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$L^p(\mathbb{R}^d)$ Boundedness for a Class of Nonstandard Singular Integral Operators

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Abstract
In this paper, let Ω be homogeneous of degree zero which has vanishing moment of order one, A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$, we consider a class of nonstandard singular integral operators, $T_{\Omega,\,A}$, with rough kernel being of the form $\frac{\Omega(x-y)}{|x-y|^{d+1}}\big(A(x) - A(y) - \nabla A(y)(x-y)\big)$. This operator is closely related to the Calderón commutator. We prove that, under the Grafakos-Stefanov minimum size condition $GS_\beta(S^{d-1})$ with $2 < \beta < \infty$ for Ω, $T_{\Omega,\,A}$ is bounded on $L^p(\mathbb{R}^d)$ for p with $1 + 1/(\beta-1) < p < \beta$.

Keywords Singular integral operator · Calderón reproducing formula · Approximation · Littlewood–Paley theory

Mathematics Subject Classification Primary 42B20

1 Introduction

We will work on $\mathbb{R}^d$, $d \geq 2$. For $x \in \mathbb{R}^d$ and $1 \leq n \leq d$, we denote by x_n the n-th variable of x, and $x' = x/|x|$. Let Ω be homogeneous of degree zero, integrable on S^{d-1}, the unit sphere in $\mathbb{R}^d$, and satisfy the vanishing moment condition that for all

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$1 \le n \le d$,

$$\int_{S^{d-1}} \Omega(x')x'_n dx = 0. \tag{1.1}$$

Define the d-dimensional Calderón commutator $\mathscr{C}_{\Omega,a}$ by

$$\mathscr{C}_{\Omega,a} f(x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} \big(a(x) - a(y)\big) f(y)dy,$$

where a is a function on $\mathbb{R}^d$ such that $\partial_n a \in L^\infty(\mathbb{R}^d)$ for all n with $1 \le n \le d$. This operator was introduced by Calderón [2] and plays an important role in the theory of singular integrals. For the progress of the study of Calderón commutator, we refer the references [1, 2, 10, 13, 17, 25–28], [14, Chapter 8] and the related references therein.

Now let A be a function on $\mathbb{R}^d$ such that $\nabla A \in \text{BMO}(\mathbb{R}^d)$, that is, $\partial_n A \in \text{BMO}(\mathbb{R}^d)$ for all n with $1 \le n \le d$. Let Ω be homogeneous of degree zero, integrable on S^{d-1}, and satisfy the vanishing moment condition (1.1). Define the operator $T_{\Omega,A}$ by

$$T_{\Omega,A} f(x) = \text{p.v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} \big(A(x) - A(y) - \nabla A(y)(x-y)\big) f(y)dy. \tag{1.2}$$

This operator is closely related to the d-dimensional Calderón commutator. For the case of $\nabla A \in L^\infty(\mathbb{R}^d)$, the $L^p(\mathbb{R}^d)$ boundedness and the endpoint estimates of $T_{\Omega,\,A}$ can be deduced from the $L^p(\mathbb{R}^d)$ boundedness of Calderón commutator. On the other hand, for the case of $\nabla A \in \text{BMO}(\mathbb{R}^d)$, $T_{\Omega,\,A}$ is not a Calderón–Zygmund operator even if $\Omega \in \text{Lip}(S^{d-1})$. Cohen [6] first considered the mapping properties of $T_{\Omega,\,A}$, and proved that if $\Omega \in \text{Lip}_\alpha(S^{d-1})$ ($\alpha \in (0,\,1]$), then for $p \in (1,\,\infty)$, $T_{\Omega,\,A}$ is a bounded operator on $L^p(\mathbb{R}^d)$ with bound $C\|\nabla A\|_{\text{BMO}(\mathbb{R}^d)}$; see also [8] for the $L^p(\mathbb{R}^d)$ boundedness of an operator related to $T_{\Omega,\,A}$. Hofmann [18] improved the result of Cohen, and showed that $\Omega \in L^\infty(S^{d-1})$ is a sufficient condition such that $T_{\Omega,\,A}$ is bounded on $L^p(\mathbb{R}^d)$. Fairly recently, Hu, Tao, Wang and Xue [22] considered the $L^p(\mathbb{R}^d)$ boundedness of $T_{\Omega,A}$ when Ω satisfies certain minimum size condition, and established the following estimates.

Theorem 1.1 *Let Ω be homogeneous of degree zero, satisfy the vanishing condition (1.1), A be a function on $\mathbb{R}^d$ such that $\nabla A \in \text{BMO}(\mathbb{R}^d)$. Suppose that $\Omega \in L(\log L)^2(S^{d-1})$. Then $T_{\Omega,\,A}$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1,\,\infty)$.*

There is, however, another typical minimum size condition for functions on S^{d-1}. Let $\Omega \in L^1(S^{d-1})$ and $\beta \in [1,\,\infty)$, we say that $\Omega \in GS_\beta(S^{d-1})$ if

$$\sup_{\zeta \in S^{d-1}} \int_{S^{d-1}} |\Omega(\theta)| \log^\beta \Big(\frac{1}{|\zeta \cdot \theta|}\Big) d\theta < \infty.$$

This size condition was introduced by Grafakos and Stefanov [15], to study the $L^p(\mathbb{R}^d)$ boundedness for the homogeneous singular integral operator defined by

$$T_\Omega f(x) = \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d} f(y) dy \tag{1.3}$$

Grafakos and Stefanov [15] proved that if Ω is homogeneous of degree zero and has mean value zero on S^{d-1} and $\Omega \in GS_\beta(S^{d-1})$ for some $\beta > 1$, then the operator T_Ω is bounded on $L^p(\mathbb{R}^d)$ for $1 + 1/\beta < p < 1 + \beta$. Fan, Guo and Pan [12] improved the result of Grafakos and Stefanov, and proved that $\Omega \in GS_\beta(S^{d-1})$ for some $\beta > 1$ is a sufficient condition such that T_Ω is bounded on $L^p(\mathbb{R}^d)$ for $2\beta/(2\beta - 1) < p < 2\beta$.

Let $P_{ry'}(x')$ be the Poission kernel on S^{d-1}, that is

$$P_{ry'}(x') = \frac{1-r^2}{|ry' - x'|^d},$$

where $0 \le r < 1$ and $x', y' \in S^{d-1}$. For a function $\Omega \in L^1(S^{d-1})$, we define the radial maximal function

$$P^+\Omega(x') = \sup_{0 \le r < 1} \left| \int_{S^{d-1}} \Omega(y') P_{rx'}(y') dy' \right|.$$

The Hardy space $H^1(S^{d-1})$, is a subspace of $L^1(S^{d-1})$ which contains all $L^1(S^{d-1})$ functions Ω with the finite norms $\|\Omega\|_{H^1(S^{d-1})} = \|P^+\Omega\|_{L^1(S^{d-1})}$, see also [9]. As is well known, for $\beta \in [1, \infty)$,

$$H^1(S^{d-1}) \subset L(\log L)^\beta(S^{d-1}) \subset GS_\beta(S^{d-1}).$$

Moreover, as Grafakos and Stefanov [15] showed,

$$\left(\cap_{\beta>1} GS_\beta(S^{d-1})\right) \backslash H^1(S^{d-1}) \neq \emptyset.$$

Thus, it is natural to ask if $T_{\Omega, A}$ enjoys a $L^p(\mathbb{R}^d)$ estimate similar to the operator T_Ω defined as (1.3) when $\Omega \in GS_\beta(S^{d-1})$ for some $\beta \in (1, \infty)$. Hu [20] considered this question and proved the following result.

Theorem 1.2 *Let Ω be homogeneous of degree zero which satisfies the vanishing moment condition (1.1), A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Suppose that $\Omega \in GS_\beta(S^{d-1})$ for some $\beta > 3$, then $T_{\Omega, A}$ is bounded on $L^2(\mathbb{R}^d)$.*

In this paper, we will improve and extend Theorem 1.2. Our main result can be stated as follows.

Theorem 1.3 *Let Ω be homogeneous of degree zero, satisfy the vanishing moment condition (1.1), A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Suppose that $\Omega \in GS_\beta(S^{d-1})$ for some $\beta > 2$. Then for p with $1 + 1/(\beta - 1) < p < \beta$, $T_{\Omega, A}$ is bounded on $L^p(\mathbb{R}^d)$.*

To prove Theorem 1.3, we will first prove that $T_{\Omega,A}$ is bounded on $L^2(\mathbb{R}^d)$ when $\Omega \in GS_\beta(S^{d-1})$ for some $\beta \in (2, \infty)$. To prove the $L^p(\mathbb{R}^d)$ boundedness of $T_{\Omega,A}$, we will show that, there exists a sequence of operators $\{R_{l,A}\}_{l\in\mathbb{N}}$ such that

(i) for $p \in (1, 2)$, $R_{l,A}$ is bounded on $L^p(\mathbb{R}^d)$ with bound Cl^2;
(ii) for any $\varepsilon \in (0, 1)$ and $l \in \mathbb{N}$,

$$\|R_{l,A} - T_{\Omega,A}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim l^{-\varepsilon\beta+2}.$$

This, via interpolation, leads to the desired $L^p(\mathbb{R}^d)$ boundedness of $T_{\Omega,A}$. We remark that in this paper, we are very much motivated by the work of Chen, Hu and Tao [4], in which the authors established a suitable approximation for the Calderón commutator with rough kernel, see also [30] for the approximation of homogeneous singular integrals with rough kernels. However, the operator we consider in this paper is more rough than the Calderón commutator, and the argument in this paper involves much more complicated estimates and refined decompositions than that in [4].

This paper is organized as follows. In Sect. 2, we establish an endpoint estimate for the operators which will be used in the approximation; we also give some facts about the Luxemburgh norms in this section. In Sect. 3, we prove that $T_{\Omega,\,A}$ with $\Omega \in GS_\beta(S^{d-1})$ for some $\beta \in (2, \infty)$ can be approximated by a sequence of operators with smooth kernels. Sect. 4 is devoted to the proof of Theorem 1.3.

Throughout this paper, we use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. Constant with subscript such as C_1, does not change in different occurrences. For any set $E \subset \mathbb{R}^d$, χ_E denotes its characteristic function. For a cube $I \subset \mathbb{R}^d$ and $\lambda \in (0, \infty)$, we use $\ell(I)$ to denote the side length of I, and λI to denote the cube with the same center as I and whose side length is λ times that of I. For $x \in \mathbb{R}^d$ and $r > 0$, $B(x, r)$ denotes the ball centered at x and having radius r. For a suitable function f, we denote $\widehat{f}$ the Fourier transform of f. For locally integrable function f and a cube $I \subset \mathbb{R}^d$, $\langle f\rangle_I$ denotes the mean value of f on I, that is, $\langle f\rangle_I = |I|^{-1}\int_I f(y)dy$.

2 A Preliminary $L^p(\mathbb{R}^d)$ Estimate

Let K be a locally integrable function on $\mathbb{R}^d\backslash\{0\}$, A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Let T_A be an $L^2(\mathbb{R}^d)$ bounded operator, and satisfy that, for bounded function f with compact support and a. e. $x \in \mathbb{R}^d\backslash\mathrm{supp}\, f$,

$$T_A f(x) = \int_{\mathbb{R}^d} K(x-y)\frac{A(x) - A(y) - \nabla A(y)(x-y)}{|x-y|} f(y)dy. \tag{2.1}$$

This operator plays a key role in the approximation of $T_{\Omega,A}$. The main purpose of this section is establish the $L^p(\mathbb{R}^d)$ boundedness for the operator T_A whose kernel K satisfies a minimum size conditions and minimum regularity conditions.

2.1 Some Facts About the Luxemburgh Norms

We list some known facts about the Luxemburgh norms. Details are given in [29]. Let $\Psi : [0, \infty) \to [0, \infty)$ be Young function, namely, Ψ is convex and continuous on $[0, \infty)$, $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$. We always assume that Ψ satisfies a doubling condition, that is, $\Psi(2t) \le C\Psi(t)$ for any $t \in (0, \infty)$.

Let Ψ be a Young function, and $Q \subset \mathbb{R}^d$ be a cube. Define the Luxemburg norm $\|\cdot\|_{L^\Psi(Q)}$ by

$$\|f\|_{L^\Psi(Q)} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Psi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1\right\}.$$

It is well known that

$$\frac{1}{|Q|}\int_Q \Psi(|f(x)|)\,dx \le 1 \Leftrightarrow \|f\|_{L^\Psi(Q)} \le 1,$$

and

$$\|f\|_{L^\Psi(Q)} \le \inf\left\{\mu + \frac{\mu}{|Q|}\int_Q \Psi\left(\frac{|f(x)|}{\mu}\right)dx : \mu > 0\right\} \le 2\|f\|_{L^\Psi(Q)}; \tag{2.2}$$

see see [29, p. 54] and [29, p. 69] respectively. For $p \in [1, \infty)$ and $\gamma \in \mathbb{R}$, set $\Psi_{p,\gamma}(t) = t^p \log^\gamma(\mathrm{e} + t)$. We denote $\|f\|_{L^{\Psi_{p,\gamma}}(Q)}$ as $\|f\|_{L^p(\log L)^\gamma,\,Q}$.

Let Ψ be a Young function. Ψ^*, the complementary function of Ψ, is defined on $[0, \infty)$ by

$$\Psi^*(t) = \sup\{st - \Psi(s) : s \ge 0\}.$$

The generalization of Hölder inequality

$$\frac{1}{|Q|}\int_Q |f(x)h(x)|dx \le \|f\|_{L^\Psi(Q)}\|h\|_{L^{\Psi^*}(Q)} \tag{2.3}$$

holds for $f \in L^\Psi(Q)$ and $h \in L^{\Psi^*}(Q)$. see [29, p. 6].

For a cube $Q \subset \mathbb{R}^d$ and $\gamma > 0$, we also define $\|f\|_{\exp L^\gamma,\,Q}$ by

$$\|f\|_{\exp L^\gamma,\,Q} = \inf\left\{t > 0 : \frac{1}{|Q|}\int_Q \exp\left(\frac{|f(y)|}{t}\right)^\gamma dy \le 2\right\}.$$

As it is well known, for $\Psi(t) = t\log(\mathrm{e}+t)$, its complementary function $\Psi^*(t) \approx \mathrm{e}^t - 1$. Let $b \in \mathrm{BMO}(\mathbb{R}^d)$. The John–Nirenberg inequality tells us that for any $Q \subset \mathbb{R}^d$ and $p \in [1, \infty)$,

$$\||b - \langle b\rangle_Q|^p\|_{\exp L^{1/p},\,Q} \lesssim \|b\|^p_{\mathrm{BMO}(\mathbb{R}^d)}.$$

This, together with the inequality (2.3), shows that

$$\frac{1}{|Q|}\int_Q |b(x)-\langle b\rangle_Q|^p|h(x)|^p dx \lesssim \|h\|^p_{L^p(\log L)^p,\,Q}\|b\|^p_{\mathrm{BMO}(\mathbb{R}^d)}. \tag{2.4}$$

2.2 The $L^p(\mathbb{R}^d)$ Estimate for T_A

We need a preliminary lemma.

Lemma 2.1 *Let A be a function on $\mathbb{R}^d$ with derivatives of order one in $L^q(\mathbb{R}^d)$ for some $q\in(d,\infty]$. Then*

$$|A(x)-A(y)|\lesssim |x-y|\left(\frac{1}{|I_x^y|}\int_{I_x^y}|\nabla A(z)|^q dz\right)^{\frac{1}{q}},$$

where I_x^y is the cube which is centered at x and has side length $2|x-y|$.

Lemma 2.1 is just Lemma 1.4 in [3].

To obtain the $L^p(\mathbb{R}^d)$ boundedness of T_A, we need the following endpoint estimate.

Theorem 2.2 *Let K be a locally integrable function on $\mathbb{R}^d\backslash\{0\}$, A be a function on $\mathbb{R}^d$ such that $\nabla A\in \mathrm{BMO}(\mathbb{R}^d)$. Let T_A be an $L^2(\mathbb{R}^d)$ bounded operator with bound no more than 1 and satisfy (2.1). Suppose that*

(i) *for each n with $1\le n\le d$, there exists an $L^2(\mathbb{R}^d)$ bounded operator T^n with bound no more than 1 and satisfies that for bounded function f with compact support and a. e. $x\in\mathbb{R}^d\backslash \mathrm{supp}\, f$,*

$$T^n f(x)=\int_{\mathbb{R}^d}K(x-y)\frac{x_n-y_n}{|x-y|}f(y)dy;$$

(ii) *for each R with $0<R<\infty$,*

$$\int_{R<|x|<2R}|K(x)|dx\le 1;$$

(iii) *for each $R>0$ and $y\in\mathbb{R}^d$ with $|y|<R/4$,*

$$\sum_{l=2}^{\infty} l\int_{2^l R<|x-y|\le 2^{l+1}R}|K(x-y)-K(x)|dx\le 1.$$

Then for $\lambda>0$ and bounded function f with compact support,

$$|\{x\in\mathbb{R}^d:\,|T_A f(x)|>\lambda\}|\lesssim\int_{\mathbb{R}^d}\frac{|f(x)|}{\lambda}\log\left(\mathrm{e}+\frac{|f(x)|}{\lambda}\right)dx.$$

Proof Theorem 2.2 can be proved by mimicking the proof of Theorem 1 in [23]. For the sake of self-contained, we present the main step of the proof here. Without loss of generality, we assume that $\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^d)} = 1$. For given bounded function f with compact support and $\lambda > 0$, we apply the Calderoń-Zygmund decomposition to f at level λ, and obtain the following decomposition of f

$$f = g + b = g + \sum_j b_j,$$

such that

(a) $\|g\|_{L^\infty(\mathbb{R}^n)} \lesssim \lambda$ and $\|g\|_{L^1(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}$;
(b) for each j, b_j is supported on a cube Q_j, and cubes $\{Q_j\}$ are pairwise disjoint, $\int_{Q_j} b_j(x)dx = 0$ and $\|b_j\|_{L^1(\mathbb{R}^n)} \lesssim \lambda|Q_j|$;
(c) $\sum_j |Q_j| \lesssim \lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}$.

The inequality (2.2) now tells us that

$$\begin{aligned}\sum_j |Q_j|\|b_j\|_{L\log L,\, Q_j} &\lesssim \sum_j |Q_j| \left(\lambda + \frac{\lambda}{|Q|}\int_{Q_j} \frac{|f(x)|}{\lambda}\log\left(\mathrm{e} + \frac{|f(x)|}{\lambda}\right)dx\right)\\ &\lesssim \int_{\mathbb{R}^d} \frac{|f(x)|}{\lambda}\log\left(\mathrm{e} + \frac{|f(x)|}{\lambda}\right)dx. \qquad (2.5)\end{aligned}$$

By the $L^2(\mathbb{R}^d)$ boundedness of T_A, we deduce that

$$|\{x \in \mathbb{R}^d : |T_A g(x)| > \lambda/2\}| \lesssim \lambda^{-2}\|T_A g\|^2_{L^2(\mathbb{R}^d)} \lesssim \lambda^{-1}\|f\|_{L^1(\mathbb{R}^d)}. \qquad (2.6)$$

To estimate $T_A b$, we set $E = \cup_j 4dQ_j$, and

$$A_j(y) = A(y) - \sum_{n=1}^{d} \langle \partial_n A\rangle_{Q_j} y_n.$$

It then follows that for $x,\ y \in \mathbb{R}^d$,

$$A(x) - A(y) - \nabla A(y)(x-y) = A_j(x) - A_j(y) - \nabla A_j(y)(x-y).$$

For $x \in \mathbb{R}^d \backslash E$, write

$$\begin{aligned}T_A b(x) &= \sum_j \int_{\mathbb{R}^d} K(x-y)\frac{A_j(x) - A_j(y)}{|x-y|} b_j(y)dy\\ &\quad - \sum_{n=1}^{d} \int_{\mathbb{R}^d} K(x-y)\frac{x_n - y_n}{|x-y|} \sum_j (\partial_n A(y) - \langle \partial_n A\rangle_{Q_j}) b_j(y)dy\\ &= \sum_j T_A^1 b_j(x) - \sum_{n=1}^{d} T^n\Big(\sum_j (\partial_n A - \langle \partial_n A\rangle_{Q_j}) b_j\Big)(x).\end{aligned}$$

Recall that T^n is bounded on $L^2(\mathbb{R}^d)$. Our assumption (ii) implies that T^n is also bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. As in [23, p. 764], an argument involving inequality (2.4) with $p = 1$ and (2.5) leads to that

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^d : \sum_{n=1}^{d}\left|T^n\Big(\sum_j(\partial_n A - \langle\partial_n A\rangle_{Q_j})b_j\Big)(x)\right| > \lambda/4\right\}\right| \\
&\le \sum_{n=1}^{d}\left|\left\{x \in \mathbb{R}^d : \left|T^n\Big(\sum_j(\partial_n A - \langle\partial_n A\rangle_{Q_j})b_j\Big)(x)\right| > \frac{\lambda}{4d}\right\}\right| \\
&\lesssim \frac{1}{\lambda}\sum_{n=1}^{d}\sum_j\|(\partial_n A - \langle\partial_n A\rangle_{Q_j})b_j\|_{L^1(\mathbb{R}^d)} \lesssim \frac{1}{\lambda}\sum_j|Q_j|\|b_j\|_{L\log L,\,Q_j} \\
&\lesssim \int_{\mathbb{R}^d}\frac{|f(x)|}{\lambda}\log\left(\mathrm{e}+\frac{|f(x)|}{\lambda}\right)dx. \qquad (2.7)
\end{aligned}$$

We now estimate $\sum_j T_A^1 b_j$. For each fixed j, we choose $x^j \in 3Q_j\backslash 2Q_j$. Observe that

$$\begin{aligned}
&\left|K(x-y)\frac{A_j(x)-A_j(y)}{|x-y|} - K(x-x^j)\frac{A_j(x)-A_j(x^j)}{|x-x^j|}\right| \\
&\le |K(x-y)-K(x-x^j)|\frac{|A_j(x)-A_j(y)|}{|x-y|} \\
&\quad + |K(x-x^j)|\left|\frac{A_j(x)-A_j(y)}{|x-y|} - \frac{A_j(x)-A_j(x^j)}{|x-x^j|}\right|.
\end{aligned}$$

For $x \in \mathbb{R}^d\backslash E$, by the vanishing moment of b_j, we have that

$$\begin{aligned}
|T_A^1 b_j(x)| \le\ &\int_{\mathbb{R}^d}|K(x-y)-K(x-x^j)|\frac{|A_j(x)-A_j(y)|}{|x-y|}|b_j(y)|dy \\
&+|K(x-x^j)||A_j(x)-A_j(x^j)|\int_{\mathbb{R}^d}\frac{|y-x^j|}{|x-y|^2}|b_j(y)|dy \\
&+|K(x-x^j)|\int_{\mathbb{R}^d}\frac{|A_j(y)-A_j(x^j)|}{|x-y|}|b_j(y)|dy \\
:=\ &\mathrm{I}_j(x)+\mathrm{II}_j(x)+\mathrm{III}_j(x).
\end{aligned}$$

For each $y \in Q_j$, we know that

$$|\langle\nabla A\rangle_{Q_j} - \langle\nabla A\rangle_{I^y_{x^j}}| \lesssim \log\left(\mathrm{e}+\frac{|x^j-y|}{\ell(Q_j)}\right).$$

It then follows from Lemma 2.1 that, for $y \in Q_j$,

$$\begin{aligned}|A_j(x^j)-A_j(y)| &\lesssim |x^j-y|\left(\frac{1}{|I^y_{x_j}|}\int_{I^y_{x^j}}|\nabla A(z)-\langle\nabla A\rangle_{Q_j}|^q dx\right)^{1/q}\\ &\le |x^j-y|\left(\frac{1}{|I^y_{x_j}|}\int_{I^y_{x^j}}|\nabla A(z)-\langle\nabla A\rangle_{I^y_{x^j}}|^q dx\right)^{1/q}\\ &\quad+|x^j-y||\langle\nabla A\rangle_{Q_j}-\langle\nabla A\rangle_{I^y_{x^j}}|\\ &\lesssim |x^j-y|\left(1+\log\left(\mathrm{e}+\frac{|x^j-y|}{\ell(Q_j)}\right)\right)\lesssim \ell(Q_j),\end{aligned}$$

since $|x^j-y|\approx\ell(Q_j)$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^d\backslash 4dQ_j}|\mathrm{III}_j(x)|dx &\lesssim \ell(Q_j)\sum_{l=2}^{\infty} l\int_{Q_j}\int_{2^{l+1}dQ_j\backslash 2^l dQ_j}|K(x-x^j)|\frac{|b_j(y)|}{|x-y|}dxdy\\ &\lesssim \|b_j\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

For $l\ge 2$, $x\in 2^{l+1}dQ_j\backslash 2^l dQ_j$ and $y\in Q_j$, another application of Lemma 2.1 leads to that

$$|A_j(x)-A_j(y)|\lesssim l|x-y|,\ \ |A_j(x)-A_j(x^j)|\lesssim l|x-x^j|.$$

This, in turn, implies that

$$\begin{aligned}\int_{\mathbb{R}^d\backslash 4dQ_j}|\mathrm{I}_j(x)|dx &\lesssim \sum_{l=2}^{\infty} l\int_{Q_j}\int_{2^{l+1}dQ_j\backslash 2^l dQ_j}|K(x-y)-K(x-x^j)|dx|b_j(y)|dy\\ &\lesssim \|b_j\|_{L^1(\mathbb{R}^d)},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^d\backslash 4dQ_j}|\mathrm{II}_j(x)|dx &\lesssim \sum_{l=2}^{\infty} l\int_{Q_j}\int_{2^{l+1}dQ_j\backslash 2^l dQ_j}|K(x-x^j)|\frac{|y-x^j|}{|x-y|}dx|b_j(y)|dy\\ &\lesssim \|b_j\|_{L^1(\mathbb{R}^d)},\end{aligned}$$

Combining the estimates for I_j, II_j and III_j leads to that

$$\begin{aligned}\left|\left\{x\in\mathbb{R}^d\backslash E:\ |\sum_j T^1_A b_j(x)|>\lambda/4\right\}\right| &\le 4\lambda^{-1}\sum_j\int_{\mathbb{R}^d\backslash 4Q_j}|T^1_A b_j(x)|dx\\ &\lesssim \lambda^{-1}\|f\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

This, along with estimates (2.6)–(2.7) and the fact $|E| \lesssim \|f\|_{L^1(\mathbb{R}^d)}$, yields our desired conclusion. □

We are now ready to give the $L^p(\mathbb{R}^d)$ boundedness for T_A.

Theorem 2.3 *Let K be a locally integrable function on $\mathbb{R}^d\backslash\{0\}$, A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Let T_A be an $L^2(\mathbb{R}^d)$ bounded operator with bound no more than 1 and satisfy (2.1). Under the hypothesis of Theorem 2.2, T_A is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, 2]$ with bound C.*

By a standard interpolation argument (see the proof of Corollary 1.3 in [22]), Theorem 2.3 follows from Theorem 2.2. We omit the details for brevity.

3 An Approximation of $T_{\Omega,A}$

In this section, we will show that $T_{\Omega,\,A}$ can be approximated by a sequences of operators with "smooth kernels". We first recall the definition of Calderón–Zygmund kernel.

Definition 3.1 Let Γ be a locally integrable function on $\mathbb{R}^d\backslash\{0\}$. We say that Γ is a Calderón–Zygmund kernel, if

(i) for all $x \in \mathbb{R}^d\backslash\{0\}$,

$$|\Gamma(x)| \lesssim \frac{1}{|x|^d};$$

(ii) for $x,\ y \in \mathbb{R}^d$ with $|x| \geq 4|y|$,

$$|\Gamma(x-y) - \Gamma(x)| \lesssim \frac{|y|}{|x-y|^{d+1}}.$$

Lemma 3.2 *Let Γ be a function on $\mathbb{R}^d\backslash\{0\}$ which satisfies the following conditions:*

(i) *Γ is a Calderón–Zygmund kernel;*
(ii) *for all $r,\ R$ with $0 < r < R < \infty$ and $1 \leq n \leq d$,*

$$\int_{r<|x|<R} \Gamma(x)x_n dx = 0.$$

Let A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$, and $T_{\Gamma,A}$ be the operator defined by

$$T_{\Gamma,A}f(x) = \mathrm{p.\,v.} \int_{\mathbb{R}^d} \Gamma(x-y)\frac{A(x)-A(y)-\nabla A(y)(x-y)}{|x-y|} f(y)dy.$$

Then for all $p \in (1, \infty)$, $T_{\Gamma,A}$ is bounded on $L^p(\mathbb{R}^d)$.

Proof Let $\mathcal{C}_A$ be the operator defined by

$$\mathcal{C}_A f(x) = \text{p. v.} \int_{\mathbb{R}^d} \Gamma(x-y) \frac{A(x)-A(y)}{|x-y|} f(y)dy.$$

As it was pointed out in Theorem 1.1 in [24] that, under the hypothesis of Lemma 3.2, the estimate

$$\|\mathcal{C}_A f\|_{L^r(\mathbb{R}^d)} \lesssim \|\nabla A\|_{L^q(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}, \tag{3.1}$$

holds true for $p \in (1, \infty)$ and $q \in (1, \infty]$ with $1/r = 1/q + 1/p$, see also [1] for the case that K is a homogeneous kernel. With this estimate, repeating the proof of Corollary 1.2 in [6], we then can deduce the $L^p(\mathbb{R}^d)$ ($p \in (1, \infty)$) boundedness of T_A. □

Lemma 3.3 *Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be a radial function such that* $\operatorname{supp} \phi \subset \{1/4 \le |\xi| \le 4\}$ *and*

$$\sum_{l \in \mathbb{Z}} \phi^3(2^{-l}\xi) = 1, \quad |\xi| > 0.$$

Let $\Phi = \widehat{\phi}$, A be a function on $\mathbb{R}^d$ such that $\nabla A \in$ BMO($\mathbb{R}^d$). *Define the operator $S_{j;A}$ by*

$$S_{j;A} f(x) = \int_{\mathbb{R}^d} 2^{jd} \Phi(2^j(x-y)) \big(A(x) - A(y) - \nabla A(y)(x-y)\big) f(y)dy.$$

Then

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} |2^j S_{j;A} f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}, \tag{3.2}$$

and

$$\Big\| \sum_{j \in \mathbb{Z}} 2^j S_{j;A} f_j \Big\|_{L^2(\mathbb{R}^d)} \lesssim \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_{L^2(\mathbb{R}^d)}. \tag{3.3}$$

Proof We only prove (3.2), since (3.3) can be deduced from (3.2) by a standard duality argument. On the other hand, by the well known randomization argument (see [11, p. 545]), to prove (3.2), it suffices to prove that for all $\{\varepsilon_j\}_{j \in \mathbb{Z}}$ with $\varepsilon_j = \pm 1$,

$$\Big\| \sum_{j \in \mathbb{Z}} \varepsilon_j 2^j S_{j;\,A} f \Big\|_{L^2(\mathbb{R}^d)} \le C \|f\|_{L^2(\mathbb{R}^d)}, \tag{3.4}$$

and the bound C is independent of $\{\varepsilon_j\}$.

Let $\varepsilon_j = \pm 1$ $(j \in \mathbb{Z})$, and

$$K_1(x) = \sum_{j\in\mathbb{Z}} 2^{j(d+1)}\varepsilon_j \Phi(2^j x). \tag{3.5}$$

By the fact that

$$|\Phi(2^j x)| \lesssim \frac{1}{(1+|2^j x|)^{d+2}},$$

we know that for each $x \in \mathbb{R}^d\backslash\{0\}$,

$$|K_1(x)| \le \sum_{j:\,2^j\le|x|^{-1}} 2^{j(d+1)}|\Phi(2^j x)| + \sum_{j:\,2^j>|x|^{-1}} 2^{j(d+1)}|\Phi(2^j x)| \lesssim |x|^{-d-1}. \tag{3.6}$$

On the other hand, by the smoothness of Φ, it is easy to verify that for $x,\ h \in \mathbb{R}^d$ with $|x| \ge 4|h|$,

$$|K_1(x+h) - K_1(x)| \lesssim \frac{|h|}{|x|^{d+2}}. \tag{3.7}$$

Since Φ is also a radial function, it certainly enjoys vanishing moment of order one. Thus, for all $0 < r < R < \infty$ and $1 \le n \le d$,

$$\int_{r<|x|<R} K_1(x)x_n dx = 0. \tag{3.8}$$

Estimates (3.6)-(3.8), via Lemma 3.2, leads to our desired conclusion. □

Lemma 3.4 *Let* $\phi \in C_0^\infty(\mathbb{R}^d)$ *be a radial function such that* $\operatorname{supp}\phi \subset \{1/4 \le |\xi| \le 4\}$ *and*

$$\sum_{l\in\mathbb{Z}} \phi^3(2^{-l}\xi) = 1,\ \ |\xi| > 0.$$

Let S_j *be the operator defined by*

$$\widehat{S_j f}(\xi) = \phi(2^{-j}\xi)\widehat{f}(\xi).$$

Then

(i) *for* $b \in \mathrm{BMO}(\mathbb{R}^d)$*, we have that*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} |[b,\ S_j]f|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)};$$

where and in what follows, for a locally integrable function b and a linear operator T, $[b,\ T]$ denotes the commutator defined

$$[b,\ T]f(x) = b(x)Tf(x) - T(bf)(x);$$

(ii) *for function a on $\mathbb{R}^d$ which satisfies that $\nabla a \in L^\infty(\mathbb{R}^d)$, it follows that*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|2^j[a,\ S_j]f|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

Proof Conclusion (i) is just [19, Lemma 1]. To prove conclusion (ii), let $\Phi = \widehat{\phi}$ and K_1 be the function defined by (3.5). Estimates (3.6)-(3.8), via (3.1), leads to conclusion (ii). □

Remark 3.5 Conclusion (ii) of Lemma 3.4 was first proved by Chen and Ding, using a different argument, see [5, Lemma 2.3].

Lemma 3.6 *Let $\delta \in (0,\ 1)$, $l \in \mathbb{Z}$ and $D > 0$ be constants, m be a multiplier such that* $\operatorname{supp} m \subset \{|\xi| \le D^{-1}2^l\}$, *and*

$$\|m\|_{L^\infty(\mathbb{R}^d)} \le D^{-1}\min\{(\delta 2^l)^2,\ \log^{-\beta}(\mathrm{e}+2^l)\},$$

and for all multi-indices $\gamma \in \mathbb{Z}_+^d$,

$$\|\partial^\gamma m\|_{L^\infty(\mathbb{R}^d)} \le D^{|\gamma|-1}\max\{1,\ 2^{-l|\gamma|}\}.$$

Let A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$, and $T_{m,\,A}$ be the operator defined by

$$T_{m,\,A}f(x) = \mathrm{p.\,v.}\int_{\mathbb{R}^d}\Theta(x-y)(A(x)-A(y)-\nabla A(y)(x-y)\big)f(y)dy,$$

with Θ the inverse Fourier transform of m. Then for any $\varepsilon \in (0,\ 1)$,

$$\|T_{m,A}f\|_{L^2(\mathbb{R}^d)} \lesssim \min\{(\delta 2^l)^{\varepsilon/2},\ \log^{-\varepsilon\beta+1}(\mathrm{e}+2^l)\}\|f\|_{L^2(\mathbb{R}^d)}. \tag{3.9}$$

Proof The argument here is a variant of the proof of Lemma 3.2 in [4], together with some refined estimates of Luxemburg norms. We assume that $\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^d)} = 1$. Set $E = \min\{(\delta 2^l)^2,\ \log^{-\beta}(\mathrm{e}+2^l)\}$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be a radial function, $\operatorname{supp}\phi \subset B(0,\ 2)$, $\phi(x) = 1$ when $|x| \le 1$. Set $\varphi(x) = \phi(x) - \phi(2x)$. Then $\operatorname{supp}\varphi \subset \{1/4 \le |x| \le 4\}$ and

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}x) \equiv 1,\ \ |x| > 0. \tag{3.10}$$

Let $\varphi_j(x) = \varphi(2^{-j}x)$ for $j \in \mathbb{Z}$. Set

$$W_j(x) = \Theta(x)\varphi_j(x), \ \ j \in \mathbb{Z}.$$

Let $T_{m,j}$ be the convolution operator with kernel W_j. Observing that for all multi-indices $\gamma \in \mathbb{Z}_+^d$, $\partial^\gamma \varphi(0) = 0$, we thus have that

$$\int_{\mathbb{R}^d} \widehat{\varphi}(\xi)\xi^\gamma d\xi = 0.$$

This, in turn, implies that for all $N \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$,

$$\begin{aligned} |\widehat{W_j}(\xi)| &= \left| \int_{\mathbb{R}^d} \Big(m(\xi - 2^{-j}\eta) - \sum_{|\gamma| \le N} \frac{1}{\gamma!} \partial^\gamma m(\xi)(2^{-j}\eta)^\gamma \Big) \widehat{\varphi}(\eta) d\eta \right| \\ &\lesssim 2^{-j(N+1)} \sum_{|\gamma|=N+1} \|\partial^\gamma m\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\eta|^{N+1} |\widehat{\varphi}(\eta)| d\eta \\ &\lesssim 2^{-j(N+1)} D^N \max\{1, \ 2^{-l(N+1)}\}. \end{aligned} \tag{3.11}$$

On the other hand, a trivial computation yields for $j \in \mathbb{Z}$,

$$\|\widehat{W_j}\|_{L^\infty(\mathbb{R}^d)} \le \|m\|_{L^\infty(\mathbb{R}^d)} \|\widehat{\varphi_j}\|_{L^1(\mathbb{R}^d)} \lesssim D^{-1}E. \tag{3.12}$$

Interpolation inequalities (3.11) and (3.12) gives us that for $\varepsilon \in (0, \ 1)$,

$$\|\widehat{W_j}\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{-j(N+1)(1-\varepsilon)} D^{N(1-\varepsilon)-\varepsilon} \max\{1, \ 2^{-l(N+1)}\}^{1-\varepsilon} E^\varepsilon. \tag{3.13}$$

We now prove (3.9). Let $T_{m,j;A}$ be the operator defined by

$$T_{m,j;A} f(x) = \int_{\mathbb{R}^d} W_j(x-y) \big(A(x) - A(y) - \nabla A(y)(x-y) \big) f(y) dy.$$

For $\varepsilon \in (0, \ 1)$, let $F_\varepsilon = \min\{(\delta 2^l)^{2\varepsilon}, \ \log^{-\varepsilon\beta+1}(\mathrm{e} + 2^l)\}$. We claim that for all $j \in \mathbb{Z}$ and $N \in \mathbb{N}$ and $\varepsilon \in (0, \ 1)$,

$$\begin{aligned} \|T_{m,j;A} f\|_{L^2(\mathbb{R}^d)} \lesssim (2^{-j}D)^{N(1-\varepsilon)-\varepsilon} \log(\mathrm{e} + 2^j D^{-1}) F_\varepsilon \\ \times \max\{1, \ 2^{-l(N+1)}\}^{1-\varepsilon} \|f\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{3.14}$$

Observe that supp $W_j \subset \{x : |x| \le 2^{j+2}\}$. If I is a cube having side length 2^j, and $f \in L^2(\mathbb{R}^d)$ with supp $f \subset I$, then $T_{m,j} f \subset 100dI$. Therefore, to prove (3.14), we may assume that supp $f \subset I$ with I a cube having side length 2^j. Let x_0 be a point on the boundary of $200dI$ and $A_I^*(y) = A(y) - \sum_{n=1}^d \langle \partial_n A \rangle_{100dI} y_n$, and

$$A_I(y) = (A_I^*(y) - A_I^*(x_0))\zeta_I(y),$$

where $\zeta_I \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp}\zeta \subset 150dI$ and $\zeta(x) \equiv 1$ when $x \in 100dI$. Observe that $\|\nabla\zeta\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{-j}$. An application of Lemma 2.1 tells us that for all $y \in 100dI$,

$$|A_I(y) - A_I(x_0)| \lesssim 2^j.$$

This shows that

$$\|A_I\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^j.$$

Write

$$T_{m,j;A}f(x) = A_I(x)T_{m,j}f(x) - T_{m,\,j}(A_I f)(x) - \sum_{n=1}^{d}[h_n,\ T_{m,j}](f\partial_n A_I)(x),$$

where $h_n(x) = x_n$ (recall that x_n denotes the nth variable of x). It then follows from (3.13) that

$$\begin{aligned}&\|A_I T_{m,j}f\|_{L^2(\mathbb{R}^d)} + \|T_{m,j}(A_I f)\|_{L^2(\mathbb{R}^d)}\\ &\quad\lesssim (2^{-j}D)^{N(1-\varepsilon)-\varepsilon}\max\{1,\ 2^{-j(N+1)}\}^{1-\varepsilon}E^{\varepsilon}\|f\|_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{3.15}$$

Applying the John–Nirenberg inequality, we know that

$$\||\partial_n A_I|^2\|_{\exp L^{1/2},\,I} \lesssim 1.$$

Recall that $\operatorname{supp}[h_n,\ T_{m,j}](f\partial_n A_I) \subset 100dI$. It then follows from inequality (2.4) that

$$\begin{aligned}\|[h_n,\ T_{m,j}](f\partial_n A_I)\|_{L^2(\mathbb{R}^d)} &= \sup_{\|g\|_{L^2(\mathbb{R}^d)}\le 1}\left|\int_{\mathbb{R}^d}\partial_n A_I(x)f(x)[h_n,\ T_{m,j}]g(x)dx\right|\\ &\le \|f\|_{L^2(\mathbb{R}^d)}\sup_{\substack{\|g\|_{L^2(\mathbb{R}^d)}\le 1\\ \operatorname{supp} g\subset 100dI}}\|\partial_n A_I[h_n,\ T_{m,j}]g\|_{L^2(I)}\\ &\le \|f\|_{L^2(\mathbb{R}^d)}\Big(|I|\sup_{\substack{\|g\|_{L^2(\mathbb{R}^d)}\le 1\\ \operatorname{supp} g\subset 100dI}}\|[h,T_{m,\,j}]g\|^2_{L^2(\log L)^2,\,I}\Big)^{1/2}.\end{aligned}$$

Now let $g \in L^2(\mathbb{R}^d)$ with $\|g\|_{L^2(\mathbb{R}^d)} \le 1$ and $\operatorname{supp} g \subset 100dI$. Observe that

$$\|W_j\|_{L^\infty(\mathbb{R}^d)} \lesssim \|m\|_{L^1(\mathbb{R}^d)} \lesssim 2^{dl}D^{-d-1}E,$$

which, via Young's inequality, implies that

$$\|T_{m,j}g\|_{L^\infty(\mathbb{R}^d)} \lesssim \|W_j\|_{L^\infty(\mathbb{R}^d)}\|g\|_{L^1(\mathbb{R}^d)} \lesssim 2^{dl}D^{-d-1}E\|g\|_{L^1(\mathbb{R}^d)},$$

and so

$$\|\chi_I[h_n,\ T_{m,j}]g\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{dl}2^jD^{-d-1}E\|g\|_{L^1(\mathbb{R}^d)} \lesssim 2^{dl}2^jD^{-d-1}E2^{dj/2},$$

since

$$\|g\|_{L^1(\mathbb{R}^d)} \lesssim |I|^{1/2}\|g\|_{L^2(\mathbb{R}^d)} \lesssim 2^{dj/2}.$$

On the other hand, we deduce from (3.13) that

$$\begin{aligned}\|[h_n,\ T_{m,j}]g\|_{L^2(\mathbb{R}^d)} &\lesssim 2^j\|T_{m,j}g\|_{L^2(\mathbb{R}^d)}\\ &\lesssim (2^{-j}D)^{N(1-\varepsilon)-\varepsilon}\max\{1,\ 2^{-l(N+1)}\}^{1-\varepsilon}E^\varepsilon\|g\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Set

$$\lambda_0 = \big[(2^{-j}D)^{N(1-\varepsilon)-\varepsilon}\log^2(\mathrm{e}+2^jD^{-1})F_\varepsilon\max\{1,\ 2^{-l(N+1)}\}^{1-\varepsilon}\big]^2 2^{-jd}.$$

A straightforward computation tells us that

$$\begin{aligned}&\int_I |[h_n,\ T_{m,j}]g(x)|^2\log^2\left(\mathrm{e}+\frac{|[h_n,\ T_{m,j}]g(x)|}{\sqrt{\lambda_0}}\right)dx\\ &\lesssim \Big(\log^2(\mathrm{e}+2^jD^{-1})+\max\{1,\ l\}\Big)\int_I |[h_n,\ T_{m,j}]g(x)|^2dx\\ &\lesssim \big[(2^{-j}D)^{N(1-\varepsilon)-\varepsilon}F_\varepsilon\max\{1,\ 2^{-l(N+1)}\}^{(1-\varepsilon)}\log(2+2^jD^{-1})\big]^2\\ &\lesssim \lambda_0 2^{jd},\end{aligned}$$

since $E^\varepsilon\max\{1,\ l\}\le F_\varepsilon$, and

$$\frac{\|\chi_I[h_n,\ T_{m,j}]g\|_{L^\infty(\mathbb{R}^d)}}{\sqrt{\lambda_0}} \lesssim (2^jD^{-1})^{d+1}(2^jD^{-1})^{N(1-\varepsilon)-\varepsilon}2^{dl}F_\varepsilon^{-1}.$$

This tells us that

$$\|[h_n, T_{m,\,j}]g\|_{L^2(\log L)^2,\,I} \lesssim \sqrt{\lambda_0},$$

and thus

$$\begin{aligned}\|[h_n,\ T_{m,j}](f\partial_nA_I)\|_{L^2(\mathbb{R}^d)} \lesssim\ & (2^{-j}D)^{N(1-\varepsilon)-\varepsilon}\log(\mathrm{e}+2^jD^{-1})\\ &\times F_\varepsilon\max\{1, 2^{-l(N+1)}\}^{1-\varepsilon}\|f\|_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{3.16}$$

The estimate (3.14) then follows from (3.15) and (3.16).

We now conclude the proof of Lemma 3.6. It suffices to consider the case $\varepsilon \in (4/5,\ 1)$. Let $G_\varepsilon = \min\{(\delta 2^l)^{\varepsilon/2},\ \log^{-\varepsilon\beta+1}(\mathrm{e}+2^l)\}$. For each fixed $\varepsilon \in (4/5,\ 1)$, we choose $N_1,\ N_2 \in \mathbb{N}$ such that

$$\frac{\varepsilon}{1-\varepsilon} < N_1 < \frac{5\varepsilon/2-1}{1-\varepsilon},\ \ N_2(1-\varepsilon)-\varepsilon < 0.$$

Observe that

$$F_\varepsilon \max\{1, 2^{-l(N_1+1)}\}^{1-\varepsilon} \lesssim G_\varepsilon,\ F_\varepsilon \max\{1, 2^{-l(N_2+1)}\}^{1-\varepsilon} \lesssim G_\varepsilon.$$

A straightforward computation shows that if

$$\begin{aligned}
\|T_{m,A} f\|_{L^2(\mathbb{R}^d)} &\lesssim \sum_{j:\,2^{-j}D\le 1} \|T_{m,\,j;\,A} f\|_{L^2(\mathbb{R}^d)} + \sum_{j:\,2^{-j}D>1} \|T_{m,j;\,A} f\|_{L^2(\mathbb{R}^d)} \\
&\lesssim \sum_{j:\,2^{-j}D\le 1} (2^{-j}D)^{N_1(1-\varepsilon)-\varepsilon} \log(\mathrm{e}+2^j D^{-1}) G_\varepsilon \|f\|_{L^2(\mathbb{R}^d)} \\
&\quad + \sum_{j:\,2^{-j}D>1} (2^{-j}D)^{N_2(1-\varepsilon)-\varepsilon} \log(\mathrm{e}+2^j D^{-1}) G_\varepsilon \|f\|_{L^2(\mathbb{R}^d)} \\
&\lesssim G_\varepsilon \|f\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

This completes the proof of Lemma 3.6. □

Lemma 3.7 *Let $\delta \in (0,\ 1)$, $l \in \mathbb{Z}$ and $D > 0$ be constants, m be a multiplier such that* $\operatorname{supp} m \subset \{|\xi| \le D^{-1}2^l\}$, *and*

$$\|m\|_{L^\infty(\mathbb{R}^d)} \le D^{-1} \min\{(\delta 2^l)^2,\ \log^{-\beta}(\mathrm{e}+2^l)\},$$

and for all multi-indices $\gamma \in \mathbb{Z}_+^d$,

$$\|\partial^\gamma m\|_{L^\infty(\mathbb{R}^d)} \le D^{|\gamma|-1} \max\{1,\ 2^{-l|\gamma|}\}.$$

Let T_m be the multiplier operator defined by

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi).$$

Then for any $b \in \mathrm{BMO}(\mathbb{R}^d)$ and $\varepsilon \in (0,\ 1)$,

$$\|[b,\ T_m] f\|_{L^2(\mathbb{R}^d)} \lesssim D^{-1} \min\{(\delta 2^l)^{2\varepsilon},\ \log^{-\varepsilon\beta+1}(\mathrm{e}+2^l)\} \|b\|_{\mathrm{BMO}(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}.$$

Proof Let $\widetilde{m}(\xi) = Dm(2^l D^{-1}\xi)$ and $T_{\widetilde{m}}$ be the multiplier operator with multiplier $\widetilde{m}$. We know that $\operatorname{supp} \widetilde{m} \subset \{|\xi| \le 1\}$, and

$$\|\widetilde{m}\|_{L^\infty(\mathbb{R}^d)} \le \min\{(\delta 2^l)^2,\ \log^{-\beta}(\mathrm{e}+2^l)\},$$

and for all multi-indices $\gamma \in \mathbb{Z}_+^d$,

$$\|\partial^\gamma \widetilde{m}\|_{L^\infty(\mathbb{R}^d)} \lesssim 1.$$

Applying Lemma 2 in [19], we then obtain that

$$\|[b,\ T_{\widetilde{m}}]f\|_{L^2(\mathbb{R}^d)} \lesssim \min\{(\delta 2^l)^{2\varepsilon},\ \log^{-\varepsilon\beta+1}(\mathrm{e}+2^l)\}\|b\|_{\mathrm{BMO}(\mathbb{R}^d)}\|f\|_{L^2(\mathbb{R}^d)}.$$

This, via dilation-invariance, implies our desired conclusion and then completes the proof of Lemma 3.7. □

Theorem 3.8 *Let $\delta \in (0,\ 1)$ be a constant, $\{\mu_j\}_{j\in\mathbb{Z}}$ be a sequence of functions on $\mathbb{R}^d\backslash\{0\}$. Suppose that for some $\beta \in (2,\ \infty)$,*

$$\|\mu_j\|_{L^1(\mathbb{R}^d)} \lesssim 2^{-j},\ |\widehat{\mu_j}(\xi)| \lesssim 2^{-j}\min\{|\delta 2^j\xi|^2,\ \log^{-\beta}(\mathrm{e}+|2^j\xi|)\},$$

and for all multi-indices $\gamma \in \mathbb{Z}_+^d$,

$$\|\partial^\gamma \widehat{\mu_j}\|_{L^\infty(\mathbb{R}^d)} \lesssim 2^{j(|\gamma|-1)}.$$

Let $\mu(x) = \sum_{j\in\mathbb{Z}} \mu_j(x)$ and $T_{\mu,A}$ be the operator defined by

$$T_{\mu,A}f(x) = \mathrm{p.\,v.}\int_{\mathbb{R}^d} \mu(x-y)\big(A(x)-A(y)-\nabla A(y)(x-y)\big)f(y)dy,$$

where A is a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Then for any $\varepsilon \in (0,\ 1)$,

$$\|T_{\mu,A}f\|_{L^2(\mathbb{R}^d)} \lesssim \log^{-\varepsilon\beta+2}(\mathrm{e}+\delta^{-1})\|f\|_{L^2(\mathbb{R}^d)}.$$

Proof It suffices to consider the case $\varepsilon \in (1/2,\ 1)$. Let T be the operator defined by

$$Tf(x) = \mathrm{p.\,v.}\int_{\mathbb{R}^d} \mu(x-y)f(y)dy.$$

It is easy to verify that for $\xi \in \mathbb{R}^d$,

$$|\widehat{\mu}(\xi)| \lesssim |\xi| \sum_{j:\,2^j>|\xi|^{-1}} \log^{-\beta}(\mathrm{e}+|2^j\xi|) + |\xi|^2 \sum_{j:\,2^j\le|\xi|^{-1}} 2^j \lesssim |\xi|.$$

This in turn, gives us that

$$\int_{\mathbb{R}^d} |\widehat{Tf}(\xi)|^2 d\xi \lesssim \|f\|^2_{\dot{L}^2_1(\mathbb{R}^d)},$$

and

$$\int_{\mathbb{R}^d} |\xi|^{-2}|\widehat{Tf}(\xi)|^2 d\xi \lesssim \|f\|^2_{L^2(\mathbb{R}^d)},$$

where $\|f\|_{\dot{L}^2_1(\mathbb{R}^d)}$ is the homogeneous Sobolev norm defined as

$$\|f\|^2_{\dot{L}^2_1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^2|\widehat{f}(\xi)|^2 d\xi.$$

Let U_j be the convolution operator with kernel μ_j, and $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \phi \le 1$, $\operatorname{supp}\phi \subset \{1/4 \le |\xi| \le 4\}$ and

$$\sum_{l\in\mathbb{Z}} \phi^3(2^{-l}\xi) = 1,\ \ |\xi| > 0.$$

Set $m_j(\xi) = \widehat{\mu_j}(\xi)$, and $m_j^l(\xi) = m_j(\xi)\phi(2^{j-l}\xi)$. Define the operator U_j^l by

$$\widehat{U_j^l f}(\xi) = m_j^l(\xi)\widehat{f}(\xi).$$

Let S_l be the multiplier operator defined as in Lemma 3.4. We claim that for functions $f,\ g \in C_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} g(x)T_{\mu,A}f(x)dx = \int_{\mathbb{R}^d} g(x)\sum_l\sum_j (S_{l-j}U_j^l S_{l-j})_A f(x)dx, \tag{3.17}$$

where and in what follows,

$$(S_{l-j}U_j^l S_{l-j})_A f(x) = \int_{\mathbb{R}^d} L(x-y)\big(A(x) - A(y) - \nabla A(y)(x-y)\big)f(y)dy,$$

with L the kernel of the convolution operator $S_{l-j}U_j^l S_{l-j}$. We define $U_{j,A}^l$ similarly. To prove this, let $R > 0$ be large enough such that $\operatorname{supp} f \cup \operatorname{supp} g \subset B(0,\ R)$. Let $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \zeta \le 1$, $\zeta \equiv 1$ on $B(0,\ R)$ and $\operatorname{supp}\zeta \subset B(0,\ 2R)$. Set

$$A_R(y) = \Big(A(y) - \sum_{n=1}^d \langle \partial_n A\rangle_{B(0,\ R)}y_n\Big)\zeta(y).$$

Then

$$\int_{\mathbb{R}^d} g(x)T_{\mu,A}f(x)dx = \int_{\mathbb{R}^d} g(x)A_R(x)Tf(x)dx - \int_{\mathbb{R}^d} g(x)T(A_R f)(x)dx$$
$$-\sum_{n=1}^d \int_{\mathbb{R}^d} g(x)[h_n,\ T](f\partial_n A_R)(x)dx,$$

where the function $h_n(x) = x_n$. Note that $h_n f \partial_n A_R \in L^2(\mathbb{R}^d)$. It then follows that

$$\int_{\mathbb{R}^d} g(x)[h_n,\ T](f\partial_n A_R)(x)dx = \int_{\mathbb{R}^d} g(x)\sum_l \sum_j [h_n,\ S_{l-j}U_j^l S_{l-j}](f\partial_n A_R)(x)dx.$$

Since $gA_R,\ fA_R \in L^2(\mathbb{R}^d)$, we also have that

$$\int_{\mathbb{R}^d} g(x)A_R(x)Tf(x)dx = \int_{\mathbb{R}^d} g(x)A_R(x)\sum_l \sum_j S_{l-j}U_j^l S_{l-j} f(x)dx,$$
$$\int_{\mathbb{R}^d} g(x)T(A_R f)(x)dx = \int_{\mathbb{R}^d} g(x)\sum_l \sum_j S_{l-j}U_j^l S_{l-j}(fA_R)(x)dx.$$

These three equalities lead to (3.17) directly.

Now we estimate $(S_{l-j}U_j^l S_{l-j})_A f$. Obviously, $\operatorname{supp} m_j^l \subset \{|\xi| \le 2^{l-j+2}\}$ and

$$|m_j^l(\xi)| \lesssim 2^{-j} \min\{(\delta 2^l)^2,\ \log^{-\beta}(\mathrm{e}+2^l)\}.$$

Furthermore, by the fact that

$$|\partial^\gamma \phi(2^{j-l}\xi)| \lesssim 2^{(j-l)|\gamma|},\ \ |\partial^\gamma m_j(\xi)| \lesssim 2^{j(|\gamma|-1)},$$

it then follows that for all $\gamma \in \mathbb{Z}_+^d$,

$$|\partial^\gamma m_j^l(\xi)| \lesssim 2^{j(|\gamma|-1)} \max\{1,\ 2^{-|\gamma|l}\}.$$

This, via Lemma 3.6, tells us that

$$\|U_{j,\,A}^l f\|_{L^2(\mathbb{R}^d)} \lesssim \min\{(\delta 2^l)^{\frac{\varepsilon}{2}},\ \log^{-\varepsilon\beta+1}(\mathrm{e}+2^l)\}\|f\|_{L^2(\mathbb{R}^d)}. \tag{3.18}$$

Also, we have that

$$\|U_j^l f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-j} \min\{(\delta 2^l)^2,\ \log^{-\beta}(\mathrm{e}+2^l)\}\|f\|_{L^2(\mathbb{R}^d)}. \tag{3.19}$$

For fixed $j,\ l \in \mathbb{Z}$, write

$$\begin{aligned}(S_{l-j}U_j^l S_{l-j})_A f(x) \;&=\; S_{l-j,A}(U_j^l S_{l-j} f)(x) + S_j(U_j^l S_{l-j})_A f(x)\\ &\quad + \sum_{n=1}^d [h_n,\ S_{l-j}]([\partial_n A,\ U_j^l S_{l-j}]f)(x)\\ &:= \mathrm{I}_j^l f(x) + \mathrm{II}_j^l f(x) + \sum_{n=1}^d \mathrm{III}_j^{l,n} f(x).\end{aligned}$$

We now estimate terms I_j^l, II_j^l and $\mathrm{III}_j^{l,n}$. Inequality (3.3) in Lemma 3.3, along with (3.19) leads to that

$$\begin{aligned}\Big\| \sum_j S_{l-j,A}(U_j^l S_{l-j} f)\Big\|_{L^2(\mathbb{R}^d)}^2 &\lesssim \sum_j 2^{2(j-l)} \|U_j^l S_{l-j} f\|_{L^2(\mathbb{R}^d)}^2 \\ &\lesssim 2^{-2l} \min\{(\delta 2^l)^2,\ \log^{-\beta}(\mathrm{e}+2^l)\}^2 \|f\|_{L^2(\mathbb{R}^d)}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_l \Big\| \sum_j \mathrm{I}_j^l f\Big\|_{L^2(\mathbb{R}^d)} &\lesssim \Big(\sum_{l>\log \frac{1}{\sqrt{\delta}}} l^{-\beta} + \delta^2 \sum_{l\le \log \frac{1}{\sqrt{\delta}}} 2^l \Big) \|f\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \log^{-\beta+1}(\mathrm{e}+\delta^{-1}) \|f\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

For each fixed $j, l \in \mathbb{Z}$ and n with $1 \le n \le d$, it follows from Lemma 3.7 and (3.19) that

$$\begin{aligned}\big\|[\partial_n A,\ U_j^l S_{l-j}]f\big\|_{L^2(\mathbb{R}^d)} &\le \big\|[\partial_n A,\ U_j^l] S_{l-j} f\big\|_{L^2(\mathbb{R}^d)} + \big\|U_j^l[\partial_n A,\ S_{l-j}]f\big\|_{L^2(\mathbb{R}^d)} \\ &\lesssim 2^{-j} \min\{(\delta 2^l)^{2\varepsilon},\ \log^{-\varepsilon\beta+1}(\mathrm{e}+2^l)\} \|S_{l-j} f\|_{L^2(\mathbb{R}^d)}^2 \\ &\quad + 2^{-j} \min\{(\delta 2^l)^2, \log^{-\beta}(\mathrm{e}+2^l)\} \|[\partial_n A, S_{l-j}]f\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

which, along with Lemma 3.4, implies that

$$\begin{aligned}&\Big\| \sum_j |[h_n,\ S_{l-j}]([\partial_n A,\ U_j^l S_{l-j}]f)|\Big\|_{L^2(\mathbb{R}^d)}^2 \lesssim \sum_j 2^{2(j-l)} \big\|[\partial_n A,\ U_j^l S_{l-j}]f\big\|_{L^2(\mathbb{R}^d)}^2 \\ &\lesssim 2^{-2l} \min\{(\delta 2^l)^{2\varepsilon},\ \log^{-\varepsilon\beta+1}(\mathrm{e}+2^l)\}^2 \|f\|_{L^2(\mathbb{R}^d)}^2.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_l \Big\| \sum_j \mathrm{III}_j^{l,n} f\Big\|_{L^2(\mathbb{R}^d)} &\lesssim \Big(\sum_{l>\log \frac{1}{\sqrt{\delta}}} l^{-\varepsilon\beta+1} + \delta^2 \sum_{l\le \log \frac{1}{\sqrt{\delta}}} 2^{(4\varepsilon-2)l} \Big) \|f\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \log^{-\varepsilon\beta+2}(\mathrm{e}+\delta^{-1}) \|f\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

As for term II_j^l, write

$$(U_j^l S_{l-j})_A f = U_{j,A}^l S_{l-j} f + U_j^l S_{l-j,\,A} f + \sum_{n=1}^d [h_n,\ U_j^l]([\partial_n A,\ S_{l-j}]f).$$

It follows from Littlewood–Paley theory and (3.18) that

$$\Big\| \sum_j S_{l-j} U^l_{j,\,A} S_{l-j} f \Big\|^2_{L^2(\mathbb{R}^d)} \lesssim \sum_j \|U^l_{j,\,A} S_{l-j} f\|^2_{L^2(\mathbb{R}^d)}$$
$$\lesssim \min\{(\delta 2^l)^{\varepsilon/2},\ \log^{-\varepsilon\beta+1}(\mathrm{e}+2^l)\}^2 \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Again by Lemma 3.3 and (3.19), we deduce that

$$\Big\| \sum_j S_{l-j} U^l_j S_{l-j,A} f \Big\|^2_{L^2(\mathbb{R}^d)} \lesssim \sum_j \|U^l_j S_{l-j,A} f\|^2_{L^2(\mathbb{R}^d)}$$
$$\lesssim 2^{-2l} \min\{(\delta 2^l)^2,\ \log^{-\beta}(\mathrm{e}+2^l)\}^2 \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Similar to the term $\mathrm{III}^{l,n}_j$, we have that for each $1 \le n \le d$,

$$\Big\| \sum_j S_{l-j}[h_n,\ U^l_j]([\partial_n A,\ S_{l-j}]f) \Big\|^2_{L^2(\mathbb{R}^d)}$$
$$\lesssim \sum_j \|[h_n,\ U^l_j]([\partial_n A,\ S_{l-j}]f)\|^2_{L^2(\mathbb{R}^d)}$$
$$\lesssim 2^{-2l} \min\{(\delta 2^l)^{2\varepsilon},\ \log^{-\varepsilon\beta+1}(\mathrm{e}+2^l)\}^2 \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Therefore,

$$\sum_l \Big\| \sum_j \mathrm{II}^l_j f \Big\|_{L^2(\mathbb{R}^d)} \lesssim \log^{-\varepsilon\beta+2}(\mathrm{e}+\delta^{-1}) \|f\|_{L^2(\mathbb{R}^d)}.$$

The estimates for I^l_j, II^l_j and $\mathrm{III}^{l,n}_j$ above, via (3.17), leads to our desired conclusion. □

We are now ready to establish our main result in this section.

Theorem 3.9 *Let Ω be homogeneous of degree zero, satisfy the vanishing moment condition (1.1) and $\Omega \in GS_\beta(S^{d-1})$ for some $\beta \in (2,\ \infty)$. Let A be a function on $\mathbb{R}^d$ such that $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$. Then there exists a sequence of operators $\{R_{l,A}\}_{l\in\mathbb{N}}$ such that*

(i) *$R_{l,\,A}$ is defined as*

$$R_{l,A} f(x) = \mathrm{p.\,v.} \int_{\mathbb{R}^d} \widetilde{K}^l(x-y) \frac{A(x) - A(y) - \nabla A(y)(x-y)}{|x-y|} f(y)dy,$$

the function $\widetilde{K}^l$ satisfies the size condition that for $0 < R < \infty$,

$$\int_{R<|x|<2R} |\widetilde{K}^l(x)| dx \lesssim 1,$$

and the regularity that for all $R > 0$ *and* $y \in \mathbb{R}^d$,

$$\sum_{m=2}^{\infty} m \int_{2^m R<|x-y|\le 2^{m+1}R} |\widetilde{K}^l(x-y) - \widetilde{K}^l(x)|dx \lesssim l^2;$$

(ii) *for each fixed n with* $1 \le n \le d$, *the operator* W_l^n *defined by*

$$W_l^n f(x) = \text{p. v.} \int_{\mathbb{R}^d} \widetilde{K}^l(x-y)\frac{x_n - y_n}{|x-y|} f(y)dy$$

is bounded on $L^2(\mathbb{R}^d)$ *with bound independent of* l;

(iii) *for each fixed* $\varepsilon \in (0,\ 1)$,

$$\|R_{l,\,A} - T_{\Omega,A}\|_{L^2(\mathbb{R}^d)} \lesssim l^{-\varepsilon\beta+2}. \tag{3.20}$$

Proof For $j \in \mathbb{Z}$, let $K_j(x) = \frac{\Omega(x)}{|x|^{d+1}}\chi_{\{2^{j-1}\le|x|<2^j\}}(x)$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a nonnegative radial function such that

$$\operatorname{supp}\psi \subset \{x:\ |x| \le 1/4\},\quad \int_{\mathbb{R}^d} \psi(x)dx = 1.$$

For $j \in \mathbb{Z}$, set $\psi_j(x) = 2^{-dj}\psi(2^{-j}x)$. For a positive integer l, define

$$H_l(x) = \sum_{j\in\mathbb{Z}} K_j * \psi_{j-l}(x).$$

Let R_l be the convolution operator with kernel H_l. For a function A on $\mathbb{R}^d$ with $\nabla A \in \mathrm{BMO}(\mathbb{R}^d)$, denote

$$R_{l,A} f(x) = \text{p. v.} \int_{\mathbb{R}^d} H_l(x-y)\big(A(x) - A(y) - \nabla A(y)(x-y)\big) f(y)dy.$$

Now we prove (3.20). Write

$$\sum_{j\in\mathbb{Z}} K_j(x) - H_l(x) = \sum_{j\in\mathbb{Z}} \big(K_j(x) - K_j * \psi_{j-l}(x)\big) =: \sum_{j\in\mathbb{Z}} \mu_{j,l}(x).$$

The fact ψ is radial, implies that, for n with $1 \le n \le d$,

$$\int_{\mathbb{R}^d} \psi(x)x_n dx = 0.$$

From this we know that for all n with $1 \le n \le d$, $\partial_n\widehat{\psi}(0) = 0$. By Taylor series expansion and the fact that $\widehat{\psi}(0) = 1$, we deduce that

$$|\widehat{\psi}(2^{j-l}\xi) - 1| \lesssim \min\{1,\ |2^{j-l}\xi|^2\}.$$

On the other hand, as it was proved in [15], we know that when $\Omega \in GS_\beta(S^{d-1})$ for some $\beta \in (1, \infty)$,

$$|\widehat{K_j}(\xi)| \lesssim 2^{-j} \min\{1, \log^{-\beta}(\mathrm{e} + |2^j\xi|)\},$$

which, in turn leads to following Fourier transform estimate

$$|\widehat{\mu_{j,l}}(\xi)| = |\widehat{K_j}(\xi)||\widehat{\psi}(2^{j-l}\xi) - 1| \lesssim 2^{-j} \min\{\log^{-\beta}(\mathrm{e} + |2^j\xi|), |2^{j-l}\xi|^2\}. \tag{3.21}$$

On the other hand, a trivial computation shows that for all multi-indices $\gamma \in \mathbb{Z}_+^d$,

$$\|\partial^\gamma \widehat{K_j}\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\Omega\|_{L^1(S^{d-1})} 2^{(|\gamma|-1)j},$$

and so for all $\xi \in \mathbb{R}^d$,

$$|\partial^\gamma \widehat{\mu_{j,l}}(\xi)| \lesssim \sum_{\gamma_1+\gamma_2=\gamma} |\partial^{\gamma_1} \widehat{K_j}(\xi)||\partial^{\gamma_2}\widehat{\psi}(2^{j-l}\xi)| \lesssim \|\Omega\|_{L^1(S^{d-1})} 2^{j(|\gamma|-1)}. \tag{3.22}$$

Let $\widetilde{K}^l(x-y) = H_l(x-y)|x-y|$. The Fourier transforms (3.21) and (3.22), via Theorem 3.8 with $\delta = 2^{-l}$, lead to (3.20) directly.

We now verify conclusion (i). For each fixed $R > 0$,

$$\int_{R<|x|<2R} |H_l(x)|dx \lesssim \sum_{j:\,2^j \approx R} \|K_j\|_{L^1(\mathbb{R}^d)} \|\psi_{j-l}\|_{L^1(\mathbb{R}^d)} \lesssim R^{-1}.$$

On the other hand, for $R > 0$ and $y \in \mathbb{R}^d$ with $|y| < R/4$,

$$\begin{aligned}
&\int_{2^m R<|x-y|\le 2^{m+1}R} \big|H_l(x-y)|x-y| - H_l(x)|x|\big|dx \\
&\quad \le \int_{2^m R<|x-y|\le 2^{m+1}R} |H_l(x-y) - H_l(x)||x-y|dx \\
&\qquad + |y| \int_{2^{m-1}R<|x|\le 2^{m+2}R} |H_l(x)|dx \\
&\quad \lesssim 2^m R \int_{2^m R<|x-y|\le 2^{m+1}R} |H_l(x-y) - H_l(x)|dx + \frac{|y|}{2^m R}.
\end{aligned}$$

Observe that

$$\|\psi_{j-l}(\cdot - y) - \psi_{j-l}(\cdot)\|_{L^1(\mathbb{R}^d)} \lesssim \min\{1, 2^{l-j}|y|\}.$$

Young's inequality now tells us that

$$\begin{aligned}
&\int_{2^m R<|x-y|\le 2^{m+1}R} |H_l(x-y)-H_l(x)|dx \\
&\lesssim \sum_{j:\,2^j\approx 2^{m+1}R} \|K_j\|_{L^1(\mathbb{R}^d)}\|\psi_{j-l}(\cdot-y)-\psi_{j-l}(\cdot)\|_{L^1(\mathbb{R}^d)} \\
&\lesssim (2^m R)^{-1}\min\{1,\ 2^{l-m}\}.
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
&\sum_{m=2}^{\infty} m \int_{2^m R<|x-y|\le 2^{m+1}R} |\widetilde{K}^l(x-y)-\widetilde{K}^l(x)|dx \\
&\lesssim \sum_{m=2}^{\infty} m\min\{1,\ 2^{l-m}\} + \sum_{m=2}^{\infty} m2^{-m} \lesssim l^2.
\end{aligned}$$

Finally, for each fixed n with $1\le n\le d$, let

$$Y_l^n f(x) = \text{p. v.}\int_{\mathbb{R}^d}\Big(\sum_{j\in\mathbb{Z}} K_j(x-y) - H_l(x-y)\Big)(x_n-y_n)f(y)dy.$$

The estimates (3.21) and (3.22), via [4, Theorem 3.4], state that

$$\|Y_l^n f\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

For $1\le n\le d$, let T_Ω^n be the operator defined by

$$T_\Omega^n h(x) = \text{p. v.}\int_{\mathbb{R}^d}\frac{\Omega(x-y)(x_n-y_n)}{|x-y|^{d+1}}h(y)dy. \tag{3.23}$$

It is well known that T_Ω^n is bounded on $L^2(\mathbb{R}^d)$. Note that

$$\text{p. v.}\int_{\mathbb{R}^d}\sum_{j\in\mathbb{Z}} K_j(x-y)(x_n-y_n)f(y)dy = T_\Omega^n f(x).$$

Therefore, W_l^n is bounded on $L^2(\mathbb{R}^d)$ with bound independent of l. This completes the proof of Theorem 3.9. □

4 Proof of Theorem 1.3

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be a radial function which satisfies (3.10), $\varphi_j(x) = \varphi(2^{-j}x)$. For each fixed $j \in \mathbb{Z}$, set

$$T_{\Omega,\,A;\,j} f(x) = \int_{\mathbb{R}^d} K_{A,\,j}(x,\, y) f(y) dy,$$

where

$$K_{A,\,j}(x,\, y) = \frac{\Omega(x-y)}{|x-y|^{d+1}} \big(A(x) - A(y) - \nabla A(y)(x-y)\big)\varphi_j(|x-y|).$$

Let $\omega \in C_0^\infty(\mathbb{R}^d)$ be a radial function, have integral zero and $\operatorname{supp}\omega \subset B(0,\, 1)$. Note that $\widehat{\omega}$ is also a radial function on $\mathbb{R}^d$. Let Q_s be the operator defined by $Q_t f(x) = \omega_t * f(x)$, where $\omega_t(x) = t^{-d}\omega(t^{-1}x)$ for $t > 0$. We assume that

$$\int_0^\infty [\widehat{\omega}(s)]^4 \frac{ds}{s} = 1.$$

The Calderón reproducing formula

$$\int_0^\infty Q_s^4 \frac{ds}{s} = I \tag{4.1}$$

then holds true. Moreover, the classical Littlewood–Paley theory tells us that for all $p \in (1,\, \infty)$,

$$\Big\|\Big(\int_0^\infty |Q_s f|^2 \frac{ds}{s}\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \tag{4.2}$$

It is well know that for $b \in \mathrm{BMO}(\mathbb{R}^d)$ and $p \in (1,\, \infty)$,

$$\Big\|\Big(\int_0^\infty |[b,\, Q_s] f|^2 \frac{ds}{s}\Big)^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^d)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}. \tag{4.3}$$

For a function $\Omega \in L^1(S^{d-1})$, define the operators $W_{\Omega,j}$ and $U_{\Omega,n,j}$ by

$$W_{\Omega,j} h(x) = \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} \varphi_j(x-y) h(y) dy,$$

$$U_{\Omega,n,j} h(x) = \int_{\mathbb{R}^d} \frac{\Omega(x-y)(x_n - y_n)}{|x-y|^{d+1}} \varphi_j(x-y) h(y) dy,\ \ 1 \le n \le d.$$

Lemma 4.1 *Let Ω be homogeneous of degree zero, and $\Omega \in GS_\beta(S^{d-1})$ for some $\beta \in (1,\, \infty)$, then for $j \in \mathbb{Z}$ and $s \in (0, 2^j]$,*

$$\|Q_s W_{\Omega,\,j} f\|_{L^2(\mathbb{R}^d)} \lesssim 2^{-j} \log^{-\beta}(\mathrm{e} + 2^j/s) \|f\|_{L^2(\mathbb{R}^d)}, \tag{4.4}$$

and

$$\|Q_s U_{\Omega,n,j} f\|_{L^2(\mathbb{R}^d)} \lesssim \log^{-\beta}(\mathrm{e}+2^j/s)\|f\|_{L^2(\mathbb{R}^d)}. \tag{4.5}$$

Furthermore, for $b \in \mathrm{BMO}(\mathbb{R}^d)$, $j \in \mathbb{Z}$ *and* $s \in (0,\ 2^j]$,

$$\big\|[b,\ Q_s U_{\Omega,n,\ j}]f\big\|_{L^2(\mathbb{R}^d)} \lesssim \log^{-\beta+1}(\mathrm{e}+2^j/s)\|f\|_{L^2(\mathbb{R}^d)}. \tag{4.6}$$

Proof Inequalities (4.4) and (4.5) were proved in [4]. We now prove (4.6). We may assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^d)} = 1$. By dilation-invariance, it suffices to consider the case $j = 0$ and $s \in (0,\ 1]$. Let $\mathbb{R}^d = \cup_l I_l$, where I_l are cubes having disjoint interiors, and side length 1. For each fixed l, let $f_l = f\chi_{I_l}$. Observing that supp $Q_s U_{\Omega,0} f_l \subset 20dI_l$, and $Q_s U_{\Omega,n,0} f_l$ have bounded overlaps, we then have that

$$\|[b,\ Q_s U_{\Omega,n,0}]f\|^2_{L^2(\mathbb{R}^d)} \lesssim \sum_l \|[b,\ Q_s U_{\Omega,n,0}]f_l\|^2_{L^2(\mathbb{R}^d)}.$$

Thus, we may assume that supp$f \subset I$, with I a cube having side length 1. An application of the inequality (2.4) gives us that

$$\int_{\mathbb{R}^d} |b(x)-\langle b\rangle_I|^2 |Q_s U_{\Omega,n,0} f(x)|^2 dx \lesssim \|Q_s U_{\Omega,n,0} f\|^2_{L^2(\log L)^2,20dI}.$$

Now let $\lambda_0 = \log^{-\beta+1}(\mathrm{e}+1/s)$, h be a function on $\mathbb{R}^d$ such that supp $h \subset 20dI$ and $\|h\|_{L^2(\mathbb{R}^d)} = 1$. Observing that $\|h\|_{L^1(\mathbb{R}^d)} \lesssim 1$, we then get that

$$\|Q_s U_{\Omega,n,0} h\|_{L^\infty(\mathbb{R}^d)} \lesssim s^{-d}\|U_{\Omega,n,0} h\|_{L^1(\mathbb{R}^d)} \lesssim s^{-d}\|h\|_{L^1(\mathbb{R}^d)} \lesssim s^{-d},$$

and for any $x \in 20dI$ and $s \in (0,\ 1]$,

$$\frac{|Q_s U_{\Omega,n,0} h(x)|}{\lambda_0} \lesssim s^{-d}\log^{\beta-1}(\mathrm{e}+1/s) \lesssim s^{-d-1}.$$

A straightforward computation involving estimate (4.5) leads to that

$$\int_{20dI} \left(\frac{|Q_s U_{\Omega,n,0} h(x)|}{\lambda_0}\right)^2 \log^2\left(\mathrm{e}+\frac{|Q_s U_{\Omega,n,0} h(x)|}{\lambda_0}\right) dx$$
$$\lesssim \frac{1}{\lambda_0^2}\|Q_s U_{\Omega,n,0} h\|^2_{L^2(\mathbb{R}^d)} \log^2(\mathrm{e}+1/s) \lesssim 1,$$

Therefore,

$$\|Q_s U_{\Omega,n,0} h\|_{L^2(\log L)^2,20dI} \lesssim \lambda_0.$$

This, via inequality (2.4), yields

$$\||b-\langle b\rangle_I|Q_sU_{\Omega,n,0}h\|_{L^2(\mathbb{R}^d)} \lesssim \lambda_0\|h\|_{L^2(\mathbb{R}^d)}. \tag{4.7}$$

We also have that

$$\|Q_sU_{\Omega,n,0}((b-\langle b\rangle_I)f)\|_{L^2(\mathbb{R}^d)} \lesssim \lambda_0\|f\|_{L^2(\mathbb{R}^d)}. \tag{4.8}$$

In fact, a standard computation leads to that

$$\begin{aligned}
&\|Q_sU_{\Omega,n,0}((b-\langle b\rangle_I)f)\|_{L^2(\mathbb{R}^d)}\\
&\quad= \sup_{\|g\|_{L^2(\mathbb{R}^d)}\le 1}\left|\int_{\mathbb{R}^d} Q_sU_{\Omega,n,0}((b-\langle b\rangle_I)f)(x)g(x)dx\right|\\
&\quad= \sup_{\|g\|_{L^2(\mathbb{R}^d)}\le 1}\left|\int_I Q_sU_{\Omega,n,0}(g\chi_{20dI})(x)(b(x)-\langle b\rangle_I)f(x)dx\right|\\
&\quad\le \sup_{\|g\|_{L^2(\mathbb{R}^d)}\le 1}\|f\|_{L^2(\mathbb{R}^d)}\|(b-\langle b\rangle_I)Q_sU_{\Omega,n,0}(g\chi_{20dI})\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

which, along with (4.7), implies (4.8). Combining estimates (4.7) and (4.8) yields (4.6) for the case of $j=0$, and completes the proof of Lemma 4.1. □

Proof of Theorem 1.3 The procedure follows two steps. At first, we prove the $L^2(\mathbb{R}^d)$ boundedness of T_A, by following the argument in the proof of Theorem 1.3 in [17], together with some refined decomposition and estimates for $T_{\Omega,A}$. Then we prove the $L^p(\mathbb{R}^d)$ boundedness, using the approximation established in Sect. 3. Again, we assume that $\|\nabla A\|_{\mathrm{BMO}(\mathbb{R}^d)}=1$.

We now prove the $L^2(\mathbb{R}^d)$ boundedness of $T_{\Omega,\,A}$. By the Calderón reproducing formula (4.1), it suffices to prove that for $f,\ g\in C_0^\infty(\mathbb{R}^d)$,

$$\left|\int_0^\infty\int_0^t\int_{\mathbb{R}^d} Q_s^4T_{\Omega,\,A}Q_t^4f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\right| \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}; \tag{4.9}$$

and

$$\left|\int_0^\infty\int_t^\infty\int_{\mathbb{R}^d} Q_s^4T_{\Omega,\,A}Q_t^4f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\right| \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{4.10}$$

We first prove (4.9). Let $\alpha\in(\frac{d+1}{d+2},\ 1)$ be a constant. For each fixed $j\in\mathbb{Z}$, let

$$F_{j,1}=\{(s,\ t):\ 0<t\le 2^j,\ 0<s\le t\},$$

$$F_{j,2}=\{(s,\ t):\ 2^j<t<\infty,\ 0<s\le(2^jt^{\alpha-1})^{1/\alpha}\},$$

and

$$F_{j,3} = \{(s,\ t) : 2^j < t < \infty,\ (2^j t^{\alpha-1})^{1/\alpha} < s \le t\}.$$

For $k = 1,\ 2,\ 3$, set

$$\mathrm{E}_k(f,\ g) = \sum_{j\in\mathbb{Z}} \int\int \chi_{F_{j,k}} \int_{\mathbb{R}^d} Q_s T_{\Omega,\ A;\,j} Q_t^4 f(x) Q_s^3 g(x) dx \frac{ds}{s}\frac{dt}{t},$$

with

$$T_{\Omega,A;j} = \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^{d+1}} \varphi_j(|x-y|)\big(A(x) - A(y) - \nabla A(y)(x-y)\big) f(y)dy.$$

It was proved in [22] that

$$\|E_3(f,\ g)\|_{L^2(\mathbb{R}^d)} \lesssim \|\Omega\|_{L^1(S^{d-1})}\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

Then the proof of (4.9) is reduced to proving that for $k = 1,\ 2$,

$$\|E_k(f,\ g)\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}. \tag{4.11}$$

The proofs of (4.11) for $k = 1$ and $k = 2$ are similar, so we only prove (4.11) for the case of $k = 2$. For each fixed $j \in \mathbb{Z}$, let $\{I_l\}$ be the sequence of cubes having disjoint interiors and side lengths 2^j, such that $\mathbb{R}^d = \bigcup_l I_l$. For fixed l, let $h_{s,l}(x) = Q_s g(x)\chi_{I_l}(x)$, $\zeta_l \in C_0^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\zeta_l \subset 48dI_l$, $0 \le \zeta_l \le 1$ and $\zeta_l(x) \equiv 1$ when $x \in 32dI_l$. Let x^l be a point on the boundary of $100dI_l$. Let

$$\tilde{A}_l(y) = A(y) - \sum_{m=1}^{d} \langle \partial_m A\rangle_{I_l} y_m,\ \ A_l(y) = (\tilde{A}_l(y) - \tilde{A}_l(x^l))\zeta_l(y).$$

It follows from Lemma 2.1 that for all $y \in \mathbb{R}^d$,

$$|A_l(y)| \lesssim 2^j,\ \ |\nabla A_l(y)| \lesssim 1 + |\nabla A(y) - \langle \nabla A\rangle_{I_l}|. \tag{4.12}$$

Note that for $x \in 48dI_l$ and $y \in \mathbb{R}^d$ with $|x-y| \le 2^{j+2}$,

$$A(x) - A(y) - \nabla A(y)(x-y) = A_l(x) - A_l(y) - \nabla A_l(y)(x-y).$$

Write

$$T_{\Omega,\ A;\,j} h(x) = A_l W_{\Omega,j} h(x) - W_{\Omega,j}(A_l h)(x) - \sum_{n=1}^{d} U_{\Omega,n,j}(\partial_n A_l h)(x).$$

Set

$$\mathrm{D}_1 = \left|\sum_j\sum_l\int\int\chi_{F_{j,2}}\int_{\mathbb{R}^d} A_l W_{\Omega,j} Q_t^4 f(x) Q_s^3 h_{s,l}(x)dx\frac{ds}{s}\frac{dt}{t}\right|,$$

$$\mathrm{D}_2 = \left|\sum_j\sum_l\int\int\chi_{F_{j,2}}\int_{\mathbb{R}^d} W_{\Omega,j}(A_l Q_t^4 f)(x) Q_s^3 h_{s,l}(x)dx\frac{ds}{s}\frac{dt}{t}\right|,$$

and for $1 \le n \le d$,

$$\mathrm{D}_{3,n} = \Big|\sum_j\sum_l\int\int\chi_{F_{j,2}}\int_{\mathbb{R}^d} U_{\Omega,n,j}(\partial_n A_l Q_t^4 f)(x) Q_s^3 h_{s,l}(x)dx\frac{ds}{s}\frac{dt}{t}\Big|.$$

It then follows that

$$|E_2(f,\ g)| \le \mathrm{D}_1 + \mathrm{D}_2 + \sum_{n=1}^d \mathrm{D}_{3,n}.$$

We first consider term D_1. To this aim, we split it into two parts as

$$\begin{aligned}
\mathrm{D}_1 &= \left|\sum_j\sum_l\int\int\chi_{F_{j,2}}\int_{\mathbb{R}^d} Q_t^4 f(x) W_{\Omega,j}(A_l Q_s^3 h_{s,l})(x)dx\frac{ds}{s}\frac{dt}{t}\right| \\
&\le \left|\sum_j\sum_l\int\int\chi_{F_{j,2}}\int_{\mathbb{R}^d} Q_t^4 f(x) W_{\Omega,j}([A_l,\ Q_s^3] h_{s,l})(x)dx\frac{ds}{s}\frac{dt}{t}\right| \\
&\quad + \left|\sum_j\sum_l\int\int\chi_{F_{j,2}}\int_{\mathbb{R}^d} Q_t^4 f(x) W_{\Omega,j} Q_s^3 (A_l h_{s,l})(x)dx\frac{ds}{s}\frac{dt}{t}\right| \\
&:= \mathrm{D}_{11} + \mathrm{D}_{12}.
\end{aligned}$$

An application of Hölder's inequality leads to that

$$\begin{aligned}
\mathrm{D}_{11} &\lesssim \left(\sum_{j\in\mathbb{Z}}\sum_l\int\int\chi_{F_{j,2}}\left\|\chi_{64dI_l} Q_t^4 f\right\|_{L^2(\mathbb{R}^d)}^2 (2^{-j}s)^{\frac{1}{2}}\frac{ds}{s}\frac{dt}{t}\right)^{1/2} \\
&\quad\times\left(\sum_{j\in\mathbb{Z}}\sum_l\int\int\chi_{F_{j,2}}\left\|W_{\Omega,\ j}([A_l,\ Q_s^3]h_{s,l})\right\|_{L^2(\mathbb{R}^d)}^2 (2^{-j}s)^{-\frac{1}{2}}\frac{ds}{s}\frac{dt}{t}\right)^{1/2} \\
&:= \mathrm{I}_1\mathrm{I}_2.
\end{aligned}$$

A straightforward computation gives that

$$\mathrm{I}_1 \lesssim \left\| \left(\int_0^\infty |Q_t^4 f|^2 \int_0^t \sum_{j:\, 2^j \geq s^\alpha t^{1-\alpha}} (2^{-j}s)^{\frac{1}{2}} \frac{ds}{s} \frac{dt}{t} \right)^{1/2} \right\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}.$$

Lemma 2.1, along with estimate (4.12), tells us that for $x,\ y \in \mathbb{R}^d$ with $|x-y| \leq s \leq 2^j$,

$$\begin{aligned} |A_l(x) - A_l(y)| &\lesssim |x-y| \left(\frac{1}{|I_x^y|} \int_{I_x^y} |\nabla A(z) - \langle \nabla A \rangle_{I_l}|^q dz \right)^{1/q} \\ &\lesssim |x-y| \left(1 + \log\left(\frac{2^j}{|x-y|} \right) \right) \lesssim 2^j (2^{-j}s)^{\frac{1}{2}}, \end{aligned}$$

since $\Phi(t) = t\log(\mathrm{e}+t)$ is increasing and $\Phi(t) \leq t^{1/2}$ when $t \leq 1$. Therefore,

$$|[A_l, Q_s^3]h(x)| \lesssim \int_{\mathbb{R}^d} \widetilde{\omega}_s(x-y)|A_l(x) - A_l(y)||h(y)|dy \lesssim 2^j (2^{-j}s)^{\frac{1}{2}} Mh(x),$$

where $\widetilde{\omega}_s(x) = s^{-d}\widetilde{\omega}(s^{-1}x)$ and $\widetilde{\omega}(x) = \omega * \omega * \omega(x)$. Let M_Ω be the operator defined by

$$M_\Omega h(x) = \sup_{r>0} r^{-d} \int_{|x-y|<r} |\Omega(x-y)h(y)|dy.$$

We then have that

$$\begin{aligned} \mathrm{I}_2 &\lesssim \left(\sum_{j\in\mathbb{Z}} \sum_l \int\int \chi_{F_{j,2}} \left\| M_\Omega M h_{s,l} \right\|^2_{L^2(\mathbb{R}^d)} (2^{-j}s)^{\frac{1}{2}} \frac{ds}{s} \frac{dt}{t} \right)^{1/2} \\ &\lesssim \left(\sum_{j\in\mathbb{Z}} \sum_l \int\int \chi_{F_{j,2}} \|h_{s,l}\|^2_{L^2(\mathbb{R}^d)} (2^{-j}s)^{\frac{1}{2}} \frac{ds}{s} \frac{dt}{t} \right)^{1/2} \\ &\lesssim \|g\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

since

$$\int_s^\infty \sum_{j:\, 2^j \geq s^\alpha t^{1-\alpha}} (2^{-j}s)^{\frac{1}{2}} \frac{dt}{t} \lesssim 1.$$

Therefore,

$$\mathrm{D}_{11} \lesssim \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}.$$

For term D_{12}, another application of Hölder's inequality yields

$$\begin{aligned}\mathrm{D}_{12} &\lesssim \left(\sum_{j\in\mathbb{Z}}\int\int \chi_{F_{j,2}}\|Q_s^2Q_t^3 f\|^2_{L^2(\mathbb{R}^d)}\log^{-\sigma_1}(\mathrm{e}+2^j/s)\frac{ds}{s}\frac{dt}{t}\right)^{1/2}\\ &\times\left\|\left(\sum_{j\in\mathbb{Z}}\int\int \chi_{F_{j,2}}\Big|Q_sW_{\Omega,j}Q_t\Big(\sum_l A_l h_{s,l}\Big)\Big|^2\log^{\sigma_1}(\mathrm{e}+2^j/s)\frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{L^2(\mathbb{R}^d)}\\ &:= \mathrm{I}_3\mathrm{I}_4,\end{aligned}$$

where $1<\sigma_1<2\beta-1$ is a constant. Observe that

$$\sum_{j:2^j\geq s}\log^{-\sigma_1}(\mathrm{e}+2^j/s)\lesssim 1.$$

It then follows that

$$\mathrm{I}_3\lesssim\left\|\left(\int_0^\infty\int_0^\infty|Q_sQ_t^3 f|^2\frac{ds}{s}\frac{dt}{t}\right)^{1/2}\right\|_{L^2(\mathbb{R}^d)}\lesssim\|f\|_{L^2(\mathbb{R}^d)}.$$

From (4.3) and (4.4) in Lemma 4.1, we know that

$$\begin{aligned}\mathrm{I}_4 &\lesssim\left(\sum_{j\in\mathbb{Z}}\int_0^{2^j}\Big\|W_{\Omega,j}Q_s\Big(\sum_l A_l h_{s,l}\Big)(x)\Big\|^2_{L^2(\mathbb{R}^d)}\log^{\sigma_1}(\mathrm{e}+2^j/s)\frac{ds}{s}\right)^{1/2}\\ &\lesssim\left(\sum_{j\in\mathbb{Z}}\int_0^{2^j}2^{-j}\Big\|\sum_l A_l h_{s,l}\Big\|^2_{L^2(\mathbb{R}^d)}\log^{-2\beta+\sigma_1}(\mathrm{e}+2^j/s)\frac{ds}{s}\right)^{1/2}\\ &\lesssim\left(\int_0^\infty\|Q_sg\|^2_{L^2(\mathbb{R}^d)}\sum_{j:\,2^j\geq s}\log^{-2\beta+\sigma_1}(\mathrm{e}+2^j/s)\frac{ds}{s}\right)^{1/2}\\ &\lesssim\|g\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

where in the third inequality, we have invoked the fact that the supports of functions $\{A_lh_{s,l}\}_l$ have bounded overlaps, and

$$\Big|\sum_l A_l h_{s,l}\Big|^2\lesssim 2^j\sum_l|h_{s,l}|^2=2^j|Q_sg|^2.$$

Therefore,

$$\mathrm{D}_{12}\lesssim\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

We turn our attention to term D_2. Observe that $Q_s W_{\Omega,j} = W_{\Omega,j} Q_s$. It then follows that

$$\begin{aligned}\mathrm{D}_2 &\le \Big|\sum_j \sum_l \int\int \chi_{F_{j,2}} \int_{\mathbb{R}^d} Q_s W_{\Omega,j}([A_l, Q_s^2]Q_t^4 f)(x) h_{s,l}(x) dx \frac{ds}{s}\frac{dt}{t}\Big| \\ &\quad + \Big|\sum_j \sum_l \int\int \chi_{F_{j,2}} \int_{\mathbb{R}^d} Q_s W_{\Omega,j}(A_l Q_s^2 Q_t^4 f)(x) h_{s,l}(x) dx \frac{ds}{s}\frac{dt}{t}\Big| \\ &:= \mathrm{D}_{21} + \mathrm{D}_{22}.\end{aligned}$$

Similar to term D_{11} and term D_{12} respectively,, we have that

$$\mathrm{D}_{21} \lesssim \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}, \ \ \mathrm{D}_{22} \lesssim \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}.$$

To consider $\mathrm{D}_{3,n}$, we write

$$\partial_n A_l Q_t^4 f(x) = [\partial_n A, Q_t] Q_t^3 f(x) + Q_t[\partial_n A, Q_t] Q_t^2 f(x) + Q_t^2(\partial_n \widetilde{A}_l Q_t^2) f(x),$$

and

$$\begin{aligned}\mathrm{D}_{3,n} &\le \sum_j \int\int \chi_{F_{j,2}} \Big|\sum_l \int_{\mathbb{R}^d} Q_s^3 h_{s,l}(x) U_{\Omega,n,j}([\partial_n A, Q_t] Q_t^3 f)(x) dx\Big| \frac{ds}{s}\frac{dt}{t} \\ &\quad + \sum_j \int\int \chi_{F_{j,2}} \Big|\sum_l \int_{\mathbb{R}^d} Q_s^3 h_{s,l}(x) U_{\Omega,n,j} Q_t([\partial_n A, Q_t] Q_t^2 f)(x) dx\Big| \frac{ds}{s}\frac{dt}{t} \\ &\quad + \sum_j \int\int \chi_{F_{j,2}} \Big|\sum_l \int_{\mathbb{R}^d} Q_s^3 h_{s,l}(x) U_{\Omega,n,j} Q_t^2(\partial_n \widetilde{A}_l Q_t^2) f(x) dx\Big| \frac{ds}{s}\frac{dt}{t} \\ &:= \sum_{i=1}^3 \mathrm{D}_{3,n}^i.\end{aligned}$$

Let $2 < \sigma_2 < 2\beta - 2$. It follows from Hölder's inequality that

$$\begin{aligned}\mathrm{D}_{3,n}^1 &= \sum_j \int\int \chi_{F_{j,2}} \Big|\sum_l \int_{\mathbb{R}^d} Q_s^2 U_{\Omega,n,j}([\partial_n A, Q_t] Q_t^3 f)(x) Q_s h_{s,l}(x) dx\Big| \frac{ds}{s}\frac{dt}{t} \\ &\lesssim \left(\sum_j \int\int \chi_{F_{j,2}} \big\|Q_s^2 U_{\Omega,n,j}([\partial_n A, Q_t] Q_t^3 f)\big\|_{L^2(\mathbb{R}^d)}^2 \log^{\sigma_2}(\mathrm{e} + \frac{2^j}{s}) \frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_j \int\int \chi_{F_{j,2}} \Big\|Q_s\big(\sum_l h_{s,l}\big)\Big\|_{L^2(\mathbb{R}^d)}^2 \log^{-\sigma_2}(\mathrm{e} + 2^j/s) \frac{ds}{s}\frac{dt}{t}\right)^{1/2} \\ &:= \mathrm{J}_1 \mathrm{J}_2.\end{aligned}$$

We have that

$$\mathrm{J}_2 \leq \left(\int_0^\infty \|Q_s^2 g\|_{L^2(\mathbb{R}^d)}^2 \int_s^\infty \sum_{j:2^j \geq s^\alpha t^{1-\alpha}} \log^{-\sigma_2}(\mathrm{e} + 2^j/s) \frac{dt}{t} \frac{ds}{s} \right)^{\frac{1}{2}}$$
$$\lesssim \left\| \left(\int_0^\infty |Q_s^2 g|^2 \frac{ds}{s} \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^d)} \lesssim \|g\|_{L^2(\mathbb{R}^d)},$$

where in the second inequality, we have invoked the fact that

$$\int_s^\infty \sum_{j:2^j \geq s^\alpha t^{1-\alpha}} \log^{-\sigma_2}(\mathrm{e} + 2^j/s) \frac{dt}{t} \lesssim \int_s^\infty \log^{-\sigma_2+1}(\mathrm{e} + t/s) \frac{dt}{t} \lesssim 1.$$

On the other hand, it follows from (4.5) in Lemma 4.1 that

$$\mathrm{J}_1 \leq \left(\sum_j \int \int \chi_{F_{j,2}} \log^{-2\beta+\sigma_2}(\mathrm{e} + 2^j/s) \|[\partial_n A, Q_t] Q_t^3 f\|_{L^2(\mathbb{R}^d)}^2 \frac{ds}{s} \frac{dt}{t} \right)^{\frac{1}{2}}$$
$$\lesssim \left(\int_0^\infty \|[\partial_n A, Q_t] Q_t^3 f\|_{L^2(\mathbb{R}^d)}^2 \frac{dt}{t} \sum_{j:2^j \leq t} \int_0^{(2^j t^{\alpha-1})^{1/\alpha}} \log^{-2\beta+\sigma_2}(\mathrm{e} + \frac{2^j}{s}) \frac{ds}{s} \right)^{\frac{1}{2}}$$
$$\lesssim \|f\|_{L^2(\mathbb{R}^d)},$$

since $\beta > 2$ and

$$\sum_{j:2^j \leq t} \int_0^{(2^j t^{\alpha-1})^{1/\alpha}} \log^{-2\beta+\sigma_2}(\mathrm{e} + 2^j/s) \frac{ds}{s} \leq \sum_{j:2^j \leq t} \log^{-2\beta+\sigma_2+1}\left(\mathrm{e} + t/2^j\right) \lesssim 1.$$

Therefore,

$$\mathrm{D}_{3,n}^1 \lesssim \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}.$$

Similarly, we have that

$$\mathrm{D}_{3,n}^2 \lesssim \|g\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}.$$

To estimate $\mathrm{D}_{3,n}^3$, write

$$\int_{\mathbb{R}^d} Q_s^3 h_{s,l}(x) U_{\Omega,n,j} Q_t^2 (\partial_n \widetilde{A}_l Q_t^2) f(x) dx$$
$$= \int_{\mathbb{R}^d} Q_t^2 h_{s,l}(x) U_{\Omega,n,j} Q_s^3 (\partial_n \widetilde{A}_l Q_t^2) f(x) dx$$

$$= -\int_{\mathbb{R}^d} Q_t^2 h_{s,l}(x) U_{\Omega,n,j} Q_s^2([\partial_n A, Q_s] Q_t^2 f)(x)dx$$
$$+ \int_{\mathbb{R}^d} Q_t^2 h_{s,l}(x) U_{\Omega,n,j} Q_s^2(\partial_n \widetilde{A}_l Q_s Q_t^2 f)(x)dx := \mathrm{I}_1^l(s, t) + \mathrm{I}_2^l(s, t).$$

For the integral corresponding to I_l^1, we choose σ_3 with $1 < \sigma_3 < 2\beta - 1$, and deduce from (4.5) in Lemma 4.1 that

$$\sum_j \int\int \chi_{F_{j,2}} \Big| \sum_l \mathrm{I}_1^l(s,t) \Big| \frac{ds}{s}\frac{dt}{t}$$
$$\lesssim \left(\sum_j \int\int \chi_{F_{j,2}} \| Q_t^2 Q_s g\|^2_{L^2(\mathbb{R}^d)} \log^{-\sigma_3}(\mathrm{e} + 2^j/s) \frac{ds}{s}\frac{dt}{t} \right)^{\frac12}$$
$$\times \left(\sum_j \int\int \chi_{F_{j,2}} \| U_{\Omega,n,j} Q_s^2 [\partial_n A, Q_s] Q_t^2 f\|^2_{L^2(\mathbb{R}^d)} \log^{\sigma_3}\big(\mathrm{e} + \frac{2^j}{s}\big) \frac{ds}{s}\frac{dt}{t} \right)^{\frac12}$$
$$\lesssim \|g\|_{L^2(\mathbb{R}^d)} \left(\int_0^\infty \int_0^\infty \big\| [\partial_n A, Q_s] Q_t^2 f\big\|^2_{L^2(\mathbb{R}^d)} \sum_{j:2^j \ge s} \log^{-2\beta+\sigma_3}(\mathrm{e} + 2^j/s) \frac{ds}{s}\frac{dt}{t} \right)^{\frac12}$$
$$\lesssim \|g\|_{L^2(\mathbb{R}^d)} \left(\int_0^\infty \int_0^\infty \big\| [\partial_n A, Q_s] Q_t^2 f\big\|^2_{L^2(\mathbb{R}^d)} \frac{ds}{s}\frac{dt}{t} \right)^{\frac12} \lesssim \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}.$$

To estimate the integral corresponding to I_2^l, write

$$\begin{aligned}
\mathrm{I}_2^l &= \int_{\mathbb{R}^d} Q_s Q_t^2 f(x) [\partial_n A, U_{\Omega,n,j} Q_s] Q_t^2 Q_s h_{s,l}(x)dx \\
&\quad + \int_{\mathbb{R}^d} Q_s Q_t^2 f(x) U_{\Omega,n,j} Q_s [\partial_n A, Q_t^2] Q_s h_{s,l}(x)dx \\
&\quad + \int_{\mathbb{R}^d} Q_s Q_t^2 f(x) U_{\Omega,n,j} Q_s Q_t^2 [\partial_n A, Q_s] h_{s,l}(x)dx \\
&\quad + \int_{\mathbb{R}^d} Q_s Q_t^2 f(x) U_{\Omega,n,j} Q_s Q_t^2 Q_s (\partial_m \widetilde{A}_l h_{s,l})(x)dx \\
&:= \mathrm{V}_{n,l}^1(s,t) + \mathrm{V}_{n,l}^2(s,t) + \mathrm{V}_{n,l}^3(s,t) + \mathrm{V}_{n,l}^4(s,t).
\end{aligned}$$

The estimates for the part of $\mathrm{V}_{n,l}^2$ and $\mathrm{V}_{n,l}^3$ are similar to the estimate for the term corresponding to I_1^l, and are omitted. As for $\mathrm{V}_{n,l}^4$, we choose $1 < \sigma_4 < 2\beta - 3$. It then follows from Lemma 4.1 that

$$\left(\sum_j \int\int \chi_{F_{j,2}} \Big\| U_{\Omega,n,j} Q_s Q_t^2 (\sum_l Q_s(\partial_n \widetilde{A}_l h_{s,l})) \Big\|^2_{L^2(\mathbb{R}^d)} \log^{\sigma_4}(\mathrm{e} + \frac{2^j}{s}) \frac{ds}{s}\frac{dt}{t} \right)^{\frac12}$$

$$\lesssim \left(\sum_j \int\int \chi_{F_{j,2}} \| Q_t^2 (\sum_l Q_s(\partial_n \widetilde{A}_l h_{s,l}))\|^2_{L^2(\mathbb{R}^d)} \log^{-2\beta+\sigma_4}(\mathrm{e}+\frac{2^j}{s})\frac{ds}{s}\frac{dt}{t}\right)^{\frac{1}{2}}$$

$$\lesssim \left(\sum_j \int_0^{2^j} \| \sum_l Q_s(\partial_n \widetilde{A}_l h_{s,l})\|^2_{L^2(\mathbb{R}^d)} \log^{-2\beta+\sigma_4}(\mathrm{e}+\frac{2^j}{s})\frac{ds}{s}\right)^{\frac{1}{2}}.$$

Let $x \in 48dI_l$ and $q \in (1, 2)$. A straightforward computation involving Hölder's inequality and the John-Nirenberg inequality gives us that

$$|Q_s(\partial_n \widetilde{A}_l h)(x)| \lesssim M_q h(x) + \log(\mathrm{e}+2^j/s)Mh(x),$$

where $I(x, s)$ is the cube centered at x and having side length s. This implies that

$$\sum_j \int_0^{2^j} \| \sum_l Q_s(\partial_n \widetilde{A}_l h_{s,l})\|^2_{L^2(\mathbb{R}^d)} \log^{-2\beta+\sigma_4}(\mathrm{e}+2^j/s)\frac{ds}{s}$$
$$\lesssim \int_0^{\infty} \|Q_s g\|^2_{L^2(\mathbb{R}^d)} \sum_{j:2^j\ge s} \log^{-2\beta+\sigma_4+2}(\mathrm{e}+2^j/s)\frac{ds}{s} \lesssim \|g\|_{L^2(\mathbb{R}^d)},$$

since $-2\beta+\sigma_4+2 < -1$. Therefore,

$$\sum_j \int\int \chi_{F_{j,2}} \sum_l |\mathrm{V}^4_{n,l}(s,t)|\frac{ds}{s}\frac{dt}{t}$$
$$\lesssim \Big(\sum_j \int\int \chi_{F_{j,2}} \|Q_s Q_t^2 f\|^2_{L^2(\mathbb{R}^d)} \log^{-\sigma_4}(\mathrm{e}+2^j/s)\frac{ds}{s}\frac{dt}{t}\Big)^{\frac{1}{2}}$$
$$\times\Big(\sum_j \int\int \chi_{F_{j,2}} \|U_{\Omega,n,j} Q_s Q_t^2 (\sum_l Q_s(\partial_n \widetilde{A}_l h_{s,l}))\|^2_{L^2(\mathbb{R}^d)} \log^{\sigma_4}(\mathrm{e}+\frac{2^j}{s})\frac{ds}{s}\frac{dt}{t}\Big)^{\frac{1}{2}}$$
$$\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

Now, we consider the part corresponding to $\mathrm{V}^1_{n,l}$. Invoking (4.6) in Lemma 4.1, we deduce that

$$\Big\|\Big(\sum_j \int\int \chi_{F_{j,2}} |[\partial_n A, U_{\Omega,n,j} Q_s] Q_t^2 Q_s^2 g|^2 \log^{\sigma_4}(\mathrm{e}+2^j/s)\frac{ds}{s}\frac{dt}{t}\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^d)}$$
$$\lesssim \Big(\sum_j \int\int \chi_{F_{j,2}} \|Q_t^2 Q_s^2 g\|^2_{L^2(\mathbb{R}^d)} \log^{-2\beta+\sigma_4+2}(\mathrm{e}+2^j/s)\frac{ds}{s}\frac{dt}{t}\Big)^{\frac{1}{2}}$$
$$\lesssim \|g\|_{L^2(\mathbb{R}^d)}.$$

Therefore,

$$\begin{aligned}&\sum_j\int\int\chi_{F_{j,2}}\sum_l|\mathrm{V}^1_{n,l}(s,t)|\frac{ds}{s}\frac{dt}{t}\\&\lesssim\Big\|\Big(\sum_j\int\int\chi_{F_{j,2}}|Q_sQ_t^2f|^2\log^{-\sigma_4}((\mathrm{e}+2^j/s)\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}\\&\quad\times\Big\|\Big(\sum_j\int\int\chi_{F_{j,2}}|[\partial_nA,U_{\Omega,n,j}Q_s]Q_t^2Q_s^2g|^2\log^{\sigma_4}(\mathrm{e}+2^j/s)\frac{ds}{s}\frac{dt}{t}\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^d)}\\&\lesssim\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Combining the estimates for I^1_l, and $\mathrm{V}^i_{n,l}$ $(i=1,\ 2,\ 3,\ 4)$, yields

$$\mathrm{D}^3_{3,n}\lesssim\|g\|_{L^2(\mathbb{R}^d)}\|f\|_{L^2(\mathbb{R}^d)}.$$

which, along with the estimates for D_1, D_2, $\mathrm{D}^1_{3,n}$ and $\mathrm{D}^2_{3,n}$ leads to (4.11) with $k=2$. This verifies the inequality (4.9).

Now we turn our attention to inequality (4.10). Let P_s be the operator defined by

$$P_s=\int_s^\infty Q_t^4\frac{dt}{t}.$$

It was proved in [16] that

$$\int_0^\infty\|P_sf\|^2_{L^2(\mathbb{R}^d)}\frac{ds}{s}\le\|f\|_{L^2(\mathbb{R}^d)}.$$

Let $\widetilde{T}_{\Omega,\,A}$ be the adjoint of $T_{\Omega,\,A}$, that is,

$$\widetilde{T}_{\Omega,\,A}f(x)=\mathrm{p.\,v.}\int_{\mathbb{R}^d}\frac{\widetilde{\Omega}(y-x)}{|x-y|^{d+1}}\big(A(x)-A(y)-\nabla A(x)(x-y)\big)f(y)dy,$$

with $\widetilde{\Omega}(x)=\Omega(-x)$. Obviously,

$$\widetilde{T}_{\Omega,\,A}h(x)=T_{\widetilde{\Omega},\,A}h(x)+\sum_{n=1}^d[\partial_nA,\ T^n_{\widetilde{\Omega}}]h(x),\tag{4.13}$$

where $T^n_{\widetilde{\Omega}}$ is defined as in (3.23), but with Ω replaced by $\widetilde{\Omega}$. As the inequality (4.9), we have that

$$\Big|\int_0^\infty\int_0^t\int_{\mathbb{R}^d}Q_s^4T_{\widetilde{\Omega},\,A}Q_t^4f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big|\lesssim\|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)};$$

For each n with $1 \le n \le d$, we know from [21, Theorem 2] that $[\partial_n A,\ T^n_\Omega]$ is bounded on $L^p(\mathbb{R}^d)$ provided that $1 + 1/(\beta - 1) < p < \beta$. A straightforward computation yields

$$\begin{aligned}
&\Big|\int_0^\infty \int_0^t \int_{\mathbb{R}^d} Q_s^4[\partial_n A,\ T^n_{\widetilde{\Omega}}]Q_t^4 f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big| \\
&= \Big|\int_0^\infty \int_s^\infty \int_{\mathbb{R}^d} Q_t^4 f(x)[\partial_n A,\ T^n_{\widetilde{\Omega}}]Q_s^4 g(x)dx\frac{ds}{s}\frac{dt}{t}\Big| \\
&\lesssim \Big(\int_0^\infty \|[\partial_n A,\ T^n_{\widetilde{\Omega}}]Q_s^4 g\|^2_{L^2(\mathbb{R}^d)}\frac{ds}{s}\Big)^{1/2}\Big(\int_0^\infty \|P_s f\|^2_{L^2(\mathbb{R}^d)}\frac{ds}{s}\Big)^{1/2} \\
&\lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Therefore,

$$\Big|\int_0^\infty \int_0^t \int_{\mathbb{R}^d} Q_s^4 \widetilde{T}_{\Omega,\,A}Q_t^4 f(x)g(x)dx\frac{ds}{s}\frac{dt}{t}\Big| \lesssim \|f\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.$$

This, via dulaity argument, gives (4.10).

With the $L^2(\mathbb{R}^d)$ boundedness of $T_{\Omega,\,A}$ in hand, we now verify the $L^p(\mathbb{R}^d)$ boundedness of $T_{\Omega,\,A}$ for the case of $1 + 1/(\beta - 1) < p < 2$. Let $R_{l,A}$ be the operator defined as in Theorem 3.9, and $\varepsilon \in (0,\ 1)$ be a constant which will be chosen later. An application of Theorem 3.9 gives us that

$$\big\|R_{2^l,A}f - R_{2^{l+1},A}f\big\|_{L^2(\mathbb{R}^d)} \lesssim 2^{(-\varepsilon\beta+2)l}\|f\|_{L^2(\mathbb{R}^d)}. \tag{4.14}$$

Therefore, the series

$$T_{\Omega,A} = R_{2,\,A} + \sum_{l=1}^{\infty}(R_{2^{l+1},A} - R_{2^l,\,A}) \tag{4.15}$$

converges in $L^2(\mathbb{R}^d)$ operator norm and for $f,\ g \in C_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}(T_{\Omega,A} - R_{2,A})f(x)g(x)dx = \sum_{l=1}^{\infty}\int_{\mathbb{R}^d}(R_{2^{l+1},A} - R_{2^l,A})f(x)g(x)dx. \tag{4.16}$$

On the other hand, from Theorem 3.9 we know that $R_{l,A}$ is bounded on $L^2(\mathbb{R}^d)$ with bound independent of l. This, via Theorem 2.3, (ii) and (iii) of Theorem 3.9, shows that for $p \in (1,\ 2]$, $R_{l,A}$ is bounded on $L^p(\mathbb{R}^d)$ with bound Cl^2. Thus, we have that

$$\big\|R_{2^l,A}f - R_{2^{l+1},A}f\big\|_{L^p(\mathbb{R}^d)} \lesssim 2^{2l}\|f\|_{L^p(\mathbb{R}^d)},\quad p \in (1,\ 2]. \tag{4.17}$$

Let $1 < p < 2$ and $\varrho \in (0,\ 1)$. Interpolation between the inequalities (4.14) and (4.17) leads to that

$$\big\|R_{2^l,A}f - R_{2^{l+1},A}f\big\|_{L^p(\mathbb{R}^d)} \lesssim 2^{(-2\varepsilon\beta/p'+2+\varrho)l}\|f\|_{L^p(\mathbb{R}^d)}.$$

For each p with $1 + 1/\beta < p < 2$, we choose $\varepsilon > 0$ close to 1 sufficiently, and $\varrho > 0$ close to 0 sufficiently, such that $2\varepsilon\beta/p' > 2 + \varrho$, and then obtain that

$$\sum_{l=1}^{\infty} \left\| R_{2^l,\,A} f - R_{2^{l+1},\,A} f \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

This, along with (4.16), shows that $T_{\Omega,\,A}$ is bounded on $L^p(\mathbb{R}^d)$.

It remains to consider the $L^p(\mathbb{R}^d)$ boundedness of $T_{\Omega,A}$ for the case of $2 < p < \beta$. Observe that the operator $T_{\widetilde{\Omega},\,A}$ is also bounded on $L^p(\mathbb{R}^d)$ for $1 + 1/\beta < p < 2$. Thus by (4.13), we know that $\widetilde{T}_{\Omega,\,A}$, the adjoint operator of $T_{\Omega,\,A}$, is also bounded on $L^p(\mathbb{R}^d)$ for $1 + 1/\beta < p < 2$, and so $T_{\Omega,\,A}$ is bounded on $L^p(\mathbb{R}^d)$ for $2 < p < \beta$. This finishes the proof of Theorem 1.3. □

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