemma 16 *The operator $\sigma_*^{1,\boxbslash T}$ is quasi-local, is of weak type (L^1, L^1) and of strong type (L^p, L^p) for each $1 < p \leq \infty$.*

Proof We prove that operator $\sigma_*^{1,\boxbslash T}$ is of strong type (L^∞, L^∞) in the first place. Introduce the finite sequence $\{\tilde{t}_{k,2^n-1} : 0 \leq k \leq 2^n - 1\} = \{t_{2^n-1-k,2^n-1} : 0 \leq$

$k \le 2^n - 1\}$ and apply Lemma 14. The point is that since the only properties for the finite sequence $\{t_{k,2^n-1} : 0 \le k \le 2^n - 1\}$ is supposed (8.1) and (8.2). No monotonicity of any kind. Of course these two properties hold for the "reversed" sequence $\{\tilde{t}_{k,2^n-1} : 0 \le k \le 2^n - 1\}$.

This property of the kernel function of the operator $\sigma_{2^n}^{1,\boxslash T}$ is $\left|\sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_{2^n-l} D^2_l\right| = \left|\sum_{l=0}^{2^n-1} \tilde{t}_{l,2^n-1} D^1_l D^2_{2^n-1}\right|$ and consequently we can estimate its L^1-norm by the help of Lemma 14 as follows. Besides, by Lemma 3 we can also use the fact that

$$\left|\sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_{2^n-l} D^2_l\right| = \left| D^1_{2^n} \sum_{l=0}^{2^n-1} t_{l,2^n-1} D^2_l - w^1_{2^n-1} \sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_l D^2_l \right| = H_n$$

on the set $\overline{I_n} \times G$. Then we have

$$\begin{aligned}
&\left\|\sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_{2^n-l} D^2_l\right\|_1 \\
&\le \int_{\overline{I_{n-1}}\times G} \left|\sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_{2^n-l} D^2_l\right| \\
&\quad + \int_{G\times\overline{I_{n-1}}} \left|\sum_{l=0}^{2^n-1} \tilde{t}_{l,2^n-1} D^1_l D^2_{2^n-l}\right| \\
&\quad + \int_{I_{n-1}\times I_{n-1}} \left|\sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_{2^n-l} D^2_l\right| \\
&\le \|H_n\|_1 + \|H_n\|_1 + \frac{4}{2^{2n}} \left|\sum_{l=0}^{2^n-1} t_{l,2^n-1} 2^{2n}\right| \le c.
\end{aligned}$$

(Of course, the second H_n is taken wit respect to sequence $\{\tilde{t}\}$.) That is, we proved that operator $\sigma_*^{1,\cdot\boxslash T}$ is of strong type (L^∞, L^∞). Then we prove that it is quasi-local. In order to get this, we have to verify that for any $f \in L^1(G^2)$ supported on some square $I_a \times I_a$ and $\int_{G^2} f = 0$

$$\int_{\overline{I_a \times I_a}} \sup_n \left| f * \left|\sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_{2^n-l} D^2_l\right|\right| \le c\|f\|_1.$$

Since

$$\left|\sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_{2^n-l} D^2_l\right| = \left|D^1_{2^n} \sum_{l=0}^{2^n-1} t_{l,2^n-1} D^2_l - w^1_{2^n-1} \sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_l D^2_l\right| = H_n$$

on the set $\overline{I_a} \times G$ and then by Lemma 15

$$\int_{\overline{I_a} \times G} \sigma_*^{1,\boxslash T}(f) \le c\|f\|_1.$$

To discuss the integral on the set $G \times \overline{I_a}$ is quite the same thing. We just use again

$$\sum_{l=0}^{2^n-1} t_{l,2^n-1} D^1_{2^n-l} D^2_l = \sum_{l=0}^{2^n-1} t_{2^n-1-l,2^n-1} D^1_l D^2_{2^n-l} = \sum_{l=0}^{2^n-1} \tilde{t}_{l,2^n-1} D^1_l D^2_{2^n-l}.$$

That is, instead of the sequence $\{t_{k,2^n-1}\}$ we use the sequence $\{\tilde{t}_{k,2^n-1}\} = \{t_{2^n-1-k,2^n-1}\}$ ($k \in \{0, \ldots, 2^n-1\}$) and again we can apply Lemma 15. The key point is that the only properties for the finite sequence $\{t_{k,2^n-1}\}$ is supposed (8.1) and (8.2). No monotonicity of any kind. And of course these two properties hold for the "reversed" sequence $\{\tilde{t}_{k,2^n-1:0\le k\le 2^n-1}\}$. That is, the operator $\sigma_*^{1,\boxslash T}$ is quasi-local. The rest of the proof of Lemma 16 follows the steps of the standard method. □

The proof of Theorem 5 The proof is an immediate consequence of Lemma 16, the inequality

$$\sigma_*^{\boxslash T}(f) \le \sigma_*^{1,\boxslash T}(|f|)$$

which gives that the operator $\sigma_*^{\boxslash T}$ is of weak type (L^1, L^1). Furthermore, the relation $0 \le t_{k,2^n-1} \le c/2^{n/2}$ (which comes from (8.2)) and the fact that we have a.e. convergence for Walsh polynomials (a set of which is dense in the Lebesgue space of two-dimensional integrable functions) complete the proof of Theorem 5. □

The following comment is due to one of the reviewers of the article. This - as you can see below raises new questions and problems.

Remark 4 Let $f \in L^1(G^2)$ and for every $n \in \mathbb{N}$ let $\{t_{k,n-1} : 0 \le k \le n-1\}$ be a finite and monotone (not necessarily in the same sense for different n's) sequence of non-negative numbers with $\sum_{k=0}^{n-1} t_{k,n-1} = 1$. Let us also assume that in the non-decreasing case it holds $t_{n-1,n-1} = O(1/n)$. Then we have the a.e. relation

$$\sigma_n^{\triangle T}(f) \to f.$$

Proof The Abel transform gives

$$\begin{aligned}\left|\sum_{k=0}^{n-1} t_{k,n-1} S_k^{\triangle T} f\right| &= \left|\sum_{k=0}^{n-2} (t_{k,n-1} - t_{k+1,n-1}) k \sigma_n^{\triangle} f + t_{n-1,n-1} n \sigma_n^{\triangle}(f)\right| \\ &\leq \sup_n \left|\sigma_n^{\triangle}(f)\right| \left(\sum_{k=0}^{n-1} |t_{k,n} - t_{k+1,n}| k + t_{n-1,n-1} n\right) \\ &\leq c \sup_n \left|\sigma_n^{\triangle}(f)\right|.\end{aligned}$$

Finally, the fact that the operator $\sup_n \left|\sigma_n^{\triangle}\right|$ is of weak type (L^1, L^1) (see [16]) implies that the operator $\sup_n \left|\sigma_n^{\triangle T}\right|$ is also of weak type (L^1, L^1). From here, by the usual density argument, the statement follows. □

Problems 1 *The following open questions might be of interest.*

- *Is it true that if the sequence* $\{t_{k,2^n-1} : 0 \leq k \leq 2^n - 1\}$ *is assumed to have properties* (8.1) *and* (8.2)*, but not monotonicity of any kind, even then for any two-variable integrable function, it holds*

$$\sigma_n^{\triangle T}(f) \to f$$

  or at least $\sigma_{2^n}^{\triangle T}(f) \to f$ *a.e. (In this case, of course, the lack of monotonicity makes the Abel transform "not helpful".)*
- *What be said about the sequence of operators* $(\sigma_n^{\boxslash T})$*?*
- *Can the Condition* (8.2) *be weakened (and still preserving a.e. convergence)?*

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References

1. Bakhvalov, A.N.: Waterman classes and triangular partial sums of double Fourier series (Russian. English, Russian summary). Anal. Math. **27**(1), 3–36 (2001)

2. Baramidze, D., Persson, L. E., Singh, H., Tephnadze, G.: Some new results and inequalities for subsequences of Nörlund logarithmic means of Walsh–Fourier series, J. Inequal. Appl. 30 (2022)
3. Baramidze, L., Persson, L. E., Tephnadze, G., Wall, P.: Sharp $H_p - L_p$ type inequalities of weighted maximal operators of Vilenkin–Nörlund means and its applications J. Inequal. Appl. 242 (2016)
4. Blahota, I., Nagy, K.: Approximation by Θ-means of Walsh–Fourier series. Anal. Math. **44**(1), 57–71 (2018)
5. Blahota, I., Nagy, K.: Approximation by matrix transform of Vilenkin–Fourier series. Publ. Math. Debrecen **99**(1–2), 223–242 (2021)
6. Blahota, I., Nagy, K.: Approximation by Marcinkiewicz type matrix transform of Vilenkin–Fourier series. Mediterr. J. Math. **19**(4), 165 (2022)
7. Blahota, I., Tephnadze, G.: A note on maximal operators of Vilenkin–Nörlund means. Acta Math. Acad. Paed. Nyíregyh. **32**, 203–213 (2016)
8. Blahota, I., Nagy, K., Tephnadze, G.: Approximation by Marcinkiewicz Θ-means of double Walsh–Fourier series. Math. Inequal. Appl. **22**(3), 837–853 (2019)
9. Bovdi, V., Salim, M., Ursul, M.: Completely simple endomorphism rings of modules. Appl. Gen. Topol. **19**(2), 223–237 (2018)
10. Chripkó, Á.: Weighted approximation via Θ-summations of Fourier–Jacobi series. Stud. Sci. Math. Hungar. **47**(2), 139–154 (2010)
11. Eisner, T.: The Θ-summation on local fields. Ann. Univ. Sci. Budapest. Sect. Comput. **33**, 137–160 (2011)
12. Fridli, S., Manchanda, P., Siddiqi, A.H.: Approximation by Walsh–Nörlund means. Acta Sci. Math. **74**, 593–608 (2008)
13. Gát, G.: On the Fejér kernel functions with respect to the Walsh–Paley system. Acta Acad. Paedagog. Agriensis Sect. Math. (N.S.) **24**, 105–110 (1997)
14. Gát, G.: Convergence of Marcinkiewicz means of integrable functions with respect to two-dimensional Vilenkin systems. Georgian Math. J. **11**(3), 467–478 (2004)
15. Gát, G.: On almost everywhere convergence and divergence of Marcinkiewicz-like means of integrable functions with respect to the two-dimensional Walsh system. J. Approx. Theory **164**(1), 145–161 (2012)
16. Gát, G.: Almost everywhere convergence of Fejér means of two-dimensional triangular Walsh–Fourier series. J. Fourier Anal. Appl. **24**, 1249–1275 (2018)
17. Glukhov, V.A.: On the summability of multiple Fourier series with respect to multiplicative systems. Mat. Zametki **39**, 665–673 (1986). ((**in Russian**))
18. Goginava, U.: On the approximation properties of Cesàro means of negative order of Walsh–Fourier series. J. Approx. Theory **115**, 9–20 (2002)
19. Goginava, U.: Almost everywhere convergence of (C, α)-means of cubical partial sums of d-dimensional Walsh–Fourier series. J. Approx. Theory **141**, 8–28 (2006)
20. Goginava, U.: Almost everywhere convergence of (C, α)-means of cubical partial sums of d-dimensional Walsh–Fourier series. J. Approx. Theory **141**, 8–28 (2006)
21. Goginava, U., Weisz, F.: Maximal operator of the Fejér means of triangular partial sums of two-dimensional Walsh–Fourier series. Georgian Math. J. **19**(1), 101–115 (2012)
22. Herriot, J.G.: Nörlund summability of multiple Fourier series. Duke Math. J. **11**, 735–754 (1944)
23. Hewitt, E., Ross, K.: Abstract Harmonic Analysis, vol. I. Springer, Heidelberg (1963)
24. Jastrebova, M.A.: On approximation of functions satisfying the Lipschitz condition by arithmetic means of their Walsh–Fourier series. Mat. Sb. **71**, 214–226 (1966). ((**Russian**))
25. Joudeh, A., Gát, G.: Almost everywhere convergence of Cesàro means with varying parameters of Walsh–Fourier series. Miskolc Math. Notices **19**(1), 303–317 (2018)
26. Karagulyan, G.A., Muradyan, K.R.: Divergent triangular sums of double trigonometric Fourier series. J. Contemp. Math. Anal. **50**(4), 196–207 (2015)
27. Memić, N.: Almost everywhere convergence of some subsequences of the Nörlund logarithmic means of Walsh–Fourier series. Anal. Math. **41**(1), 45–54 (2015)
28. Memić, N., Persson, L.E., Tephnadze, G.: A note on the maximal operators of Vilenkin–Nörlund means with non-increasing coefficients. Stud.Sci. Math. Hungar. **53**(4), 545–556 (2016)
29. Móricz, F., Rhoades, B.E.: Approximation by weighted means of Walsh–Fourier series. Int. J. Math. Sci. **19**(1), 1–8 (1996)
30. Móricz, F., Siddiqi, A.: Approximation by Nörlund means of Walsh–Fourier series. J. Approx. Theory **70**, 375–389 (1992)

31. Nagy, K.: Approximation by Nörlund means of quadratical partial sums of double Walsh–Fourier series. Anal. Math. **36**(4), 299–319 (2010)
32. Nagy, K.: Approximation by Nörlund means of Walsh–Kaczmarz–Fourier series. Georgian Math. J. **18**(1), 147–162 (2011)
33. Nagy, K.: Approximation by Nörlund means of double Walsh–Fourier series for Lipschitz functions. Math. Ineq. Appl. **15**(2), 301–322 (2012)
34. Schipp, F., Wade, W.R., Simon, P., Pál, J.: Walsh Series. An Introduction to Dyadic Harmonic Analysis. Adam Hilger, Bristol (1990)
35. Shavardenidze, G.: On the convergence of Cesàro means of negative order of Vilenkin-Fourier series. Stud. Sci. Math. Hungar. **56**(1), 22–44 (2019)
36. Skvortsov, V.A.: Certain estimates of approximation of functions by Cesàro means of Walsh-Fourier series. Mat. Zametki **29**, 539–547 (1981). ((**Russian**))
37. Tepnadze, T.: On the approximation properties of Cesàro means of negative order of Vilenkin-Fourier series. Stud. Sci. Math. Hungar. **53**(4), 532–544 (2016)
38. Toledo, R.: On the boundedness of the L_1-norm of Walsh-Fejér kernels. J. Math. Anal. Appl. **457**(1), 153–178 (2018)
39. Weisz, F.: Convergence of double Walsh-Fourier series and Hardy spaces. Approx. Theory Appl.(N.S.) **17**(2), 32–44 (2001)
40. Weisz, F.: Θ-summability of Fourier series. Acta Math. Hungar. **103**(1–2), 139–175 (2004)
41. Weisz, F.: Triangular Cesàro summability of two-dimensional Fourier series. Acta Math. Hungar. **132**, 27–41 (2011)
42. Weisz, F.: Triangular summability and Lebesgue points of two-dimensional Fourier transforms. Banach J. Math. Anal. **11**, 223–238 (2017)
43. Weisz, F.: Lebesgue Points and Summability of Higher Dimensional Fourier Series. Applied and Numerical Harmonic Analysis, Springer, Birkhäuser (2021)
44. Weisz, F.: Triangular Cesàro summability and Lebesgue points of two-dimensional Fourier series. Math. Inequal. Appl. **25**(3), 631–646 (2022)
45. Yano, Sh.: On Walsh–Fourier series. Tohoku Math. J. **3**, 223–242 (1951)
46. Yano, Sh.: On approximation by Walsh functions. Proc. Am. Math. Soc. **2**, 962–967 (1951)
47. Zygmund, A.: Trigonometric Series, vol. I, 3rd edn. Cambridge University Press, Cambridge (2002)