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The Fourier Transform on Rearrangement-Invariant Spaces

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Abstract

Let ρ be a rearrangement-invariant (r.i.) norm on the set $M(\mathbb{R}^n)$ of Lebesgue-measurable functions on $\mathbb{R}^n$ such that the space $L_\rho(\mathbb{R}^n) = \{f \in M(\mathbb{R}^n) : \rho(f) < \infty\}$ is an interpolation space between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$. The principal result of this paper asserts that given such a ρ, the inequality

$$\rho(\hat{f}) \leq C\sigma(f)$$

holds for any r.i. norm σ on $M(\mathbb{R}^n)$ if and only if

$$\bar{\rho}\left(Uf^*\right) \leq C\bar{\sigma}(f^*).$$

Here, $\bar{\rho}$ is the unique r.i. norm on $M(\mathbb{R}_+)$, $\mathbb{R}_+ = (0, \infty)$, satisfying $\bar{\rho}(f^*) = \rho(f)$ and $Uf^*(t) = \int_0^{1/t} f^*$, in which f^* is the nonincreasing rearrangement of f on $\mathbb{R}_+$. Further, in this case the smallest r.i. norm σ for which $\rho(\hat{f}) \leq C\sigma(f)$ holds is given by

$$\sigma(f) = \bar{\sigma}(f^*) = \bar{\rho}\left(Uf^*\right),$$

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where, necessarily, $\bar{\rho}\left(\int_0^{1/t} \chi_{(0,a)}\right) = \bar{\rho}\,(\min\{1/t,\ a\}) < \infty$, for all $a > 0$. We further specialize and expand these results in the contexts of Orlicz and Lorentz Gamma spaces.

Keywords Fourier transform · Weighted inequalities · Lorentz Gamma spaces · Orlicz spaces · Interpolation spaces

Mathematics Subject Classification Primary 42B10; Secondary 46M35 · 46E30 · 46B70

1 Introduction

Given f an $L_1(\mathbb{R}^n)$ function, its Fourier transform, defined by

$$(\mathscr{F} f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n,$$

satisfies the inequality

$$\|\hat{f}\|_\infty \le \|f\|_1.$$

Plancherel, in 1910, proved the n-dimensional version of the Riesz–Fischer theorem, namely

$$\|\hat{f}\|_2 = \|f\|_2.$$

Standard interpolation theorems yield that $L_{p'}(\mathbb{R}^n)$, $p' = \frac{p}{p-1}$, is an interpolation space (defined in Sect. 2) between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$ for $1 < p < 2$, leading to the Hausdorff–Young inequality (1926),

$$\|\hat{f}\|_{p'} \le C_p \|f\|_p,$$

in this case.

Inspired by the work of Jodeit and Torchinsky [13], in which the authors have generalized the Hausdorff–Young inequality, replacing the L_p spaces with Orlicz spaces, we prove the following theorem which is central to the rest of the results in this paper.

Theorem 1.1 *Let $\rho(f) = \bar{\rho}(f^*)$ be an r.i. norm such that the Banach space $L_\rho(\mathbb{R}^n)$ is an interpolation space between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$. Then,*

$$\rho(\hat{f}) \le C\sigma(f), \tag{1.1}$$

for any r.i. norm σ if and only if

$$\bar{\rho}(Uf^*) \le C\,\bar{\sigma}(f^*), \tag{1.2}$$

where $C > 0$ *is independent of* $f \in L_\sigma(\mathbb{R}^n)$.

For r.i norms $\rho = \rho_{p'}$ and $\sigma = \rho_p$, where $\rho_p(f) = \|f\|_p$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, the space $L_{\rho_{p'}}(\mathbb{R}^n) = L_{p'}(\mathbb{R}^n)$ is an interpolation space between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$ when $1 < p < 2$ and the inequality (1.2), amounts to

$$C\,\bar{\rho}_p(f^*) \geq \bar{\rho}_{p'}(Uf^*) = \left[\int_{\mathbb{R}_+} \left(\int_0^{1/t} f^*\right)^{p'} dt\right]^{\frac{1}{p'}} = \left[\int_{\mathbb{R}_+} \left(\int_0^{t} f^*\right)^{p'} \frac{dt}{t^2}\right]^{\frac{1}{p'}},$$

which is a special case of Hardy's inequality; see [9, p. 124]. Therefore, Theorem 1.1 leads to the Hausdorff–Young inequality in this case.

The Orlicz spaces, $L_{\rho_\Phi}(\mathbb{R}^n)$, are defined in terms of a nondecreasing convex (Orlicz) function Φ mapping $\mathbb{R}_+$ onto itself with the norm being given by

$$\rho_\Phi(f) = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1\right\}.$$

Our reformulation of the result in [13] asserts that, given an Orlicz function Φ, one has

$$\rho_{\Phi_1}(\hat{f}) \leq C\rho_{\Phi_2}(f),$$

in which

$$\rho_{\Phi_1}(f) = \rho_\Phi(f^2)^{1/2}$$

and ρ_{Φ_2} defined in terms of $\widetilde{\Phi}_2$, with

$$\widetilde{\Phi}_2(t) = \frac{1}{\Phi_1(t^{-1})}, \quad t \in \mathbb{R}_+.$$

We discuss this and related results on Orlicz spaces in detail in Sect. 4. Theorem 1.1 tells us that, for $1 < p \leq 2$, the smallest r.i. norm σ for which

$$\rho_{p'}(\hat{f}) \leq C\sigma(f),$$

is given by

$$\sigma(f) = \bar{\sigma}(f^*) = \rho_{p'}\left(Uf^*\right) = \left[\int_{\mathbb{R}_+} \left(\int_0^t f^*\right)^{p'} \frac{dt}{t^2}\right]^{\frac{1}{p'}} = \left[\int_{\mathbb{R}_+} \left(t^{1/p} f^{**}(t)\right)^{p'} \frac{dt}{t}\right]^{\frac{1}{p'}},$$

the so-called Lorentz norm $\rho_{p,p'}$, which is smaller than ρ_p.

In the next section we provide material on r.i. spaces and interpolation theory. Theorem 1.1 and some of its consequences are proved in Sect. 3. Section 4 deals with the

Fourier transform in the context of Orlicz spaces and Sect. 5 considers the boundedness of the Fourier transform between Lorentz Gamma spaces. Section 6 concludes with some remarks on other related work.

Throughout this article, we write $A \simeq B$ to abbreviate $C_1 A \le B \le C_2 A$ for some constants $C_1, C_2 > 0$ independent of A and B.

2 Rearrangement Invariant Spaces and the *K*-Functional

Definition 2.1 A rearrangement-invariant (r.i.) Banach function norm ρ on $M(\Omega)$, $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+$, satisfies

(1) $\rho(f) \ge 0$, with $\rho(f) = 0$ if and only if $f = 0$ a.e.;
(2) $\rho(cf) = c\,\rho(f)$, $c > 0$;
(3) $\rho(f+g) \le \rho(f) + \rho(g)$;
(4) $0 \le f_n \nearrow f$ implies $\rho(f_n) \nearrow \rho(f)$;
(5) $\rho(\chi_E) < \infty$ for all measurable $E \subset \Omega$ such that $|E| < \infty$;
(6) $\int_E f \le C_E\, \rho(f)$, with $E \subset \Omega$, $|E| < \infty$ and $C_E > 0$ independent of $f \in M(\Omega)$;
(7) $\rho(f) = \rho(g)$ whenever $\mu_f = \mu_g$. Here, μ_h, for $h \in M(\Omega)$, denotes the distribution function of h defined as $\mu_h(\lambda) = |\{x \in \Omega : |h(x)| > \lambda\}|$, $\lambda \in \mathbb{R}_+$.

Corresponding to an r.i. norm ρ on $M(\Omega)$ is the class

$$L_\rho(\Omega) := \{f \in M(\Omega) : \rho(f) < \infty\},$$

which becomes a Banach space of Lebesgue measurable functions under the norm $\rho(f)$, $f \in L_\rho(\Omega)$. The space $L_\rho(\Omega)$ is then a rearrangement-invariant space.

According to a fundamental result of Luxemberg [9, Chapter 2, Theorem 4.10], there corresponds to every r.i. norm ρ on $M(\mathbb{R}^n)$ an r.i. norm $\bar{\rho}$ on $M(\mathbb{R}_+)$ such that

$$\rho(f) = \bar{\rho}(f^*), \quad f \in M(\mathbb{R}^n). \tag{2.1}$$

Here,

$$f^*(t) = \mu_f^{-1}(t) = \inf\left\{\lambda \in \mathbb{R}_+ : \mu_f(\lambda) \le t\right\},\ t \in \mathbb{R}_+.$$

There is only one such $\bar{\rho}$ since both $\mathbb{R}^n$ and $\mathbb{R}_+$ are nonatomic and have infinite Lebesgue measure, see [9, p. 64].

A theorem of Hardy and Littlewood asserts that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le \int_{\mathbb{R}_+} f^*(t)g^*(t)dt, \qquad f, g \in M(\mathbb{R}^n). \tag{2.2}$$

The operation of rearrangement, though not sublinear itself, is sublinear in the average, namely,

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \qquad f, g \in M(\mathbb{R}^n), \qquad t \in \mathbb{R}_+, \tag{2.3}$$

in which

$$h^{**}(t) = t^{-1} \int_0^t h^*, \quad 0 \le h \in M(\mathbb{R}_+), \quad t \in \mathbb{R}_+.$$

A basic technique for working with r.i. norms involves the Hardy–Littlewood–Polya (HLP) Principle which asserts that

$$f^{**} \le g^{**} \quad \text{implies} \quad \rho(f) \le \rho(g);$$

see [9, Chapter 3, Proposition 4.6]. This principle is based on a result of Hardy, a generalized form of which reads

$$\int_0^t f \le \int_0^t g \tag{2.4}$$

implies

$$\int_0^t fh^* \le \int_0^t gh^*, \quad t \in \mathbb{R}_+,$$

for all $0 \le f, g \in M(\mathbb{R}_+)$ and $h \in M(\mathbb{R}_+)$. The Köthe dual of an r.i. norm ρ on $M(\Omega)$ is another such norm, ρ', with

$$\rho'(g) := \sup_{\rho(f) \le 1} \int_\Omega |f(x)g(x)|dx, \quad f, g \in M(\Omega).$$

It obeys the Principle of Duality

$$\rho'' = (\rho')' = \rho. \tag{2.5}$$

Further, one has the Hölder inequality

$$\int_\Omega |f(x)g(x)|dx \le \rho(f)\rho'(g), \quad f, g \in M(\Omega).$$

Finally,

$$\bar{\rho'} = (\bar{\rho})'.$$

The Orlicz and Lorentz Gamma spaces studied in sections 5 and 6, respectively, are examples of such r.i. spaces.

The dilation operator E_s, $s \in \mathbb{R}_+$, is defined at $f \in M(\mathbb{R}_+)$, $t \in \mathbb{R}_+$, by

$$(E_s f)(t) = f(st), \quad s, t \in \mathbb{R}_+.$$

The operator E_s is bounded on any r.i. space $L_\rho(\mathbb{R}_+)$. We denote its norm by $h_\rho(s)$. Using h_ρ we define the lower and upper indices of $L_\rho(\mathbb{R}_+)$ as

$$i_\rho = \sup_{s>1} \frac{-\log h_\rho(s)}{\log s} \quad \text{and} \quad I_\rho = \inf_{0<s<1} \frac{-\log h_\rho(s)}{\log s}, \tag{2.6}$$

respectively. One has

$$i_\rho = \lim_{s\to\infty} \frac{-\log h_\rho(s)}{\log s} \quad \text{and} \quad I_\rho = \lim_{s\to 0^+} \frac{-\log h_\rho(s)}{\log s}.$$

Further, $0 \le i_\rho \le I_\rho \le 1$ and, moreover,

$$i_{\rho'} = 1 - I_\rho \quad \text{and} \quad I_{\rho'} = 1 - i_\rho.$$

For all this, see [8, pp. 1250–1252].

If we denote by $k_\rho(s)$ the norm of E_s on the characteristic functions χ_F, $F \subset \mathbb{R}_+$, $|F| < \infty$, and define j_ρ and J_ρ by replacing $h_\rho(s)$ in (2.6) by $k_\rho(s)$, we obtain the fundamental indices of $L_\rho(\mathbb{R}_+)$. It turns out that when $L_\rho(\mathbb{R}_+)$ is an Orlicz space or Lorentz Gamma space $i_\rho = j_\rho$ and $I_\rho = J_\rho$. For ρ an Orlicz norm see [9]; for ρ a Lorentz Gamma norm see [10].

Finally, we describe that part of Interpolation Theory which is relevant to this paper.

Let X_1 and X_2 be Banach spaces compatible in the sense that both are continuously imbedded in the same Hausdorff topological space H, written

$$X_i \hookrightarrow H, \quad i = 1, 2.$$

The spaces $X_1 \cap X_2$ and $X_1 + X_2$ are the sets

$$X_1 \cap X_2 := \{x : x \in X_1 \ \text{ and } \ x \in X_2\}$$

and

$$X_1 + X_2 := \{x : x = x_1 + x_2, \ \text{ for some } x_1 \in X_1, \ x_2 \in X_2\},$$

with norms

$$\|x\|_{X_1\cap X_2} = \max\left[\|x\|_{X_1}, \|x\|_{X_2}\right]$$

and

$$\|x\|_{X_1+X_2} = \inf\left\{ \|x_1\|_{X_1} + \|x_2\|_{X_2} : x = x_1 + x_2, \ x_1 \in X_1, \ x_2 \in X_2\right\}.$$

Recall that given Banach spaces X_1 and X_2 imbedded in a common Hausdorff topological vector space, their Peetre K-functional is defined for $x \in X_1 + X_2$, $t > 0$, by

$$K(t, x; X_1, X_2) = \inf_{x=x_1+x_2} \left[\|x_1\|_{X_1} + t\, \|x_2\|_{X_2} \right].$$

We observe that, for $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+$, $p \in [1, \infty)$, $L_p(\Omega)$ and $L_\infty(\Omega)$ are compatible, each being continuously imbedded in the Hausdorff topological space $M(\Omega)$ equipped with the topology of convergence in measure. One has

$$K(t, f; L_p(\Omega), L_\infty(\Omega)) \simeq \left[\int_0^{t^p} f^*(s)^p ds \right]^{1/p}, \; t > 0, \tag{2.7}$$

$f \in \left(L_p + L_\infty\right)(\Omega)$, see [12].

The inequality

$$\int_0^t (\hat{f})^*(s)^2 ds \leq C_1 \int_0^t (Uf^*)(s)^2 ds, \quad t \in \mathbb{R}_+, \tag{2.8}$$

from [13] reads

$$K\left(t, (\hat{f})^*; L_2(\mathbb{R}_+), L_\infty(\mathbb{R}_+)\right) \leq K\left(t, C_2\, Uf^*; L_2(\mathbb{R}_+), L_\infty(\mathbb{R}_+)\right). \tag{2.9}$$

Definition 2.2 A Banach space Y is said to be intermediate between X_1 and X_2 if

$$X_1 \cap X_2 \hookrightarrow Y \hookrightarrow X_1 + X_2.$$

Definition 2.3 A Banach space Y intermediate between the compatible spaces X_1 and X_2 is said to be an interpolation space between X_1 and X_2 if every linear operator T on $X_1 + X_2$ satisfying

$$T : X_i \to X_i, \quad i = 1, 2,$$

also satisfies $T : Y \to Y$.

Suppose now that μ is an r.i. norm on $M(\mathbb{R}_+)$ satisfying $\mu\left(\frac{1}{1+t}\right) < \infty$. Denote by X_μ the set of all $x \in X_1 + X_2$ for which

$$\rho_\mu(x) = \mu\left(\frac{K(t, x; X_1, X_2)}{t}\right) < \infty.$$

Then, X_μ, with the norm ρ_μ, is an interpolation space between X_1 and X_2, see [4]. Therefore, from (2.7), we have that the space $X_{\rho_{\mu,p}}$, with the norm

$$\rho_{\mu,p}(f) = \overline{\rho_{\mu,p}}(f^*) = \bar{\mu}\left(t^{-1}\left[\int_0^{t^p} f^*(s)^p ds\right]^{1/p}\right), \quad f \in M(\Omega), \tag{2.10}$$

is an interpolation space between $L_p(\Omega)$ and $L_\infty(\Omega)$.

Definition 2.4 A Banach space Y intermediate between the compatible spaces X_1 and X_2 is said to be monotone if, given $x, y \in X_1 + X_2$, with

$$K(t, x; X_1, X_2) \leq K(t, y; X_1, X_2), \quad t \in \mathbb{R}_+, \tag{2.11}$$

one has $y \in Y$ implies $x \in Y$ and $\|x\|_Y \leq \|y\|_Y$.

The result of Lorentz–Shimogaki in [17, Theorem 2 and Lemma 3] asserts that the r.i. interpolation spaces between $L_p(\Omega)$ and $L_\infty(\Omega)$ are precisely the monotone spaces in that context. Further, the inequality (2.9) is a special case of (2.11). Thus, for $L_\rho(\mathbb{R}^n)$ between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$, there holds

$$\begin{aligned} \rho(\hat{f}) = \bar{\rho}\left((\hat{f})^*\right) &\leq C_2 \bar{\rho}\left(Uf^*\right) \\ &\leq MC_2 \bar{\sigma}(f^*) \\ &= MC_2 \sigma(f), \end{aligned}$$

whenever the r.i. norms ρ and σ on $M(\mathbb{R}^n)$ satisfy

$$\bar{\rho}(Uf^*) \leq M\,\bar{\sigma}(f^*), \quad f \in M(\mathbb{R}^n). \tag{2.12}$$

Remark 2.1 We have, for simplicity, chosen to restrict attention to functions $f \in L_1(\mathbb{R}^n)$, since then $\hat{f}$ is defined as a classical Lebesgue integral. Again, it is well known that for $f \in L_2(\mathbb{R}^n)$

$$\lim_{R \to \infty} \int_{\{x \in \mathbb{R}^n \,:\, |x| \leq R\}} f(x)\, e^{-2\pi i x \cdot \xi}\, dx, \quad \xi \in \mathbb{R}^n,$$

exists in the norm of $L_2(\mathbb{R}^n)$, which can be used to define $\hat{f}$. Thus, the Fourier transform can be defined as a function for all $f \in (L_1 + L_2)(\mathbb{R}^n)$. Indeed, it is shown in [3] that $(L_1 + L_2)(\mathbb{R}^n)$ is the largest r.i. space of functions that is mapped by $\mathscr{F}$ into a space of locally integrable functions.

The Editor has referred us to the paper [28], among others, where it is shown that, essentially the set of functions f for which $\hat{f}$ is defined as a function is the amalgam space $\ell_2(L_1(\mathbb{R}^n))$, which in the case $n = 1$ has the norm

$$\left(\sum_{k=-\infty}^{\infty} \left(\int_k^{k+1} |f(x)|\, dx \right)^2 \right)^{1/2}.$$

This is a Banach function norm on $M(\mathbb{R}^n)$ that is not rearrangement-invariant, namely, it satisfies (1)–(6) in Definition 2.1, but not (7). Thus, we need spaces other than the r.i. ones to study the Fourier transform in the context of this space.

3 Proof of Theorem 1.1

Proof The "if" part was proved towards the end of the Sect. 2. For the "only if" part, let B be the unit ball in $\mathbb{R}^n$ centered at the origin. Then $\widehat{\chi_B}$ is real-valued, radial and continuous, with $\widehat{\chi_B}(0) = |B|$. Also, $0 \le \chi_B * \chi_B \le |B|\chi_{2B} \in L^1$ and $(\widehat{\chi_B * \chi_B}) = (\widehat{\chi_B})^2$.

Choose $r > 0$ such that $\widehat{\chi_B} \ge |B|/2$ on rB. Let $0 \le f \in L^1(\mathbb{R}^n)$ be radial and radially decreasing. For $t \in \mathbb{R}_+$, choose $s > 0$ such that $|sB| = t^{-1}$. Then

$$
\begin{aligned}
(Uf^*)(t) &= \int_0^{1/t} f^* = \int_{sB} f(y)dy \\
&= \left(r^{-1}s\right)^n \int_{rB} f\left(r^{-1}sy\right) dy \\
&\le \left(r^{-1}s\right)^n \int_{rB} \frac{4}{|B|^2} (\widehat{\chi_B}(y))^2 f\left(r^{-1}sy\right) dy \\
&\le \left(r^{-1}s\right)^n \frac{4}{|B|^2} \int_{\mathbb{R}^n} (\widehat{\chi_B * \chi_B})(y) f\left(r^{-1}sy\right) dy \\
&= \left(r^{-1}s\right)^n \frac{4}{|B|^2} (rs^{-1})^n \int_{\mathbb{R}^n} (\chi_B * \chi_B)(\xi) \hat{f}\left(rs^{-1}\xi\right) d\xi \\
&\le \frac{4}{|B|^2}|B| \int_{2B} |\hat{f}\left(rs^{-1}\xi\right)| d\xi \\
&= \frac{4}{|B|}\left(r^{-1}s\right)^n \int_{|\eta|\le 2rs^{-1}} |\hat{f}(\eta)| d\eta \\
&= \frac{4}{|B|}\left(r^{-1}s\right)^n \int_0^{(2rs^{-1})^n|B|} \left(\hat{f}\right)^* \\
&= \frac{4}{|B|^2} r^{-n}\frac{1}{t} \int_0^{2^n r^n |B|^2 t} \left(\hat{f}\right)^* \\
&\le \frac{4}{|B|^2} r^{-n}\frac{1}{t} \int_0^{t} \left(\hat{f}\right)^* = C_n \frac{1}{t}\int_0^t \left(\hat{f}\right)^*,
\end{aligned}
$$

where we further shrink r to be such that $2^n r^n |B|^2 < 1$ and the constant $C_n > 0$ depends only on n.

Therefore, for $f \in (L_1 \cap L_\sigma)(\mathbb{R}^n)$ such that $f(x) = g(|x|)$, $x \in \mathbb{R}^n$, with $g \downarrow$ on $\mathbb{R}_+$,

$$
\begin{aligned}
\bar{\rho}(Uf^*) &\le C_n\, \bar{\rho}\left(\frac{1}{t}\int_0^t \left(\hat{f}\right)^*\right) \\
&\le C_n'\, \bar{\rho}\left(\left(\hat{f}\right)^*\right) \\
&= C_n'\, \rho\left(\hat{f}\right) \\
&\le C\, C_n'\, \sigma(f), \quad \text{by assumption}, \\
&= C\, C_n'\, \bar{\sigma}\left(f^*\right),
\end{aligned}
$$

where the second inequality is the boundedness of the averaging operator, $P : g \mapsto \frac{1}{t}\int_0^t g$, on $L_{\bar{\rho}}(\mathbb{R}_+)$, which follows from our hypothesis on $L_{\bar{\rho}}(\mathbb{R}_+)$ that it is the interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$, and the Hardy's inequality.

Given $h \in (L_1 \cap L_{\bar{\sigma}})(\mathbb{R}_+)$, let $g(t) = h^*(|B|t^n)$ and set $f(x) = g(|x|)$. Then, the rearrangement of f with respect to n-dimensional Lebesgue measure is equal to the rearrangement of h with respect to 1-dimensional Lebesgue measure. The foregoing argument then yields

$$\bar{\rho}(Uh) \le \bar{\rho}\left(Uh^*\right) = \bar{\rho}\left(Uf^*\right) \le C\bar{\sigma}(f^*) = C\bar{\sigma}(h^*) = C\bar{\sigma}(h).$$

The space $(L_1 \cap L_{\bar{\sigma}})(\mathbb{R}_+)$ includes all bounded functions of compact support whence the monotone convergence theorem and the Fatou property of $\bar{\rho}$ and $\bar{\sigma}$ completes the proof. □

Boyd in [7, pp. 92–98] associates to each r.i. norm ρ on $M(\Omega)$, $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+$, and each $p > 1$ the functional

$$\rho^{(p)}(f) = \rho(|f|^p)^{\frac{1}{p}}, \quad f \in M(\Omega). \tag{3.1}$$

He shows that $\rho^{(p)}$ is an r.i. norm on $M(\Omega)$ and that $\bar{\rho}(f^{**}) \le C\bar{\rho}(f^*)$ holds with $\bar{\rho} = \overline{\rho^{(p)}} = \bar{\rho}^{(p)}$.

The space defined by the norm $\rho^{(p)}$ is now referred to as the p-convexification of $L_\rho(\mathbb{R}^n)$. It was studied in a series of papers by G. Lozanovskiĭ about the time Boyd, independently, introduced his spaces. See the references to G. Lozanovskiĭ's work in [19]. This latter paper treats the K-functional of p-convexifications, as does the paper [1]. These papers should shed light on the work involving $\rho^{(2)}$ in this and the next two sections.

Theorem 3.1 *Let ρ be an r.i. norm on $M(\Omega)$. For fixed $p > 1$, define $\rho^{(p)}$ as in (3.1). Then, $L_{\rho^{(p)}}(\Omega)$ is an interpolation space between $L_p(\Omega)$ and $L_\infty(\Omega)$.*

Proof Suppose the linear operator T satisfies

$$T : L_p(\Omega) \to L_p(\Omega) \quad \text{and} \quad T : L_\infty(\Omega) \to L_\infty(\Omega).$$

Then, according to [9, Theorem 1.11, pp. 301–304], there exists $C > 0$, such that

$$\int_0^t (Tf)^*(s)^p ds \le C\int_0^t f^*(s)^p ds, \quad f \in (L_p + L_\infty)(\Omega),\ t \in \mathbb{R}_+. \tag{3.2}$$

The HLP Principle involving $\bar{\rho}$ yields

$$\begin{aligned} \rho\left((Tf)^p\right) &= \bar{\rho}\left(\left[(Tf)^*\right]^p\right) \\ &\le C\bar{\rho}\left(\left[f^*\right]^p\right) \\ &= C\rho\left(|f|^p\right), \quad f \in L_{\rho^{(p)}}(\Omega), \end{aligned}$$

and hence

$$\rho^{(p)}(Tf) \leq C^{1/p}\rho^{(p)}(f), \quad f \in L_{\rho^{(p)}}(\Omega).$$

□

Theorem 3.2 *Let ρ and σ be r.i. norms on $M(\mathbb{R}^n)$ determined, respectively, by the r.i. norms $\bar{\rho}$ and $\bar{\sigma}$ on $M(\mathbb{R}_+)$ by $\rho(f) = \bar{\rho}(f^*)$ and $\sigma(f) = \bar{\sigma}(f^*)$, $f \in M(\mathbb{R}^n)$. Then,*

$$\rho^{(2)}(\hat{f}) \leq C\sigma(f), \quad f \in (L_\sigma \cap L_1)(\mathbb{R}^n),$$

if and only if

$$\overline{\rho^{(2)}}(Ug) \leq C\bar{\sigma}(g), \quad g \in M(\mathbb{R}_+).$$

Proof The result is a consequence of Theorems 3.1 and 1.1. □

From our discussion on the spaces $X_{\rho_{\mu,p}}$, with the norm $\rho_{\mu,p}$ given by (2.10), Theorem 1.1 guarantees

Theorem 3.3 *Let μ and σ be r.i. norms on $M(\mathbb{R}^n)$ determined, respectively, by the r.i. norms $\bar{\mu}$ and $\bar{\sigma}$ on $M(\mathbb{R}_+)$. Suppose $\bar{\mu}\left(\frac{1}{1+t}\right) < \infty$. Set*

$$\rho_{\mu,2}(f) = \overline{\rho_{\mu,2}}(f^*) = \bar{\mu}\left(t^{-1}\left[\int_0^{t^2} f^*(s)^2 ds\right]^{1/2}\right).$$

Then,

$$\rho_{\mu,2}(\hat{f}) \leq C\sigma(f), \quad f \in (L_\sigma \cap L_1)(\mathbb{R}^n),$$

if and only if

$$\overline{\rho_{\mu,2}}(Ug) \leq C\bar{\sigma}(g), \quad g \in M(\mathbb{R}_+).$$

Finally, consider an r.i. norm ρ on $M(\mathbb{R}^n)$ determined by the r.i. norm $\bar{\rho}$ on $M(\mathbb{R}_+)$ and set

$$\rho_U(f) := (\bar{\rho} \circ U)(f^*) = \bar{\rho}(Uf^*), \quad f \in M(\mathbb{R}^n).$$

One has ρ_U an r.i. norm if $(\bar{\rho} \circ U)(\chi_{(0,t)}) < \infty$ for all $t > 0$, or, equivalently, $\bar{\rho}\left(\frac{1}{1+t}\right) < \infty$. In that case, $L_{\bar{\rho} \circ U}(\mathbb{R}_+)$ is the largest r.i. space to be mapped into $L_{\bar{\rho}}(\mathbb{R}_+)$ by U.

With this background we now have

Theorem 3.4 *Let ρ be an r.i. norm on $M(\mathbb{R}^n)$ defined in terms of an r.i. norm $\bar{\rho}$ on $M(\mathbb{R}_+)$ such that*

$$\bar{\rho}\left(\frac{1}{1+t}\right) < \infty.$$

Assuming $L_{\bar{\rho}}(\mathbb{R}_+)$ is an interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$, one has that $L_{\rho_U}(\mathbb{R}^n)$ is the largest r.i. space of functions on $\mathbb{R}^n$ to be mapped into $L_\rho(\mathbb{R}^n)$ by $\mathscr{F}$.

4 $\mathscr{F}$ in the Context of Orlicz Spaces

An Orlicz gauge norm is given in terms of an N-function

$$\Phi(x) = \int_0^x \phi, \quad x \in \mathbb{R}_+;$$

here ϕ is a nondecreasing function mapping $\mathbb{R}_+$ onto itself. These N-functions are convex functions of the type from [13] referred to in the Introduction. Specifically, the gauge norm ρ_Φ is defined at $f \in M(\Omega)$, $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+$, by

$$\rho_\Phi(f) = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

One can show $\rho_\Phi(f) = \bar{\rho}_\Phi(f^*)$, so that the Orlicz space

$$L_\Phi(\Omega) = \{f \in M(\Omega) : \rho_\Phi(f) < \infty\}$$

is an r.i. space. The norm $(\rho_\Phi)'$ dual to ρ_Φ is equivalent to the gauge norm $\rho_{\tilde{\Phi}}$, where $\tilde{\Phi}(t) = \int_0^t \phi^{-1}$, $t \in \mathbb{R}_+$, see [9].

The definitive work on $\mathscr{F}$ between Orlicz spaces is due to Jodeit and Torchinsky. See, in particular, [13, Theorem 2.16]. This theorem asserts that if A and B are N-functions with $L_A(\mathbb{R}^n) \subset (L_1 + L_2)(\mathbb{R}^n)$, $L_B(\mathbb{R}^n) \subset (L_2 + L_\infty)(\mathbb{R}^n)$ and $\mathscr{F} : L_A(\mathbb{R}^n) \to L_B(\mathbb{R}^n)$, then there exist N-functions A_1 and B_1 with $L_{A_1}(\mathbb{R}^n) \supset L_A(\mathbb{R}^n)$ and $L_{B_1}(\mathbb{R}^n) \subset L_B(\mathbb{R}^n)$ for which $\mathscr{F} : L_{A_1}(\mathbb{R}^n) \to L_{B_1}(\mathbb{R}^n)$. Moreover, $B_1(t) = 1/\tilde{A}_1(t^{-1})$; $A_1(t)/t^2 \downarrow$ on $\mathbb{R}_+$ and so $B_1(t)/t^2 \uparrow$ on $\mathbb{R}_+$.

Using the results in the previous sections we now show $L_{A_1}(\mathbb{R}_+)$ is an interpolation space between $L_1(\mathbb{R}_+)$ and $L_2(\mathbb{R}_+)$, while $L_{B_1}(\mathbb{R}_+)$ is an interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$.

To begin, we observe that $B_1(t)/t^2 \uparrow$ is equivalent to $B_1(t) = \Phi(t^2)$ for some N-function Φ. Indeed, given the latter, one has $B_1(t)/t^2 = \Phi(t^2)/t^2 \uparrow$ on $\mathbb{R}_+$. Again $B_1(t)/t^2 \uparrow$ implies $B_1(t^{1/2})/t \uparrow$, so that $\Phi(t) = B_1(t^{1/2})$ is such that $B_1(t) = \Phi(t^2)$. Next,

$$\rho_\Phi^{(2)}(f) = \rho_\Phi(f^2)^{\frac{1}{2}} = \rho_{B_1}(f).$$

According to Theorem 3.1, then, $L_{B_1}(\mathbb{R}^n)$ is an interpolation space between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$.

Now, $B_1(t) = 1/\tilde{A}_1(t^{-1})$ is equivalent to $\tilde{A}_1(t) = 1/B_1(t^{-1})$, whence $\tilde{A}_1(t)/t^2 = (t^{-1})^2/B_1(t^{-1})$ and so $B_1(t)/t^2 \uparrow$ amounts to $\tilde{A}_1(t)/t^2 \uparrow$, that is, $L_{\tilde{A}_1}(\mathbb{R}^n)$ is an interpolation space between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$. Since $L_{\tilde{A}_1}(\mathbb{R}^n)$ is the Köthe dual of $L_{A_1}(\mathbb{R}^n)$ we conclude that $L_{A_1}(\mathbb{R}^n)$ is an interpolation space between $L_1(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n)$.

The monotonicity conditions on A_1 and B_1 translate into conditions on their associated fundamental functions. For example, $L_{B_1}(\mathbb{R}^n)$ has fundamental function $\phi_{B_1}(t) = \rho_{B_1}\left(\chi_{(0,t)}\right) = 1/B_1^{-1}(t^{-1})$, $t \in \mathbb{R}_+$. Thus setting $t = 1/B_1(y)$ in $\frac{\phi_{B_1}(t)}{t^{1/2}} = \frac{1}{B_1^{-1}(t^{-1})t^{1/2}}$ we arrive at $\left(\frac{B_1(y)}{y^2}\right)^{1/2}$, which increases in y and therefore decreases in t, so $\frac{\phi_{B_1}(t)}{t^{1/2}} \downarrow$.

We observe that $L_{A_1}(\mathbb{R}^n)$ is not the largest r.i. space that $\mathscr{F}$ maps into $L_{B_1}(\mathbb{R}^n)$; that space has norm $\rho_{B_1}(Uf^*)$. In the Lebesgue context, in which, say, $B_1(t) = t^{p'}$, $1 < p < 2$,

$$\rho_{p'}(Uf^*) = \left[\int_{\mathbb{R}_+} (Uf^*)(t)^{p'} dt\right]^{1/p'} \simeq \left[\int_{\mathbb{R}_+} \left[t^{\frac{1}{p}} f^{**}(t)\right]^{p'} \frac{dt}{t}\right]^{1/p'},$$

which is the so-called Lorentz norm $\rho_{p,p'}$. This norm is smaller than $\rho_{A_1} = \rho_p$. For more details see the next section.

The foregoing argument can be used to associate a pair of N-functions (A, B) to a given N-function Φ such that $\mathscr{F} : L_A(\mathbb{R}^n) \to L_B(\mathbb{R}^n)$. Moreover, $L_A(\mathbb{R}^n)$ is an interpolation space between $L_1(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n)$, while $L_B(\mathbb{R}^n)$ is an interpolation space between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$. Indeed, we have

Theorem 4.1 *Let Φ be an N-function. Set $B(t) = \Phi(t^2)$ and $\tilde{A}(t) = 1/B(t^{-1})$. Then, essentially, $\frac{A(t)}{t^2} \downarrow$,*

$$\mathscr{F} : L_A(\mathbb{R}^n) \to L_B(\mathbb{R}^n)$$

or, equivalently,

$$\mathscr{F} : L_{\tilde{B}}(\mathbb{R}^n) \to L_{\tilde{A}}(\mathbb{R}^n)$$

with $L_A(\mathbb{R}^n)$ an interpolation space between $L_1(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n)$, while $L_B(\mathbb{R}^n)$ is an interpolation space between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$.

Proof The preceding discussion shows $\rho_B = \rho_{\Phi^{(2)}}$ whence $L_B(\mathbb{R}^n)$ is between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$. Again $B(t)/t^2 \uparrow$ and $\tilde{A}(t)/t^2 \uparrow$ which means $L_{\tilde{A}}$ is an interpolation space between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$, whence $L_A(\mathbb{R}^n)$ is an interpolation space between $L_1(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n)$. Finally, Theorem 3.10 in [13] ensures

$$\mathscr{F} : L_A(\mathbb{R}^n) \to L_B(\mathbb{R}^n),$$

since $B(t) = 1/\tilde{A}(t^{-1})$ is equivalent to $\tilde{A}(t) = 1/B(t^{-1})$. ◻

5 $\mathscr{F}$ Between Lorentz Gamma Spaces

In this section, we make use of the operators P and Q defined by

$$(Pf)(t) = \frac{1}{t}\int_0^t f \quad \text{and} \quad (Qg)(t) = \int_t^\infty g(s)\frac{ds}{s}, \quad f, g \in M(\mathbb{R}_+),\ t \in \mathbb{R}_+.$$

These operators satisfy the equations

$$\int_{\mathbb{R}_+} g\, Pf = \int_{\mathbb{R}_+} f\, Qg, \quad f, g \in M(\mathbb{R}_+),$$

and

$$PQ = QP = P + Q.$$

Fix an index $p \in (1, \infty)$ and a weight $0 \le u \in M(\mathbb{R}_+)$. The Lorentz Gamma norm $\rho_{p,u}$ defined in terms of the Lorentz norm $\lambda_{p,u}(f) = \lambda_{p,u}(f^*) = \left(\int_{\mathbb{R}_+} f^*(t)^p dt\right)^{1/p}$ by

$$\rho_{p,u}(f) = \lambda_{p,u}(f^{**}), \quad f \in M(\Omega),$$

where, once again, $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+$.

To guarantee $\rho_{p,u}(\chi_E) < \infty$ for all measurable sets $E \subset \Omega$ with $|E| < \infty$, we require

$$\int_{\mathbb{R}_+} \frac{u(t)}{1 + t^p} dt < \infty. \tag{5.1}$$

The Lorentz Gamma space

$$\Gamma_{p,u}(\Omega) = \left\{ f \in M(\Omega) : \rho_{p,u}(f) < \infty \right\},$$

is then an r.i. space. The norm, $\rho_{p',v'}$, dual to $\rho_{p,u}$ is given by

$$\rho_{p',v'}(g) = \left(\int_{\mathbb{R}_+} g^{**}(t)^p v'(t) dt\right)^{1/p'}, \quad g \in M(\Omega)$$

where, $p' = p/(p-1)$ and

$$v'(t) = \frac{t^{p'+p-1} \int_0^t u \ \int_t^\infty u(s)s^{-p} ds}{\left[\int_0^t v + t^p \int_t^\infty u(s)s^{-p} ds\right]^{p'+1}}.$$

This is shown, for example, in [10].

In this section we study the inequality

$$\rho_{p,u}(\hat{f}) \leq C\, \rho_{q,v}(f), \quad f \in (L_1 \cap \Gamma_{q,v})(\mathbb{R}^n). \tag{5.2}$$

We begin by assuming $\Gamma_{p,u}(\mathbb{R}_+)$ is an interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$, then address the question of when this is the case later in the section.

Recall that Theorem 1.1 ensures that (5.2) holds if and only if

$$\bar{\rho}_{p,u}(Uf^*) \leq C\bar{\rho}_{q,v}(f^*), \quad f \in M(\mathbb{R}_+). \tag{5.3}$$

Theorem 5.1 *Let the indices p, q and weights u, v be as described above. Then, given that $\Gamma_{p,u}(\mathbb{R}_+)$ is an interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$, one has (5.2) if and only if*

$$\bar{\rho}_{q',v'}(g^{**}) \leq C\bar{\rho}_{p',u_{p}'}(g^*), \quad g \in M(\mathbb{R}_+). \tag{5.4}$$

where, as usual, $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$, $\int_{\mathbb{R}_+} v = \infty$,

$$v'(t) = \frac{t^{q'+q-1} \int_0^t v \int_t^\infty v(s)s^{-q}ds}{\left[\int_0^t v + t^q \int_t^\infty v(s)s^{-q}ds\right]^{q'+1}}$$
$$u_p(t) = u(t^{-1})t^{p-2}, \quad \int_{\mathbb{R}_+} u_p = \infty,$$

and

$$u_p{}'(t) = \frac{t^{p'+p-1} \int_0^t u_p \int_t^\infty u_p(s)s^{-p}ds}{\left[\int_0^t u_p + t^p \int_t^\infty u_p(s)s^{-p}ds\right]^{p'+1}}, \quad t \in \mathbb{R}_+.$$

Proof The inequality (5.3) tells that the space determined by the r.i. norm $\bar{\rho}_{p,u}(Uf^*)$ is the largest one mapped into $\Gamma_{p,u}(\mathbb{R}_+)$ by U. Now,

$$\bar{\rho}_{p,u}(Uf^*) = \left[\int_{\mathbb{R}_+} P(Uf^*)(t)^p u(t)\, dt\right]^{\frac{1}{p}},$$
$$P\left(Uf^*\right)(t) = t^{-1}\left(P(Qf^*)\right)(t^{-1}).$$

Further,

$$\int_{\mathbb{R}_+} \left[t^{-1}\left(P(Qf^*)\right)(t^{-1})\right]^p u(t)dt = \int_{\mathbb{R}_+} \left[t\left(P(Qf^*)\right)(t)\right]^p u(t^{-1})t^{-2}dt.$$

We have shown that

$$\bar{\rho}_{p,u}(Uf^*) = \left[\int_{\mathbb{R}_+} \left(P(Qf^*)\right)(t)^p u_p(t)dt\right]^{\frac{1}{p}}$$
$$= \bar{\rho}_{p,u_p}(Qf^*).$$

Therefore, any $\Gamma_{q,v}(\mathbb{R}_+)$ mapped into $\Gamma_{p,u}(\mathbb{R}_+)$ by U must be embedded into this largest domain; that is,

$$\bar{\rho}_{p,u_p}(Qf^*) \leq C\bar{\rho}_{q,v}(f^*), \quad f \in M(\mathbb{R}_+). \tag{5.5}$$

But, since (5.5) is equivalent to (5.4), its dual inequality, (5.5) may be tested over any $0 \leq f \in M(\mathbb{R}_+)$, as is seen in

$$\int_{\mathbb{R}_+} g^* Qf = \int_{\mathbb{R}_+} fPg^* \leq \int_{\mathbb{R}_+} f^* Pg^* = \int_{\mathbb{R}_+} g^* Qf^*.$$

□

The inequality (5.4), and hence (5.2), amounts to

$$\left[\int_{\mathbb{R}_+} (Ph)^{q'} v'\right]^{\frac{1}{q'}} \leq C\left[\int_{\mathbb{R}_+} h^{p'} {u_p}'\right]^{\frac{1}{p'}}, \tag{5.6}$$

with $h = g^{**}$ belonging to

$$\Omega_{0,1}(\mathbb{R}_+) = \{0 \leq h \in M(\mathbb{R}_+) : h(t) \downarrow \text{ and } t\,h(t) \uparrow \text{ on } \mathbb{R}_+\}.$$

Such inequalities are shown in Theorem 4.4 of [10] to be equivalent to a pair of weighted norm inequalities involving general non-negative measurable functions. In the case of (5.6) this leads to

Theorem 5.2 *Let the indices p, q and weights u, v be as in Theorem 5.1. Then, (5.6) holds if and only if*

$$\left[\int_{\mathbb{R}_+} [(P+Q)g]^{q'} v'\right]^{\frac{1}{q'}} \leq C\left[\int_{\mathbb{R}_+} g^{p'} {u_p}^{1-p'}\right]^{\frac{1}{p'}} \tag{5.7}$$

and

$$\left[\int_{\mathbb{R}_+} (P^2 g)^{q'} v'\right]^{\frac{1}{q'}} \leq C\left[\int_{\mathbb{R}_+} g^{p'} {u_p}^{1-p'}\right]^{\frac{1}{p'}}, \quad 0 \leq g \in M(\mathbb{R}_+).$$

Proof According to Theorem 4.4 in [10], one has (5.6) if and only if

$$\left[\int_{\mathbb{R}_+} [(P+Q)Qg]^p u_p\right]^{\frac{1}{p}} \leq C\left[\int_{\mathbb{R}_+} g^q {v'}^{1-q}\right]^{\frac{1}{q}}$$

and

$$\left[\int_{\mathbb{R}_+} [P(P+Q)g]^{q'} v'\right]^{\frac{1}{q'}} \leq C\left[\int_{\mathbb{R}_+} g^{p'} {u_p}^{1-p'}\right]^{\frac{1}{p'}},$$

holds for all $0 \leq g \in M(\mathbb{R}_+)$.

These are dual inequalities. We choose the second one, which easily reduces to (5.7). □

To deal with the case $q \leq p$ we will use special instances of the following combination of Theorems 1.7 and 4.1 from [6].

Theorem 5.3 *Consider* $0 \leq K(x, y) \in M(\mathbb{R}_+ \times \mathbb{R}_+)$*, which, for fixed* $y \in \mathbb{R}_+$*, increases in* x *and, for fixed* $x \in \mathbb{R}_+$*, decreases in* y *and which, moreover, satisfies the growth condition*

$$K(x, y) \leq K(x, z) + K(z, y), \quad 0 < y < z < x.$$

Let t, u, v *and* w *be nonnegative, measurable (weight) functions on* $\mathbb{R}_+$ *and suppose* $\Phi_1(x) = \int_0^x \phi_1$ *and* $\Phi_2(x) = \int_0^x \phi_2$ *are* N*-functions having complementary functions* $\Psi_1(x) = \int_0^x {\phi_1}^{-1}$ *and* $\Psi_2(x) = \int_0^x {\phi_2}^{-1}$*, respectively, with* $\Phi_1 \circ {\Phi_2}^{-1}$ *convex. Then there exists* $c > 0$ *such that*

$$\begin{aligned}{\Phi_1}^{-1}\left(\int_{\mathbb{R}_+} \Phi_1\left(cw(x)\int_0^x K(x, y)f(y)dy\right)t(x)dx\right) \\ \leq {\Phi_2}^{-1}\left(\int_{\mathbb{R}_+} \Phi_2\left(u(y)f(y)\right)v(y)dy\right),\end{aligned}$$

$0 \leq f \in M(\mathbb{R}_+)$*, if and only if*

$$\int_0^x \frac{K(x, y)}{u(y)}{\phi_2}^{-1}\left(\frac{c\alpha(\lambda, x)K(x, y)}{\lambda u(y)v(y)}\right)dy \leq c^{-1}\lambda$$

and

$$\int_0^x \frac{1}{u(y)}\phi_2^{-1}\left(\frac{c\beta(\lambda, x)}{\lambda u(y)v(y)}\right)dy \leq c^{-1}\lambda,$$

where

$$\alpha(\lambda, x) = \Phi_2 \circ {\Phi_1}^{-1}\left(\int_x^\infty \Phi_1\left(\lambda w(y)\right)t(y)dy\right)$$

and

$$\beta(\lambda, x) = \Phi_2 \circ {\Phi_1}^{-1}\left(\int_x^\infty \Phi_1\left(\lambda w(y)K(y, x)\right)t(y)dy\right).$$

Theorem 5.4 *Let p, q, u, u', u_p, v, v' be as in the Theorem 5.1, with $1 < q \leq p < \infty$. Then, given that $\Gamma_{p,u}(\mathbb{R}_+)$ is an interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$, one has (5.2) if and only if*

(1)

$$\left(\int_0^x u_p(y)dy\right)^{\frac{1}{p}} \left(\int_x^\infty v'(y)y^{-q'}dy\right)^{\frac{1}{q'}} \leq C$$

(2)

$$\left(\int_0^x v'(y)dy\right)^{\frac{1}{q'}} \left(\int_x^\infty u_p(y)y^{-p}dy\right)^{\frac{1}{p}} \leq C$$

(3)

$$\left(\int_0^x \left(\log\frac{x}{y}\right)^p u_p(y)dy\right)^{\frac{1}{p}} \left(\int_x^\infty v'(y)y^{-q'}dy\right)^{\frac{1}{q'}} \leq C$$

(4)

$$\left(\int_0^x u_p(y)dy\right)^{\frac{1}{p}} \left(\int_x^\infty v'(y)\left(\frac{1}{y}\log\frac{y}{x}\right)^{q'}dy\right)^{\frac{1}{q'}} \leq C.$$

Indeed (1) and (3) can be combined into

$$\left(\int_0^\infty \left(\log\left(1+\frac{x}{y}\right)\right)^p u_p(y)dy\right)^{\frac{1}{p}} \left(\int_x^\infty v'(y)y^{-q'}dy\right)^{\frac{1}{q'}} \leq C.$$

Proof The first inequality in (5.7) amounts to

$$\left[\int_{\mathbb{R}_+} (Pg)^{q'} v'\right]^{\frac{1}{q'}} \leq C \left[\int_{\mathbb{R}_+} g^{p'} u_p^{1-p'}\right]^{\frac{1}{p'}}$$

and

$$\left[\int_{\mathbb{R}_+} (Qg)^{q'} v'\right]^{\frac{1}{q'}} \leq C \left[\int_{\mathbb{R}_+} g^{p'} u_p^{1-p'}\right]^{\frac{1}{p'}}, \quad 0 \leq g \in M(\mathbb{R}_+),$$

the latter inequality being, by duality, equivalent to

$$\left[\int_{\mathbb{R}_+} (Pf)^p u_p\right]^{\frac{1}{p}} \leq C \left[\int_{\mathbb{R}_+} f^q v'^{1-q}\right]^{\frac{1}{q}}, \quad 0 \leq f \in M(\mathbb{R}_+).$$

We illustrate the method of proof with the second inequality in (6.7) involving

$$(P^2 g)(x) = \frac{1}{x}\int_0^x \log\left(\frac{x}{y}\right) g(y)dy.$$

Thus, taking, in Theorem 6.3, $K(x,y) = \log_+ \frac{x}{y}$, $\Phi_1(x) = x^{q'}$, $\Phi_2(x) = x^{p'}$ (observe that $(\Phi_1 \circ \Phi_2{}^{-1})(x) = x^{\frac{q'}{p'}}$, which is convex when $q \le p$), $w(y) = y^{-1}$, $t(y) = v'(y)$, $u(y) = u_p(y)^{-1}$, $v(y) = u_p(y)$ we get

$$\alpha(\lambda, x) = \lambda^{p'}\left(\int_x^\infty v'(y) y^{-q'} dy\right)^{\frac{p'}{q'}}$$

and

$$\beta(\lambda, x) = \lambda^{p'}\left(\int_x^\infty v'(y)\left(y^{-1}\log\frac{y}{x}\right)^{q'} dy\right)^{\frac{p'}{q'}},$$

from which the conditions in Theorem 5.3 yields (3) and (4). We point out that λ cancels. □

The inequality (5.2) is much easier to deal with when

$$\rho_{p,u}(f) \simeq \lambda_{p,u}(f) = \left(\int_{\mathbb{R}_+} f^*(t)^p\, u(t)\, dt\right)^{\frac{1}{p}}, \tag{5.8}$$

which equivalence is not all that uncommon, as we will see later in this section. Indeed, given (5.8),

$$\begin{aligned}\rho_{p,u}(Uf^*) &\simeq \left(\int_{\mathbb{R}_+} \left(Uf^*\right)(t)^p\, u(t)\, dt\right)^{\frac{1}{p}}\\ &= \left(\int_{\mathbb{R}_+} f^{**}(t)^p\, u_p(t)\, dt\right)^{\frac{1}{p}}\\ &= \rho_{p,u_p}(f).\end{aligned}$$

We thus have

Theorem 5.5 *Let p, q, u, u_p and v be as in Theorem 5.1. Then, given that $\Gamma_{p,u}(\mathbb{R}_+)$ is an interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$, with $\rho_{p,u}$ satisfying (5.8), one has*

$$\rho_{p,u}(\hat{f}) \le C\, \rho_{p,u_p}(f). \tag{5.9}$$

Moreover, there is no essentially smaller r.i.-norm that can replace ρ_{p,u_p} in (5.9).

Finally, there is a relatively simple condition sufficient to guarantee (5.2). It comes out of working with the inequality (5.5) and involves the norm of the dilation operator E_s as a mapping from $\Gamma_{q,v}(\mathbb{R}_+)$ to $\Gamma_{p,u_p}(\mathbb{R}_+)$, namely,

$$h(\Gamma_{q,v}, \Gamma_{p,u_p})(t) = \inf \left\{ M > 0 : \bar{\rho}_{p,u_p}(f(ts)) = \bar{\rho}_{p,u_p}((E_t f)(s)) \leq M \bar{\rho}_{q,v}(f) < \infty \right\}.$$

The argument in the proof of Theorem 4.1 of [16] ensures (5.5) provided

$$\int_1^\infty h(\Gamma_{q,v}, \Gamma_{p,u_p})(t) \frac{dt}{t} < \infty.$$

Again the argument in the proof of Theorem 5.2 in [10] yields

$$h(\Gamma_{q,v}, \Gamma_{p,u_p})(t) = \sup_{s>0} \frac{\left[\int_0^{s/t} u_p + (s/t)^p \int_{s/t}^\infty u_p(y) y^{-p} dy\right]^{\frac{1}{p}}}{\left[\int_0^s v + s^q \int_s^\infty v(y) y^{-q} dy\right]^{\frac{1}{q}}},$$

when $1 < q \leq p < \infty$.

Altogether, we have

Theorem 5.6 *Let p, q, u, u_p and v be as in Theorem 5.1, with $1 < q \leq p < \infty$. Then, given that $\Gamma_{p,u}(\mathbb{R}_+)$ is an interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$, one has (5.9) provided*

$$\int_1^\infty \sup_{s>0} \frac{\left[\int_0^{s/t} u_p + (s/t)^p \int_{s/t}^\infty u_p(y) y^{-p} dy\right]^{\frac{1}{p}}}{\left[\int_0^s v + s^q \int_s^\infty v(y) y^{-q} dy\right]^{\frac{1}{q}}} \frac{dt}{t} < \infty.$$

Proof The result follows from the preceeding discussion, since (5.9) and (5.5) are equivalent when $\Gamma_{p,u}(\mathbb{R}_+)$ is an interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$. □

We now consider the question of when $\Gamma_{p,u}(\mathbb{R}_+)$ is an interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$.

To begin, recall that $\rho_{p,u} \simeq \lambda_{p,u}$ was shown in [2] to be equivalent to the B_p condition

$$t^p \int_t^\infty u(s) \frac{ds}{s^p} \leq C \int_0^t u, \quad t \in \mathbb{R}_+. \tag{5.10}$$

We have

Theorem 5.7 *Fix $p \in (2, \infty)$ and a weight $0 \leq u \in M(\mathbb{R}_+)$. Suppose u satisfies the $B_{p/2}$ condition. Then, $\Gamma_{p,u}(\mathbb{R}_+)$ is an interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$.*

Proof The $B_{p/2}$ condition is necessary and sufficient in order that $\rho_{p/2,u} \simeq \lambda_{p/2,u}$. Thus,

$$\rho^{(2)}_{p/2,u}(f^*) \simeq \left[\int_{\mathbb{R}_+} (f^*(t)^2)^{\frac{p}{2}}\, u(t)\, dt\right]^{\frac{2}{p}\cdot\frac{1}{2}} = \lambda_{p,u}(f^*).$$

But, $B_{p/2}$ condition implies

$$t^p \int_t^\infty u(s)\frac{ds}{s^p} = \int_t^\infty u(s)\left(\frac{t}{s}\right)^p ds \le \int_t^\infty u(s)\left(\frac{t}{s}\right)^{\frac{p}{2}} ds \le C\int_0^t u, \quad t \in \mathbb{R}_+,$$

and so

$$\lambda_{p,u}(f^*) \simeq \rho_{p,u}(f^*), \quad f \in M(\mathbb{R}_+).$$

We conclude $\Gamma_{p,u}(\mathbb{R}_+) = L_{\rho^{(2)}_{p/2,u}}(\mathbb{R}_+)$ and hence, in view of Theorem 3.1, $\Gamma_{p,u}(\mathbb{R}_+)$ is an interpolation space between $L_2(\mathbb{R}_+)$ and $L_\infty(\mathbb{R}_+)$. □

Remark 5.1 G. Sinnamon in [27] proved that, given $u \in B_{p/2}$ and provided $0 < q \le 2 \le p < \infty$, one has (5.2) if and only if

$$\bar{\rho}_{p,u}(\chi_{(0,t)}) \le C\bar{\rho}_{q,v_q}(\chi_{(0,t)}), \quad t \in \mathbb{R}_+,$$

with $v_q(t) = v(t^{-1})\,t^{q-2}$, $t \in \mathbb{R}_+$. Theorem 5.7 and the fact that $\bar{\rho}_{p,u}(Uf^*) \simeq \bar{\rho}_{p,u_p}(f^*)$, ensures that, for $p \in [2,\infty)$ and *any* $q \in (1,\infty)$, one has (5.2) if and only if

$$\bar{\rho}_{p,u_p}(\chi_{(0,t)}) \le C\bar{\rho}_{q,v}(\chi_{(0,t)}), \quad t \in \mathbb{R}_+.$$

In the proof of Theorem 5.10 below we require a corollary of the following result of R. Sharpley from [25, Lemma 3.1, Corollary 3.2]

Theorem 5.8 *Let ρ be an r.i. norm on $M(\mathbb{R}^n)$. Suppose the fundamental indices of $L_{\bar{\rho}}(\mathbb{R}_+)$ lie in $(0,1)$. Given $p \in (1,\infty)$, set $\mu_p(t) = \frac{\bar{\rho}(\chi_{(0,t)})^p}{t}$, $t \in \mathbb{R}_+$. Then, $\bar{\rho}_{p,\mu_p}\left(\chi_{(0,t)}\right) = \bar{\rho}\left(\chi_{(0,t)}\right)$, $t \in \mathbb{R}_+$. Moreover,*

$$\bar{\rho}_{p,\mu_p}(f^*) \simeq \bar{\lambda}_{p,\mu_p}(f^*), \quad f \in M(\mathbb{R}^n).$$

Corollary 5.9 *Let $\rho = \rho_{p,u}$ be as in Theorem 5.8. Then, $\rho = \rho_{p,\mu_p}$, where $\mu_p(t) = \frac{\bar{\rho}(\chi_{(0,t)})^p}{t}$, $t \in \mathbb{R}_+$.*

Proof The spaces $\Gamma_{p,u}(\mathbb{R}^n)$ and $\Gamma_{p,\mu_p}(\mathbb{R}^n)$ have $\bar{\rho}_{p,\mu_p}\left(\chi_{(0,t)}\right) = \bar{\rho}_{p,u}\left(\chi_{(0,t)}\right)$, $t \in \mathbb{R}_+$. As such, the spaces are identical, in view of [10, Theorem 5.1]. □

The principal result of this section is

Theorem 5.10 *Fix $p \in [2, \infty)$ and $0 \le u \in M(\mathbb{R}_+)$, with $\int_{R_+} \frac{u(t)}{1+t^p} dt < \infty$. Suppose the fundamental indices of $\Gamma_{p,u}(\mathbb{R}^n)$ lie in $(0, 1)$. Then, $\Gamma_{p,u}(\mathbb{R}^n)$ is an interpolation space between $L_p(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$ (and hence between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$) if and only if*

$$\sup_{s \ge t} \frac{\bar{\rho}_{p,u}\left(\chi_{(0,s)}\right)^p}{s} \le C \frac{\bar{\rho}_{p,u}\left(\chi_{(0,t)}\right)^p}{t}, \tag{5.11}$$

for some $C > 0$ independent of $t \in \mathbb{R}_+$. Moreover, the optimal r.i. domain for $\mathscr{F}$ corresponding to $\Gamma_{p,u}(\mathbb{R}^n)$ has the norm

$$\bar{\rho}_{p,u}(Uf^*) \simeq \bar{\lambda}_{p,u}(Uf^*) = \bar{\rho}_{p,u_p}(f^*).$$

Proof Suppose first that $p = 2$. Given $T : L_2(\mathbb{R}^n), L_\infty(\mathbb{R}^n) \to L_2(\mathbb{R}^n), L_\infty(\mathbb{R}^n)$ one has, according to [9, Theorem 1.11, p. 301] and [12],

$$\int_0^t (Tf)^*(s)^2 ds \le C' M_2^2 \int_0^{Mt} f^*(s)^2 ds = C' M_2 M_\infty \int_0^t f^*(Ms)^2 ds,$$

$f \in (L_2 + L_\infty)(\mathbb{R}_+)$, in which $M = M_\infty/M_2$, M_k being the norm of T on $L_k(\mathbb{R}_+)$, $k = 2, \infty$. In view of (5.11), HLP yields

$$\begin{aligned}
\int_{\mathbb{R}_+} (Tf)^*(t)^2 \frac{\bar{\rho}_{2,u}\left(\chi_{(0,t)}\right)^2}{t} dt &\le \int_{\mathbb{R}_+} (Tf)^*(t)^2 \sup_{s \ge t} \frac{\bar{\rho}_{2,u}\left(\chi_{(0,s)}\right)^2}{s} dt \\
&\le C' M_2 M_\infty \int_{\mathbb{R}_+} f^*(Mt)^2 \sup_{s \ge t} \frac{\bar{\rho}_{2,u}\left(\chi_{(0,s)}\right)^2}{s} dt \\
&\le C C' M_2 M_\infty \int_{\mathbb{R}_+} f^*(Mt)^2 \frac{\bar{\rho}_{2,u}\left(\chi_{(0,t)}\right)^2}{t} dt,
\end{aligned}$$

$f \in (L_2 + L_\infty)(\mathbb{R}_+)$. Theorem 5.8 now ensures the latter is equivalent to

$$\int_{\mathbb{R}_+} (Tf)^{**}(t)^2 \frac{\bar{\rho}_{2,u}\left(\chi_{(0,t)}\right)^2}{t} dt \le C C' M_2 M_\infty h(M)^2 \int_{\mathbb{R}_+} f^{**}(t)^2 \frac{\bar{\rho}_{2,u}\left(\chi_{(0,t)}\right)^2}{t} dt,$$

where $h(t)$ is the norm of the dilation operator E_t on $\Gamma_{2,\mu_2}(\mathbb{R}_+) = \Gamma_{2,u}(\mathbb{R}_+)$, $\mu_2(s) = \frac{\bar{\rho}_{2,u}\left(\chi_{(0,s)}\right)^2}{s}$, by Corollary 5.9, that is, $T : \Gamma_{2,u}(\mathbb{R}^n) \to \Gamma_{2,u}(\mathbb{R}^n)$. Thus, $\Gamma_{2,u}(\mathbb{R}^n)$ is between $L_2(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$.

Suppose, next, $p > 2$. The "if" part of our theorem will follow in this case if we can show (5.11) implies the weight $w(t) = \frac{\bar{\rho}_{p,u}(\chi_{(0,t)})^p}{t}$ satisfies $B_{p/2}$ condition. But,

$$\begin{aligned}
t^{p/2}\int_t^\infty \frac{\bar{\rho}_{p,u}\left(\chi_{(0,s)}\right)^p}{s}\frac{ds}{s^{p/2}} &\le t^{p/2}\int_t^\infty \sup_{y\ge s}\frac{\bar{\rho}_{p,u}\left(\chi_{(0,y)}\right)^p}{y}\frac{ds}{s^{p/2}}\\
&\le t^{p/2}\sup_{y\ge t}\frac{\bar{\rho}_{p,u}\left(\chi_{(0,y)}\right)^p}{y}\int_t^\infty\frac{ds}{s^{p/2}}\\
&\le C\,t\,\frac{\bar{\rho}_{p,u}\left(\chi_{(0,t)}\right)^p}{t}\\
&\le C^2\int_0^t\frac{\bar{\rho}_{p,u}\left(\chi_{(0,s)}\right)^p}{s}ds, \quad t\in\mathbb{R}_+.
\end{aligned}$$

This completes the proof of "if" part.

As for the "only if" part we rely on a result of L. Maligranda [18] asserting that if $L_\rho(\mathbb{R}^n)$ is an interpolation space between $L_p(\mathbb{R}^n)$ and $L_\infty(\mathbb{R}^n)$, then

$$\frac{\bar{\rho}(\chi_{(0,s)})}{\bar{\rho}(\chi_{(0,t)})} \le C\max\left[\left(\frac{s}{t}\right)^{\frac{1}{p}}, 1\right]. \tag{5.12}$$

Indeed, for $t \le s$, (5.12) yields

$$\frac{\bar{\rho}(\chi_{(0,s)})}{\bar{\rho}(\chi_{(0,t)})} \le C\left(\frac{s}{t}\right)^{\frac{1}{p}}$$

or

$$\frac{\bar{\rho}(\chi_{(0,s)})^p}{s} \le C\,\frac{\bar{\rho}(\chi_{(0,t)})^p}{t},$$

from which (5.11) follows. □

To this point the Lorentz Gamma range norms have been equivalent to functionals of the form

$$\lambda_{p,u}(f) = \left[\int_{\mathbb{R}_+} f^*(s)^p u(t)dt\right]^{\frac{1}{p}}.$$

This need not be the case for the $\rho_{2p,u}$ in Theorem 5.12 below.

Lemma 5.11 *Fix* $p \in (1,\infty)$ *and* $0 \le u \in M(\mathbb{R}_+)$, *with*

$$\int_{\mathbb{R}_+}\frac{u(t)}{1+t^p}dt < \infty.$$

Then,

$$\left(\int_{\mathbb{R}_+} f^{**}(t)^p\, u(t)\, dt\right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}_+} f^{*}(t)^p\, u^{(p)}(t)\, dt\right)^{\frac{1}{p}}, \quad f \in M(\mathbb{R}_+), \tag{5.13}$$

where

$$u^{(p)}(t) = p\, t^{p-1} \int_t^\infty u(s) s^{-p} ds, \quad t \in \mathbb{R}_+;$$

moreover, $u^{(p)}$ is essentially the smallest weight for which (5.13) holds.

Proof It is shown in [21] that

$$\left(\int_{\mathbb{R}_+} f^{**}(t)^p\, u(t)\, dt\right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}_+} f^{*}(t)^p\, v(t)\, dt\right)^{\frac{1}{p}}, \quad f \in M(\mathbb{R}_+),$$

if and only if

$$\int_0^t u(s)ds + t^p \int_t^\infty u(s)s^{-p}ds \leq C \int_0^t v, \quad t \in \mathbb{R}_+.$$

But,

$$\begin{aligned} \int_0^t u^{(p)}(s)ds &= \int_0^t p\, s^{p-1} \int_s^\infty u(y)y^{-p}dy\, ds \\ &= \int_0^t p\, s^{p-1} \int_s^t u(y)y^{-p}dy\, ds + \left[\int_0^t p\, s^{p-1}ds\right]\left[\int_t^\infty u(s)s^{-p}ds\right] \\ &= \int_0^t \left(\int_0^y p\, s^{p-1}ds\right) u(y)y^{-p}dy + t^p \int_t^\infty u(s)s^{-p}ds \\ &= \int_0^t u + t^p \int_t^\infty u(s)s^{-p}ds. \end{aligned}$$

We conclude that

$$\left(\int_{\mathbb{R}_+} f^{**}(t)^p\, u(t)\, dt\right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{R}_+} f^{*}(t)^p\, u^{(p)}(t)\, dt\right)^{\frac{1}{p}}, \quad f \in M(\mathbb{R}_+).$$

□

Theorem 5.12 *Let p and u be as in Lemma 5.11. Then,*

$$\rho_{2p,u}(\widehat{f}) = \bar{\rho}_{2p,u}((\widehat{f})^*) \leq C\bar{\rho}_{2p,u_{2p}^{(p)}}(f^*) = \rho_{2p,u_{2p}^{(p)}}(f), \tag{5.14}$$

where

$$u_{2p}^{(p)}(t) = u^{(p)}(t^{-1})\, t^{2p-2} = p(t^{-1})^{p-1}\left(\int_{t^{-1}}^{\infty} u(s)s^{-p}ds\right) t^{2p-2} = p\, t^{p-1}\int_{t^{-1}}^{\infty} u(s)s^{-p}ds.$$

Proof Applying the construction in (3.1) to the functionals in (5.13) yields

$$\left(\int_{\mathbb{R}_+}\left(t^{-1}\int_0^t f^*(s)^2ds\right)^p u(t)\,dt\right)^{\frac{1}{2p}} \le \left(\int_{\mathbb{R}_+} f^*(t)^{2p}\, u^{(p)}(t)\,dt\right)^{\frac{1}{2p}} = \lambda_{2p,u^{(p)}}(f).$$

Again,

$$\left(t^{-1}\int_0^t f^*(s)ds\right)^{2p} \le \left(t^{-1}\int_0^t f^*(s)^2ds\right)^p$$

by Hölder's inequality.

Hence, using HLP in (2.8), yields

$$\begin{aligned}
\rho_{2p,u}(\widehat{f}) &\le \rho_{p,u}(|\widehat{f}|^2)^{1/2} \\
&\le C\rho_{p,u}((Uf^*)^2)^{1/2} \\
&\le C\lambda_{2p,u^{(p)}}(Uf^*) \\
&= C\left(\int_{\mathbb{R}_+}(Uf^*)(t)^{2p}\, u^{(p)}(t)\,dt\right)^{\frac{1}{2p}} \\
&= C\left(\int_{\mathbb{R}_+} f^{**}(t)^{2p}\, u_{2p}^{(p)}(t)\,dt\right)^{\frac{1}{2p}} \\
&= C\rho_{2p,u_{2p}^{(p)}}(f).
\end{aligned}$$

□

Example 5.1 Fix p, $1 < p < \infty$, and set

$$u(t) = \begin{cases} t^{2p-1}\left(\log\frac{1}{t}\right)^{-\alpha}, & 0 < t < 1, \\ t^{p-1-\alpha}, & t > 1, \end{cases}$$

with $0 < \alpha < 1$. Then, one has

$$\rho_{2p,u}(f) \not\simeq \lambda_{2p,u}(f), \quad f \in M(\mathbb{R}_+),$$

or, equivalently,

$$t^{2p}\int_t^{\infty} u(s)s^{-2p}ds \le C\int_0^t u, \quad t \in \mathbb{R}_+, \tag{5.15}$$

does not hold. Indeed, the left hand side of (5.15) is equal to $Ct^{2p}\left(\log \frac{1}{t}\right)^{-\alpha+1}$, while the right hand side is

$$\int_0^t u = \int_0^t s^{2p-1}\left(\log \tfrac{1}{s}\right)^{-\alpha} \simeq t^{2p}\left(\log \tfrac{1}{t}\right)^{-\alpha}, \quad 0 < t < 1,$$

in view of L'Hôspital rule. The ratio of the left side to the right side in (5.15) is, essentially, $\log \frac{1}{t}$ which $\to \infty$ as $t \to 0^+$.

6 Other Work

Inequalities involving Fourier transform other than those considered in this paper are weighted Lebesgue inequalities

$$\left(\int_{\mathbb{R}^n} |\hat{f}(x)\, w(x)|^q\, dx\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n} |f(x)\, v(x)|^p\, dt\right)^{\frac{1}{p}}$$

and weighted Lorentz inequalities

$$\left(\int_{\mathbb{R}_+} (\hat{f})^*(t)^q\, w(t)\, dt\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}_+}\left(\int_0^{1/t} f^*\right)^p v(t)\, dt\right)^{\frac{1}{p}},$$

in which $0 \le v, w \in M(\mathbb{R}^n)$ and $1 < p, q < \infty$.

In both [15, 20] conditions are for the Lebesgue inequalities that apply not just to w and v but to all weights equimeasurable with them. The extreme cases of these are the decreasing rearrangement, W, of w and the increasing rearrangement, V, of v. This reduces the considerations to the case $w \downarrow$ and $v \uparrow$.

The weighted Lebesgue inequalities are shown in [15, 20] to be equivalent to inequalities of the form, for example when $1 \le p \le q \le \infty$,

$$\left(\int_0^{t^{-1}} w\right)^{\frac{1}{q}}\left(\int_0^t w^{-\frac{1}{p-1}}\right)^{\frac{q}{p'}} \le B, \quad t \in \mathbb{R}_+. \tag{6.1}$$

In [15] the sufficiency is proved using the inequality (2.8) from [13]. The necessity comes out of the inequality

$$\rho_q\left((\hat{f})^* w\right) \simeq \rho_2\left(Uf^* w\right)$$

from [15]. The proofs in [20] are more complicated. The conditions for the weighted Lorentz inequalities are similar to (6.1).

A brief survey of papers on these inequalities, from the pioneering work of Benedetto and Heinig [5] through that of G. Sinnamon [26] and Rastegari and Sinnamon [24], is given in the paper [22] of Nursultanov and Tikhonov.

In this paper we have seen the behaviour of $\mathscr{F}$ on r.i. spaces depends on its action on radially decreasing functions. But what about the size of f if $\hat{f}$ is radially decreasing? This question is taken up in [11] in the context of Fourier series where functions with a cosine series having decreasing coefficients as $|n| \to \infty$ are studied.

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