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A New Class of FBI Transforms and Applications

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Abstract
We introduce a class of FBI transforms using weight functions (which includes the subclass of Sjöstrand's FBI transforms used by Christ in (Commun Partial Differ Equ 22(3–4):359–379, 1997)) that is well suited when dealing with ultradifferentiable functions (see Definition 2.3) and ultradistributions (see Definition 2.15) defined by weight functions in the sense of Braun, Meise and Taylor (BMT). We show how to characterize local regularity of BMT ultradistributions using this wider class of FBI transform and, as an application, we characterize the BMT vectors (see Definition 1.2) and prove a relation between BMT local regularity and BMT vectors.

Keywords FBI transform · Ultradifferentiable functions · Ultradistribution · Iterates of operators

Mathematics Subject Classification 35A22 · 35A23 · 42B10 · 46F05

1 Introduction

The purpose of this paper is twofold: (i) to explore a new class of FBI transform and show that it can be used to characterize regularity in the classes of ultradifferentiable functions in the sense of Braun et al. [12]; and (ii) use these techniques to study regularity of iterated hypoelliptic constant coefficient partial differential operators. This new systematic approach has led us to a plethora of unanticipated results. We show that the FBI transforms introduced in (i) are not only fundamental to obtain the

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results in (ii) but also allows us to extend similar results that appeared recently where a similar program was developed with the use of the Fourier transform.

1.1 The Augmented Class of FBI Transforms

The Fourier transform can be used to characterize smoothness of distributions (Paley–Wiener Theorem) and also can be used to characterize analyticity of distributions. However, the analyticity characterization is significantly more difficult than the smoothness characterization (see [22]). An alternative tool to characterize regularity (smooth and analytic) is the FBI transform (see [4]).

In the next paragraphs we will recall several variations and generalizations of the classical FBI transform. Given $n \in \mathbb{N}$ and $0 < \tau \leq 1$ consider the following form in $\mathbb{R}^n \times \mathbb{R}^n$

$$F = dx_1 \wedge \cdots \wedge dx_n \wedge d\left(\xi_1 + ix_1\langle\xi\rangle^\tau\right) \wedge \cdots \wedge d\left(\xi_n + ix_n\langle\xi\rangle^\tau\right),$$

where $\xi = (\xi_1, \ldots, \xi_n)$, $x = (x_1, \ldots, x_n)$ and $\langle\xi\rangle = \sqrt{1 + \sum_{j=1}^n \xi_j^2}$. Define the function $a_\tau : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ with the property that

$$F = a_\tau(x, \xi) dx_1 \wedge \cdots \wedge dx_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n.$$

Using the above form in [13] M. Christ defined the following variation of the FBI transform (in the original article τ is actually λ)

$$\mathcal{F}_\tau u(x, \xi) = \left\langle u(x'); e^{i(x-x')\xi - \langle\xi\rangle^\tau (x-x')} a_\tau(x - x', \xi)\right\rangle,$$

where $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$, $u \in \mathcal{E}'(\mathbb{R}^n)$ (i.e., u is a compactly supported distribution) and the pairing refers to the interaction between distributions and test functions with respect to the variable $x' \in \mathbb{R}^n$. Moreover, among other interesting results he proved that for a given $s > 1$ a *distribution* $u \in \mathcal{D}'(\mathbb{R}^n)$ is G^s (Gevrey of order s, see next paragraph) in a neighborhood of x_0 if and only if there exist $\tau \geq \frac{1}{s}$, $v \in \mathcal{E}'(\mathbb{R}^n)$ with $v \equiv u$ in a neighborhood of x_0, positive constants a, C and an open neighborhood V of x_0 such that

$$|\mathcal{F}_\tau v(x, \xi)| \leq Ce^{-a\|\xi\|^{1/s}}, \quad \forall (x, \xi) \in V \times \mathbb{R}^n.$$

Observe that the "limit" choice of τ to study G^s regularity is $\tau = 1/s$.

In [18] we considered a more general class of FBI transform first introduced in [3] (see also [2] where this FBI transform plays a fundamental role) and we showed that it can be used to characterize Denjoy–Carleman regularity as we now explain. Consider a positive sequence $M = (M_j)$ satisfying some special properties. If $U \subset \mathbb{R}^n$ is an open set and $f \in C^\infty(U)$ we say that f is in $\mathcal{E}^M(U)$ (f is M-Denjoy–Carleman in U) if for each compact set $K \subset U$ there exist positive constants C, h such that $|\partial^\alpha f(x)| \leq Ch^{|\alpha|} M_{|\alpha|}$, for each $x \in K$ and $\alpha \in \mathbb{N}_0^n$. The space of compactly supported functions in $\mathcal{E}^M(U)$ is denoted by $\mathcal{D}^M(U)$. We equip $\mathcal{E}^M(U)$ and $\mathcal{D}^M(U)$ with their usual topologies, the topological duals of these spaces are denoted by $\mathcal{E}^{M'}(U)$ and $\mathcal{D}^{M'}(U)$ respectively (see [26] for more information). The Gevrey spaces G^s of order s are given by choosing $M = (j!^s)$. For a fixed sequence $M = (M_j)$ its associated function is defined by $M(t) \doteq \sup_j \log(t^j/M_j)$.

The main result in [18] can now be stated as follows: for u in $\mathcal{E}^{M'}(U)$, $0 < \tau \leq 1$ and $k \in \mathbb{N}$ denote

$$\mathcal{F}^k_\tau u(x,\xi) = \left\langle u(x'); e^{i(x-x')\xi - \|\xi\|^\tau (x-x')^{2k}} \right\rangle, \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{1.1}$$

Then u is $\mathcal{E}^M$ in a neighborhood of x_0 if and only if there exist $0 < \tau \leq 1$ so that $M(t) = O(t^\tau)$ as $t \to \infty$, $v \in \mathcal{E}^{M'}(\mathbb{R}^n)$ with $v \equiv u$ in a neighborhood of x_0, positive constants $a, C > 0$ and an open neighborhood V of x_0 such that

$$\left|\mathcal{F}^k_\tau v(x,\xi)\right| \leq Ce^{-aM(\|\xi\|)}, \quad \forall (x,\xi) \in V \times \mathbb{R}^n. \tag{1.2}$$

A natural question that arises is the following: is there a “limit" choice of τ? When dealing with the Gevrey class of order s it follows that the choice is $\tau = 1/s$. However this is far from being trivial when M is not a Gevrey sequence. Note that the choice $\tau \geq \frac{1}{s}$ is equivalent to the inclusion $G^{1/\tau} \subset G^s$.

Since $m(t) = t^\tau$ is equivalent to the associated function M of the sequence $(M_j) = (j!^{1/\tau})$, meaning there exist constants C_1, C_2 and $a > 0$ such that $C_1 M(t) \leq m(t) \leq C_2 M(t)$, for all $t > a$ (consequently, $\mathcal{E}^M = \mathcal{E}^m$ as per Definition 2.3), a naive approach to try to answer this question is to *enlarge* the class of the FBI transform considered by allowing the term $\|\xi\|^\tau$ in (1.1) to be any *possible* weight function as given in Definition 2.1.

In order to justify the previous sentence we need to recall the definition of ultradifferentiable functions by means of *weight functions* as introduced in [12] (see Definitions 2.1 and 2.3).

Recall that, given two weight functions ω and σ the condition $\omega(t) = O(\sigma(t))$ as $t \to \infty$ implies the inclusion $\mathcal{E}^\sigma(\Omega) \subset \mathcal{E}^\omega(\Omega)$, for any open set $\Omega \subset \mathbb{R}^n$.

Moving on, given any ultradifferentiable class $\mathcal{E}^\omega$ as in Definition 2.3 the augmented class of FBI transforms announced earlier which will be suitable to study regularity problems in $\mathcal{E}^\omega$ would be those allowing weight functions $\sigma(\|\xi\|)$ instead of $\|\xi\|^\tau$ in (1.2) as long as $\omega(t) = O(\sigma(t))$ as $t \to \infty$. However, the lack of regularity on general weight functions σ make it very difficult to deal with these FBI transforms. In order to avoid this problem we will prove that for each weight function there exists an *equivalent* one possessing the desired regularity (see Definition 2.18 for the precise meaning of equivalence used here) and therefore define the same class of ultradifferentiable functions, see Remark 2.19.

To be more precise given a weight function σ there exists $\mu_\sigma \in C^1((1,\infty))$ such that σ and μ_σ are equivalent (see Proposition 2.20). Let $\kappa > 0$, following [13] we consider the differential form in $\mathbb{R}^n \times \mathbb{R}^n$

$$F \doteq dx'_1 \wedge \cdots \wedge dx'_n \wedge d\left(\xi_1 + ix'_1\kappa\mu_\sigma(\xi)\right) \wedge \cdots \wedge d\left(\xi_n + ix'_n\kappa\mu_\sigma(\xi)\right), \quad (x', \xi) \in \mathbb{R}^n \times \mathbb{R}^n$$

and the function $a^\kappa_{\mu_\sigma} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ defined by the property

$$F = a^\kappa_{\mu_\sigma}(x', \xi)dx'_1 \wedge \cdots \wedge dx'_n \wedge d\xi_1 \wedge \cdots \wedge d\xi_n. \tag{1.3}$$

Note that, fixing $\xi \in \mathbb{R}^n$ it follows that $x' \mapsto a^\kappa_{\mu_\sigma}(x', \xi)$ is a polynomial function (consequently it is a $\mathcal{E}^\omega$ function). Hence, for each $u \in \mathcal{D}^{\sigma\prime}(\mathbb{R}^n)$ and $\phi \in \mathcal{D}^\sigma(\mathbb{R}^n)$ we can define

$$\mathcal{F}^\kappa_{\mu_\sigma}(\phi\, u)(x, \xi) \doteq \left\langle u_{x'}; \phi(x')\, e^{i(x-x')\cdot\xi - \kappa\mu_\sigma(\xi)(x-x')^2} a^\kappa_{\mu_\sigma}(x - x', \xi) \right\rangle, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{1.4}$$

Here the notation $u_{x'}$ is to emphasize that the ultradistribution u is acting on the function in x' and the other variables are thought as parameters. When $\kappa = 1$ we simply denote $\mathcal{F}_{\mu_\sigma} = \mathcal{F}^\kappa_{\mu_\sigma}$.

Our main result is the following FBI characterization of ultradifferentiable functions that can be viewed as a Paley-Wiener type theorem.

Theorem 1.1 *Fix a weight function ω (see Definition 2.1), $\Omega \subset \mathbb{R}^n$ an open set and $u \in \mathcal{D}^{\omega\prime}(\mathbb{R}^n)$. In order that $u \in \mathcal{E}^\omega$ in a neighborhood of $x_0 \in \Omega$ it is necessary and sufficient that there exist a weight function σ with $\omega(t) = O(\sigma(t))$ as $t \to +\infty$ so that for each $\phi \in \mathcal{D}^\sigma(\Omega)$ there exist $C, c > 0$ and a neighborhood $V \subset \Omega$ of x_0 such that*

$$|\mathcal{F}_{\mu_\sigma}(\phi u)(x, \xi)| \leq Ce^{-c\omega(\xi)}, \quad (x, \xi) \in V \times \mathbb{R}^n. \tag{1.5}$$

To prove Theorem 1.1 we actually show a slightly stronger result, see Theorem 4.2.

As it is customary in these Paley-Wiener type results one of the main ingredients to prove the sufficient part of Theorem 1.1 is the inversion formula of the FBI transform. We provide two different versions for the FBI inversion formula, see Lemmas 5.3 and 5.5. It turns out that we use the first inversion formula, Lemma 5.3, to prove the second Lemma 5.5 and this was inspired by [32, Lemma IX.4.1].

Observe that using [12, 8.9 Remark] it follows that for each sequence (M_j) considered in [18] there exists a function ω_M satisfying Definition 2.1 such that the space of ultradifferentiable functions defined by (M_j) coincides with the space defined by ω_M. However, given a function ω satisfying Definition 2.1 it is necessary to impose on ω an additional stronger condition (the existence of a constant $H > 0$ such that $2\omega(t) \leq \omega(Ht) + H$, for each $t > 0$) to obtain a sequence (M_j) such that $\mathcal{E}^M = \mathcal{E}^\omega$ (see [11]). Hence, for general weight functions one can think that the local regularity characterization results presented here as natural extensions from the ones given in [18].

Previous Paley–Wiener type results for ultradifferentiable functions appeared in the literature but with the use of the Fourier transform, [12], and this is an obstacle to work with elements in their dual. The use of FBI transforms allow us to work also with ultradistributions, the natural ambient.

1.2 Application to Ultradifferentiable Vectors

As an application of Theorem 1.1 we study BMT vectors (or *iterates*) of constant coefficients partial differential operators $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$. To be more precise recall that in [29] Nelson introduced the set of *analytic vectors* of a partial differential operator with analytic coefficients $P(x, D)$ in an open set $U \subset \mathbb{R}^n$ and proved that analytic vectors are real analytic functions exactly when $P(x, D)$ is elliptic.

Later Komatsu, in [25], as well as Kotake and Narasimhan, in [27], obtained a slightly improvement of Nelson's result as follows: let $U \subset \mathbb{R}^n$ and $P(x, D)$ as before then a function $f \in L^2_{loc}(U)$ is real analytic in U if and only if

1. $P^j f \in L^2_{loc}(U)$ (in the sense of the distributions), for each j, and
2. $\forall K \subset U$ compact there exist $C, h > 0$ such that

$$\|P^j f\|_{L^2(K)} \leq Ch^j (hj)^{jm}, \quad j \in \{0, 1, \dots\}. \tag{1.6}$$

Here and throughout these notes $P^j f \doteq (P \circ \cdots \circ P) f$, for $j \in \mathbb{N}$, and $P^0 f \doteq f$.

Newberger and Zielezny initiated an investigation in the Gevrey category G^s (replacing $j!^m$ by $j!^{sm}$ in (1.6)), see [30]. It is worth mentioning the work of Baouendi and Métivier [1] that deals with the case when $P(D)$ is of principal type and hypoelliptic with analytic coefficients. They observed that if u is an s-Gevrey vector of P (in a smaller neighborhood) then there exist $s' > s$ such that u is s'-Gevrey. There is a vast literature concerning optimal regularity for such Gevrey vectors, G^s, $s \geq 1$, for instance [15, 16, 31] to cite just a few. Also, for elliptic operators a proof of the Kotake–Narasimhan theorem in some classes of ultradifferentiable functions is given in [7].

Recently, regularity problems for ultradifferentiable vectors defined by weight functions in the sense of Braun et al. [12] for constant coefficients operators (see Definition 1.2) appeared in the literature. It turns out that completeness of these spaces is equivalent to the hypoellipticity of P [23]. Additionally, there are others characterizations in terms of the decay of the Fourier transform, [5, 8, 24]. It is worth mentioning that in all of these results, the authors prove Paley–Wiener theorems only for distributions when the natural object to deal with are ultradistributions (see implication (2) $\Rightarrow$ (1) at Theorem 1.3). We define BMT-vectors as follows:

Definition 1.2 For each weight function ω, open set $\Omega \subset \mathbb{R}^n$ and polynomial function $P(\xi)$ of degree m we define $\mathcal{E}^\omega(\Omega; P)$ as the set of $u \in C^\infty(\Omega)$ such that for each compact set $K \subset \Omega$ there exists $\lambda > 0$ satisfying

$$\sup_{j\in\mathbb{N}_0} \|P(D)^j u\|_{L^2(K)} e^{-\frac{1}{\lambda}\varphi^*(\lambda m j)} < +\infty,$$

where we are denoting $\varphi^*(x) = \varphi^*_\omega(x) \doteq \sup\{xy - \varphi(y) : y \geq 0\}$, for each $x > 0$ (see (2.1)).

We will call the elements of $\mathcal{E}^\omega(\Omega; P)$ BMT vectors.

To state our main result for ultradifferentiable vectors we need to introduce some ingredients. Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ be a hypoelliptic constant coefficients partial differential operator. Then, it is well known (see [21]) that there exist constants $C, K > 0$ and a positive number $\tilde{\rho} \leq 1$ such that

$$|P(\xi)| \geq C\|\xi\|^{\tilde{\rho}m}, \quad \forall \|\xi\| \geq K. \tag{1.7}$$

Define the hypoelliptic index by

$$\rho = \sup\left\{\tilde{\rho} \leq 1 \text{ so that(1.7)is valid for some positive constants} C, K\right\}.$$

It is known that this supremum is attained and it is a rational number. In other words there exist constants $C, K > 0$ such that

$$|P(\xi)| \geq C\|\xi\|^{\rho m}, \quad \|\xi\| \geq K. \tag{1.8}$$

We will refer to the number ρ as the *Hörmander hypoellipticity index* of $P(D)$. Is is important to mention that this condition also plays an important role in [10]. For a fixed weight function ω we will define $\sigma(t) = \sigma_{\omega,\rho}(t) = \omega(t^\rho)$. Note that, if ω is a weight function and $0 < \rho \leq 1$ then σ is a weight function.

The next result gives a characterization of the spaces $\mathcal{E}^\omega(\Omega; P)$ (see Definition 1.2) in terms of the FBI transforms introduced here. The characterization of vectors of partial differential operators using Fourier transform was first given in [9].

Theorem 1.3 *Fix a weight function ω, a hypoelliptic constant coefficient partial differential operator $P(D)$ of order m on an open set $\Omega \subset \mathbb{R}^n$ along with his hypoellipticity index ρ and define the weight function σ as before. Let $\varphi(t) = \omega(e^t)$ and φ^* is Young conjugate, see* (2.1). *Given $u \in \mathcal{D}^{\sigma\prime}(\Omega)$ and $x_0 \in \Omega$ the following conditions are equivalent:*

1. *There exists a neighborhood U of x_0 such that $u \in \mathcal{E}^\omega(U; P)$.*
2. *There exist $\phi \in \mathcal{D}^\omega(\mathbb{R}^n)$ (such that $\phi \equiv 1$ in a neighborhood of x_0), $C, \lambda, c > 0$ and a neighborhood V of x_0 such that*

$$|\mathcal{F}_{\mu_\sigma}\left(\phi\,[P(D)]^N(u)\right)(x,\xi)| \leq Ce^{\frac{1}{\lambda}\varphi^*(Nm\lambda)}e^{-c\sigma(\|\xi\|)}, \\ (x,\xi,N) \in V \times \mathbb{R}^n \times \mathbb{N}_0. \tag{1.9}$$

One of the main tools to prove Theorem 1.3 is a version of [20, Theorem 4.1] specialized to balls on which we obtain a better estimate (see Lemma A.1). This allows us to get rid of a strong restriction on the weight functions treated in [6, Theorem 3.3]. Specifically, in their theorem they consider weight functions ω so that $\omega(t^\gamma) = o(\sigma(t))$, as $t \to +\infty$, where γ is a constant that arises from the hypoellipticity of P

and $\sigma(t) = t^{1/s}$ for some $s > 1$. This is equivalent to $\omega(t) = o(t^{1/\gamma s})$, as $t \to +\infty$ which in turn, implies that $G^{s\gamma} \subset \mathcal{E}^{\omega}$.

Another important improvement is that we allow $u \in \mathcal{D}^{\sigma\prime}(\Omega)$ while in [6, Theorem 3.3] the authors considered only $u \in \mathcal{D}'(\Omega)$. Note that both, inequality (1.5) in Theorem 1.1 and condition (2) in Theorem 1.3 can be checked if u is only an ultradistribution and this will be used in a forthcoming paper to define wave front-sets for ultradistributions.

Another application is given in Corollary 6.4 where we recover the Kotake-Narasimhan theorem in this context. Also, it is important to recall that similar results for global Gevrey classes were studied in [19].

In the following we will use the notation $B(x, r)$ to denote the ball of radius $r > 0$ and centered at $x \in \mathbb{R}^n$. We will also use the notation $\mathbb{N}_0^n$ to denote the set of all multiindices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j \in \{0, 1, 2, \dots\}$ for each $j \in \{1, \dots, n\}$ and denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

The organization of this paper is as follows: in the second chapter we recall the definitions of weight functions, ultradifferentiable functions and ultradistributions as well as we introduce the function μ_ω. We prove the necessity of Theorem 1.1 in chapter 3. In chapter 4 we present inversion formulas of our class of FBI transforms and we use them to prove the sufficiency of Theorem 1.1. A characterization of iterates of constant coefficients operators and a relation between ultradifferentiable functions and iterates of constant coefficients operators using our class of FBI transforms is given in chapter 5.

2 Ultradifferentiable Functions Defined by Weight Functions

This section is devoted to recalling basic definitions concerning ultradifferentiable functions and ultradistributions. First we establish the concept of weight functions used throughout this work.

Definition 2.1 A continuous function $\omega : [0, +\infty[\to [0, +\infty[$ with $\omega \equiv 0$ in $[0, 1]$ and increasing in $[0, +\infty)$ is called a weight function when the following conditions are satisfied:

$$\text{there exists } L > 1 \text{ such that } \omega(e\,t) \leq L(\omega(t) + 1), \quad \forall t > 0; \tag{α}$$

$$\int_1^\infty \frac{\omega(t)}{t^2} dt < +\infty; \tag{β}$$

$$\lim_{t \to +\infty} \frac{\log t}{\omega(t)} = 0; \tag{γ}$$

$$\varphi(t) = \varphi_\omega(t) \doteq \omega(e^t) \quad \text{is convex.} \tag{δ}$$

Moreover, given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we will write $\omega(x) = \omega\left(\left(\sum_{j=1}^n |x_j|\right)^{1/2}\right)$.

Example 2.2 For each $s > 1$, let $\omega_s \equiv 0$ in $[0, 1]$ and $\omega_s(t) = t^{\frac{1}{s}} - 1$, for $t > 1$. Note that, ω_s is a weight function.

2.1 Ultradifferentiable Functions

Before we introduce the notion of ultradifferentiable functions we need to recall the notion of Young conjugate that will be used throughout this paper. Let ω be a weight function and let φ be defined by (δ). The Young conjugate of φ, φ^*, is defined by

$$\varphi^*(x) = \varphi^*_\omega(x) \doteq \sup\{xy - \varphi(y) : y \geq 0\}. \tag{2.1}$$

It is important to recall that, by (γ) and (δ), the Young conjugate of φ is well defined. Moreover, φ^* is an increasing convex function, $\varphi^*(0) = 0$, $\lim_{t \to +\infty} \frac{\varphi^*(t)}{t} = +\infty$ and $(\varphi^*)^* = \varphi$ (see [12, 1.3 Remark]).

Definition 2.3 Let $K \subset \mathbb{R}^n$ be a regular compact[1] set and $C^\infty(K)$ as in [26]. For each weight function ω and $\lambda > 0$ we define

$$\mathcal{E}^\omega_\lambda(K) = \left\{ f \in C^\infty(K) : \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^n} |\partial^\alpha f(x)| e^{-\frac{1}{\lambda}\varphi^*(\lambda|\alpha|)} < +\infty \right\}$$

and

$$\mathcal{D}^\omega_\lambda(K) = \left\{ f \in C^\infty(\mathbb{R}^n) : \sup_{x \in K} \sup_{\alpha \in \mathbb{N}_0^n} |\partial^\alpha f(x)| e^{-\frac{1}{\lambda}\varphi^*(\lambda|\alpha|)} < +\infty \text{ and } \operatorname{supp} f \subset K \right\}.$$

Let Ω be an open subset of $\mathbb{R}^n$, we also define

$$\mathcal{E}^\omega(\Omega) = \underset{\overleftarrow{K \subset\subset \Omega}}{proj} \ \underset{\overrightarrow{\lambda > 0}}{\text{ind}} \ \mathcal{E}^\omega_\lambda(K) \text{ and } \mathcal{D}^\omega(\Omega) = \underset{\overrightarrow{K \subset\subset \Omega}}{ind} \ \underset{\overrightarrow{\lambda > 0}}{\text{ind}} \ \mathcal{D}^\omega_\lambda(K).$$

Remark 2.4 Let Ω be an open subset of $\mathbb{R}^n$. It follows that $\mathcal{E}^\omega(\Omega)$ is the set of smooth functions $f : \Omega \to \mathbb{C}$ such that for each compact set $K \subset \Omega$ there exist $\lambda, C > 0$ satisfying

$$\sup_{x \in K} |\partial^\alpha f(x)| \leq C e^{\frac{1}{\lambda}\varphi^*(\lambda|\alpha|)}, \quad \forall \alpha \in \mathbb{N}_0^n.$$

And $\mathcal{D}^\omega(\Omega)$ is the subset of $\mathcal{E}^\omega(\Omega)$ consisting of all compactly supported functions.

[1] A compact set with a finite number of connected components each of which has the property (P) of Whitney, i.e., "there is a number C such that any two points x and y of a connected component L are joined by an arc in L of length less than or equal to $C|x - y|$".

Remark 2.5 For each open set $\Omega \subset \mathbb{R}^n$ condition (β) guarantees that $\mathcal{D}^{\omega}(\Omega) \neq \{0\}$ (see [12, 2.5 Corollary]).

Example 2.6 If $s > 1$ and ω_s is given as in Example 2.2, then the space $\mathcal{E}^{\omega_s}(\Omega)$ coincides with the Gevrey class of order s, $G^s(\Omega)$, for each open set $\Omega \subset \mathbb{R}^n$.

2.2 Technical Results on Weight Functions

This subsection is dedicated to the presentation of some technical results on weight functions and its Young conjugate. In the end of this subsection we will prove an important characterization of ultradistributions.

Throughout this section we will consider a fixed weight function ω (see Definition 2.1).

Remark 2.7 Using (γ) we see that there exists $A > 1$ such that

$$1 \leq \omega(t), \quad t \geq A.$$

Thus, applying (α) it follows that

$$\omega(et) \leq 2L\omega(t), \quad t \geq A. \tag{2.2}$$

Next we will present some properties of φ^* (see (2.1)).

Remark 2.8 Note that, φ^* is an increasing function, is a convex function, $\varphi^*(0) = 0$, $\lim_{t\to+\infty} \frac{\varphi^*(t)}{t} = +\infty$, $\frac{\varphi^*(t)}{t}$ is increasing and $(\varphi^*)^* = \varphi$ (see [12, 1.3 Remark]).

Thus,

$$2\lambda\varphi^*\left(\frac{s+t}{2\lambda}\right) \leq \lambda\varphi^*\left(\frac{s}{\lambda}\right) + \lambda\varphi^*\left(\frac{t}{\lambda}\right) \leq \lambda\varphi^*\left(\frac{s+t}{\lambda}\right), \quad \text{for every } \lambda, s, t > 0. \tag{2.3}$$

Remark 2.9 Using condition (α) and [12, 1.4 Lemma] it follows that,

$$L^k\varphi^*\left(\frac{t}{L^k}\right) + kt \leq \varphi^*(t) + \sum_{j=1}^{k} L^j, \quad t \geq 0, \quad k \in \mathbb{N}.$$

Thus, for any $\lambda, \alpha > 0$ choosing $t = \alpha\lambda L^k$ we obtain

$$\frac{1}{\lambda}\varphi^*(\alpha\lambda) + k\alpha \leq \frac{1}{\lambda L^k}\varphi^*\left(\alpha\lambda L^k\right) + \frac{1}{\lambda L^k}\sum_{j=1}^{k} L^j. \tag{2.4}$$

Note that for each $h > 0$ there exists $k \in \mathbb{N}$ such that $\log h < k$. Thus, considering $\alpha = j$ (for any $j \in \mathbb{N}$), it follows that

$$\frac{1}{\lambda}\varphi^*(j\lambda) + j\log h \le \frac{1}{\lambda}\varphi^*(j\lambda) + jk \le \frac{1}{\lambda L^k}\varphi^*(j\lambda L^k) + \frac{1}{\lambda L^k}\sum_{q=1}^{k} L^q, \quad \lambda > 0 \text{ and } j \in \mathbb{N}.$$

Thus, denoting $C_h = \exp\{\frac{1}{L^k}\sum_{q=1}^{k} L^q\}$ and $\lambda_h = \lambda L^k$, we conclude that

$$h^j e^{\frac{1}{\lambda}\varphi^*(j\lambda)} \le C_h e^{\frac{1}{\lambda_k}\varphi^*(j\lambda_h)}, \tag{2.5}$$

for each $h, \lambda > 0$ and $j \in \mathbb{N}$.

Remark 2.10 If ω and σ are weight functions such that there exists $A, C > 0$ satisfying

$$\omega(x) \le C\sigma(x), \quad x \ge A;$$

then there exists constants $\lambda_\sigma, \tilde{L} > 0$ such that $\varphi^*_\sigma(t) \le \frac{1}{\lambda_\sigma}\varphi^*_\omega(\lambda_\sigma t) + \tilde{L}, \forall t \ge 0$.

In order to prove the above claim we will consider $L > 1$ such that $\sigma(ex) \le L(\sigma(x) + 1)$, for each $x > 0$. In particular,

$$\sigma(x) \ge \frac{\sigma(ex)}{L} - L \ge \frac{\frac{\sigma(e^2x)}{L} - L}{L} - L \ge \frac{\sigma(e^2x)}{L^2} - 2L \ge \cdots \ge \frac{\sigma(e^m x)}{L^m} - mL,$$

for each $m \in \mathbb{N}$ and $x > 0$. Thus, considering m such that $e^m > A$

$$\begin{aligned}\varphi^*_\sigma(t) &= \sup_{y\ge0}\{yt - \sigma(e^y)\} \le \sup_{y\ge0}\left\{yt - \frac{\sigma(e^m e^y)}{L^m} + mL\right\} \le \sup_{y\ge0}\left\{yt - \frac{\omega(e^m e^y)}{CL^m} + mL\right\} \\ &\le \sup_{y\ge0}\left\{yt - \frac{\omega(e^y)}{CL^m}\right\} + mL \le \frac{1}{\lambda_\sigma}\sup_{y\ge0}\left\{yt\lambda_\sigma - \omega(e^y)\right\} + mL = \frac{1}{\lambda_\sigma}\varphi^*_\omega(\lambda_\sigma t) + mL.\end{aligned}$$

for $\lambda_\sigma = L^m C$ and for each $t > 0$.

Remark 2.11 Since ω is increasing it follows that,

$$\omega(t) = t\frac{\omega(t)}{t} = t\int_t^{+\infty}\frac{\omega(t)}{s^2}ds \le t\int_t^{+\infty}\frac{\omega(s)}{s^2}ds, \quad t > 1, \tag{2.6}$$

hence, using (β), it follows that $\omega(t) = o(t)$ $(t \to +\infty)$.

Remark 2.12 Using the definition of φ and φ^*, for each $b \ge 1$ and $\lambda > 0$ we have,

$$\frac{1}{\lambda}\varphi^*(\lambda b) \ge \frac{1}{\lambda}\{\lambda b y - \varphi(y)\} = by - \frac{1}{\lambda}\omega(e^y), \quad \forall\, y \ge 0.$$

Thus, considering $x \ge 1$ and $y = \log x \ge 0$,

$$e^{\frac{1}{\lambda}\varphi^*(\lambda b)} \ge x^b e^{-\frac{1}{\lambda}\omega(x)}, \quad \forall\, b \ge 1 \quad \text{and} \quad \forall\, x \ge 0. \tag{2.7}$$

Moreover, since $\omega(x) = 0$ (when $x \in [0, 1]$), (2.7) also follows for $x \in [0, 1]$. Furthermore, choosing $b = x$, it follows that, $b^b \leq e^{\frac{1}{\lambda}\varphi^*(\lambda b)+\frac{1}{\lambda}\omega(b)}$. Thus, using Remark 2.11, there exists $b^* > 0$ (independent of λ) such that

$$b^b \leq e^{\frac{1}{\lambda}\varphi^*(\lambda b)} e^{\frac{b}{\lambda}}, \quad \forall b \in [b^*; +\infty) \quad \lambda > 0.$$

Therefore, for each $\lambda > 0$, there exists C_λ such that

$$j! \leq j^j \leq C_\lambda^j e^{\frac{1}{\lambda}\varphi^*(\lambda j)}, \quad \forall j \in \mathbb{N}. \tag{2.8}$$

Remark 2.13 Using (2.8) and (2.5) it follows that the space of all real analytic functions is contained in $\mathcal{E}^\omega$.

Remark 2.14 Observe that, using (2.7), for $b = n + 1$, it follows that

$$\int_{\|\xi\|>r} e^{-a\omega(c\|\xi\|)} d\xi < +\infty, \tag{2.9}$$

for each $a, c > 0$ and $r \geq 1$.

2.3 Ultradistributions

Definition 2.15 The continuous dual of $\mathcal{D}^\omega(\Omega)$ is denoted by $\mathcal{D}^{\omega\prime}(\Omega)$ and its elements are called ultradistributions. Also, the continuous dual of $\mathcal{E}^\omega(\Omega)$ is denoted by $\mathcal{E}^{\omega\prime}(\Omega)$.

Remark 2.16 It follows from the definition that a linear functional $u : \mathcal{D}^\omega(\Omega) \to \mathbb{C}$ is continuous if and only if u is sequentially continuous.

Lemma 2.17 *Let $u : \mathcal{D}^\omega(\Omega) \to \mathbb{C}$ be a linear functional. The following statements are equivalent:*

1. *u is (sequentially) continuous;*
2. *For all compact $K \subset \Omega$ and for each $\delta > 0$ there exists a real positive constant $C_{\delta,K}$ such that*

$$|\langle u, \phi\rangle| \leq C_{\delta,K} \sum_{\alpha\in\mathbb{N}_0^n} e^{-\delta\varphi^*(|\alpha|/\delta)} \|\partial^\alpha\phi\|_{L^\infty(K)}, \qquad \forall\phi \in \mathcal{D}^\omega(K) \doteq \mathcal{E}^\omega(\Omega) \cap C_0^\infty(K). \tag{2.10}$$

Proof Consider u such that (2.10) is satisfied. If $\phi_j \to 0$ in $\mathcal{D}^\omega(\Omega)$, then there exists a compact set $K \subset \Omega$ such that $\operatorname{supp}\phi_j \subset K$ (for all $j \in \mathbb{N}$) and there exists $\lambda > 0$ such that

$$\sup_{\alpha\in\mathbb{N}_0^n} \sup_{x\in K} |\partial^\alpha\phi_j(x)| \exp\left[-\frac{1}{\lambda}\varphi^*(|\alpha|\lambda)\right] \to 0, \quad j \to +\infty.$$

Hence, for each $\epsilon > 0$ there exists $j_\epsilon \in \mathbb{N}$ such that

$$\sup_{\alpha\in\mathbb{N}_0^n} \sup_{x\in K} |\partial^\alpha \phi_j(x)| \exp\left[-\frac{1}{\lambda}\varphi^*(|\alpha|\lambda)\right] < \epsilon,$$

when $j > j_\epsilon$. Thus, for each $\delta > 0$ there exists $C_{\delta,K} > 0$ such that,

$$\left|\langle u, \phi_j\rangle\right| \le C_{\delta,K} \sum_{\alpha\in\mathbb{N}_0^n} e^{-\delta\varphi^*(|\alpha|/\delta)} \epsilon e^{\frac{1}{\lambda}\varphi^*(|\alpha|\lambda)},$$

where $j > j_\epsilon$. Considering $\delta = 1/(\lambda L)$ (where L is the constant considered in (α) and using (2.4) it follows that

$$e^{-\delta\varphi^*(|\alpha|/\delta)} e^{\frac{1}{\lambda}\varphi^*(|\alpha|\lambda)} = e^{-\frac{1}{\lambda L}\varphi^*(|\alpha|\lambda L)+\frac{1}{\lambda}\varphi^*(|\alpha|\lambda)} \le e^A e^{-|\alpha|},$$

where $A = A(\lambda) = \frac{1}{\lambda}$. Hence,

$$|\langle u, \phi_j\rangle| \le \epsilon C_{\delta,K} e^A \sum_{\alpha\in\mathbb{N}_0^n} e^{-|\alpha|}, \quad j > j_\epsilon.$$

Since $\sum_{\alpha\in\mathbb{N}_0^n} e^{-|\alpha|} < +\infty$, we can conclude that $\langle u, \phi_j\rangle \to 0$. Therefore, u is continuous.

Conversely, suppose by contradiction that there exist a compact set $K \subset \Omega$ and $\delta > 0$ such that for each $C > 0$ there exists $\phi = \phi_C \in \mathcal{D}^\omega(K)$ where

$$|\langle u, \phi\rangle| > C \sum_{\alpha\in\mathbb{N}_0^n} e^{-\delta\varphi^*(|\alpha|/\delta)} \|\partial^\alpha \phi\|_{L^\infty(K)}.$$

Thus, for each $j \in \mathbb{N}$ there exists $\phi_j \in \mathcal{D}^\omega(K)$ such that

$$r_j \doteq |\langle u, \phi_j\rangle| > j \sum_{\alpha\in\mathbb{N}_0^n} e^{-\delta\varphi^*(|\alpha|/\delta)} \|\partial^\alpha \phi_j\|_{L^\infty(K)}.$$

Set $\psi_j = \phi_j/r_j$. Note that $\operatorname{supp}\psi_j \subset K$ and $e^{-\delta\varphi^*(|\alpha|/\delta)}\|\partial^\alpha\psi_j\|_{L^\infty(K)} \le \frac{1}{j}$ ($\forall\alpha \in \mathbb{N}_0^n$). Hence, $\psi_j \to 0$ in $\mathcal{D}^\omega(\Omega)$ and $|\langle u, \psi_j\rangle| = 1$. This fact means that a contradiction was obtained. Therefore, u is not continuous. □

2.4 A Regular Equivalent Weight Function

Definition 2.18 Let $\omega : [0, +\infty[\to [0, +\infty[$ and $\rho : [0, +\infty[\to [0, +\infty[$ be functions. We say that ω and ρ are equivalent ($\omega \sim \rho$) when there exist $\delta, A, C > 0$ such that $A\omega(t) \le \rho(t) \le C\omega(t)$, for $t > \delta$.

Remark 2.19 Let $\omega : [0, +\infty[\to [0, +\infty[$ and $\rho : [0, +\infty[\to [0, +\infty[$ be weight functions. If $\omega \sim \rho$ then $\mathcal{E}^{\omega} = \mathcal{E}^{\rho}$ and $\mathcal{D}^{\omega} = \mathcal{D}^{\rho}$ (see [12, 3.4 Proposition]).

The next proposition guarantees that for each weight function ω there exists a C^1 function μ such that $\omega \sim \mu$. Therefore, we recall that by (2.2) there exist $A, H > 1$ such that $\omega(et) \leq H\,\omega(t)$, when $t \geq A$. Fix $h > \max\{2; \log(He)\}$.

Proposition 2.20 *There exists a function $Q \in C^1((1, +\infty))$ satisfying:*

1. $\omega \sim Q$ *and* $\lim_{y\to+\infty} Q'(y) = 0$.
2. *There exist $\delta_\omega > 0$ and $D > 0$ such that, if $\mu_\omega(y) \doteq (h-1)Q\left(\left(\delta_\omega + \|y\|^2\right)^{1/2}\right)$, for $y \in \mathbb{R}^n$, then,*

$$\mu_\omega(\xi) \geq \omega\left(\left(\delta_\omega + \|\xi\|^2\right)^{1/2}\right) > 0, \quad \forall \xi \in \mathbb{R}^n; \tag{2.11}$$

$$\mu_\omega(\xi) \leq D\,\omega((\delta_\omega + \|\xi\|^2)^{1/2}), \quad \forall \xi \in \mathbb{R}^n; \tag{2.12}$$

$$|\partial_{\xi_j}\mu_\omega(\xi)| < 1, \qquad \forall \xi \in \mathbb{R}^n \text{ and } j \in \{1, \ldots, n\}. \tag{2.13}$$

3. *Denoting $a^{\kappa}_{\mu_\omega}$ as in (1.3) it follows that for each compact set $K \subset \mathbb{R}^n$ there exists $C_K > 0$ (independent of ξ) such that $\sup_{x\in K} |\partial_x^{\alpha}\{a^{\kappa}_{\mu_\omega}(x, \xi)\}| \leq C_K$, for each $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$.*

Proof 1. Define

$$Q(y) = \int_1^{+\infty} \frac{\omega(yt)}{t^h}dt = y^{h-1}\int_y^{+\infty}\frac{\omega(s)}{s^h}ds, \quad y > 0.$$

Since $h > 2$ it follows that $0 \leq \frac{\omega(s)}{s^h} \leq \frac{\omega(s)}{s^2}$, when $s > 1$. Recalling that $\omega \equiv 0$ in $[0, 1]$ and using (β) we can see that the function Q is well defined. And

$$Q'(y) = (h-1)y^{h-2}\int_y^{+\infty}\frac{\omega(s)}{s^h}ds - y^{h-1}\frac{\omega(y)}{y^h}.$$

Therefore, Q' is a continuous function. Moreover, since ω is increasing and $h > 2$ we have

$$Q(y) \geq \omega(y)\int_1^{+\infty}\frac{1}{t^h}dt = \frac{1}{h-1}\omega(y), \quad y > 0. \tag{2.14}$$

And, since $h > \log(He)$ we obtain $e^h/(He) > 1$. Consequently, $C = \sum_{j=0}^{+\infty}\left(\frac{He}{e^h}\right)^j < +\infty$. Thus,

$$Q(y) = \int_1^{+\infty} \frac{\omega(yt)}{t^h} dt = \sum_{j=0}^{+\infty} \int_{e^j}^{e^{j+1}} \frac{\omega(yt)}{t^h} dt \leq \sum_{j=0}^{+\infty} \frac{\omega(e^{j+1}y)}{e^{jh}} e^j(e-1) \tag{2.15}$$

$$\leq \sum_{j=0}^{+\infty} \frac{H^{j+1}\omega(y)}{e^{jh}} e^j(e-1) \leq H(e-1)\omega(y) \sum_{j=0}^{+\infty}\left(\frac{He}{e^h}\right)^j = CH(e-1)\omega(y) \tag{2.16}$$

when $y \geq A$. Moreover, recalling that $h > 2$

$$|Q'(y)| \leq (h-1)y^{h-2} \int_y^{+\infty} \frac{\omega(s)}{s^2} s^{2-h} ds + \frac{\omega(y)}{y} \leq (h-1) \int_y^{+\infty} \frac{\omega(s)}{s^2} ds + \frac{\omega(y)}{y}, \quad y > 1.$$

Thus using (β) and (2.6) it follows that $\lim_{y\to+\infty} |Q'(y)| = 0$.
2. Since $\lim_{y\to+\infty} |Q'(y)| = 0$ there is $\delta_\omega > A$ such that $|Q'(y)| < \frac{1}{h-1}$ when $y > \delta_\omega^{1/2}$. Hence, if μ_ω is as being defined above then, $|\partial_{\xi_j}\mu_\omega(\xi)| < 1$, for each $\xi \in \mathbb{R}^n$ and $j \in \{1, \ldots, n\}$. Moreover, using (2.14) it follows that $\mu_\omega(\xi) = (h-1)Q\left(\left(\delta_\omega + \|\xi\|^2\right)^{1/2}\right) \geq \omega\left(\left(\delta_\omega + \|\xi\|^2\right)^{1/2}\right)$, $\forall \xi \in \mathbb{R}^n$. And, using (2.15) we obtain $\mu_\omega(\xi) \leq D\,\omega((\delta_\omega + \|\xi\|^2)^{1/2})$ where $D = (h-1)CH(e-1)$.
3. It follows from (2.13) and the definition of $a^\kappa_{\mu_\omega}$. □

Remark 2.21 Since ω satisfies (δ), it follows that the Q function defined in the proof of Proposition 2.20 also satisfies (δ).

Remark 2.22 The Q function defined in the proof of Proposition 2.20 was inspired by the proof of [28, 1.3. Proposition]. The function presented there is $\chi(y) = \int_1^{+\infty} \frac{\omega(yt)}{t^2} dt$. In their work, the authors establish a similar inequality to (2.15) by assuming a stronger hypothesis than ours: the existence of $K > 1$, $T > 0$, and $0 < \epsilon < 1$ such that $\omega(Kt) \leq (K - \epsilon)\omega(t)$ for each $t \geq T$. However, since we do not have this hypothesis, we considered the power h instead of 2.

3 Technical Results on FBI Transforms

This chapter is dedicated to the presentation of some technical results. Using Faà di Bruno's formula we will obtain estimates for derivatives of FBI terms. At the end of this section we will present a general inequality for the FBI transform acting on ultradistributions.

Lemma 3.1 *Let ω be a weight function (see Definition 2.1) and $\kappa > 0$. If μ_ω and $\delta = \delta_\omega$ are defined as in Proposition 2.20 then, there exist δ, $D > 0$ such that,*

$$\left|\partial_y^m \left\{e^{-\kappa\mu_\omega(\xi)(x-y)^2}\right\}\right| \leq D^m \theta^{-m/2} m! e^{-\frac{\kappa}{2}\omega((\|\xi\|^2+\delta)^{1/2})\|x-y\|^2} e^{\frac{\theta}{2}\omega((\|\xi\|^2+\delta)^{1/2})}, \tag{3.1}$$

for all $\xi \in \mathbb{R}^n$ $x, y \in \mathbb{R}$, $m \in \mathbb{N}_0$ *and* $\theta > 0$.

Proof Note that, using Faà di Bruno's formula (see [14, Corollary 2.11]) it follows that

$$\partial_y^m \left\{ e^{-\kappa \mu_\omega(\xi)(x-y)^2} \right\} = \sum_{r=1}^{m} e^{-\kappa \mu_\omega(\xi)(x-y)^2} \sum_{\mathfrak{p}(m,r)} m! \prod_{j=1}^{m} \frac{[\partial_y^j \{-\kappa \mu_\omega(\xi)(x-y)^2\}]^{k_j}}{k_j! [j!]^{k_j}},$$

where $\mathfrak{p}(m,r) = \left\{ (k_1, \dots, k_m) \in \mathbb{N}_0^m : \sum_{j=1}^m k_j = r, \sum_{j=1}^m j k_j = m \right\}$.

Since $\partial_y^j \{-\kappa \mu_\omega(\xi)(x-y)^2\} = 0$, for $j \geq 3$, we will be able to consider the sum over a subset of $\mathfrak{p}(q,r)$, considering only derivatives of order less than three. Hence, denoting $\mathfrak{p}_2(m,r) = \left\{ (k_1, k_2) \in \mathbb{N}_0^2 : k_1 + k_2 = r, k_1 + 2k_2 = m \right\}$, we have

$$\left| \partial_y^m \left\{ e^{-\kappa \mu_\omega(\xi)(x-y)^2} \right\} \right| \leq \sum_{r=1}^{m} \left| e^{-\kappa \mu_\omega(\xi)(x-y)^2} \right| \sum_{\mathfrak{p}_2(m,r)} m! \kappa^r \frac{|2\mu_\omega(\xi)(x-y)|^{k_1} |2\mu_\omega(\xi)|^{k_2}}{k_1! k_2! 1!^{k_1} 2!^{k_2}}.$$

And, using that $r = k_1 + k_2 = \frac{k_1}{2} + \frac{m}{2}$, $k_1 \leq m$, $m! \leq 2^m r! k_2!$ and $k_2! k_1!^{1/2} = (k_2!^2 k_1!)^{1/2} \leq m!^{1/2}$ we obtain,

$$\begin{aligned}
\left| \partial_y^m \{ e^{-\kappa \mu_\omega(\xi)(x-y)^2} \} \right| &\leq \sum_{r=1}^{m} \kappa^r \sum_{\mathfrak{p}_2(m,r)} \frac{m! 2^{k_1}}{k_1! k_2!} \left[e^{-2\kappa \mu_\omega(\xi)(x-y)^2} \left| \mu_\omega(\xi)(x-y)^2 \right|^{k_1} \right]^{\frac{1}{2}} |\mu_\omega(\xi)|^{\frac{m}{2}} \\
&\leq D^m \sum_{r=1}^{m} \kappa^{\frac{m}{2}} \sum_{\mathfrak{p}_2(m,r)} \frac{m! 2^{k_1}}{k_1! k_2!} \left[e^{-2\kappa \mu_\omega(\xi)(x-y)^2} e^{\frac{\kappa}{D} |\mu_\omega(\xi)(x-y)^2|} k_1! \right]^{\frac{1}{2}} |\mu_\omega(\xi)|^{\frac{m}{2}} \\
&\leq [D\kappa^{\frac{1}{2}}]^m m!^{1/2} 4^m \left[e^{-2\kappa \mu_\omega(\xi)(x-y)^2} e^{\frac{\kappa}{D} |\mu_\omega(\xi)|(x-y)^2} \right]^{\frac{1}{2}} |\mu_\omega(\xi)|^{\frac{m}{2}} \\
&\quad \times \sum_{r=1}^{m} r! \sum_{\mathfrak{p}_2(m,r)} \frac{1}{k_1! k_2!} \\
&\leq [D\kappa^{\frac{1}{2}}]^m m!^{1/2} 4^m \left[e^{-2\kappa \mu_\omega(\xi)(x-y)^2} e^{\frac{\kappa}{D} |\mu_\omega(\xi)|(x-y)^2} \right]^{\frac{1}{2}} |\mu_\omega(\xi)|^{\frac{m}{2}} \\
&\quad \times \sum_{r=1}^{m} r! \sum_{\mathfrak{p}(m,r)} \prod_{j=1}^{m} \frac{1}{k_j!},
\end{aligned}$$

where $D > 1$ is as in (2.12). Recalling that (see [14, p. 515]),

$$r! \sum_{\mathfrak{p}(m,r)} \prod_{j=1}^{m} \frac{1}{k_j!} = \binom{m-1}{r-1} \leq \binom{m}{r}$$

and

$$\sum_{r=0}^{m} \binom{m}{r} = 2^m,$$

using (2.11) and (2.12) it follows that

$$\left|\partial_y^m \left\{e^{-\kappa\mu_\omega(\xi)(x-y)^2}\right\}\right| \leq [D^{\frac{3}{2}}\kappa^{\frac{1}{2}}]^m 4^m m!^{1/2} e^{-\frac{\kappa}{2}\omega((\|\xi\|^2+\delta)^{1/2})(x-y)^2}[\omega((\|\xi\|^2+\delta)^{1/2})]^{\frac{m}{2}} 2^m. \tag{3.2}$$

Thus, for each $\theta > 0$ it follows that,

$$\left|\partial_y^m \left\{e^{-\kappa\mu_\omega(\xi)(z-y)^2}\right\}\right| \leq [D^{\frac{3}{2}}\kappa^{\frac{1}{2}}16]^m \theta^{-\frac{m}{2}} m! e^{-\frac{\kappa}{2}\omega((\|\xi\|^2+\delta)^{1/2})(x-y)^2} e^{(\theta/2)\omega((\|\xi\|^2+\delta)^{1/2})}.$$

□

Remark 3.2 Observe that $\partial_y \left\{e^{-\kappa\mu_\omega(\xi)(x-y)^2}\right\} = -\partial_x \left\{e^{-\kappa\mu_\omega(\xi)(x-y)^2}\right\}$, for $\xi \in \mathbb{R}^n$ and $y, x \in \mathbb{R}$. Thus, we can rewrite (3.1) by

$$\left|\partial_x^\ell \partial_y^m \left\{e^{-\kappa\mu_\omega(\xi)(x-y)^2}\right\}\right| \leq D^{m+\ell}\theta^{-(m+\ell)/2}(m+\ell)! e^{-\frac{\kappa}{2}\omega((\|\xi\|^2+\delta)^{1/2})\,|x-y|^2} e^{(\theta/2)\omega((\|\xi\|^2+\delta)^{1/2})}, \tag{3.3}$$

for all $\xi \in \mathbb{R}^n$, $x, y \in \mathbb{R}$, $\ell, m \in \mathbb{N}_0$ and $\theta > 0$.

Remark 3.3 Observe that

$$\partial_y^\alpha \left\{e^{-\kappa\mu_\omega(\xi)(x-y)^2}\right\} = \prod_{j=1}^n \partial_{y_j}^{\alpha_j} \left\{e^{-\kappa\mu_\omega(\xi)(x_j-y_j)^2}\right\},$$

for each $x, \xi, y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0$. Thus, it follows from Lemma 3.1,

$$\left|\partial_y^\alpha \left\{e^{-\kappa\mu_\omega(\xi)(x-y)^2}\right\}\right| \leq D^{|\alpha|}\theta^{-|\alpha|/2}\alpha! e^{-\frac{\kappa}{2}\omega((\|\xi\|^2+\delta)^{1/2})\|x-y\|^2} e^{\frac{\theta n}{2}\omega((\|\xi\|^2+\delta)^{1/2})}, \tag{3.4}$$

Next we will present bounds on the derivatives of the FBI phase.

Lemma 3.4 *If ω is a weight function, μ_ω and $\delta = \delta_\omega$ are defined as in Proposition 2.20, $\kappa > 0$ and $Q_\omega^\kappa(z, y, \xi) = i(x-y)\cdot\xi - \kappa\mu_\omega(\xi)(x-y)^2$, then for each $\theta, \lambda > 0$ there exists $D = D_{\theta,\lambda}$ such that*

$$|\partial_x^\gamma \partial_y^\alpha \left\{e^{Q_\omega^\kappa(x,y,\xi)}\right\}| \leq D^{|\alpha|+|\gamma|} e^{\frac{1}{\lambda}\varphi^*(\lambda|(\alpha,\gamma)|)} \times \\ \times e^{\frac{1}{\lambda}\omega((\|\xi\|^2+\delta)^{1/2})+\frac{n\theta}{2}\omega((\|\xi\|^2+\delta)^{1/2})} e^{-\frac{\kappa}{2}\omega((\|\xi\|^2+\delta)^{1/2})\,(x-y)^2}, \tag{3.5}$$

for $(\gamma, \alpha, x, y, \xi) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$.

Proof From Leibniz rule it follows that,

$$\partial_x^\gamma \partial_y^\alpha \left\{ e^{Q_\omega^\kappa(x,y,\xi)} \right\} = \sum_{\beta \le (\gamma,\alpha)} \binom{\gamma}{\beta'} \binom{\alpha}{\beta''} (i\xi)^{\gamma-\beta'} (-i\xi)^{\alpha-\beta''} e^{i(x-y)\xi} \times \prod_{k=1}^{n} \partial_{x_k}^{\beta_k'} \partial_{y_k}^{\beta_k''} \left\{ e^{-\kappa \mu_\omega(\xi)(x_k-y_k)^2} \right\},$$

where $\beta = (\beta', \beta'')$, $\beta' = (\beta_1', \ldots, \beta_n') \in \mathbb{N}_0^n$ and $\beta'' = (\beta_1'', \ldots, \beta_n'') \in \mathbb{N}_0^n$.

Therefore, using (3.3), (2.3), (2.7) and (2.8) there exists $C, D > 1$ such that,

$$\begin{aligned}
|\partial_x^\gamma \partial_y^\alpha \left\{ e^{Q_\omega(z,y,\xi)} \right\}| &\le \sum_{\beta \le (\gamma,\alpha)} \binom{\gamma}{\beta'} \binom{\alpha}{\beta''} \|\xi\|^{|(\gamma,\alpha)-\beta|} D^{|\beta|} \theta^{-\frac{|\beta|}{2}} \beta!\, e^{-\frac{\kappa}{2}\omega((\|\xi\|^2+\delta)^{1/2})\,(x-y)^2} \\
&\quad \times e^{\frac{n\theta}{2}\omega((\|\xi\|^2+\delta)^{1/2})} \\
&\le \sum_{\beta \le (\gamma,\alpha)} \binom{\gamma}{\beta'} \binom{\alpha}{\beta''} e^{\frac{1}{\lambda}\varphi^*(\lambda|(\gamma,\alpha)-\beta|)} e^{\frac{1}{\lambda}\omega((\|\xi\|^2+\delta)^{1/2})} \\
&\quad \times C^{|\beta|} \theta^{-\frac{|\beta|}{2}} e^{\frac{1}{\lambda}\varphi^*(\lambda|\beta|)} e^{\frac{|\beta|}{\lambda}} \\
&\quad \times e^{-\frac{\kappa\omega((\|\xi\|^2+\delta)^{1/2})}{2}(x-y)^2} e^{\frac{n\theta}{2}\omega((\|\xi\|^2+\delta)^{1/2})} \\
&\le (1 + C\theta^{-1/2} e^{\frac{1}{\lambda}})^{|\gamma|+|\alpha|} e^{\frac{1}{\lambda}\varphi^*(\lambda|(\gamma,\alpha)|)} e^{\frac{1}{\lambda}\omega((\|\xi\|^2+\delta)^{1/2})} \\
&\quad \times e^{-\frac{\kappa\omega((\|\xi\|^2+\delta)^{1/2})}{2}(x-y)^2} e^{\frac{n\theta}{2}\omega((\|\xi\|^2+\delta)^{1/2})}
\end{aligned}$$

for each $\lambda, \theta > 0$. □

The following result consists of a general domination of the FBI transform for ultradistributions.

Lemma 3.5 *Let $\Omega \subset \mathbb{R}^n$ be an open subset. If ω is a weight function, μ_ω and $\delta = \delta_\omega$ are defined as in Proposition 2.20, $\kappa > 0$, $u \in \mathcal{D}^{\omega\prime}(\Omega)$ and $\phi \in \mathcal{D}^\omega(\Omega)$ with $K = supp\,\phi$. Then, there exists $\lambda > 0$ such that for each $\theta > 0$ we can find $C = C_\theta > 0$ such that*

$$|\partial_x^\gamma \mathcal{F}_{\mu_\omega}^\kappa(\phi u)(x,\xi)| \le C e^{(1/\lambda)\varphi^*(\lambda|\gamma|)} e^{\theta\omega((\|\xi\|^2+\delta)^{1/2})} \sup_{y\in K} e^{-\frac{\kappa}{2}(x-y)^2\omega((\|\xi\|^2+\delta)^{1/2})},$$

for each $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\gamma \in \mathbb{N}_0^n$.

Proof Observe that denoting $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ and considering a real sequence $\{h_j\}_{j=1}^\infty$ such that $h_j \to 0$ it follows that, if $\psi \in \mathcal{E}^\omega(\mathbb{R}^n \times \mathbb{R}^n)$ is such that $\psi(x,y) = \psi_1(y)\psi_2(x,y)$ (where $\psi_1 \in \mathcal{D}^\omega(\mathbb{R}^n)$ and $\psi_2 \in \mathcal{E}^\omega(\mathbb{R}^n \times \mathbb{R}^n)$), then

$$\frac{\psi(x + h_j e_k, \cdot) - \psi(x, \cdot)}{h_j} \to \partial_{x_k}\psi(x, \cdot),$$

in $\mathcal{D}^{\omega}$, for each $k \in \{1, \dots n\}$ and $x \in \mathbb{R}^n$. Thus, using the continuity of u

$$\lim_{h\to 0} \frac{\langle u_y; \psi(x+he_k, y)\rangle - \langle u_y; \psi(x,y)\rangle}{h} = \lim_{h\to 0} \left\langle u_y; \frac{\psi(x+he_k, y) - \psi(x,y)}{h} \right\rangle$$
$$= \langle u_y; \partial_{x_k}\psi(x,y)\rangle.$$

Hence, it follows that

$$\partial_x^{\gamma} \mathcal{F}^{\kappa}_{\mu_\omega}(\phi u)(x,\xi) = \left\langle u(y); \phi(y)\partial_x^{\gamma}\left\{ e^{i(x-y)\cdot\xi - \kappa\mu_\omega(\xi)(x-y)^2} a_{\mu_\omega}(x-y,\xi)\right\}\right\rangle$$

where $\gamma \in \mathbb{N}_0^n$. Moreover, from Lemma 2.17 for each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\partial_x^{\gamma} \mathcal{F}^{\kappa}_{\mu_\omega} u(x,\xi)| \le C_\epsilon \sum_{\alpha} e^{-\epsilon\varphi^*(|\alpha|/\epsilon)} \sup_{y\in K} \left| \partial_y^{\alpha}\partial_x^{\gamma}\left\{\phi(y) e^{i(x-y)\cdot\xi - \kappa\mu_\omega(\xi)(x-y)^2} a_{\mu_\omega}(x-y,\xi)\right\}\right|.$$

Since $\phi \in \mathcal{D}^{\omega}$, there exists $\lambda_1 > 0$ such that $\|\partial^{\beta''}\phi\|_\infty \le e^{\frac{1}{2\lambda_1}\varphi^*(2\lambda_1|\beta''|)}$, for each $\beta'' \in \mathbb{N}_0^n$. Moreover, using Leibniz rule, (2.3), Lemma 3.4 and Proposition 2.20, for each $\lambda, \theta > 0$ there exist $D > 0$ such that

$$\begin{aligned}
|\partial_x^{\gamma} \mathcal{F}^{\kappa}_{\mu_\omega} u(x,\xi)| &\le CC_\epsilon \sum_{\alpha} e^{-\epsilon\varphi^*\left(\frac{|\alpha|}{\epsilon}\right)} \sup_{y\in K} \sum_{\beta\le(\alpha,\gamma)} \binom{(\alpha,\gamma)}{\beta} D^{|\beta|} \\
&\quad \times e^{\frac{1}{2\lambda_2}\varphi^*(2\lambda_2|(\alpha,\gamma)|) + \left(\frac{1}{\lambda}+\frac{n\theta}{2}\right)\omega((\|\xi\|^2+\delta)^{1/2})} e^{-\frac{\kappa}{2}\omega((\|\xi\|^2+\delta)^{1/2})(x-y)^2} \\
&\le CC_\epsilon e^{\frac{1}{\lambda_2}\varphi^*(\lambda_2|\gamma|)} \sum_{\alpha} e^{-\epsilon\varphi^*\left(\frac{|\alpha|}{\epsilon}\right)} e^{\frac{1}{\lambda_2}\varphi^*(\lambda_2|\alpha|)} (1+D)^{|\alpha+\gamma|} e^{\left(\frac{1}{\lambda}+\frac{n\theta}{2}\right)\omega((\|\xi\|^2+\delta)^{1/2})} \\
&\quad \times \sup_{y\in K} e^{-\frac{\kappa}{2}\omega((\|\xi\|^2+\delta)^{1/2})(x-y)^2},
\end{aligned}$$

where $\lambda_2 = \max\{\lambda, \lambda_1\}$. Next we will consider $\epsilon = \frac{1}{\lambda_2 L^k}$, where $k \in \mathbb{N}$ is chosen such that $e^{-k}(1+D) < 1$ and L is as in (α). From, (2.4),

$$\begin{aligned}
|\partial_x^{\gamma} \mathcal{F}^{\kappa}_{\mu_\omega} u(x,\xi)| &\le CC_\epsilon e^{\frac{1}{\lambda_2}\varphi^*(\lambda_2|\gamma|)} e^{\frac{1}{\lambda L^k}\sum_{j=1}^{k} L^j} \sum_{\alpha} e^{-k|\alpha|}(1+D)^{|\alpha+\gamma|} e^{\left(\frac{1}{\lambda}+\frac{n\theta}{2}\right)\omega((\|\xi\|^2+\delta)^{1/2})} \\
&\quad \times \sup_{y\in K} e^{-\frac{\kappa}{2}\omega((\|\xi\|^2+\delta)^{1/2})(x-y)^2} \\
&\le C' e^{\frac{1}{\lambda_2}\varphi^*(\lambda_2|\gamma|)} (1+D)^{|\gamma|} e^{\left(\frac{1}{\lambda}+\frac{n\theta}{2}\right)\omega((\|\xi\|^2+\delta)^{1/2})} \sup_{y\in K} e^{-\frac{\kappa}{2}\omega((\|\xi\|^2+\delta)^{1/2})(x-y)^2},
\end{aligned}$$

for some $C' > 0$. From the arbitrariness of λ and θ the proof is completed. □

4 Proof Theorem 1.1: Necessity

The aim of this section is to prove the necessary condition of Theorem 1.1 i.e, when given an ultradistribution u such that u is $\mathcal{E}^{\omega}$ regular in a neighborhood of a given point x_0, then (1.5) is satisfied. To do so we will prove a more general result, Theorem 4.2, in which the necessity part of Theorem 1.1 will be a particular case (just choose $\alpha = 0$ and $\kappa = 1$ in Theorem 4.2).

The reasons to present this more general result are the following. First, we need the result for arbitrary $\alpha \neq 0$ for the FBI inversion formula for ultradifferentiable functions presented in Sect. 5.1, see Lemma 5.2 and its consequences, Lemma 5.3 and Lemma 5.5.

Second, we deal with any parameter $\kappa > 0$ since in the proof of Lemma 5.5 we consider a general κ and use Lemma 5.3 with $\kappa/2$ (see (5.14)). For this reason it is required to allow arbitrary $\kappa > 0$. Moreover, Lemma 5.5 is important in the proof of the sufficiency part of Theorem 1.1.

Following the main ideas of [18] we start by presenting a characterization of ultradistributions vanishing in a neighborhood of a given point. And then we will use this fact to prove the main result of this section.

Lemma 4.1 *Let σ be a weight function, $\kappa > 0$, $x_0 \in \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ be an open neighborhood of x_0 and $u \in \mathcal{D}^{\sigma}{}'(\Omega)$. If u vanishes in a neighborhood of x_0, then for each $\phi \in \mathcal{D}^{\sigma}(\Omega)$ there exist constants $C, c, \lambda > 0$ and a neighborhood $V \subset \Omega$ of x_0 such that*

$$|\partial_x^{\alpha} \mathcal{F}^{\kappa}_{\mu_\sigma}(\phi u)(x,\xi)| \leq C e^{\frac{1}{\lambda}\varphi^*_\sigma(\lambda|\alpha|)} e^{-c\sigma(\xi)}, \quad (x,\xi,\alpha) \in V \times \mathbb{R}^n \times \mathbb{N}_0^n. \tag{4.1}$$

Proof Given $\phi \in \mathcal{D}^{\sigma}(\Omega)$ it follows that $\phi \in \mathcal{D}^{\sigma}(\mathbb{R}^n)$ and there exists $R > 0$ such that $\operatorname{supp}\phi \subset B(x_0, R)$. Considering $u \in \mathcal{D}^{\sigma}{}'(\Omega)$ vanishing in a neighborhood of x_0, we can assume that there exists $0 < r < R/3$ such that $u \equiv 0$ in $B(x_0, 3r)$. Also, consider a function $\psi \in \mathcal{D}^{\sigma}(B(x_0, 3r))$ such that $\psi \equiv 1$ in $B(x_0, 2r)$. Then, $supp\{\phi\psi\} \subset B(x_0, 3r)$ and $supp\{\phi(1-\psi)\} \subset B(x_0, R)\backslash\overline{B(x_0, r)}$. Thus, using Lemma 3.5, there exists $\lambda > 0$ such that for each $\theta > 0$ there exists $C = C_\theta > 0$ such that

$$\begin{aligned}|\partial_x^{\alpha} \mathcal{F}^{\kappa}_{\mu_\sigma}(\phi u)(x,\xi)| =& |\partial_x^{\alpha} \mathcal{F}^{\kappa}_{\mu_\sigma}(\phi(1-\psi)u)(x,\xi)| \\ \leq& C e^{\frac{1}{\lambda}\varphi^*_\sigma(\lambda|\alpha|)} e^{\theta\sigma((\|\xi\|^2+\delta)^{1/2})} \sup_{r<\|y-x_0\|<R} e^{-a\sigma((\|\xi\|^2+\delta)^{1/2})(x-y)^2},\end{aligned}$$

for every $(x, \xi, \alpha) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N}_0^n$ where $a = \frac{\kappa}{2}$. Moreover, if $\|x - x_0\| < r/2$ and $\|y - x_0\| > r$ then $\|x - y\| > r/2$. Thus, since $\delta > 0$ (here δ denotes the constant δ_σ as defined in Proposition 2.20.), recalling that σ is an increasing function and considering $\theta = \frac{ar^2}{8}$ we obtain

$$|\partial_x^{\alpha} \mathcal{F}^{\kappa}_{\mu_\sigma}(\phi u)(x,\xi)| \leq C e^{\frac{1}{\lambda}\varphi^*_\sigma(\lambda|\alpha|)} e^{-\frac{ar^2}{8}\sigma((\|\xi\|^2+\delta)^{1/2})} \leq C e^{\frac{1}{\lambda}\varphi^*_\sigma(\lambda|\alpha|)} e^{-\frac{ar^2}{8}\sigma(\xi)},$$
$$\text{for all } \|x - x_0\| < \tfrac{r}{2}, \xi \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{N}_0^n.$$

Thus setting $V = B(x_0, r/2)$ finishes the proof. □

Next we shall present the main result of this section whose consequence is the necessity part of Theorem 1.1. Note that if σ and ω are weight functions with $\omega(t) = O(\sigma(t))$ as $t \to +\infty$ then $\mathcal{D}^\sigma(\Omega) \subset \mathcal{D}^\omega(\Omega)$ (see Remark 2.10) and $\mathcal{D}^{\omega\prime}(\Omega) \subset \mathcal{D}^{\sigma\prime}(\Omega)$.

Theorem 4.2 *Let ω and σ be weight functions such that $\omega(t) = O(\sigma(t))$ as $t \to +\infty$, $\kappa > 0$, $\Omega \subset \mathbb{R}^n$ be an open set and $u \in \mathcal{D}^{\sigma\prime}(\Omega)$. If u is $\mathcal{E}^\omega$ in a neighborhood of $x_0 \in \Omega$ then for each $\phi \in \mathcal{D}^\sigma(\Omega)$ there exist $C, c, \lambda > 0$ and a neighborhood $V \subset \Omega$ of x_0 such that*

$$|\partial_x^\alpha \mathcal{F}_{\mu_\sigma}^\kappa(\phi u)(x,\xi)| \le C e^{\frac{1}{\lambda}\varphi^*(\lambda|\alpha|)} e^{-c\omega(\xi)}, \quad (x,\xi,\alpha) \in V \times \mathbb{R}^n \times \mathbb{N}_0^n, \tag{4.2}$$

where $\varphi^* = \varphi_\omega^*$ is the Young conjugate of $\varphi(t) = \omega(e^t)$.

Proof Fix $R > 0$ such that $u \in \mathcal{E}^\omega(B(x_0, R))$ and $B(x_0, R) \subset \Omega$. Consider $\psi \in \mathcal{D}^\sigma(B(x_0, R))$ such that $\psi \equiv 1$ in $B(x_0, \frac{R}{2})$ and write

$$\mathcal{F}_{\mu_\sigma}^\kappa(\phi u)(x,\xi) = \mathcal{F}_{\mu_\sigma}^\kappa(\psi\phi u)(x,\xi) + \mathcal{F}_{\mu_\sigma}^\kappa((1-\psi)\phi u)(x,\xi), \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{4.3}$$

Now we use Lemma 4.1 to obtain constants $C, a, \lambda > 0$ and a bounded neighborhood V of x_0 such that

$$|\partial_x^\alpha \mathcal{F}_{\mu_\sigma}^\kappa((1-\psi)\phi u)(x,\xi)| \le C e^{\frac{1}{\lambda}\varphi_\sigma^*(\lambda|\alpha|)} e^{-a\sigma(\xi)}, \quad (x,\xi) \in V \times \mathbb{R}^n.$$

Furthermore, since $\omega(t) = O(\sigma(t))$ as $t \to +\infty$ there exist $A, c > 0$ so that $\omega(\xi) \le c\sigma(\xi)$ for $\|\xi\| > A$ and by the continuity of ω, for each $c > 0$ there exists $C_c > 0$ such that $1 = e^{c\omega(\xi)}e^{-c\omega(\xi)} \le C_c e^{-c\omega(\xi)}$ for $\|\xi\| \le A$. Thus, we use Remark 2.10, define $a_1 = \frac{a}{c}$, increase $\lambda, C > 0$, if necessary, and obtain

$$|\partial_x^\alpha \mathcal{F}_{\mu_\sigma}^\kappa((1-\psi)\phi u)(x,\xi)| \le C e^{\frac{1}{\lambda}\varphi^*(\lambda|\alpha|)} e^{-a_1\omega(\xi)}, \quad (x,\xi) \in V \times \mathbb{R}^n. \tag{4.4}$$

Next we will fix $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$ and consider $j \in \{1, \dots, n\}$ such that $\|\xi\| \le n|\xi_j|$. Moreover, given an arbitrary $N \in \mathbb{N}_0$, integrating by parts with respect to y_j, we can write

$$\mathcal{F}_{\mu_\sigma}^\kappa(\psi\phi u)(x,\xi) = \frac{(-1)^N}{(-i\xi_j)^N} \int e^{i(x-y)\xi} \partial_{y_j}^N \left\{ u(y)\psi(y)\phi(y) e^{-\kappa\mu_\sigma(\xi)(x-y)^2} a_{\mu_\sigma}(x-y,\xi) \right\} dy,$$

for $x, \xi \in \mathbb{R}^n$, where it was used that $\psi\phi u \in \mathcal{D}^\omega(B(x_0, R)) \subset \mathcal{D}^\omega(\mathbb{R}^n)$. Given $\alpha \in \mathbb{N}_0^n$, we have

$$\begin{aligned} \partial_x^\alpha \mathcal{F}_{\mu_\sigma}^\kappa(\psi\phi u)(x,\xi) = &\frac{(-1)^N}{(-i\xi_j)^N} \int \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} (i\xi)^{\alpha-\gamma} e^{i(x-y)\xi} \times \\ &\times \partial_x^\gamma \partial_{y_j}^N \left\{ u(y)\psi(y)\phi(y) e^{-\kappa\mu_\sigma(\xi)(x-y)^2} a_{\mu_\sigma}(x-y,\xi) \right\} dy \end{aligned} \tag{4.5}$$

Now, Remark 2.8, (2.5), (2.8) together with Leibniz's rule and Proposition 2.20 (where it is proved that the derivatives of a_{μ_σ} are uniformly bounded in compact sets) show that there exist $C_2, \lambda_2 > 0$ (independent of x and ξ) such that

$$|\partial_x^{\beta_1}\partial_y^{\beta_2}\left\{u(y)\psi(y)\phi(y)a_{\mu_\sigma}(x-y,\xi)\right\}| \leq C_2 e^{\frac{1}{\lambda_2}\varphi^*(\lambda_2|\beta_2|)}, \quad \forall \beta_1, \beta_2 \in \mathbb{N}_0^n \text{ and } \forall y, \xi \in \mathbb{R}^n \text{ and } x \in V. \tag{4.6}$$

Furthermore, it follows from inequality (3.2) that there exist $D > 0$ such that

$$\begin{aligned}\left|\partial_x^{\beta_1}\partial_y^{\beta_2}\left\{e^{-\kappa\mu_\sigma(\xi)(x-y)^2}\right\}\right| &= \left|\prod_{j=1}^n \partial_{x_j}^{\beta_1^j}\partial_{y_j}^{\beta_2^j}\left\{e^{-\kappa\mu_\sigma(\xi)(x_j-y_j)^2}\right\}\right| \\ &\leq D^{|\beta_1|+|\beta_2|}e^{-\frac{\kappa}{2}\sigma((\|\xi\|^2+\delta_\sigma)^{1/2})(x-y)^2}[\sigma((\|\xi\|^2+\delta_\sigma)^{\frac{1}{2}})]^{\frac{|\beta_1+\beta_2|}{2}}|\beta_1+\beta_2|!^{\frac{1}{2}}, \\ &\text{for all } \beta_1,\beta_2 \in \mathbb{N}_0^n \text{ and } x, y, \xi \in \mathbb{R}^n.\end{aligned} \tag{4.7}$$

Denote $\xi_\sigma \doteq \sigma((\|\xi\|^2 + \delta_\sigma)^{1/2})$, increase D (if necessary) and use (2.7) to obtain

$$\begin{aligned}&\left|\partial_x^{\beta_1}\partial_y^{\beta_2}\left\{e^{-\kappa\mu_\sigma(\xi)(x-y)^2}\right\}\right| \leq D^{|\beta_1+\beta_2|}|\beta_1|^{\frac{|\beta_1|}{2}}|\beta_2|^{\frac{|\beta_2|}{2}}(\xi_\sigma^{|\beta_1+\beta_2|})^{\frac{1}{2}} \\ &\leq e^{\frac{\varphi^*(\lambda_0|\beta_1|)}{2\lambda_0}}e^{\frac{1}{2\lambda_0}\omega(D^2|\beta_1|)}e^{\frac{\varphi^*(\lambda_1|\beta_2|)}{2\lambda_1}}e^{\frac{1}{2\lambda_1}\omega(D^2|\beta_2|)}e^{\frac{\varphi^*(\lambda_1|\beta_2|)}{2\lambda_1}}e^{\frac{1}{2\lambda_1}\omega(\xi_\sigma)}e^{\frac{\varphi^*(\lambda_0|\beta_1|)}{2\lambda_0}}e^{\frac{1}{2\lambda_0}\omega(\xi_\sigma)}, \\ &\text{for all } \lambda_0, \lambda_1 > 0,\ \beta_1, \beta_2 \in \mathbb{N}_0^n \text{ and } x, y \in \mathbb{R}^n..\end{aligned} \tag{4.8}$$

Using Remark 2.11 we can find $D_1 > D$ such that $\omega(t) \leq t$, for each $t > D_1^2$ and since ω is increasing we can write

$$\begin{aligned}\left|\partial_x^{\beta_1}\partial_y^{\beta_2}\left\{e^{-\kappa\mu_\sigma(\xi)(x-y)^2}\right\}\right| &\leq e^{\frac{\varphi^*(\lambda_0|\beta_1|)}{\lambda_0}}e^{\frac{\varphi^*(\lambda_1|\beta_2|)}{\lambda_1}}e^{\frac{D_1^2|\beta_1|}{2\lambda_0}}e^{\frac{D_1^2|\beta_2|}{2\lambda_1}}e^{\left(\frac{1}{2\lambda_0}+\frac{1}{2\lambda_1}\right)\omega(\xi_\sigma)}, \\ &\text{for all } \lambda_0, \lambda_1 > 0,\ \beta_1, \beta_2 \in \mathbb{N}_0^n \text{ and } x, y \in \mathbb{R}^n.\end{aligned} \tag{4.9}$$

Thus, using Leibniz's rule, Remark 2.8, (4.6) and (4.9) there exists $m = m(n) > 0$ such that

$$\begin{aligned}&\left|\partial_x^{\beta_1}\partial_y^{\beta_2}\left\{u(y)\psi(y)\phi(y)e^{-\kappa\mu_\sigma(\xi)(x-y)^2}a_{\mu_\sigma}(x-y,\xi)\right\}\right| \\ &\leq \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}C_2e^{\frac{1}{\lambda_2}\varphi^*(\lambda_2|\beta_2-\gamma_2|)}e^{\frac{\varphi^*(\lambda_0|\gamma_1|)}{\lambda_0}}e^{\frac{\varphi^*(\lambda_1|\gamma_2|)}{\lambda_1}}e^{\frac{D_1^2|\gamma_1|}{2\lambda_0}}e^{\frac{D_1^2|\gamma_2|}{2\lambda_1}}e^{\left(\frac{1}{2\lambda_0}+\frac{1}{2\lambda_1}\right)\omega(\xi_\sigma)} \\ &\leq C_2\,e^{\frac{1}{\lambda_1}\varphi^*(\lambda_1|\beta_2|)}e^{\frac{1}{\lambda_0}\varphi^*(\lambda_0|\beta_1|)}\left(m+me^{\frac{D_1^2}{2\lambda_0}}\right)^{|\beta_1|}\left(m+me^{\frac{D_1^2}{2\lambda_1}}\right)^{|\beta_2|}e^{\left(\frac{1}{2\lambda_0}+\frac{1}{2\lambda_1}\right)\omega(\xi_\sigma)} \\ &\leq C_2\,e^{\frac{1}{\lambda_1}\varphi^*(\lambda_1|\beta_2|)}e^{\frac{1}{\lambda_0}\varphi^*(\lambda_0|\beta_1|)}C_{\lambda_0}^{|\beta_1|}C_{\lambda_1}^{|\beta_2|}e^{\left(\frac{1}{2\lambda_0}+\frac{1}{2\lambda_1}\right)\omega(\xi_\sigma)}, \\ &\text{for all } \lambda_0 > 0,\ \beta_1, \beta_2 \in \mathbb{N}_0^n, x \in V \text{ and } y \in \mathbb{R}^n,\end{aligned} \tag{4.10}$$

where we considered $\lambda_1 = \lambda_2$, used the notation $\beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ and denoted $C_\lambda = m + me^{\frac{D_1^2}{2\lambda}}$ (for $\lambda > 0$).

Hence, recalling that $\|\xi\| \leq n|\xi_j|$, denoting $\tilde{C} = C_2\,|\text{supp}\,\psi\,\phi|$ (where $|\text{supp}\,\psi\,\phi| = \int_{\text{supp}\,(\psi\,\phi)} 1\,dy$) and using (4.10) one can estimate the expression in (4.5) by

$$\left|\partial_x^\alpha \mathcal{F}_{\mu_\sigma}^\kappa(\psi\phi u)(x,\xi)\right| \leq \frac{\tilde{C}n^N}{\|\xi\|^N}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma} C_2\|\xi\|^{|\alpha-\gamma|}\, e^{\frac{1}{\lambda_1}\varphi^*(\lambda_1 N)} e^{\frac{1}{\lambda_0}\varphi^*(\lambda_0|\gamma|)} C_{\lambda_0}^{|\gamma|} C_{\lambda_1}^N e^{\left(\frac{1}{2\lambda_0}+\frac{1}{2\lambda_1}\right)\omega(\xi_\sigma)}$$
$$\text{for all } N \in \mathbb{N}_0,\ \lambda_0 > 0,\ x \in V \text{ and } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{4.11}$$

Denoting $a = \frac{1}{nC_{\lambda_1}}$ and using Remark 2.8, (2.7) and (2.3) there exists $k_{\lambda_0} = k(\lambda_, a) > 0$ such that

$$\left|\partial_x^\alpha \mathcal{F}_{\mu_\sigma}^\kappa(\psi\phi u)(x,\xi)\right| \leq \tilde{C}\frac{1}{(a\|\xi\|)^N} k_{\lambda_0}^{|\alpha|}\, e^{\frac{1}{\lambda_1}\varphi^*(\lambda_1 N)} e^{\frac{1}{\lambda_0}\omega(a\|\xi\|)} e^{\frac{1}{\lambda_0}\varphi^*(\lambda_0|\alpha|)} e^{\left(\frac{1}{2\lambda_0}+\frac{1}{2\lambda_1}\right)\omega(\xi_\sigma)}$$
$$\text{for all } \lambda_0 > 0,\ x \in V \text{ and } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Taking the infimum in $N \in \mathbb{N}_0$ in the last inequality we obtain

$$\left|\partial_x^\alpha \mathcal{F}_{\mu_\sigma}^\kappa(\psi\phi u)(x,\xi)\right| \leq \tilde{C}\inf_{N\in\mathbb{N}_0}\frac{e^{\frac{1}{\lambda_1}\varphi^*(\lambda_1 N)}}{(a\|\xi\|)^N} k_{\lambda_0}^{|\alpha|}\, e^{\frac{1}{\lambda_0}\varphi^*(\lambda_0|\alpha|)} e^{\frac{1}{\lambda_0}\omega(a\|\xi\|)} e^{\left(\frac{1}{2\lambda_0}+\frac{1}{2\lambda_1}\right)\omega(\xi_\sigma)}$$
$$\text{for all } \lambda_0 > 0,\ x \in V \text{ and } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{4.12}$$

Recalling that $\inf_{N\in\mathbb{N}_0} t^{-N} e^{\frac{1}{\lambda_1}\varphi^*(\lambda_1 N)} \leq e^{\log t} e^{-\frac{1}{\lambda_1}\omega(t)}$ for each $t \geq 1$, see [12, page 218],then by increasing $\tilde{C}$ if necessary, it follows that one can further estimate the expression in (4.12) by

$$\left|\partial_x^\alpha \mathcal{F}_{\mu_\sigma}^\kappa(\psi\phi u)(x,\xi)\right| \leq \tilde{C}e^{\log(a\|\xi\|)-\frac{1}{\lambda_1}\omega(a\|\xi\|)} k_{\lambda_0}^{|\alpha|} e^{\frac{1}{\lambda_0}\varphi^*(\lambda_0|\alpha|)} e^{\frac{1}{\lambda_0}\omega(a\|\xi\|)} e^{\left(\frac{1}{2\lambda_0}+\frac{1}{2\lambda_1}\right)\omega(\xi_\sigma)}$$
$$\text{for all } \lambda_0 > 0,\ x \in V \text{ and } \|\xi\| \geq 1.$$

Now, we take advantage of γ) and Remark 2.11 to obtain $A_1 > 0$ such that
$\xi_\sigma = \sigma((\|\xi\|^2 + \delta_\sigma)^{1/2}) \leq a\|\xi\|$ and $\log(a\|\xi\|) \leq \frac{1}{4\lambda_1}\omega(a\xi)$, when $\|\xi\| > A_1$.
Thus, choosing $\lambda_0 = 12\lambda_1$

$$\left|\partial_x^\alpha \mathcal{F}_{\mu_\sigma}^\kappa(\psi\phi u)(x,\xi)\right| \leq \tilde{C}e^{-\frac{1}{8\lambda_1}\omega(a\|\xi\|)} k_{\lambda_0}^{|\alpha|}\, e^{\frac{1}{\lambda_0}\varphi^*(\lambda_0|\alpha|)}, \tag{4.13}$$

for each $x \in V$ and $\|\xi\| > A_1$. Now, we choose $k \in \mathbb{N}$ satisfying $e^{-k} \leq a$. Hence, using (2.2) we see that there exists $A_2 \geq A_1$ such that

$$-\omega(a\xi) \leq -\omega(e^{-k}\xi) \leq -(2L)^{-k}\omega(\xi), \quad \|\xi\| > A_2.$$

Thus, using (2.5) there exist $a_1 > 0$ and $\lambda_3 > 0$ such that one can further estimate (4.13) by

$$\left|\partial_x^\alpha \mathcal{F}^\kappa_{\mu_\sigma}(\psi\phi u)(x,\xi)\right| \leq \tilde{C} e^{-a_1\omega(\xi)} e^{\frac{1}{\lambda_3}\varphi^*(\lambda_3|\alpha|)}, \quad \forall x \in V, \ \|\xi\| > A_2. \tag{4.14}$$

This concludes the proof when $\|\xi\| > A_2$. For ξ small, we go back to (4.5) and use (4.10) to get (possibly by increasing m)

$$\begin{aligned}|\partial_x^\alpha \mathcal{F}^\kappa_{\mu_\sigma}(\psi\phi u)(x,\xi)| = &\left|\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\int e^{i(x-y)\xi}\partial_x^\gamma\partial_y^{\alpha-\gamma}\left\{u(y)\psi(y)\phi(y)e^{-\kappa\mu_\sigma(\xi)(x-y)^2}a_{\mu_\sigma}(x-y,\xi)\right\}dy\right|\\ \leq &\tilde{C}e^{\frac{1}{\lambda_1}\varphi^*(\lambda_1(|\alpha|))}\left(m+me^{\frac{D_1^2}{2\lambda_1}}\right)^{|\alpha|} e^{\frac{1}{2\lambda_1}\omega(\xi_\sigma)}.\end{aligned}$$

Since ω and σ are continuous function (consequently $\sup_{\|\xi\|\leq A_2} e^{\frac{1}{2\lambda_1}\omega(\xi_\sigma)} < +\infty$), using (2.5), there exists $\lambda_2, \ \tilde{C}_1, \ \tilde{C}_2 > 0$ such that

$$|\partial_x^\alpha \mathcal{F}^\kappa_{\mu_\sigma}(\psi\phi u)(x,\xi)| \leq \tilde{C}_1 e^{\frac{1}{\lambda_2}\varphi^*(\lambda_2(|\alpha|))} \leq \tilde{C}_2 e^{\frac{1}{\lambda_2}\varphi^*(\lambda_2(|\alpha|))} e^{-a_1\omega(\xi)} \tag{4.15}$$

for $\|\xi\| \leq A_2$, $\tilde{C}_2 \geq \tilde{C}_1 \sup_{\|\xi\|\leq A_2} e^{a_1\omega(\xi)}$. Therefore, it follows from (4.3), (4.4), (4.14) and (4.15) that the proof is completed. □

An immediate consequence is the necessity of Theorem 1.1.

Corollary 4.3 *Fix a weight function ω, an open set $\Omega \subset \mathbb{R}^n$ and $u \in \mathcal{D}^{\omega\prime}(\Omega)$. If $u \in \mathcal{E}^\omega$ in a neighborhood of $x_0 \in \Omega$ then there exist a weight function σ such that $\omega(t) = O(\sigma(t))$ as $t \to +\infty$, so that for each $\phi \in \mathcal{D}^\sigma(\Omega)$ there exist $C, c > 0$ and a neighborhood $x_0 \in V \subset \Omega$ such that*

$$|\mathcal{F}_{\mu_\sigma}(\phi u)(x,\xi)| \leq Ce^{-c\omega(\xi)}, \quad (x,\xi) \in V \times \mathbb{R}^n. \tag{4.16}$$

Proof Under the corollary hypothesis one can select any weight function $\omega(t) = O(\sigma(t))$ as $t \to +\infty$ then the proof of Theorem 4.2 will work for $u \in \mathcal{D}^{\omega\prime}(\Omega) \subset \mathcal{D}^{\sigma\prime}(\Omega)$. □

5 Proof Theorem 1.1: Sufficiency

In order to prove the sufficiency part of Theorem 1.1 we shall prove an inversion formula for the FBI transform $\mathcal{F}^\kappa_{\mu_\omega}$ defined in (1.4).

5.1 Inversion Formula for Ultradifferentiable Functions

Throughout this section we will consider a weight function ω and we will denote by μ_ω the function defined in Proposition 2.20. In order to prove the sufficient condition of Theorem 1.1 in this section we will present two inversion formulas of $\mathcal{F}^{\kappa}_{\mu_\omega} u$ when u is in $\mathcal{E}^{\omega}$, $\kappa > 0$ and ω is any weight function.

Lemma 5.1 *If* $u \in C_c^{n+1}(\mathbb{R}^n)$ *then*

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathcal{F}^{\kappa}_{\mu_\omega} u(x,\xi) d\xi, \qquad \forall x \in \mathbb{R}^n.$$

Proof Let $R > 1$ be such that $\operatorname{supp} u \subset B(0,R)$. Since $u \in C_c^{n+1}(\mathbb{R}^n)$ it follows that $\sup_{\xi \in \mathbb{R}^n}\{(1+\|\xi\|)^{n+1}|\hat{u}(\xi)|\} < +\infty$. Thus, using the Fourier inversion formula we can write

$$u(x) = (2\pi)^{-n} \lim_{\epsilon \to 0} \iint u(x') e^{i(x-x')\cdot\xi - \epsilon\xi^2} d\xi dx'. \tag{5.1}$$

Define $\Gamma = \Gamma(x,x',\xi,t) \doteq \xi + i\, t\, \kappa\, \mu_\omega(\xi)\,(x-x')$, for $x, x', \xi \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Since $\mu_\omega \geq 0$,

$$\mathcal{R}\{i(x-x')\cdot\Gamma - \epsilon\cdot\Gamma^2\} \leq -\epsilon\left[\xi^2 - t^2\kappa^2[\mu_\omega(\xi)]^2\|x-x'\|^2\right],$$
$$\text{for all } (x,x',\xi,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1].$$

This, together with the fact that the function ω is increasing and (2.12) we see that there exist constants $D, \delta > 0$ such that

$$\begin{aligned}&\mathcal{R}\{i(x-x')\cdot\Gamma(x,x',\xi,t) - \epsilon\cdot[\Gamma(x,x',\xi,t)]^2\}\\ &\leq -\epsilon\left(\xi^2 - \kappa^2[D\omega(\|\xi\| + \delta^{\frac{1}{2}})]^2|(R+r)^2\right),\\ &\text{for } 0 \leq t \leq 1,\ \|x'\| \leq R,\ \|x\| < r \text{ and } \xi \in \mathbb{R}^n,\end{aligned}$$

for an arbitrary $r > 0$.

Moreover, using (2.6) there exists a positive constant $\tilde{A}$ such that one can further estimate the last expression as follows

$$\mathcal{R}\{i(x-x')\cdot\Gamma(x,x',\xi,t) - \epsilon\cdot[\Gamma(x,x',\xi,t)]^2\} \leq -\epsilon\left(1-\frac{1}{2}\right)\|\xi\|^2 = -\frac{\epsilon}{2}\|\xi\|^2,$$
$$\text{for } 0 \leq t \leq 1,\ \|x'\| \leq R,\ \|x\| < r \text{ and } \|\xi\| > \tilde{A}. \tag{5.2}$$

Since $\xi \mapsto e^{-\epsilon\tilde{a}\|\xi\|^2}$ is a L^1 function for each $\epsilon > 0$, using (5.2) it follows that

$$u(x) = (2\pi)^{-n} \lim_{\epsilon \to 0} \lim_{S \to +\infty} \int_{\mathbb{R}^n} \int_{\|\xi\| \leq S} u(x') e^{i(x-x')\cdot\xi - \epsilon\xi^2} d\xi dx'.$$

Now, using that $\zeta \mapsto u(x')e^{i(x-x')\cdot\zeta - \epsilon\zeta^2}$ is holomorphic, we may apply Stokes' theorem together with the definition of a_{μ_ω} (see (1.3)) to obtain,

$$u(x) = (2\pi)^{-n} \lim_{\epsilon \to 0^+} \lim_{S \to +\infty} \int_{\mathbb{R}^n} \int_{\|\xi\| \leq S} u(x') e^{i(x-x')\cdot\Gamma(x,x',\xi,1) - \epsilon[\Gamma(x,x',\xi,1)]^2} a^{\kappa}_{\mu_\omega}(x-x',\xi) d\xi dx'. \tag{5.3}$$

Thus, as a consequence of the third item in Proposition 2.20, inequality (5.2) and Fubini's theorem, we can further express (5.3) as

$$u(x) = (2\pi)^{-n} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x')e^{i(x-x')\cdot\Gamma(x,x',\xi,1)-\epsilon[\Gamma(x,x',\xi,1)]^2} a^{\kappa}_{\mu_\omega}(x-x',\xi)dx'd\xi$$
$$= (2\pi)^{-n} \lim_{\epsilon\to 0^+} \int\int u(x')e^{i(x-x')\xi-\kappa\mu_\omega(\xi)(x-x')^2} a^{\kappa}_{\mu_\omega}(x-x',\xi)e^{-\epsilon(\xi+i\kappa\mu_\omega(\xi)(x-x'))^2}dx'd\xi. \tag{5.4}$$

Now, we will show that we can use the dominated convergence theorem in the right-hand side of (5.4).

The goal is to bound the integral in x' uniformly in ϵ by an L^1 function in ξ. Let $Q_\kappa(x,x',\xi) \doteq -\epsilon(\xi+i\kappa\mu_\omega(\xi)(x-x'))^2$ and note that it follows from (2.6) that there exist constants $\tilde{A}, a > 0$ so that

$$\mathcal{R}\{Q_\kappa(x,x',\xi)\} = -\epsilon(\xi^2 - [\kappa\mu_\omega(\xi)]^2\|x-x'\|^2) \le -\epsilon a\|\xi\|^2, \quad \text{for } x' \in B(0,R),\ x\in B(0,r),\ \|\xi\| > \tilde{A}. \tag{5.5}$$

Therefore, the trivial bound is not good enough for our purpose. The trick here is to use integration by parts and to do so fix $\xi \in \mathbb{R}^n$ and assume without loss of generality that $|\xi_1| = \max\{|\xi_k| : k \in \{1,\dots,n\}\}$. Then it follows that

$$\|\xi\| = 2\left(\|\xi\| - \frac{\|\xi\|}{2}\right) \le 2\left(\sqrt{n}|\xi_1| - \frac{\|\xi\|}{2}\frac{|x_1-x_1'|}{R+r}\right) \tag{5.6}$$

for $x' \in B(0,R)$ and $x \in B(0,r)$. Moreover, using (2.6) and (2.12) and increasing $\tilde{A} > 0$ (if necessary) we have

$$\frac{\|\xi\|}{2}\frac{1}{R+r} \ge \kappa 2\sqrt{n} D\,\omega\big((\|\xi\|^2+\delta)^{1/2}\big) \ge \kappa 2\sqrt{n}\,|\mu_\omega(\xi)|, \qquad \|\xi\| \ge \tilde{A} \tag{5.7}$$

where D is the constant appearing in (2.12). Thus one can use (5.7) to continue estimating (5.6) as

$$\|\xi\| \le 2\left(\sqrt{n}|i\xi_1| - \kappa 2\sqrt{n}|\mu_\omega(\xi)||x_1'-x_1|\right) \le 2\sqrt{n}\big|-i\xi_1 + 2(x_1-x_1')\kappa\mu_\omega(\xi)\big|, \quad \text{for } x' \in B(0,R),\ x \in B(0,r) \text{ and } \|\xi\| > \tilde{A} \text{ (increasing } \tilde{A}, \text{ if necessary)}. \tag{5.8}$$

Thus, integrating by parts the integral in x' in the right-hand side of (5.4) multiplied by $\|\xi\|^{n+1}$, we have

$$\|\xi\|^{n+1}\left|\int_{\mathbb{R}^n} u(x')e^{i(x-x')\xi-\kappa\mu_\omega(\xi)(x-x')^2} a^{\kappa}_{\mu_\omega}(x-x',\xi)e^{-\epsilon(\xi+i\kappa\mu_\omega(\xi)(x-x'))^2}dx'\right|$$
$$\le \|\xi\|^n(4n)^{\frac{1}{2}}\left|\int_{\mathbb{R}^n}\partial_{x_1'}\left\{e^{i(x-x')\xi-\kappa\mu_\omega(\xi)(x-x')^2}\right\}u(x')a^{\kappa}_{\mu_\omega}(x-x',\xi)e^{-\epsilon(\xi+i\kappa\mu_\omega(\xi)(x-x'))^2}dx'\right|$$
$$\le (4n)^{\frac{n+1}{2}}\left|\int_{\mathbb{R}^n} e^{i(x-x')\xi-\kappa\mu_\omega(\xi)(x-x')^2}\partial^{n+1}_{x_1'}\left\{u(x')a^{\kappa}_{\mu_\omega}(x-x',\xi)e^{-\epsilon(\xi+i\kappa\mu_\omega(\xi)(x-x'))^2}\right\}dx'\right|, \tag{5.9}$$

for $x' \in B(0,R)$, $x \in B(0,r)$ and $\|\xi\| > \tilde{A}$.

Moreover, for each $q \in \{1,\ldots,n+1\}$, using Faà di Bruno's formula (see [14, Corollary 2.11])

$$\partial^q_{x_1'}\left\{e^{Q_\kappa(x,x',\xi)}\right\} = \sum_{r=1}^{q} e^{Q_\kappa(x,x',\xi)} \sum_{\mathfrak{p}(q,r)} q! \prod_{p=1}^{q} \frac{[\partial^p_{x_1'} Q_\kappa(x,x',\xi)]^{k_p}}{k_p![p!]^{k_p}}$$

and

$$\mathfrak{p}(q,r) = \left\{(k_1,\ldots,k_q) : k_p \geq 0,\ \sum_{p=1}^{q} k_p = r,\ \sum_{p=1}^{q} pk_p = q\right\}.$$

It is easy to see that there exists a constant $c > 0$ such that

$$|\partial_{x_1'} Q_\kappa(x,x',\xi)| \leq \epsilon c\|\xi\|^2, \quad |\partial^2_{x_1'} Q_\kappa(x,x',\xi)| \leq \epsilon c\|\xi\|^2, \quad \text{and} \quad \partial^p_{x_1'} Q_\kappa(x,x',\xi) = 0\ (p \geq 3),$$

for $x' \in B(0,R)$, $x \in B(0,r)$ and $\|\xi\| > \tilde{A}$.

Consequently, we will be able to consider the sum over a subset of $\mathfrak{p}(q,r)$, considering only derivatives of order less than three. Moreover, since $r!\sum_{\mathfrak{p}(q,r)}\prod_{p=1}^{q}\frac{1}{k_p!} = \binom{q-1}{r-1}$ (see [14, p. 515]), it follows that

$$\begin{aligned}
\left|\partial^q_{x_1'}\left\{e^{Q_\kappa(x,x',\xi)}\right\}\right| &\leq \sum_{r=1}^{q} e^{-\epsilon a\|\xi\|^2} \sum_{\mathfrak{p}_2(q,r)} q! \frac{(\epsilon c\|\xi\|^2)^r}{k_1!k_2!2^{k_2}} \\
&\leq q!\sum_{r=1}^{q} r! \sum_{\mathfrak{p}_2(q,r)} \frac{(c/a)^r}{k_1!k_2!2^{k_2}} \\
&\leq q!\sum_{r=1}^{q} (c/a)^r r! \sum_{\mathfrak{p}(q,r)} \prod_{p=1}^{q} \frac{1}{k_p!} \\
&= q!\sum_{r=1}^{q} (c/a)^r \binom{q-1}{r-1} \\
&\leq (1+c/a)^q q!
\end{aligned} \tag{5.10}$$

where $\mathfrak{p}_2(q,r) = \{(k_1,k_2) : k_1 + k_2 = r \text{ and } k_1 + 2k_2 = q\}$, for $x' \in B(0,R)$, $x \in B(0,r)$ and $\|\xi\| > \tilde{A}$. Thus using the Leibniz rule and (5.10) one can bound the term in (5.9) by an uniform constant $C_n > 0$ independent of ϵ,

$$\|\xi\|^{n+1}\left|\int_{\mathbb{R}^n} u(x')e^{i(x-x')\xi-\kappa\mu_\omega(\xi)(x-x')^2} a_{\mu_\omega}(x-x',\xi)e^{-\epsilon(\xi+it\kappa\mu_\omega(\xi)(x-x'))^2}dx'\right| \leq C_n, \quad \text{for all} x \in B(0,r) \text{and } \|\xi\| > \tilde{A}. \tag{5.11}$$

Since u is compactly supported, considering (5.4) and taking into account (5.11) we can use the dominated convergence theorem in the right-hand side of (5.4) to conclude that

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathcal{F}^{\kappa}_{\mu_\omega} u(x, \xi) d\xi, \quad \|x\| < r$$

for a arbitrary $r > 0$ as we wished to prove. □

Lemma 5.2 *If* $\kappa > 0$ *and* $u \in \mathcal{D}^{\omega}(\mathbb{R}^n)$ *then, for* $x \in \mathbb{R}^n$:

$$u(x) = (2\pi)^{-n} \lim_{j \to +\infty} \int e^{-\frac{\|\xi\|^2}{j}} \mathcal{F}^{\kappa}_{\mu_\omega} u(x, \xi) d\xi, \quad in \quad \mathcal{E}^{\omega}(\mathbb{R}^n).$$

Proof Observe that, from Theorem 4.2 for each compact subset $K \subset \mathbb{R}^n$ there exist an open neighborhood V of K and $a, \lambda, C > 0$ such that

$$|\partial^{\alpha}_x \mathcal{F}^{\kappa}_{\mu_\omega} u(x, \xi)| \leq C e^{-a\omega(\xi)} e^{\frac{1}{\lambda}\varphi^*(|\alpha|\lambda)}, \quad x \in V \text{ and } \xi \in \mathbb{R}^n. \tag{5.12}$$

Moreover, using Lemma 5.1 it follows that $u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathcal{F}^{\kappa}_{\mu_\omega} u(x, \xi) d\xi$. Thus, from (5.12),

$$\begin{aligned}\left|\partial^{\alpha}_x \left\{u(x) - (2\pi)^{-n} \int e^{-\frac{\|\xi\|^2}{j}} \mathcal{F}^{\kappa}_{\mu_\omega} u(x, \xi) d\xi\right\}\right| &= \left|\partial^{\alpha}_x \left\{(2\pi)^{-n} \int \left(1 - e^{-\frac{\|\xi\|^2}{j}}\right) \mathcal{F}^{\kappa}_{\mu_\omega} u(x, \xi) d\xi\right\}\right| \\ &\leq (2\pi)^{-n} \int (1 - e^{-\frac{\|\xi\|^2}{j}}) C e^{\frac{1}{\lambda}\varphi^*(|\alpha|\lambda)} e^{-a\omega(\xi)} d\xi \\ &\leq (2\pi)^{-n} C e^{\frac{1}{\lambda}\varphi^*(|\alpha|\lambda)} \int (1 - e^{-\frac{\|\xi\|^2}{j}}) e^{-a\omega(\xi)} d\xi, \quad x \in K.\end{aligned}$$

The lemma now follows from (2.9) and the dominated convergence theorem. □

5.2 FBI Inversion Formulas for Ultradistributions

The aim of this section is to present two inversion formulas for ultradistributions.

Lemma 5.3 *Let* $\kappa > 0$ *be arbitrary. If* $u \in \mathcal{D}^{\omega\prime}(\mathbb{R}^n)$ *and* $\psi \in \mathcal{D}^{\omega}(\mathbb{R}^n)$ *then*

$$\langle u_j; \phi \rangle \to \langle u; \psi\phi \rangle, \quad j \to +\infty \quad for\ each \quad \phi \in \mathcal{D}^{\omega}(\mathbb{R}^n)$$

where $u_j(x) \doteq (2\pi)^{-n} \int e^{-\frac{\|\xi\|^2}{j}} \mathcal{F}^{\kappa}_{\mu_\omega}(\psi u)(x, \xi) d\xi$.

Remark 5.4 Note that from Lemma 3.5 and Remark 2.11 it follows that u_j is well defined.

Proof For each $\phi \in \mathcal{D}^{\omega}(\mathbb{R}^n)$ using Lemma 5.2 and the notation $\check{\phi}(x) = \phi(-x)$ we have

$$
\begin{aligned}
\langle u_j, \phi\rangle =&(2\pi)^{-n} \int\int e^{-\frac{\|\xi\|^2}{j}} \left\langle u_{x'}, \psi(x')e^{i(x-x')\xi-\kappa\mu_\omega(\xi)(x-x')^2} a_{\mu_\omega}(x-x',\xi)\right\rangle d\xi\phi(x)dx \\
=&\left\langle u_{x'}, \psi(x')(2\pi)^{-n} \int\int e^{-\frac{\|\xi\|^2}{j}} e^{i(x-x')\xi-\kappa\mu_\omega(\xi)(x-x')^2} a_{\mu_\omega}(x-x',\xi)d\xi\phi(x)dx\right\rangle \\
=&\left\langle u_{x'}, \psi(x')(2\pi)^{-n} \int e^{-\frac{\|\xi\|^2}{j}} \mathcal{F}^{\kappa}_{\mu_\omega}\check{\phi}(-x',\xi)d\xi\right\rangle \\
&\to \langle u, \psi\phi\rangle, \quad \text{as} \quad j \to +\infty.
\end{aligned}
$$

Where in the second equality we can apply similar arguments as in the proof of Lemma 3.4 to obtain the convergence of the Riemann integral in $\mathcal{D}^{\omega}-$topology.

Also, in the second and third equality, it was used that $e^{-\frac{\|\cdot\|^2}{j}} \in L^1(\mathbb{R}^n)$ and that the support of ψ and ϕ are compact subsets of $\mathbb{R}^n$. □

Next we will use Lemma 5.3 to obtain another inversion formula which will be used in the proof of Theorem 5.6.

Lemma 5.5 *Let $\kappa > 0$ be arbitrary. If $u \in \mathcal{D}^{\omega\prime}(\mathbb{R}^n)$ $\psi \in \mathcal{D}^{\omega}(\mathbb{R}^n)$ then*

$$
\langle \tilde{u}_j; \phi\rangle \to \langle u\psi; \phi\rangle, \quad j \to +\infty \quad \textit{for each } \phi \in \mathcal{D}^{\omega}(\mathbb{R}^n);
$$

where $\tilde{u}_j(x) \doteq (2\pi^3)^{-n/2} \int\int e^{i\xi(x-t)-\kappa\mu_\omega(\xi)(x-t)^2} e^{-\frac{\|\xi\|^2}{j}} \mathcal{F}^{\kappa}_{\mu_\omega}(u\psi)(t,\xi) \big(\kappa\mu_\omega(\xi)\big)^{n/2} dt d\xi$.

Proof Since, $e^{-\mu_\omega(\xi)(x-\cdot)^2} \in L^1(\mathbb{R}^n)$ (for each fixed $x, \xi \in \mathbb{R}^n$) we can rewrite $\tilde{u}_j(x)$ as

$$
\begin{aligned}
\tilde{u}_j(x) &= (2\pi^3)^{-\frac{n}{2}} \iint e^{-\kappa\mu_\omega(\xi)(x-t)^2} e^{-\frac{\|\xi\|^2}{j}} \left\langle u, \psi(\cdot)e^{i(x-\cdot)\xi-\kappa\mu_\omega(\xi)(t-\cdot)^2} a^{\kappa}_{\mu_\omega}(t-\cdot,\xi)\right\rangle \big(\kappa\mu_\omega(\xi)\big)^{\frac{n}{2}} dt d\xi \\
&= (2\pi^3)^{-\frac{n}{2}} \int e^{-\frac{\|\xi\|^2}{j}} \left\langle u, \psi(\cdot) \int e^{i(x-\cdot)\xi-\kappa\mu_\omega(\xi)[(t-\cdot)^2+(x-t)^2]} a^{\kappa}_{\mu_\omega}(t-\cdot,\xi)dt\right\rangle \big(\kappa\mu_\omega(\xi)\big)^{\frac{n}{2}} d\xi,
\end{aligned}
$$

where it was used that the Riemann sum converges to the Riemann integral in the $\mathcal{D}^{\omega}$-topology with respect to the variable t. Using the equations

$$
2\left[t - \frac{1}{2}(x+x')\right]^2 = \frac{1}{2}[(t-x)+(t-x')]^2 = \frac{1}{2}(t-x)^2 + (t-x)(t-x') + \frac{1}{2}(t-x')^2
$$

and

$$
\frac{1}{2}(x-x')^2 = \frac{1}{2}(x-t)^2 + (x-t)(t-x') + \frac{1}{2}(t-x')^2
$$

it follows that

$$\tilde{u}_j(x) = (2\pi^3)^{\frac{-n}{2}} \int e^{\frac{-\|\xi\|^2}{j}} \left\langle u, \psi \int e^{i(x-\cdot)\xi - \kappa\mu_\omega(\xi)\left[\frac{1}{2}(x-\cdot)^2 + 2(t-\frac{1}{2}(x+\cdot))^2\right]} a^\kappa_{\mu_\omega}(t-\cdot,\xi)dt \right\rangle (\kappa\mu_\omega(\xi))^{\frac{n}{2}} d\xi. \tag{5.13}$$

Since $e^{-2\kappa\mu_\omega(\xi)(\cdot - \frac{1}{2}(x+x'))^2} \in L^1(\mathbb{R}^n)$ (for each ξ, x, x', κ) we consider the change of variables, $-w = t - \frac{1}{2}\left(x + x'\right)$ to write

$$\int e^{-\kappa\mu_\omega(\xi)\left\{2[t-\frac{1}{2}(x+x')]^2\right\}} a^\kappa_{\mu_\omega}(t-x',\xi)dt = \int e^{-\kappa\mu_\omega(\xi)2w^2} a^\kappa_{\mu_\omega}\left(\tfrac{1}{2}(x-x') - w, \xi\right)dw.$$

Moreover, using the identity (see [32, (IX.4.6)])

$$\left(\frac{z}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-zy^2}[A + B(\lambda - y)]dy = A + B\lambda, \quad z, A, B, \lambda \in \mathbb{C} \text{ and } \mathcal{R}z > 0,$$

Observe that $a^\kappa_{\mu_\omega}\left(z, \xi\right)$ is a polynomial of degree n with respect to $z \in \mathbb{R}^n$ (see (1.3)). Additionally, it is a polynomial of degree 1 as a function of z_j for each $j \in \{1, \ldots, n\}$, depending on the parameters z_k for $k \neq j$. Since $\kappa > 0$ and $\mu_\omega(y) > 0$ we obtain

$$\left(\frac{2\kappa\mu_\omega(\xi)}{\pi}\right)^{\frac{n}{2}} \int e^{-\kappa\mu_\omega(\xi)2w^2} a^\kappa_{\mu_\omega}\left(\tfrac{1}{2}(x-x') - w, \xi\right)dw = a^\kappa_{\mu_\omega}\left(\tfrac{1}{2}(x-x'), \xi\right).$$

This together with (5.13) allow us to rewrite $\tilde{u}_j(x)$ as

$$\begin{aligned}
\tilde{u}_j(x) &= (2\pi)^{-n} \int e^{-\frac{\|\xi\|^2}{j}} \left\langle u_{x'}, \psi(x') e^{i(x-x')\xi - \frac{\kappa}{2}\mu_\omega(\xi)(x-x')^2} a^\kappa_{\mu_\omega}\left(\tfrac{x}{2} - \tfrac{x'}{2}, \xi\right)\right\rangle d\xi. \\
&= (2\pi)^{-n} \int e^{-\frac{\|\xi\|^2}{j}} \left\langle u_{x'}, \psi(x') e^{i(x-x')\xi - \frac{\kappa}{2}\mu(\xi)(x-x')^2} a^{\kappa/2}_\mu\left(x - x', \xi\right)\right\rangle d\xi \\
&= (2\pi)^{-n} \int e^{-\frac{\|\xi\|^2}{j}} \mathcal{F}^{\kappa/2}_{\mu_\omega}(\psi u)(x,\xi)d\xi.
\end{aligned} \tag{5.14}$$

Where in the second equality we used that $a^\kappa_{\mu_\omega}\left(\frac{x}{2}, \xi\right) = a^{\kappa/2}_{\mu_\omega}(x, \xi)$. Therefore the result follows from Lemma 5.3. □

5.3 Sufficient Condition of Theorem 1.1

In this section we will use the inversion formula presented in the previous section to prove that a certain decay of the FBI transform in all directions implies $\mathcal{E}^\omega$ local regularity.

Theorem 5.6 *Let $x_0 \in \mathbb{R}^n$, ω and σ be weight functions such that $\omega(t) = O(\sigma(t))$ (for $t \to +\infty$), and $u \in \mathcal{D}^{\sigma\prime}(\mathbb{R}^n)$. If there exist $\psi \in \mathcal{D}^\sigma(\mathbb{R}^n)$ and $C, c, \kappa, r > 0$ such* *that*

$$|\mathcal{F}^{\kappa}_{\mu_\sigma}(\psi u)(t,\xi)| \le Ce^{-c\omega(\xi)}, \quad (t,\xi) \in B(x_0,r)\times\mathbb{R}^n \tag{5.15}$$

then $u \in \mathcal{E}^{\omega}$ *in a neighborhood of* x_0.

Proof We first note that it follows from Remark 2.11 and inequality (2.12) that there exists $A_1 > 1$ such that $|\sigma((\|\xi\|^2+\delta_\sigma)^{1/2})| \le e^{-c}\|\xi\|$ and $|\mu_\sigma(\xi)| \le \|\xi\|$, for $\|\xi\| > A_1$. Moreover, using (2.7) we obtain

$$\int_{\|\xi\|>A_1} e^{-c\omega(\xi)}d\xi \le e^{c\varphi^*\left(\frac{n+1}{c}\right)} \int_{\|\xi\|>A_1} \frac{1}{\|\xi\|^{n+1}}d\xi < +\infty.$$

Moving on we want to apply Lemma 5.5 and in order to do so we write

$$\Psi(x,t,\xi,j) = (2\pi^3)^{-n/2}e^{i\xi(x-t)-\kappa\mu_\sigma(\xi)(x-t)^2}e^{-\frac{\|\xi\|^2}{j}}\mathcal{F}^{\kappa}_{\mu_\sigma}(\psi u)(t,\xi)\ [\kappa\mu_\sigma(\xi)]^{n/2},$$

and

$$\int\int \Psi(x,t,\xi,j)dtd\xi = \int_{U_1}\Psi(x,t,\xi,j)dtd\xi + \int_{U_2}\Psi(x,t,\xi,j)dtd\xi + \int_{U_3}\Psi(x,t,\xi,j)dtd\xi$$

where, $U_1 = \{(t,\xi) : \|t-x_0\| < r,\ \|\xi\| > A_1\}$, $U_2 = \{(t,\xi) : \|t-x_0\| < r,\ \|\xi\| \le A_1\}$, $U_3 = \{(t,\xi) : \|t-x_0\| \ge r,\ \xi \in \mathbb{R}^n\}$. Next we will prove that for each $\ell \in \{1,2,3\}$ there exists a function $I_\ell \in \mathcal{E}^{\omega}$ in a neighborhood U_0 of x_0 such that $\lim\limits_{j\to+\infty}\displaystyle\int_{U_\ell}\Psi(x,t,\xi,j)dtd\xi = I_\ell(x)$ in $\mathcal{E}^{\omega}(U_0)$. Therefore, using Lemma 5.5 we will obtain $u = I_1+I_2+I_3$ in a neighborhood of x_0, concluding the proof.

Observe that $\int_{\|t-x_0\|<r}\int_{\|\xi\|>A_1} e^{-c\omega(\xi)}d\xi dt < +\infty$. Thus, we can use the dominated convergence theorem to conclude that

$$\int_{U_1}\Psi(x,t,\xi,j)dtd\xi \to I_1(x) \doteq \int_{U_1}(2\pi^3)^{-n/2}e^{i\xi(x-t)-\kappa\mu_\sigma(\xi)(x-t)^2}\mathcal{F}^{\kappa}_{\mu_\sigma}(\psi u)(t,\xi)\,[\kappa\mu_\sigma(\xi)]^{n/2}dtd\xi,$$

as $j\to\infty$. Moreover, using (3.2) and (5.15) there exists $D > 0$ such that, for every $x \in \mathbb{R}^n$ and every $\alpha \in \mathbb{N}_0^n$ it holds

$$\begin{aligned}|\partial_x^\alpha I_1(x)| &= \left|\int_{U_1}(2\pi^3)^{-n/2}\partial_x^\alpha\left\{e^{i\xi(x-t)-\kappa\mu_\sigma(\xi)(x-t)^2}\right\}\mathcal{F}^{\kappa}_{\mu_\sigma}(\psi u)(t,\xi)\ [\kappa\mu_\sigma(\xi)]^{n/2}dtd\xi\right|\\ &\le \int_{U_1}(2\pi^3)^{-\frac{n}{2}}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\|\xi\|^{|\alpha-\beta|}D^{|\beta|}\beta!^{\frac{1}{2}}[\sigma((\|\xi\|^2+\delta_\sigma)^{\frac{1}{2}})]^{\frac{|\beta|}{2}}Ce^{-c\omega(\xi)}[\kappa\mu_\sigma(\xi)]^{\frac{n}{2}}dtd\xi.\end{aligned}$$

Since $|\sigma((\|\xi\|+\delta_\sigma)^{1/2})| \le e^{-c}\|\xi\|$ and $|\mu_\sigma(\xi)| \le \|\xi\|$, for $\|\xi\| > A_1$, using (2.3), (2.7) and (2.8) there exists $D_0 > 0$ such that

$$\begin{aligned} D^{|\beta|}|\beta|!^{\frac{1}{2}}[\sigma((\|\xi\|+\delta_\sigma)^{1/2})]^{\frac{|\beta|}{2}} e^{-c\omega(\xi)}|\kappa\mu_\sigma(\xi)|^{\frac{n}{2}} &\le D^{|\beta|}\kappa^{\frac{n}{2}}|\beta|!^{\frac{1}{2}} e^{-|\beta|c/2}\|\xi\|^{\frac{|\beta|+n}{2}} e^{-c\omega(\xi)} \\ &\le D^{|\beta|}\kappa^{\frac{n}{2}}(|\beta|+n)!^{\frac{1}{2}} e^{-|\beta|c/2} e^{\frac{c}{2}\varphi^*(\frac{|\beta|+n}{c})} e^{-\frac{c}{2}\omega(\xi)} \\ &\le D_0^{|\beta|+n} e^{c\varphi^*(\frac{|\beta|+n}{c})} e^{-\frac{c}{2}\omega(\xi)} \\ &\le D_0^{|\beta|+n} e^{(c/2)\varphi^*(\frac{n}{c/2})} e^{(c/2)\varphi^*(\frac{|\beta|}{c/2})} e^{-\frac{c}{2}\omega(\xi)}. \end{aligned} \tag{5.16}$$

Thus, using (2.7) there exists $C_1, D_1 > 0$ such that

$$\begin{aligned} |\partial_x^\alpha I_1(x)| &\le C_1 \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D_1^{|\beta|} \int_{\|\xi\|>A_1} \|\xi\|^{|\alpha-\beta|} e^{(c/2)\varphi^*\left(\frac{|\beta|}{c/2}\right)} e^{-\frac{c}{2}\omega(\xi)} d\xi \\ &\le C_1 \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D_1^{|\beta|} \int_{\|\xi\|>A_1} e^{\frac{c}{4}\varphi^*\left(\frac{4}{c}|\alpha-\beta|\right)} e^{\frac{c}{2}\varphi^*\left(\frac{2}{c}|\beta|\right)} e^{-\frac{c}{4}\omega(\xi)} d\xi, \\ &\text{for every } x \in \mathbb{R}^n \text{ and every } \alpha \in \mathbb{N}_0^n. \end{aligned}$$

Since $t \mapsto \frac{\varphi^*(t)}{t}$ is increasing, using (2.3), the last inequality can be further estimated as follows

$$\begin{aligned} |\partial_x^\alpha I_1(x)| \le & C_1 \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \int_{\|\xi\|>A_1} D_1^{|\beta|} e^{\frac{c}{4}\varphi^*\left(\frac{4}{c}|\alpha-\beta|\right)} e^{\frac{c}{4}\varphi^*\left(\frac{4}{c}|\beta|\right)} e^{-\frac{c}{4}\omega(\xi)} d\xi \\ \le & C_1(D_1+1)^{n|\alpha|} e^{\frac{c}{4}\varphi^*\left(\frac{4}{c}|\alpha|\right)} \int_{\|\xi\|>A_1} e^{-\frac{c}{4}\omega(\xi)} d\xi, \\ & \text{for every } x \in \mathbb{R}^n \text{ and every } \alpha \in \mathbb{N}_0^n. \end{aligned}$$

Now we invoke (2.5) to obtain $D_2 > 0$ and $\lambda_1 > 0$ such that $|\partial_x^\alpha I_1(x)| \le D_2 e^{\frac{1}{\lambda_1}\varphi^*(\lambda_1|\alpha|)}$, for each $(x,\alpha) \in \mathbb{R}^n \times \mathbb{N}_0^n$. Thus $I_1 \in \mathcal{E}^\omega(\mathbb{R}^n)$. Moreover, reasoning analogously we see that,

$$\left|\partial_x^\alpha \left\{\int_{U_1} \Psi(x,t,\xi,j)dtd\xi - I_1(x)\right\}\right| \le C_1(D_1+1)^{n|\alpha|} e^{\frac{c}{4}\varphi^*\left(\frac{4}{c}|\alpha|\right)} \int_{\|\xi\|>A_1} e^{-\frac{c}{4}\omega(\xi)} \left|e^{-\frac{\|\xi\|^2}{j}} - 1\right| d\xi.$$

Therefore,

$$\int_{U_1} \Psi(\cdot,t,\xi,j)dtd\xi \to I_1, \quad \text{as } j \to +\infty, \quad \text{in } \mathcal{E}^\omega(\mathbb{R}^n). \tag{5.17}$$

Next we will study U_2. Since U_2 is bounded for each $x \in \mathbb{R}^n$ it follows that

$$\int_{U_2} \Psi(x,t,\xi,j)dtd\xi \to I_2(x) \doteq \int_{U_2} (2\pi^3)^{-\frac{n}{2}} e^{i\xi(x-t)-\kappa\mu_\sigma(\xi)(x-t)^2} \mathcal{F}_{\mu_\sigma}^\kappa(\psi u)(t,\xi)\ [\kappa\mu_\sigma(\xi)]^{\frac{n}{2}} dtd\xi.$$

Also using (3.2) and (5.15) there exist constants $C, D > 0$ such that for every $\alpha \in \mathbb{N}_0^n$ we have

$$
\begin{aligned}
|\partial_x^\alpha I_2(x)| &= \left| \int_{U_2} (2\pi^3)^{-n/2} \partial_x^\alpha \left\{ e^{i\xi(x-t)-\kappa\mu_\sigma(\xi)(x-t)^2} \right\} \mathcal{F}_{\mu_\sigma}^\kappa(\psi u)(t,\xi)\ [\kappa\mu_\sigma(\xi)]^{n/2} dt d\xi \right| \\
&\le \int_{U_2} (2\pi^3)^{-\frac{n}{2}} C e^{-c\omega(\xi)} [\mu_\sigma(\xi)]^{\frac{n}{2}} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \|\xi\|^{|\alpha-\beta|} D^{|\beta|} \beta!^{\frac{1}{2}} [\sigma((\|\xi\|^2+\delta_\sigma)^{\frac{1}{2}})]^{\frac{|\beta|}{2}} dt d\xi \\
&\le \int_{U_2} (2\pi^3)^{-n/2} C [\sup_{\|\xi\|\le A_1} |\mu_\sigma(\xi)|]^{n/2} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} A_1^{|\alpha-\beta|} D^{|\beta|} \beta!^{1/2} [\sigma((A_1^2+\delta_\sigma)^{1/2})]^{|\beta|/2} dt d\xi.
\end{aligned}
$$

Hence, there exist $D_3, D_4 > 0$ such that $\|\partial_x^\alpha I_2\| \le D_3 D_4^{|\alpha|} |\alpha|!$, for every $\alpha \in \mathbb{N}_0^n$. Therefore, using Remark 2.12 it follows that $I_2 \in \mathcal{E}^\omega(\mathbb{R}^n)$. Moreover, one can see that

$$
\left| \partial_x^\alpha \left\{ \int_{U_2} \Psi(x,t,\xi,j) dt d\xi - I_2(x) \right\} \right| \le D_3 D_4^{|\alpha|} |\alpha|! \int_{\|\xi\|\le A_1} \left| e^{-\frac{\|\xi\|^2}{j}} - 1 \right| d\xi.
$$

Thus,

$$
\int_{U_2} \Psi(\cdot,t,\xi,j) dt d\xi \to I_2, \quad \text{as} j \to +\infty \quad \text{in } \mathcal{E}^\omega(\mathbb{R}^n). \tag{5.18}
$$

Next, in order to study the integral in the region U_3, observe that

$$
\|x-t\| \ge \|t-x_0\| - \|x-x_0\| \ge r - \frac{r}{2} = \frac{r}{2}
$$

for $\|t-x_0\| \ge r$ and $\|x-x_0\| < \frac{r}{2}$.

Thus, using (2.11), (2.12) and Lemma 3.5 there exists $D > 0$ such that,

$$
\begin{aligned}
\left| e^{i\xi(x-t)-\kappa\mu_\sigma(\xi)(x-t)^2} e^{-\frac{\|\xi\|^2}{j}} \mathcal{F}_{\mu_\sigma}^\kappa(\psi u)(t,\xi)\ [\kappa\mu_\sigma(\xi)]^{\frac{n}{2}} \right| &\le D e^{-\frac{\kappa}{2}\sigma(\delta^{\frac{1}{2}})(x-t)^2} e^{-\frac{\kappa}{2}\tilde{\sigma}(\xi)\frac{r^2}{4}} e^{\frac{\kappa r^2}{16}\tilde{\sigma}(\xi)} |\tilde{\sigma}(\xi)|^{\frac{n}{2}} \\
&= D \left[\frac{(\kappa r^2 \tilde{\sigma}(\xi))^n}{16^n n!} \frac{n! 16^n}{(\kappa r^2)^n} \right]^{\frac{1}{2}} e^{-\frac{\kappa r^2}{16}\tilde{\sigma}(\xi)} e^{-\frac{\kappa}{2}\sigma(\delta^{\frac{1}{2}})(x-t)^2} \\
&\le \frac{n!^{\frac{1}{2}} 16^{\frac{n}{2}}}{\kappa^{\frac{n}{2}} r^n} D e^{-\kappa\tilde{\sigma}(\xi)\frac{r^2}{32}} e^{-\kappa\sigma(\delta^{\frac{1}{2}})(x-t)^2}
\end{aligned}
$$

for $\tilde{\sigma}(\xi) = \sigma((\delta_\sigma + \|\xi\|^2)^{1/2})$, $(t,\xi) \in U_3$ and $\|x-x_0\| < \frac{r}{2}$, where it was used that $\left[\frac{(\kappa r^2 \tilde{\sigma}(\xi))^n}{16^n n!} \right]^{\frac{1}{2}} \le \left[e^{\frac{\kappa r^2 \tilde{\sigma}(\xi)}{16}} \right]^{\frac{1}{2}}$. Hence, using the dominated convergence theorem (recalling that $\int e^{-\kappa\frac{r^2}{16}\tilde{\sigma}(\xi)} d\xi < +\infty$ and $\int e^{-\kappa\sigma(\delta^{1/2})(x-t)^2} dt < +\infty$),

$$\int_{U_3} \Psi(x,t,\xi,j)dtd\xi \to I_3(x)$$
$$= \int_{U_3} (2\pi^3)^{-n/2} e^{i\xi(x-t)-\kappa\mu_\sigma(\xi)(x-t)^2} \mathcal{F}^\kappa_{\mu_\sigma}(\psi u)(t,\xi)\ \ [\mu_\sigma(\xi)]^{n/2} dtd\xi,$$
$$\text{whenever } \|x-x_0\| < \frac{r}{2}. \tag{5.19}$$

Moreover, using the Leibniz rule, Remark 3.3, Lemma 3.5, (2.3), (2.7) and (2.8), for each $\theta, \lambda > 0$ to be chosen there exist $D_3, D_4, D_5 > 0$ such that

$$\left|\partial^\alpha_x \left\{ e^{i\xi(x-t)-\kappa\mu_\sigma(\xi)(x-t)^2} \mathcal{F}^\kappa_{\mu_\sigma}(\psi u)(t,\xi)\ \ [\kappa\mu_\sigma(\xi)]^{\frac{n}{2}} \right\}\right|$$
$$\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \|\xi\|^{|\alpha-\beta|} D_3^{|\beta|} \beta! e^{-\frac{\kappa}{2}\sigma((\|\xi\|^2+\delta_\sigma)^{1/2})\|x-t\|^2} e^{\theta\sigma((\|\xi\|^2+\delta_\sigma)^{1/2})} [\mu_\sigma(\xi)]^{\frac{n}{2}}$$
$$\leq \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} e^{\frac{1}{\lambda}\varphi^*(\lambda|\alpha|)} e^{\frac{1}{\lambda}\omega(\xi)} D_4^{|\beta|} [\mu_\sigma(\xi)]^{\frac{n}{2}} e^{-\frac{\kappa r^2}{16}\sigma((\|\xi\|^2+\delta_\sigma)^{1/2})}$$
$$e^{-\frac{\kappa}{16}\sigma(\sqrt{\delta_\sigma})\|x-t\|^2} e^{\theta\sigma((\|\xi\|^2+\delta_\sigma)^{1/2})}$$
$$\leq D_5^{|\alpha|} e^{\frac{\varphi^*(\lambda|\alpha|)}{\lambda}} e^{\frac{\omega(\xi)}{\lambda}} e^{-\frac{\kappa r^2}{32}\sigma((\|\xi\|^2+\delta_\sigma)^{1/2})} e^{-\kappa_\sigma\|x-t\|^2} [\mu_\sigma(\xi)]^{\frac{n}{2}},$$
$$\text{for } \|t-x_0\| \geq r \text{ and } \|x-x_0\| < \frac{r}{2}$$

where in the last inequality we fixed $\theta \doteq \frac{\kappa r^2}{32}$ and $\kappa_\sigma \doteq \frac{\kappa}{16}\sigma(\sqrt{\delta})$. Moving on using (2.5) we see that there exist $D_6 > 0$ and $\lambda_* > 0$ such that the last inequality can be further estimated as

$$\left|\partial^\alpha_x \left\{ e^{i\xi(x-t)-\kappa\mu_\sigma(\xi)(x-t)^2} \mathcal{F}^\kappa_{\mu_\sigma}(\psi u)(t,\xi)\ \ [\kappa\mu_\sigma(\xi)]^{\frac{n}{2}} \right\}\right|$$
$$\leq D_6\, e^{\frac{\varphi^*(\lambda_*|\alpha|)}{\lambda_*}} e^{\frac{\omega(\xi)}{\lambda}} e^{-\frac{\kappa r^2}{32}\sigma((\|\xi\|^2+\delta)^{1/2})} e^{-\kappa_\sigma\|x-t\|^2} |\mu_\sigma(\xi)|^{\frac{n}{2}}$$
$$\text{for } \|t-x_0\| \geq r \text{ and } \|x-x_0\| < \frac{r}{2}. \tag{5.20}$$

Moreover, from the fact that

$$\int_{\|t-x_0\|>r} e^{-\kappa_\sigma\|x-t\|^2} dt \leq \int_{\mathbb{R}^n} e^{-\kappa_\sigma y^2} dy < +\infty$$

and considering $A_3, c_1 > 0$ so that $\omega(\xi) \leq c_1\sigma(\xi)$, for $\|\xi\| > A_3$, it follows that

$$\int e^{\frac{\omega(\xi)}{\lambda}} e^{-\frac{\kappa r^2 n}{32}\sigma((\|\xi\|^2+\delta)^{1/2})} |\mu_\sigma(\xi)|^{\frac{n}{2}} d\xi \leq \int_{\|\xi\|\leq A_3} e^{\frac{\omega(\xi)}{\lambda}} e^{-\frac{\kappa r^2}{32}\sigma((\|\xi\|^2+\delta)^{1/2})} |\mu_\sigma(\xi)|^{\frac{n}{2}} d\xi$$
$$+ \int_{\|\xi\|>A_3} e^{\frac{c_1\sigma((\|\xi\|^2+\delta)^{1/2})}{\lambda}} e^{-\frac{\kappa r^2}{32}\sigma((\|\xi\|^2+\delta)^{1/2})} |\mu_\sigma(\xi)|^{\frac{n}{2}} d\xi.$$

Note that, the first integral is finite and choosing $\lambda \doteq \frac{64\,c_1}{\kappa r^2}$ it follows from (2.12), (2.7) and Remark 2.11, that

$$\int_{\|\xi\|>A_3} e^{\frac{c_1\sigma((\|\xi\|^2+\delta)^{1/2})}{\lambda}} e^{-\frac{\kappa r^2}{32}\sigma((\|\xi\|^2+\delta)^{1/2})} |\mu_\sigma(\xi)|^{\frac{n}{2}} d\xi$$
$$= \int_{\|\xi\|>A_3} e^{-\frac{\kappa r^2}{64}\sigma((\|\xi\|^2+\delta)^{1/2})} |\mu_\sigma(\xi)|^{\frac{n}{2}} d\xi < +\infty. \tag{5.21}$$

Thus, summing up (5.19), (5.20) and (5.21) we see that there exists $D_7, \lambda_* > 0$ such that

$$|\partial_x^\alpha I_3(x)| \leq D_7 e^{\frac{1}{\lambda_*}\varphi^*(\lambda_*|\alpha|)}, \quad \forall (x,\alpha) \in B\left(x_0, \tfrac{r}{2}\right) \times \mathbb{N}_0^n.$$

This shows that $I_3 \in \mathcal{E}^\omega$. In addition to that and similarly as before it follows that

$$\left|\partial^\alpha \left\{\int_{U_3} \Psi(x,t,\xi,j)dtd\xi - I_3(x)\right\}\right| \leq D_7 e^{\frac{1}{\lambda_*}\varphi^*(\lambda_*|\alpha|)} \int_{\|\xi\|\leq A_1} \left|e^{-\frac{\|\xi\|^2}{j}} - 1\right| d\xi.$$

Hence

$$\int_{U_3} \Psi(\cdot,t,\xi,j)dtd\xi \to I_3, \quad \text{as } j \to +\infty, \quad \text{in } \mathcal{E}^\omega(B(x_0,\tfrac{r}{2})). \tag{5.22}$$

Therefore, using (5.17), (5.18), (5.22) and Lemma 5.5 it follows that $u = I_1 + I_2 + I_3$ in $B(x_0, \frac{r}{2})$, which concludes the proof. □

6 A Characterization of Ultradifferentiable Iterates of Constant Coefficients Operators

Let $P(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$ be a polynomial function of degree m. This section is dedicated to the characterization of the space $\mathcal{E}^\omega(\Omega; P)$ (see Definition 1.2) using a FBI transform.

Remark 6.1 From now on we will consider $0 < \rho \leq 1$ such that (1.8) is satisfied and denote $\sigma(t) = \sigma_{\omega,\rho}(t) = \omega(t^\rho)$. It is important to note that, if ω is a weight function and $0 < \rho \leq 1$ then σ is a weight function.

Theorem 6.2 *Let $x_0 \in \mathbb{R}^n$, ω be a weight function and $P(D)$ be a constant coefficient hypoelliptic linear operator of order m together with its hypoelliptic index ρ (satisfying* (1.8)*). Let $\sigma(t) = \omega(t^\rho)$ and $u \in \mathcal{D}^{\sigma\prime}(\mathbb{R}^n) \subset \mathcal{D}^{\omega\prime}(\mathbb{R}^n)$. The following conditions are equivalent:*

1. *There exists a neighborhood Ω of x_0 such that $u \in \mathcal{E}^{\omega}(\Omega; P)$.*
2. *There exist $\phi \in \mathcal{D}^{\omega}(\mathbb{R}^n)$ (such that $\phi \equiv 1$ in a neighborhood of x_0), $C, \lambda, c > 0$ and a neighborhood V of x_0 such that*

$$|\mathcal{F}_{\mu_\sigma}\left(\phi\,[P(D)]^N(u)\right)(x,\xi)| \leq Ce^{\frac{1}{\lambda}\varphi^*(Nm\lambda)}e^{-c\omega(\|\xi\|^\rho)}, \quad (x,\xi,N) \in V \times \mathbb{R}^n \times \mathbb{N}_0, \tag{6.1}$$

where we are denoting $\varphi^*(x) = \varphi^*_\omega(x) \doteq \sup\{xy - \varphi(y) : y \geq 0\}$, for each $x > 0$ and μ_σ is the function obtained from Proposition 2.20 when applied to σ instead of ω.

Proof (1) $\Rightarrow$ (2): Let $R \in (0, 1/3)$ be such that $B(x_0, 3R) \subset \Omega$. Set $0 < r < R$ and $\phi \in \mathcal{D}^{\omega}(B(x_0, R))$ such that $\phi \equiv 1$ in $B(x_0, r)$. Also, considering $\psi \in \mathcal{D}^{\omega}(B(x_0, 2R))$ such that $\psi \equiv 1$ in $B(x_0, R)$, it follows that

$$\begin{aligned}
\mathcal{F}_{\mu_\sigma}\left(\phi[P(D)]^N(u)\right)(x,\xi) &= \int \phi(y)[P(D)]^N(u)(y)e^{i(x-y)\xi-\mu_\sigma(\xi)(x-y)^2}a_{\mu_\sigma}(x-y,\xi)dy\\
&= \int \phi(y)[P(D)]^N(u)(y)\,\psi(y)e^{i(x-y)\xi-\mu_\sigma(\xi)(x-y)^2}a_{\mu_\sigma}(x-y,\xi)dy\\
&= \int [\phi(y)-1][P(D)]^N(u)(y)\,\psi(y)e^{i(x-y)\xi-\mu_\sigma(\xi)(x-y)^2}a_{\mu_\sigma}(x-y,\xi)dy\\
&\quad + \int [P(D)]^N u(y)\,\psi(y)e^{i(x-y)\xi-\mu_\sigma(\xi)(x-y)^2}a_{\mu_\sigma}(x-y,\xi)dy\\
&=: I_1(x,\xi) + I_2(x,\xi),
\end{aligned}$$

where $x, \xi \in \mathbb{R}^n$,

$$I_1(x,\xi) \doteq \int [\phi(y)-1][P(D)]^N(u)(y)\,\psi(y)e^{i(x-y)\xi-\mu_\sigma(\xi)(x-y)^2}a_{\mu_\sigma}(x-y,\xi)dy \tag{6.2}$$

and

$$I_2(x,\xi) \doteq \int [P(D)]^N u(y)\,\psi(y)e^{i(x-y)\xi-\mu_\sigma(\xi)(x-y)^2}a_{\mu_\sigma}(x-y,\xi)dy. \tag{6.3}$$

Next we will study I_1 and I_2. First, denoting $K = \operatorname{supp}\psi$ and using Hölder inequality,

$$|I_1(x,\xi)| \leq \|[P(D)]^N u\|_{L^2(K)}\left[\int_{r\leq|y-x_0|\leq 2R}\left|[\phi(y)-1]\,\psi(y)e^{i(x-y)\xi-\mu_\sigma(\xi)(x-y)^2}a_{\mu_\sigma}(x-y,\xi)\right|^2 dy\right]^{\frac{1}{2}}$$

Since $u \in \mathcal{E}^{\omega}(\Omega; P)$ and $K \subset \Omega$, recalling Definition 1.2, there exists $C_1, C > 0$ and $\lambda > 0$ such that

$$|I_1(x,\xi)| \leq C_1 e^{\frac{1}{\lambda}\varphi^*(\lambda m N)} \left[\int_{r\leq|y-x_0|\leq 2R} \left|[\phi(y)-1]\,\psi(y)e^{-\mu_\sigma(\xi)(x-y)^2} a_{\mu_\sigma}(x-y,\xi)\right|^2 dy\right]^{1/2}$$
$$\leq C e^{\frac{1}{\lambda}\varphi^*(\lambda m N)} \left[\sup_{r\leq|y-x_0|\leq 2R} e^{-2\sigma(\|\xi\|)(x-y)^2}\right]^{1/2} \leq C e^{\frac{1}{\lambda}\varphi^*(\lambda m N)} e^{-\omega(\|\xi\|^\rho)\frac{r^2}{4}}$$

for each x satisfying $|x-x_0| \leq \frac{r}{2}$ and $\xi \in \mathbb{R}^n$; where we also use Proposition 2.20.

Next we will consider the term I_2 given by (6.3). Let ℓ be an arbitrary positive integer. Using (1.8) we obtain for $\|\xi\| \geq K$, with K the constant appearing in (1.8), and $u_n := [P(D)]^N u$

$$\begin{aligned}|I_2(x,\xi)| &= \left|\int e^{i(x-y)\xi} u_N(y)\,\psi(y)e^{-\mu_\sigma(\xi)(x-y)^2} a_{\mu_\sigma}(x-y,\xi)dy\right| \\ &\leq \frac{1}{(C\|\xi\|^{\rho m})^\ell}\left|\int [P(\xi)]^\ell e^{i(x-y)\xi} u_N(y)\psi(y)e^{-\mu_\sigma(\xi)(x-y)^2} a_{\mu_\sigma}(x-y,\xi)dy\right| \\ &= \frac{1}{(C\|\xi\|^{\rho m})^\ell}\left|\int [P(-D_y)]^\ell \left\{e^{i(x-y)\xi}\right\} u_N(y)\psi(y)e^{-\mu_\sigma(\xi)(x-y)^2} a_{\mu_\sigma}(x-y,\xi)dy\right| \\ &= \frac{1}{(C\|\xi\|^{\rho m})^\ell}\left|\int e^{i(x-y)\xi}[P(D_y)]^\ell \left\{u_N(y)\psi(y)e^{-\mu_\sigma(\xi)(x-y)^2} a_{\mu_\sigma}(x-y,\xi)\right\} dy\right|. \end{aligned} \tag{6.4}$$

In order to study the above integral we first recall that

$$P(D)(fg) = \sum_{|\beta|\leq m} \frac{1}{\beta!} P^{(\beta)} g \times D^\beta f, \quad \forall f, g \in C^\infty,$$

where we denote $P^{(\beta)} g \doteq (\partial_\xi^\beta P)(D) g$ for each $\beta \in \mathbb{N}_0^n$. Using the linearity of $P(D)$ it follows that

$$P(D)\circ P(D)(fg) = \sum_{|\beta|\leq m} \frac{1}{\beta!} P(D)\left\{P^{(\beta)} g \times D^\beta f\right\}, \quad \forall f, g \in C^\infty.$$

Hence,

$$[P(D)]^\ell(fg) = \sum_{|\beta_1|\leq m} \frac{1}{\beta_1!}\cdots \sum_{|\beta_\ell|\leq m} \frac{1}{\beta_\ell!}[P^{(\beta_1)}\circ\cdots\circ P^{(\beta_\ell)}]g$$
$$\times D^{\beta_1+\cdots+\beta_\ell} f, \quad \forall f, g \in C^\infty.$$

Thus, inserting the last expression in (6.4) it follows that,

$$\begin{aligned} |I_2(x,\xi)| &\le \frac{1}{(C\|\xi\|^{\rho m})^{\ell}} \sum_{|\beta_1|\le m} \frac{1}{\beta_1!} \cdots \sum_{|\beta_\ell|\le m} \frac{1}{\beta_\ell!} \int_{B(x_0,2R)} \left| [P^{(\beta_1)} \circ \cdots \circ P^{(\beta_\ell)} u_N(y) \right| \\ &\quad \times \left| \partial_y^{\beta_1+\cdots+\beta_\ell} \left\{ \psi(y) e^{-\mu_\sigma(\xi)(x-y)^2} a_{\mu_\sigma}(x-y,\xi) \right\} \right| dy \\ &\le \frac{1}{(C\|\xi\|^{\rho m})^{\ell}} \sum_{|\beta_1|\le m} \frac{1}{\beta_1!} \cdots \sum_{|\beta_\ell|\le m} \frac{1}{\beta_\ell!} \|[P^{(\beta_1)} \circ \cdots \circ P^{(\beta_\ell)}]u_N\|_{L^2(B(x_0,2R))} \\ &\quad \times \left\| \partial_y^{\beta_1+\cdots+\beta_\ell} \left\{ \psi(\cdot) e^{-\mu_\sigma(\xi)(x-\cdot)^2} a_{\mu_\sigma}(x-\cdot,\xi) \right\} \right\|_{L^2(B(x_0,2R))} \end{aligned} \tag{6.5}$$

where $\|\xi\| > K$. Observe that, since σ is increasing it follows that $\sigma((\|\xi\|^2+\delta_\sigma)^{1/2}) \le \sigma(\|\xi\|+\delta_\sigma^{1/2})$, for each $\xi \in \mathbb{R}^n$. Thus, using Remark 3.3 for σ instead of ω, Leibniz's rule and the fact that $\psi \in \mathcal{D}^{\omega}$ we obtain that for each $\theta > 0$ there exist $\lambda_1, D > 0$ such that

$$\left| \partial_y^{\eta} \left\{ \psi(y) e^{-\mu_\sigma(\xi)(x-y)^2} a_{\mu_\sigma}(x-y,\xi) \right\} \right| \le \sum_{\beta\le\eta} \binom{\eta}{\beta} e^{\frac{1}{\lambda_1}\varphi^*(\lambda_1|\eta-\beta|)} D^{|\beta|} \beta! e^{\theta\sigma(\|\xi\|+\delta_\sigma^{1/2})},$$

for each $\eta \in \mathbb{N}_0^n$. Thus, using (2.3), (2.5) and (2.8), there exist $C_2, \lambda_2 > 0$ such that

$$\left| \partial_y^{\eta} \left\{ \psi(y) e^{-\mu_\sigma(\xi)(x-y)^2} a_{\mu_\sigma}(x-y,\xi) \right\} \right| \le C_2 e^{\frac{1}{\lambda_2}\varphi^*(\lambda_2|\eta|)} e^{\theta\sigma(\|\xi\|+\delta_\sigma^{1/2})}.$$

Hence, using (2.3)

$$\left\| \partial_y^{\beta_1+\cdots+\beta_\ell} \left\{ \psi(\cdot) e^{-\mu_\sigma(\xi)(x-\cdot)^2} a_{\mu_\sigma}(x-\cdot,\xi) \right\} \right\|_{L^2(x_0,2R)} \le C_2 e^{\sum_{j=1}^{\ell} \frac{1}{2^j\lambda_2}\varphi^*(2^j\lambda_2|\beta_j|)} e^{\theta\sigma(\|\xi\|+\delta_\sigma^{1/2})}. \tag{6.6}$$

Moreover, using Corollary 6.7 there exists $C > 1$ such that,

$$\begin{aligned} &\|[P^{(\beta_1)} \circ \cdots \circ P^{(\beta_\ell)}]u_N\|_{L^2(B(x_0,2R))} \\ &\le C\epsilon_1^{|\beta_1|} \sup_{j\in\{0,1\}} \left\{ \|[P^j \circ P^{(\beta_2)} \circ \ldots \circ P^{(\beta_\ell)}]u_N\|_{L^2(B(x_0,2R+\epsilon_1))} \right\} \\ &\le C\epsilon_1^{|\beta_1|} \sup_{j\in\{0,1\}} \left\{ \|[P^{(\beta_2)} \circ \cdots \circ P^{(\beta_\ell)} \circ P^j]u_N\|_{L^2(B(x_0,2R+\epsilon_1))} \right\} \\ &\le C^{\ell}\epsilon_1^{|\beta_1|} \ldots \epsilon_\ell^{|\beta_\ell|} \sup_{j_1,\ldots j_\ell\in\{0,1\}} \|[P^{j_\ell} \circ \cdots \circ P^{j_1}]u_N\|_{L^2(B(x_0,2R+\epsilon_1+\cdots+\epsilon_\ell))} \\ &\le C^{\ell}\epsilon_1^{|\beta_1|} \ldots \epsilon_\ell^{|\beta_\ell|} \sup_{j_1,\ldots j_\ell\in\{0,1\}} \|[P(D)]^{j_\ell+\cdots+j_1+N}u\|_{L^2(B(x_0,2R+\epsilon_1+\cdots+\epsilon_\ell))} \end{aligned}$$

for $\epsilon_j = \left[\frac{\beta_j!}{e^{[1/(2^j\lambda_2)]\varphi^*(2^j\lambda_2|\beta_j|)}}\right]^{1/|\beta_j|} \frac{R}{\ell e^{1/(2^j\lambda_2)} C_\omega^{1/(2^j\lambda_2|\beta_j|)}}$, where $j \in \{1, \ldots, \ell\}$ and $C_\omega > 1$ is such that $e^{\omega(t)} \le C_\omega e^t$ (for each $t \ge 0$). Observe that using (2.7) it follows

$$\epsilon_j \le \left[e^{\frac{1}{2^j\lambda_2}\omega(|\beta_j|)}\right]^{\frac{1}{|\beta_j|}} \frac{R}{\ell e^{\frac{1}{2^j\lambda_2}} C_\omega^{\frac{1}{2^j\lambda_2|\beta_j|}}} \le \left[C_\omega^{\frac{1}{2^j\lambda_2}} e^{\frac{1}{2^j\lambda_2}|\beta_j|}\right]^{\frac{1}{|\beta_j|}} \frac{R}{\ell e^{\frac{1}{2^j\lambda_2}} C_\omega^{\frac{1}{2^j\lambda_2|\beta_j|}}} = \frac{R}{\ell},$$

where $j \in \{1, \ldots, \ell\}$. Thus, $\frac{R}{\ell e^{1/(2^j\lambda_2)} C_\omega^{1/(2^j\lambda_2|\beta_j|)}} < 1$ (for $j \in \{1, \ldots, \ell\}$). Moreover, since $B(x_0, 3R) \subset \Omega$ and $u \in \mathcal{E}^\omega(\Omega; P)$ there exist $C_1, \lambda > 0$ such that

$$\begin{aligned} \|[P^{(\beta_1)} \circ \cdots \circ P^{(\beta_\ell)}]u_N\|_{L^2(B(x_0,2R))} &\le C^\ell \epsilon_1^{|\beta_1|} \cdots \epsilon_\ell^{|\beta_\ell|} \sup_{j_1,\ldots j_\ell \in \{0,1\}} \|[P(D)]^{j_\ell+\cdots+j_1+N} u\|_{L^2(B(x_0,3R))} \\ &\le C^\ell \frac{\beta_1!}{e^{[1/(2^1\lambda_2)]\varphi^*(2^1\lambda_2|\beta_1|)}} \cdots \frac{\beta_\ell!}{e^{[1/(2^\ell\lambda_2)]\varphi^*(2^\ell\lambda_2|\beta_\ell|)}} C_1 e^{\frac{1}{\lambda}\varphi^*(\lambda m(\ell+N))}. \end{aligned} \tag{6.7}$$

Hence, putting (6.5), (6.6) and (6.7) together we obtain,

$$\begin{aligned} |I_2(x,\xi)| \le{}& \frac{1}{(C\|\xi\|^{\rho m})^\ell} \sum_{|\beta_1|\le m} \frac{1}{\beta_1!} \cdots \sum_{|\beta_\ell|\le m} \frac{1}{\beta_\ell!} C_1 C^\ell \frac{\beta_1!}{e^{[1/(2^1\lambda_2)]\varphi^*(2^1\lambda_2|\beta_1|)}} \cdots \frac{\beta_\ell!}{e^{[1/(2^\ell\lambda_2)]\varphi^*(2^\ell\lambda_2|\beta_\ell|)}} \\ &\times e^{\frac{1}{\lambda}\varphi^*(\lambda m(\ell+N))} C_2 e^{\sum_{j=1}^\ell \frac{1}{2^j\lambda_2}\varphi^*(2^j\lambda_2|\beta_j|)} e^{\theta\sigma(\|\xi\|+\delta_\sigma^{1/2})} \\ ={}& \frac{1}{\|\xi\|^{\rho m\ell}} \sum_{|\beta_1|\le m} \cdots \sum_{|\beta_\ell|\le m} C_1 e^{\frac{1}{\lambda}\varphi^*(\lambda m(\ell+N))} C_2 e^{\theta\sigma(\|\xi\|+\delta_\sigma^{1/2})}. \end{aligned} \tag{6.8}$$

Moreover, using (2.3) and (2.5) we see that denoting $C_4 = C_1 C_2$ and $C_5 = \sum_{|\beta|\le m} 1$ there exist positive constants $C_6, \lambda_3, \lambda_4$ such that,

$$\begin{aligned} |I_2(x,\xi)| \le{}& \frac{1}{\|\xi\|^{\rho m\ell}} C_4 C_5^\ell e^{\frac{1}{\lambda}\varphi^*(\lambda m(\ell+N))} e^{\theta\sigma(\|\xi\|+\delta_\sigma^{1/2})} \le \frac{e^{\frac{1}{\lambda_3}\varphi^*(\lambda_3 m\ell)}}{\|\xi\|^{\rho m\ell}} C_6 e^{\frac{1}{\lambda_4}\varphi^*(\lambda_4 mN)} e^{\theta\sigma(\|\xi\|+\delta_\sigma^{1/2})} \\ ={}& \left(\frac{e^{\frac{1}{m\lambda_3}\varphi^*(m\lambda_3\ell)}}{\|\xi\|^{\rho\ell}}\right)^m C_6 e^{\frac{1}{\lambda_4}\varphi^*(\lambda_4 mN)} e^{\theta\sigma(\|\xi\|+\delta_\sigma^{1/2})}. \end{aligned} \tag{6.9}$$

Since (6.9) holds true for every $\ell \in \mathbb{N}$ one can take the infimum in ℓ and use [17, Lemma 1.4] to obtain,

$$|I_2(x,\xi)| \le \left(e^{\log\|\xi\|^\rho} e^{-\frac{1}{m\lambda_3}\omega(\|\xi\|^\rho)}\right)^m C_6 e^{\frac{1}{\lambda_4}\varphi^*(\lambda_4 mN)} e^{\theta\sigma(\|\xi\|+\delta_\sigma^{1/2})}.$$

Now one can use (γ) to get $A > 0$ such that $\log\|\xi\|^\rho \le \frac{\omega(\|\xi\|^\rho)}{2m\lambda_3}$, when $\|\xi\| > A$. Thus $m\log(\|\xi\|^\rho) - \frac{1}{\lambda_3}\omega(\|\xi\|^\rho) \le -\frac{1}{2\lambda_3}\omega(\|\xi\|^\rho)$, when $\|\xi\| > A$. This, together with the fact that $0 < \rho \le 1$, that ω is a increasing and (α)implies that $\sigma(\|\xi\| +$

$\delta_\sigma^{1/2}) = \omega([\|\xi\| + \delta_\sigma^{1/2}]^\rho) \leq \omega(\|\xi\|^\rho + \delta_\sigma^{\rho/2}) \leq \omega(e\|\xi\|^\rho) \leq L[\omega(\|\xi\|^\rho) + 1]$, when $\|\xi\| \geq \frac{\delta_\sigma^{1/2}}{(e-1)^{1/\rho}}$. Hence, choosing $\theta = \frac{1}{4\lambda_3 L}$ it follows that

$$|I_2(x,\xi)| \leq e^{-\frac{1}{2\lambda_3}\omega(\|\xi\|^\rho)} C_6 e^{\frac{1}{\lambda_4}\varphi^*(\lambda_4 mN)} e^{\theta L[\omega(\|\xi\|^\rho)+1]} = C_6 e^{\theta L} e^{\frac{1}{\lambda_4}\varphi^*(\lambda_4 mN)} e^{\theta L \omega(\|\xi\|^\rho)} e^{-\frac{1}{2\lambda_3}\omega(\|\xi\|^\rho)}$$
$$\leq C_6 e^{\theta L} e^{\frac{1}{\lambda_4}\varphi^*(\lambda_4 mN)} e^{-\frac{1}{4\lambda_3}\omega(\|\xi\|^\rho)},$$

for $\|\xi\| \geq \max\left\{A;\ \frac{\delta}{(e-1)^{1/\rho}}\right\}$. Therefore, there exists $C_7 > 0$ such that

$$|I_2(x,\xi)| \leq C_7 e^{-a\omega(\|\xi\|^\rho)} e^{\frac{1}{\lambda_4}\varphi^*(\lambda_4 mN)}, \quad \forall \xi \in \mathbb{R}^n.$$

(2) $\Rightarrow$ (1) : Consider $0 < r < R, \phi \in \mathcal{D}^\omega(B(x_0, R))$ (such that $\phi \equiv 1$ in $B(x_0, r)$) and $C, \lambda, c > 0$ such that

$$|\mathcal{F}_{\mu_\sigma}\left(\phi[P(D)]^N(u)\right)(x,\xi)| \leq Ce^{\frac{1}{\lambda}\varphi^*(Nm\lambda)} e^{-c\omega(\|\xi\|^\rho)}, \quad (x,\xi,N) \in B(x_0,r) \times \mathbb{R}^n \times \mathbb{N}_0. \tag{6.10}$$

First, observe that,

from Theorem 1.1 we can conclude that there exists $0 < \delta < r$ such that $u \in \mathcal{E}^\sigma(B(x_0,\delta))$. In order to prove that $u \in \mathcal{E}^\omega(B(x_0,\delta); P)$ we will consider a compact set $K \subset B(x_0,\delta)$. Since $\phi \equiv 1$ in K it follows that $[P(D)]^N(u)(y) = \phi(y)[P(D)]^N(u)(y)$, for each $y \in K$.

Moreover, given $v \in \mathcal{D}^{\sigma\prime}(\mathbb{R}^n)$ and denoting

$$v_j^\phi(x) = (2\pi)^{-n} \int e^{-\frac{\|\xi\|^2}{j}} \mathcal{F}_{\mu_\sigma}(\phi v)(x,\xi) d\xi \tag{6.11}$$

it follows from Lemma 5.3 that $\langle v_j^\phi; \psi\rangle \to \langle v\phi; \psi\rangle$ (for each $\psi \in \mathcal{D}^\sigma(\mathbb{R}^n)$).

Furthermore for each $N \in \mathbb{N}_0$ it follows from (6.10) that

$$(2\pi)^{-n} \int e^{-\frac{\|\xi\|^2}{j}} \mathcal{F}_{\mu_\sigma}(\phi[P(D)]^N(u))(x,\xi) d\xi$$
$$\to (2\pi)^{-n} \int \mathcal{F}_{\mu_\sigma}(\phi[P(D)]^N(u))(x,\xi) d\xi,\ \forall x \in K.$$

Thus,

$$[P(D)]^N(u)(x) = \phi(x)[P(D)]^N(u)(x)$$
$$= (2\pi)^{-n} \int \mathcal{F}_{\mu_\sigma}(\phi[P(D)]^N(u))(x,\xi) d\xi,\ \ x \in K.$$

Hence, using (6.10),

$$\left|[P(D)]^N(u))(x)\right| \leq (2\pi)^{-n} C \int e^{-c\omega(\|\xi\|^\rho)} d\xi\ e^{\frac{1}{\lambda}\varphi^*(Nm\lambda)} = C_1 e^{\frac{1}{\lambda}\varphi^*(Nm\lambda)}, \quad x \in K$$

where $C_1 = (2\pi)^{-n} C \int e^{-c\omega(\|\xi\|^\rho)} d\xi$. Therefore,

$$\left\|[P(D)]^N(u))\right\|_{L^2(K)} \le C_1 \int_K 1\, dx\; e^{\frac{1}{\lambda}\varphi^*(Nm\lambda)} = C_2 e^{\frac{1}{\lambda}\varphi^*(Nm\lambda)}$$

where $C_2 = C_1|K|$ and $|K| = \int_K 1dx$ denotes the Lebesgue measure of K, hence concluding that $\mathcal{E}^\omega(B(x_0, \delta); P)$. □

Next, as an immediate consequence of the previous theorem we prove that an iterate of a constant coefficient hypoelliptic operator is a ultradifferentiable function.

Corollary 6.3 *Let ω be a weight function, $\Omega \subset \mathbb{R}^n$ be an open set and $P(D)$ be a constant coefficient hypoelliptic linear partial differential operator of order m. Then, denoting $\sigma(t) = \omega(t^\rho)$, it follows that $\mathcal{E}^\omega(\Omega; P) \subset \mathcal{E}^\sigma(\Omega)$.*

Proof If $u \in \mathcal{E}^\omega(\Omega; P)$ then, using Theorem 6.2 it follows that u satisfies (6.1). Using Theorem 5.6, for $N = 0$, it follows that $u \in \mathcal{E}^\sigma$. □

The next result is the so called Denjoy-Carleman Kotake-Narasimhan theorem for constant coefficients operator and the proof given here is different from the one in [7].

Corollary 6.4 *Let ω be a weight function and $\Omega \subset \mathbb{R}^n$ be an open set. If $P(D)$ is a constant coefficient elliptic linear partial differential operator of order m, then $\mathcal{E}^\omega(\Omega; P) \subset \mathcal{E}^\omega(\Omega)$.*

Proof Since $P(D)$ is a constant coefficient elliptic linear partial differential operator it follows that $\rho = 1$ in (1.8). Thus, the result now follows from Corollary 6.3. □

Theorem 6.5 *Let ω be a weight function and $\Omega \subset \mathbb{R}^n$ be an open set. If $P(D)$ is a constant coefficient linear partial differential operator of order m (non necessarily hypoelliptic), then $\mathcal{E}^\omega(\Omega) \subset \mathcal{E}^\omega(\Omega; P)$.*

Proof Considering $u \in \mathcal{E}^\omega(\Omega)$, for each $K \subset \Omega$ compact there exist $C, \lambda > 0$ such that

$$|\partial^\alpha u(x)| \le C e^{\frac{1}{\lambda}\varphi^*(\lambda|\alpha|)},$$

for each $x \in K$ and $\alpha \in \mathbb{N}_0^n$. Moreover, denoting $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha u$ it follows that

$$\begin{aligned}\left\|[P(D)]^j u\right\|_{L^2(K)} &\le \sum_{|\alpha_1| \le m} |a_{\alpha_1}| \cdots \sum_{|\alpha_j| \le m} |a_{\alpha_j}| \|\partial^{\alpha_1+\cdots+\alpha_j} u\|_{L^2(K)} \\ &\le \sum_{|\alpha_1| \le m} |a_{\alpha_1}| \cdots \sum_{|\alpha_j| \le m} |a_{\alpha_j}|\, C\, |K|\, e^{\frac{1}{\lambda}\varphi^*(\lambda|\alpha_1+\cdots+\alpha_j|)}.\end{aligned}$$

where $|K| = \int_K 1dx$ denotes the Lebesgue measure of K. Since $t \mapsto \frac{\varphi^*(t)}{t}$ is increasing, by denoting $h = \sum_{|\alpha| \le m} |a_\alpha|$ it follows that

$$\|[P(D)]^j u\|_{L^2(K)} \le h^j C \|K\| e^{\frac{1}{\lambda}\varphi^*(\lambda jm)}.$$

Therefore, using (2.5) there exist $C_1 > 0$ and $\lambda_* > 0$ such that

$$\left\|[P(D)]^j u\right\|_{L^2(K)} \le C|K| \left(h^{1/m}\right)^{mj} e^{\frac{1}{\lambda}\varphi^*(\lambda jm)} \le C|K| e^{\frac{1}{\lambda_*}\varphi^*(\lambda_* jm)}.$$

Thus we conclude that $u \in \mathcal{E}^\omega(\Omega; P)$. □

Remark 6.6 It is important to observe that using Corollary 6.4 and Theorem 6.5 we can prove that if $f \in \mathcal{E}^\omega(\Omega)$, $P(D)$ is a constant coefficient elliptic linear partial differential operator of order m and $u \in L^2_{loc}(\Omega)$ is a solution of the equation

$$P(D)u = f, \quad \text{in} \quad \Omega \tag{6.12}$$

then, $u \in \mathcal{E}^\omega(\Omega)$.

Analogously, we can prove that if $P(D)$ is a non-elliptic operator but it is a constant coefficient hypoelliptic linear partial differential operator of order m satisfying (1.8) and $u \in L^2_{loc}(\Omega)$ satisfies (6.12) then $u \in \mathcal{E}^\sigma(\Omega)$ (where $\sigma(t) = \omega(t^\rho)$).

Appendix: Some Properties for a Constant Coefficient Hypoelliptic Linear Operator

Throughout this section we will consider $x_0 \in \mathbb{R}^n$, $0 < R < 1$, $m > 0$ and a constant coefficient hypoelliptic linear operator $P(D)$ of order m, defined in $B(x_0, R)$. Also, we will denote $\partial^\alpha P = P^{(\alpha)}$ and $P^{(\alpha)}(D)f = P^{(\alpha)}f$, for each $\alpha \in \mathbb{N}_0^n$ and $f \in C^\infty(B(x_0, R))$. Moreover, in order to simplify the notation we will denote $\|f\|_r = \|f\|_{L^2(B(x_0,r))} < \infty$, for each $0 < r < R$. In the proof of the next theorem we shall use [20, inequality (4.5)']. To be more precise, there exist $C, \gamma > 0$ such that,

$$\epsilon^{-|\alpha|}\|P^{(\alpha)}\phi\|_R \le C\left(\|P\phi\|_R + (1+\epsilon^{-\gamma})\|\phi\|_R\right), \tag{A.1}$$

for each $\epsilon > 0$, $\phi \in C_c^m(B(x_0, R))$ and $\alpha \in \mathbb{N}_0^n$ (such that $0 < |\alpha| \le m$). Furthermore, following [20, Lemma 4.1] for each $\epsilon, \epsilon_1 > 0$ such that $R > \epsilon_1 + \epsilon$ there exists $\phi_{(\epsilon_1,\epsilon)} \in C_c^\infty(B(x_0, R-\epsilon))$ such that $\phi_{\epsilon_1,\epsilon} \equiv 1$ in $B(x_0, R-\epsilon_1-\epsilon)$ and there exists $C > 0$ dependent of $R > 0$ but not on (ϵ_1, ϵ) satisfying,

$$\|\partial^\alpha \phi_{\epsilon_1,\epsilon}\|_{L^\infty} \le C\epsilon_1^{-|\alpha|}, \tag{A.2}$$

for $|\alpha| \le m$. We will use the following notation,

$$N_{k,r}(f) = \sup_{r>\delta>0} \delta^k \|f\|_{r-\delta},$$

for $r \le R$, $f \in L^2(B(x_0, r))$ and $k \in \mathbb{R}$.

Lemma A.1 *Let $u \in C^m(B(x_0, R))$ and $\gamma > 0$. Then there exists $D > 0$ (dependent of γ, P, R) such that,*

$$N_{\gamma-|\alpha|,R-\delta_1}(P^{(\alpha)}u) \leq D[N_{\gamma,R-\delta_1}(Pu) + \|u\|_{R-\delta_1}],$$

for each $R > \delta_1 > 0$ and $\alpha \in \mathbb{N}_0^n$, where $|\alpha| \leq m$.

Proof Consider arbitrary $0 < \delta$ and δ_1 such that $\delta_1 + \delta < R$. By (A.2) there exists $\phi_{\delta,\delta_1} \in C_c^\infty\left(B(x_0, R - \delta_1 - \frac{\delta}{2})\right)$, $\phi_{\delta,\delta_1} \equiv 1$ in $B(x_0, R - \delta_1 - \delta)$ and

$$\|\partial^\alpha \phi_{\delta,\delta_1}\|_{L^\infty} \leq C\left(\frac{\delta}{2}\right)^{-|\alpha|},$$

for each $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq m$. Moreover, for each $\epsilon > 0$ and $\beta \in \mathbb{N}_0^n$ such that $|\beta| \leq m$ it follows that,

$$\epsilon^{-|\beta|}\|P^{(\beta)}u\|_{R-\delta_1-\delta} = \epsilon^{-|\beta|}\|P^{(\beta)}(\phi_{\delta,\delta_1}u)\|_{R-\delta_1-\delta} \leq \epsilon^{-|\beta|}\|P^{(\beta)}(\phi_{\delta,\delta_1}u)\|_R.$$

Using (A.1) it follows that,

$$\epsilon^{-|\beta|}\|P^{(\beta)}u\|_{R-\delta_1-\delta} \leq C(\|P(\phi_{\delta,\delta_1}u)\|_R + (1+\epsilon^{-\gamma})\|\phi_{\delta,\delta_1}u\|_R).$$

Next we recall that, $P(D)(fg) = \sum_{|\alpha|\leq m} \frac{1}{\alpha!}P^{(\alpha)}g \times \partial^\alpha f$ for $f, g \in C^\infty$. Hence,

$$\begin{aligned}
\epsilon^{-|\beta|}\|P^{(\beta)}u\|_{R-\delta_1-\delta} &\leq C\left(\left\|\sum_{|\alpha|\leq m}\frac{1}{\alpha!}P^{(\alpha)}u\ \partial^\alpha\phi_{\delta,\delta_1}\right\|_R + (1+\epsilon^{-\gamma})\|\phi_{\delta,\delta_1}\|_\infty\|u\|_{R-\delta_1-\frac{\delta}{2}}\right)\\
&\leq C^2\left(\sum_{|\alpha|\leq m}\frac{\delta^{-|\alpha|}2^{|\alpha|}}{\alpha!}\left\|P^{(\alpha)}u\right\|_{R-\delta_1-\frac{\delta}{2}} + (1+\epsilon^{-\gamma})\|u\|_{R-\delta_1-\frac{\delta}{2}}\right)\\
&\leq C^2\left(\sum_{|\alpha|\leq m}\left(\frac{\delta}{2}\right)^{-\gamma}\left(\frac{\delta}{2}\right)^{\gamma-|\alpha|}\left\|P^{(\alpha)}u\right\|_{R-\delta_1-\frac{\delta}{2}} + (1+\epsilon^{-\gamma})\|u\|_{R-\delta_1}\right)\\
&\leq C^2\left(\sum_{|\alpha|\leq m}\left(\frac{\delta}{2}\right)^{-\gamma}N_{\gamma-|\alpha|,R-\delta_1}(P^{(\alpha)}u) + (1+\epsilon^{-\gamma})\|u\|_{R-\delta_1}\right).
\end{aligned}$$

Thus,

$$\delta^\gamma\epsilon^{-|\beta|}\|P^{(\beta)}u\|_{R-\delta_1-\delta} \leq C^2\left(\sum_{|\alpha|\leq m}2^\gamma N_{\gamma-|\alpha|,R-\delta_1}(P^{(\alpha)}u) + \left(\delta^\gamma + \left(\frac{\delta}{\epsilon}\right)^\gamma\right)\|u\|_{R-\delta_1}\right).$$

Next, considering an arbitrary $\chi > 0$ and defining $\epsilon = \frac{\delta}{\chi}$ it follows that,

$$\chi^{|\beta|}\delta^{\gamma-|\beta|}\|P^{(\beta)}u\|_{R-\delta_1-\delta} \leq C^2\left(\sum_{|\alpha|\leq m}2^\gamma N_{\gamma-|\alpha|,R-\delta_1}(P^{(\alpha)}u) + \left(\delta^\gamma + \chi^\gamma\right)\|u\|_{R-\delta_1}\right).$$

Since $0 < \delta < R < 1$ and $\gamma > 0$ we have $\delta^\gamma < 1$. Considering the supremum in $\delta > 0$ in the above inequality, we have

$$\chi^{|\beta|} N_{\gamma-|\beta|;R-\delta_1}\left(P^{(\beta)}u\right) \leq C^2 \left(\sum_{|\alpha|\leq m} 2^\gamma N_{\gamma-|\alpha|,R-\delta_1}(P^{(\alpha)}u) + \left(1+\chi^\gamma\right) \|u\|_{R-\delta_1} \right),$$

for each $\beta \in \mathbb{N}_0^n$ such that $0 < |\beta| \leq m$. Summing in β the last inequality we see that, there exists $C_1 > 0$ such that

$$\sum_{0<|\alpha|\leq m} \chi^{|\alpha|} N_{\gamma-|\alpha|;R-\delta_1}\left(P^{(\alpha)}u\right) \leq C_1 \left(\sum_{|\alpha|\leq m} 2^\gamma N_{\gamma-|\alpha|,R-\delta_1}(P^{(\alpha)}u) + \left(1+\chi^\gamma\right) \|u\|_{R-\delta_1} \right).$$

Thus,

$$\sum_{0<|\alpha|\leq m} \left(\chi^{|\alpha|} - C_1 2^\gamma\right) N_{\gamma-|\alpha|;R-\delta_1}\left(P^{(\alpha)}u\right) \leq C_1 \left(2^\gamma N_{\gamma,R-\delta_1}(u) + \left(1+\chi^\gamma\right) \|u\|_{R-\delta_1}\right).$$

Choosing $\chi > 1$, such that $\chi - C_1 2^\gamma > 1$, it follows that,

$$N_{\gamma-|\alpha|;R-\delta_1}\left(P^{(\alpha)}u\right) \leq C_1 \left(2^\gamma N_{\gamma,R-\delta_1}(u) + \left(1+\chi^\gamma\right) \|u\|_{R-\delta_1}\right),$$

for each $\alpha \in \mathbb{N}_0^n$ such that $0 < |\alpha| \leq m$. Therefore, denoting $C_2 = \max\{2^\gamma; (1+\chi^\gamma)\}$ and $D = C_1 C_2$, it follows that,

$$N_{\gamma-|\alpha|;R-\delta_1}\left(P^{(\alpha)}u\right) \leq D\left(N_{\gamma,R-\delta_1}(u) + \|u\|_{R-\delta_1}\right),$$

for each $\alpha \in \mathbb{N}_0^n$ such that $0 < |\alpha| \leq m$. □

Corollary 6.7 *Let $u \in C^m(B(x_0, R))$. There exists $C_P > 0$ such that,*

$$\|P^{(\alpha)}u\|_r \leq C_P \epsilon^{|\alpha|} \left(\|Pu\|_{r+\epsilon} + \|u\|_{r+\epsilon}\right),$$

for each $\epsilon, r > 0$ (such that $r + \epsilon \leq R < 1$) and $\alpha \in \mathbb{N}_0^n$ where $|\alpha| < m$. Let us recall the notation $\|u\|_{r+\epsilon} = \|u\|_{L^2(B(x_0,r+\epsilon))}$.

Proof Using Lemma A.1 it follows that,

$$\delta^{\gamma-|\alpha|}\|P^{(\alpha)}u\|_{R-\delta_1-\delta} \leq D\left[\sup_{0<\eta<R-\delta_1} \eta^\gamma \|Pu\|_{R-\delta_1-\eta} + \|u\|_{R-\delta_1}\right] \leq D\left(R^\gamma \|Pu\|_{R-\delta_1} + \|u\|_{R-\delta_1}\right)$$

for each $\delta, \delta_1 > 0$ such that $\delta + \delta_1 < R$ and $\alpha \in \mathbb{N}_0^n$, such that $0 < |\alpha| \leq m$. Therefore, for each $0 < r < R$ and $0 < \epsilon \leq R - r$ it follows that

$$\epsilon^{\gamma-|\alpha|}\|P^{(\alpha)}u\|_r = \epsilon^{\gamma-|\alpha|}\|P^{(\alpha)}u\|_{R-(R-r-\epsilon)-\epsilon} \leq D\left(R^\gamma \|Pu\|_{r+\epsilon} + \|u\|_{r+\epsilon}\right).$$

Hence, using that $0 < R < 1$

$$\|P^{(\alpha)}u\|_r \le \epsilon^{-\gamma} D\epsilon^{|\alpha|} \left(\|Pu\|_{r+\epsilon} + \|u\|_{r+\epsilon}\right) \le C_P \epsilon^{|\alpha|} \left(\|Pu\|_{r+\epsilon} + \|u\|_{r+\epsilon}\right),$$

where $C_P = \epsilon^{\gamma} D$. □

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Declarations

Conflict of interest The authors declare that there is no conflict of interest.

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