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The Hadamard–Bergman Convolution on the Half-Plane

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Abstract
We introduce the Hadamard–Bergman convolution on the half-plane. We show that it exists in terms of the Hadamard product and it is commutative on the Bergman space (more appropriately called the Bergman–Jerbashian space) in the half-plane. Further, we explore mapping properties of the generalized Bergman-type operators with exponential weights in weighted Bergman spaces in the half-plane. Finally, we deduce sharp inclusions for weighted Bergman spaces, from corresponding Sobolev-type inequalities.

Keywords Hadamard–Bergman convolution · Spaces of holomorphic functions · Laplace–Fourier transform

Mathematics Subject Classification 47G10 · 47B38 · 46E30

1 Introduction

1.1 The Hadamard–Bergman Convolution on the Half-Plane

Denote by Π the open right half-plane,

$$\Pi = \{z \in \mathbb{C}\colon Re(z) > 0\}.$$

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Denote by $Hol(\Pi)$ the class of functions F that are holomorphic in the right half-plane. Recall, that weighted Bergman spaces are defined as follows (see e.g. [2], Definition 1.1 on p. 1):

Definition 1.1 For $\alpha > -1$ the weighted Bergman space $\mathcal{A}_\alpha$ is the space of functions F holomorphic in the right half-plane, $F \in Hol(\Pi)$, equipped with the norm

$$||F||_{\mathcal{A}_\alpha} = \left[\iint_\Pi |F(x+iy)|^2 x^\alpha dxdy \right]^{1/2}.$$

For brevity we denote by $\mathcal{A} = \mathcal{A}_0$ the non-weighted Bergman space.

Remark 1.2 After this article was completed, the authors came across the newly published book [5] and the survey article [6], where it was stated that the weighted Bergman spaces are more appropriately called the Bergman–Jerbashian spaces.

Definition 1.3 We recall that the Hadamard–Bergman convolution operator (or the Hadamard–Bergman product) on the disc was defined in [9] as follows:

$$\begin{aligned} \mathbb{K}_g f(z) &= \int_{\mathbb{D}} g(w) f(z\overline{w}) \mathrm{dA}(w) \\ &= \lim_{r\to 1} \int_{|w|<r} g(w) f(z\overline{w}) \mathrm{dA}(w), \quad f \in Hol(\mathbb{D}), \end{aligned} \tag{1.1}$$

where

$$\mathrm{dA}(w) = \frac{1}{\pi} dxdy, \quad w = x + iy$$

is the normalized Lebesgue area measure on the unit disc $\mathbb{D}$. The integral in (1.1) is understood in improper sense.

See also [7, 8, 10] for a study of more general constructions (variable operators) than that of (1.1), and [11] for the developing of the theory for general weighted Lebesgue (Bergman) spaces and some other classical spaces of complex analysis.

In an analogy to Definition 1.3, we formally introduce the Hadamard–Bergman convolution $I(F, G)$ on the half-plane Π as follows.

Definition 1.4 Given $F, G \in Hol(\Pi)$ the Hadamard–Bergman product (or convolution) $I(F, G)$ of two functions $F, G \in Hol(\Pi)$ on the half-plane Π is formally defined as:

$$I(F, G)(z) = \iint_\Pi G(w) F(z + \overline{w})\, dw d\overline{w}, \quad \text{for } z \in \Pi. \tag{1.2}$$

The expression (1.2) written formally, will be interpreted in a precise way in Definition 3.6, in terms of the Hadamard product of Laplace preimages of functions F and G.

We can also fix the function G referring to it as a kernel and hence we can consider (1.2) as the corresponding Hadamard–Bergman convolution operator $F \mapsto I(F, G)$.

The idea of introducing and studying the Hadamard–Bergman operators is basically due to the importance of generalizing the classical Bergman operator and operators of fractional integro-differentiation (Bergman-type operators) to a more general class of integral operators. In addition to its purely fundamental significance, we note that the consideration of such integral operators has a direct application to the description of classes of functions, for example, Hölder functions. These studies have been done for the case of a disc [7, 10], and we plan to further develop the topic in the context of a half plane.

We first explore properties of the Hadamard–Bergman convolution in Bergman spaces. Specifically, in Sect. 4.1 we prove:

Theorem 1.5 *Let the functions F, G belong to the Bergman space $\mathcal{A}$. Then the Hadamard–Bergman convolution $I(F, G)$ exists in terms of the Hadamard product, and it is commutative $I(F, G) = I(G, F)$.*

This theorem provides specific conditions under which a convolution exists, in a tangible manner. At the same time, the article also presents a broader result on the existence of the convolution (see Theorem 3.7). Yet, these conditions lack constructiveness, leaving open the intriguing problem of devising a constructive description for the most comprehensive set of holomorphic functions where convolution is both defined and exists pointwise.

1.2 Generalized Bergman-Type Operators

We now introduce and explore two more integral operators. Namely, we discuss the generalized Bergman-type power operator R_β and the generalized Bergman-type exponential operator $\mathrm{S}_{\beta,\sigma}$ (see Definitions 1.6 and 1.7, correspondingly). The operator R_β, generally speaking, is a realization of the operator of fractional integration or differentiation, depending on whether $\beta < 2$ or $\beta > 2$ (and it gives the holomorphic projection for $\beta = 2$).

Further study of such operators is promising for describing spaces of holomorphic functions on the half-plane, as well as in the context of the action of these operators in classes of holomorphic Hölder functions. Meanwhile, here in the paper, the operator $\mathrm{S}_{\beta,\sigma}$ is apparently being introduced and studied for the first time.

Definition 1.6 For a parameter $\beta > 0$ and a function

$$F : \Pi \to \mathbb{C}, \tag{1.3}$$

formally define the generalized Bergman-type power operator R_β by the following formula:

$$\left(\mathrm{R}_\beta F\right)(z) = \iint_\Pi \frac{F(w)}{(z + \overline{w})^\beta} dw d\overline{w}, \quad \text{for } z \in \Pi.$$

Definition 1.7 For parameters $\beta > 0$ and $\sigma \geq 0$, and a function

$$F\colon \Pi \to \mathbb{C}, \tag{1.4}$$

formally define the generalized Bergman-type exponential operator $\mathrm{S}_{\beta,\sigma}$ by the following formula:

$$\left(\mathrm{S}_{\beta,\sigma} F\right)(z) = \iint_{\Pi} \frac{F(w)e^{-\sigma w}}{(z+\overline{w})^{\beta}} dw d\overline{w}, \quad \text{for } z \in \Pi.$$

Remark 1.8 We justify studying operators $\mathrm{S}_{\beta,\sigma}$ whose kernels involve *exponential* weight $e^{-\sigma w}$ as follows. For a fixed $\sigma \geq 0$ consider the curves Γ_c where the exponential weight $e^{-\sigma w}$ is constant,

$$\Gamma_c = \{w \in \Pi\colon e^{-\sigma w} = c\}.$$

Observe that the distance from a point on the curve Γ_c to the boundary $\partial\Pi$ of the domain Π does not depend on the actual choice of the point; it only depends on parameters σ and c. This situation is analogous to the case of power weights on the unit disc in complex plane $\mathbb{C}$.

Indeed, for a fixed $n \in \mathbb{N}$ consider the curves γ_c that consist of those points z in the unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$, for whom the absolute value of the power function z^n is equal to some constant c,

$$\gamma_c = \{z \in \mathbb{D}\colon \left|z^n\right| = c\}.$$

Observe that the distance from a point on the curve γ_c to the boundary $\partial\mathbb{D}$ of the disc $\mathbb{D}$ does not depend on the actual choice of the point; it only depends on parameters n and c.

Our aim is to explore the mapping properties of integral operators R_β and $\mathrm{S}_{\beta,\sigma}$ in weighted Bergman spaces $\mathcal{A}_\alpha$. Namely, concerning the operator $\mathrm{S}_{\beta,\sigma}$, in Sect. 4.2 we prove:

Theorem 1.9 *Let $\sigma \geq 0, \alpha > -1, \alpha + 2\beta - 4 > -1$. Then the operator $\mathrm{S}_{\beta,\sigma}$ maps the Banach space $\mathcal{A}_\alpha$ into the Banach space $\mathcal{A}_{\alpha+2\beta-4}$.*

In particular, by taking $\sigma = 0$ in Theorem 1.9, we obtain:

Corollary 1.10 *Let $\alpha > -1, \alpha+2\beta-4 > -1$. Then the operator R_β maps the Banach space $\mathcal{A}_\alpha$ into the Banach space $\mathcal{A}_{\alpha+2\beta-4}$.*

1.3 Sharp Inclusions for Weighted Bergman Spaces

It is typical for novel studies to encounter unexpected yet comprehensible effects and consequences that establish connections between the subject of the inquiry and the classical theory. For instance, in our case, we deduce Corollary 2.17 on embedding of

Sobolev spaces from a generalized Cafarelli-Kohn-Nirenberg inequality 2.16 proved by Rabier [13].

Further, we showcase how this phenomena intertwines modern research with classical frameworks. Specifically, in Sect. 4.3, from the mentioned Corollary 2.17, we deduce the following theorem regarding sharp inclusions for weighted Bergman spaces:

Theorem 1.11 *Let for* $\alpha, \alpha' > -1$ *we have*

$$F(z) \in \mathcal{A}_{\alpha},$$
$$zF(z) \in \mathcal{A}_{\alpha'}.$$

Then the inclusion

$$F(z) \in \mathcal{A}_{\alpha''}$$

holds with a non-trivial $\alpha'' \neq \alpha$ *if and only if*

- α *and* α' *are on the same side of* 0 *(including* 0*);* $\alpha \neq \alpha' + 2$ *;* α'' *is strictly between* α *and* α'*; and*

$$\frac{\alpha - \alpha''}{\alpha - \alpha'} > 0,$$

- *or* α *and* α' *are on strictly opposite sides of* 0 *;* α'' *is strictly in between* α *and* 0 *; and*

$$\frac{\alpha - \alpha''}{\alpha - \alpha'} \geq 0,$$

- *or* $\alpha \geq 0$ *and* $\alpha'' = \alpha'$*.*

The remainder of this paper is structured as follows: the next Sect. 2 surveys some well-known definitions and results that are necessary for the succeeding Sect. 3; the latter will discuss the continuous version of the Hadamard product; the proofs of the results announced in Introduction are provided in Sect. 4.

2 Auxiliary Results

2.1 Laplace Transform

Here we collect some standard definitions and known results regarding the action of Laplace transform on: weighted L^2 spaces, the class $\mathcal{E}$ of functions of sub-exponential growth, and the class of $\mathcal{E}^g$ of functionals of sub-exponential growth. These results will be used to phrase and carry out our proofs.

Definition 2.1 Recall that the Laplace transform $\mathrm{L}f$ of a complex valued function

$$f : [0, +\infty) \to \mathbb{C}$$

is formally defined as follows:

$$(\mathrm{L}f)(z) = \int_0^{+\infty} f(t)e^{-zt}\,dt. \tag{2.1}$$

The following theorem (see e.g. [4], Theorem 1, p. 2) describes weighted Bergman spaces in terms of the Laplace transform.

Theorem 2.2 *For $\alpha > -1$ the Laplace transform is an isometric isomorphism between the weighted space*

$$L^2\left([0, +\infty), t^{-(\alpha+1)}dt\right)$$

and the weighted space $\mathcal{A}_\alpha$.

Being motivated by [14, Chap. 6.1], we introduce the class of sub-exponential functions $\mathcal{E}$—a rather large space of functions whose Laplace transform turns to be holomorphic in the open right half-plane:

Definition 2.3 Denote by $\mathcal{E}$ the class of sub-exponential functions. That is, the class of those locally-integrable complex valued functions

$$f \in L^1_{loc}\left([0, +\infty)\right)$$

that satisfy the following sub-exponential growth condition:

$$f(t)e^{-\varepsilon t} \in L^1([0, +\infty)), \quad \text{for all } \varepsilon > 0. \tag{2.2}$$

The following example of a function in $\mathcal{E}$ is borrowed from [14, p. 257]:

Example 2.4 For a parameter $\beta > 0$, consider the function $g_\beta : [0, +\infty) \to \mathbb{R}$ defined by

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}.$$

We have $g_\beta \in \mathcal{E}$, and the Laplace transform $\mathrm{L}g_\beta$ may be written down explicitly,

$$\left(\mathrm{L}g_\beta\right)(z) = \frac{1}{z^\beta}, \quad \text{for } z \in \Pi.$$

Having introduced the class $\mathcal{E}$, consider $\mathrm{L}\left(\mathcal{E}\right)$, i.e. the class of images of functions from $\mathcal{E}$ under the Laplace transform. Remarks 2.5, 2.6, 2.7 highlight the way $\mathrm{L}\left(\mathcal{E}\right)$ compares with some familiar classes of functions.

Remark 2.5 ([14, p. 247]). The following inclusion holds:

$$\mathrm{L}\,(\mathcal{E}) \subset Hol(\Pi).$$

Remark 2.6 ([12, Theorem 4, p. 502]). Let for a function F the following three conditions hold:

- $F \in Hol(\Pi)$,
- $F(z) \to 0$ as $|z| \to \infty$ in any halfspace $Re(z) \geq \varepsilon$ uniformly with respect to $\arg(z)$. In other words, for every $\epsilon > 0$ we have

$$\sup_{z:\ Re(z)>\varepsilon,\ |z|=r} |F(z)| \to 0, \quad \text{as } r \to \infty,$$

- for any $a > 0$ the restriction of the function F to the axis $Re(z) = a$ is in $L_1(\mathbb{R})$. In other words

$$\int_{a+i\mathbb{R}} |F(z)|\, dz < +\infty.$$

Then $F \in \mathrm{L}\,(\mathcal{E})$.

We point out that the following remark holds:

Remark 2.7 Let $f \in \mathcal{E}$. Assume that the limit of the function f (finite or infinite) at 0 exists. Then the initial value theorem for the Laplace transform (as phrased in e.g. [1, Sect. 13.2]) claims

$$\lim_{z\in\Pi,\ z\to\infty} z \cdot (\mathrm{L}f)(z) = \lim_{t\to 0^+} f(t). \tag{2.3}$$

Consequently, if the latter limit is finite, then formula (2.3) imposes the following restriction on the growth of the function $F = \mathrm{L}f \in \mathrm{L}\,(\mathcal{E})$ at infinity: there exist constants $C > 0$ and $A > 0$ such that

$$|F(z)| \leq \frac{C}{|z|}, \quad \text{for all } Re(z) \geq A.$$

2.2 Generalized Functions

We now briefly recall some concepts from the theory of generalized functions.

Definition 2.8 Being motivated by [12, Chap. 6.1, p. 532], define the class of test functions $\mathcal{T}$ to consist of complex-valued infinitely-differentiable functions of a single positive real variable, whose supports are compact subsets of the positive semi-axis $(0, +\infty)$. Convergence $\varphi_n \to \varphi$ in the class of test functions $\mathcal{T}$ means that all functions $\varphi_n - \varphi$ vanish outside of a compact subset of the positive semi-axis $(0, +\infty)$, and that for all $m \in Z_+$ we have

$$\varphi_n^{(m)}(t) \to \varphi^{(m)}(t), \quad \text{as } n \to +\infty \tag{2.4}$$

where the convergence in (2.4) is uniform on the positive semi-axis $t \in (0, +\infty)$.

Definition 2.9 Correspondingly, define the class of test functions $\mathcal{T}_2$ to consist of complex-valued infinitely-differentiable functions of two positive real variables, whose supports are compact subsets in the first quadrant $(0, +\infty) \times (0, +\infty)$.

Definition 2.10 Following [12, Chap. 6.1, p. 536], define yet another class of test functions $\mathcal{B}$ to consist of complex-valued infinitely-differentiable functions of one real variable φ such that

$$\lim_{t\to+\infty} t^l \varphi^{(m)}(t) = 0, \quad \text{for all} \quad l, m \in \mathbb{Z}_+.$$

Further, following [12, Chap. 6.1, p. 536], convergence $\varphi_n \to \varphi$ in the class of test functions $\mathcal{B}$ means that for all $l, m \in \mathbb{Z}_+$ we have

$$t^l \varphi_n^{(m)}(t) \to t^l \varphi^{(m)}(t), \qquad \text{as} \qquad n \to +\infty \tag{2.5}$$

where the convergence in (2.5) is uniform on any ray $t \in [a, +\infty)$, $a > 0$.

Definition 2.11 Denote by $\mathcal{T}^*$ and $\mathcal{B}^*$ the duals of spaces $\mathcal{T}$ and $\mathcal{B}$.

Definition 2.12 Finally, introduce the space $\mathcal{E}^g$ of sub-exponential distributions (functionals, generalized functions) as the space of those linear bounded functionals $f \in \mathcal{T}^*$ for whom

$$f e^{-\varepsilon t} \in \mathcal{B}^*, \quad \text{for all } \varepsilon > 0. \tag{2.6}$$

It turns out that the operator L is well-defined on $\mathcal{E}^g$ and that it maps $\mathcal{E}^g$ into $Hol(\Pi)$ (see e. g., [12, Chap. 6.1, p. 536] for details). Moreover, regarding the operator L, we have the following uniqueness result in the class $\mathcal{E}^g$ (see [14, p. 257]).

Theorem 2.13 (Schwartz, [14, p. 2577]) *Let $f \in \mathcal{E}^g$. Then the condition*

$$(\mathrm{L} f)(z) = 0, \quad \textit{for all } z \in \Pi$$

implies that f is the zero of the space $\mathcal{T}^*$.

Recall that Remarks 2.5, 2.6, 2.7 describe the class $\mathrm{L}(\mathcal{E})$ in terms of inclusions into some well-known spaces. In comparison, the class $\mathrm{L}\,(\mathcal{E}^g)$ is completely described as follows (see [14, p. 258]):

Theorem 2.14 (Schwartz, [14, p. 258]) *The class $\mathrm{L}\,(\mathcal{E}^g)$ coincides with the class of functions $F \in Hol(\Pi)$ satisfying the sub-polynomial growth condition. That is, a function F, holomorphic in the right half-plane, $F \in Hol(\Pi)$, is the Laplace transform of a distribution $f \in \mathcal{E}^g$ if and only if there exists $k \in \mathbb{N}$ such that for all $\varepsilon > 0$ we have*

$$\sup_{z:\ Re(z)>\varepsilon} \frac{|F(z)|}{|z|^k} < +\infty.$$

2.3 Sobolev Spaces

We recall the definition of Sobolev spaces and formulate a generalized Cafarelli–Kohn–Nirenberg inequality proved by Rabier.

Definition 2.15 Given $a, b \in \mathbb{R}$ and $1 \leq p, q < \infty$, denote

$$\mathbb{R}_* = \mathbb{R} \setminus \{0\}$$

and consider the weighted Sobolev space

$$W^{1,(q,p)}_{\{a,b\}}$$

that consists of functions $u \in L^1_{loc}\left(R^N_*\right)$ who satisfy the following two conditions:

$$u \in L^q\left(\mathbb{R}^N : |x|^\alpha dx\right),$$
$$\nabla u \in \left(L^p\left(\mathbb{R}^N : |x|^b dx\right)\right)^N,$$

and that is equipped with the norm

$$||u||_{a,q} + ||\nabla u||_{b,p}\,.$$

In Sect. 4.3 we will need a result on embedding of the weighted Sobolev space $W^{1,(q,p)}_{\{a,b\}}$ into the weighted space $L^p\left(\mathbb{R}^N : |x|^c dx\right)$. A central result of that type is the Cafarelli–Kohn–Nirenberg inequality [3]. For our aims, we need to utilize a generalized Cafarelli–Kohn–Nirenberg inequality proved by Rabier [13]. Specifically, we single out points (i), (ii), (iv)-1 of Theorem 1.1 in [13] (as those are the points that, when specified to our assumptions, provide a non-trivial result) and phrase it as a separate Theorem 2.16. We preface Theorem 2.16 by the following notations, borrowed from formulas (1.3), (1.4) of [13, p. 2]. For given values of a, b, c, r define:

$$p^* = \infty, \quad \text{if } p \geq N,$$
$$p^* = \frac{Np}{N-p}, \quad \text{if } 1 \leq p < N,$$
$$c^0 = \frac{r(a+N)}{q} - N,$$
$$c^1 = \frac{b-p+N}{p} - N,$$
$$\theta_c = \frac{c-c^0}{c^1-c^0}, \quad \text{if points } c_0 \text{ and } c_1 \text{ are distinct.}$$

Theorem 2.16 (Rabier, [13]) *Let $a, b, c \in \mathbb{R}$ and $1 \le p, q, r < \infty$ ($1 \le p < \infty$ and $0 < q, r < \infty$ if $N = 1$). Then the following continuous embedding of Banach spaces holds:*

$$W^{1,(q,p)}_{\{a,b\}}\left(\mathbb{R}^N_*\right) \hookrightarrow L^r\left(\mathbb{R}^N,\ |x|^c dx\right). \tag{2.7}$$

with a non-trivial constant $c \ne a$ if and only if $r \le max\{p^*, q\}$ and

- *either a and $b - p$ are on the same side of $-N$ (including $-N$), $\frac{a+N}{q} \ne \frac{b-p+N}{p}$, c is in the open interval with endpoints c^0 and c^1 and*

$$\theta_c\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \le \frac{1}{r} - \frac{1}{q},$$

- *or a and $b - p$ are strictly on the opposite sides of $-N$, c is in the open interval with endpoints c^0 and $-N$ and*

$$\theta_c\left(\frac{1}{p} - \frac{1}{N} - \frac{1}{q}\right) \le \frac{1}{r} - \frac{1}{q},$$

- *or $p \le r \le p^*, a \le -N$ and $b - p < -N$, $c = c^1$,*

Corollary 2.17 *Apply Theorem 2.16 with*

$$\begin{aligned}
&N = 1,\\
&p = q = r = 2,\\
&a = -(\alpha + 1) < 0,\ b = -\left(\alpha' + 1\right) < 0,\ c = -\left(\alpha'' + 1\right) < 0,\\
&c^0 = a,\ c^1 = b,\\
&p^* = +\infty,\\
&\theta_c = \frac{c - a}{b - a}, \quad \text{if } a \ne b,
\end{aligned}$$

to get that the conditions

$$u \in L^2\left([0, +\infty), t^{-(\alpha+1)} dt\right)$$

and

$$u' \in L^2\left([0, +\infty), t^{-(\alpha'+1)} dt\right)$$

imply

$$u \in L^2\left([0, +\infty), t^{-(\alpha''+1)} dt\right)$$

if and only if the conditions of Theorem 1.11 hold.

3 Symbolic Calculus

The following auxiliary lemma will be later used in Lemma 3.4.

Lemma 3.1 *The following two expressions are equal as functionals in* $\mathcal{T}_2^*$:

$$\frac{\delta(t-s)}{t+s} = \left(\int_0^{+\infty} e^{-(t+s)x}dx\right)\left(\int_{-\infty}^{+\infty} e^{i(t-s)y}dy\right), \quad \text{for } t, s > 0, \tag{3.1}$$

where δ *is the Dirac delta function.*

Proof First note that the following two expressions are equal as functions of variables $t, s > 0$:

$$\frac{1}{t+s} = \int_0^{+\infty} e^{-(t+s)x}dx. \tag{3.2}$$

Assumption 3.2 Assume that $h \in \mathcal{T}_2$ is a test function. In particular, this implies that the support of the function h is a compact subset in the first quadrant $(0, +\infty) \times (0, +\infty)$.

Under Assumption 3.2, the following Fourier inversion formula holds:

$$h(t,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} h(t,s)e^{-isy}ds\right)e^{ity}dy, \quad \text{for } t > 0. \tag{3.3}$$

We integrate both sides of the Fourier inversion formula (3.3) from $t = 0$ to $t = +\infty$ to get

$$\int_0^{+\infty} h(t,t)dt = \int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} h(t,s)e^{-isy}ds\right)e^{ity}dydt. \tag{3.4}$$

On the one hand, we may express the left side of Eq. (3.4) in terms of the Dirac delta function as follows:

$$\int_0^{+\infty} h(t,t)dt = \int_0^{+\infty}\int_0^{+\infty} h(t,s)\delta(t-s)dtds. \tag{3.5}$$

On the other hand, due to Assumption 3.2, the function $h(t, s)$ is absolutely integrable on $[0, +\infty) \times [0, +\infty)$. Additionally,

$$\left|e^{i(t-s)y}\right| \leq 1, \quad \text{for } 0 < s, t < +\infty, -\infty < y < +\infty.$$

Thus, we may apply Fubini's theorem to the right handside of (3.4) to get

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty} h(t,s)e^{-isy}ds\right)e^{ity}dydt =$$

$$= \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{0}^{+\infty} h(t, s) e^{i(t-s)y} ds dt dy. \tag{3.6}$$

Using (3.5) and (3.6), we rewrite the equality (3.4) as follows:

$$\int_{0}^{+\infty} \int_{0}^{+\infty} h(t, s)\delta(t-s) dt ds = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{0}^{+\infty} h(t, s) e^{i(t-s)y} ds dt dy. \tag{3.7}$$

We now proved that the equality (3.7) holds under Assumption 3.2. Therefore, the following two functionals in $\mathcal{T}_2^*$ are equal:

$$\delta(t-s) = \int_{-\infty}^{+\infty} e^{i(t-s)y} dy. \tag{3.8}$$

The two claims (3.2) and (3.8) prove Lemma 3.1. □

Our arguments will make use of the following definition that introduces the continuous version of the discrete Hadamard product.

Definition 3.3 Let $f, g \colon [0, +\infty) \to \mathbb{C}$ be such that

$$\frac{f(t)g(t)}{t} \in \mathcal{E}. \tag{3.9}$$

Define (the continuous version of) the Hadamard product of functions f and g by the following formula:

$$P(f, g)(z) = \int_{0}^{+\infty} f(t)g(t)e^{-zt} \frac{dt}{2t}, \quad \text{for } z \in \Pi. \tag{3.10}$$

Note that the function $P(f, g)$ is holomorphic in the right half-plane Π.

The following lemma relates the Hadamard–Bergman convolution $I(F, G)$, to the Hadamard product $P(f, g)$ defined by formula (3.10):

Lemma 3.4 *For test functions*

$$f, g \in \mathcal{T} \tag{3.11}$$

we have

$$P(f, g)(z) = I(\mathrm{L}f, \mathrm{L}g)(z), \quad z \in \Pi. \tag{3.12}$$

Proof Under Assumption (3.11) the condition (3.9) applies, and the Hadamard product $P(f, g)$ is well-defined by formula (3.10). Now consider the measure

$$\frac{\delta(t-s)}{t+s} dt ds$$

on $\mathbb{R} \times \mathbb{R}$. In terms of that measure, we can rewrite the Hadamard product $P(f, g)$ as a double integral,

$$P(f,g)(z) = \int_0^{+\infty} \int_0^{+\infty} f(t)g(s)e^{-tz}\frac{\delta(t-s)}{t+s}dtds, \quad \text{for } z \in \Pi. \tag{3.13}$$

Further, we may interpret the expression

$$\frac{\delta(t-s)}{t+s}, \quad \text{for } t, s > 0.$$

that appears on the right side of (3.13) as a functional in $\mathcal{T}_2^*$. Correspondingly, we may interpret the whole right side of (3.13) as the value of that functional when evaluated on the function

$$e^{-zt} f(t)g(s) \in \mathcal{T}_2.$$

In other words,

$$P(f,g)(z) = \left\langle \frac{\delta(t-s)}{t+s}, e^{-tz} f(t)g(s) \right\rangle, \quad \text{for } z \in \Pi. \tag{3.14}$$

We now apply (3.1) to rewrite the right had side of (3.14) as follows:

$$\left\langle \left(\int_0^{+\infty} e^{-(t+s)x} dx \right) \left(\int_{-\infty}^{+\infty} e^{i(t-s)y} dy \right), e^{-tz} f(t)g(s) \right\rangle. \tag{3.15}$$

By Fubini's Theorem for distributions (see e.g. [15, pp. 416–419]), we rewrite the expression in (3.15) as follows

$$\int_{-\infty}^{+\infty} \int_0^{+\infty} \left(\int_0^{+\infty} f(t)e^{-t(z+x-iy)} dt \right) \left(\int_0^{+\infty} g(s)e^{-s(x+iy)} ds \right) dxdy. \tag{3.16}$$

Recalling the formula for the Laplace transform (2.1), we rewrite (3.16) in a concise form as

$$\int_{-\infty}^{+\infty} \int_0^{+\infty} (\mathrm{L}f)(z+x-iy) \cdot (\mathrm{L}g)(x+iy) dxdy \overset{w=x+iy}{=}$$
$$= \iint_\Pi (\mathrm{L}f)(z+\overline{w}) \cdot (\mathrm{L}g)(w) dw d\overline{w} = I(\mathrm{L}f, \mathrm{L}g).$$

This finishes the proof. □

Corollary 3.5 *Due to commutativity of the Hadamard product defined in 3.3, under the assumption of Lemma 3.4, we conclude that the following four functions are equal on* Π:

$$I(\mathrm{L}f, \mathrm{L}g) = P(f,g) = P(g,f) = I(\mathrm{L}g, \mathrm{L}f).$$

Being motivated by Lemma 3.4, we introduce the following precise definition of the Hadamard–Bergman convolution operator:

Definition 3.6 Assume that

$$f(t),\, g(t),\, \frac{f(t)g(t)}{t} \in \mathcal{E}, \tag{3.17}$$

and denote

$$F = \mathrm{L}f,\;\; G = \mathrm{L}g.$$

Then we interpret the Hadamard–Bergman convolution operator $I(F, G)$, that was introduced formally in Definition 1.4, as the corresponding Hadamard product $P(f, g)$ defined in (3.10).

Recall that in Lemma 3.4 the equality (3.12) was proved when f and g are test functions. We now prove the same equality for the general case:

Theorem 3.7 *Under Assumptions* (3.17) *the equality* (3.12) *holds.*

Proof Suppose that Assumptions (3.17) hold. Construct sequences of test functions $f_n, g_n \in \mathcal{T}$ such that

$$\begin{aligned} f_n &\to f, \\ g_n &\to g, \end{aligned} \tag{3.18}$$

where convergence is understood to be both pointwise and in the sense of the weighted space $L^2\left([0, +\infty), t^{-(\alpha+1)}dt\right)$. Denote $F_n = \mathrm{L}f_n$ and $G_n = \mathrm{L}g_n$. For all $z \in \Pi$ we have

$$\begin{aligned} F_n(z) &= \int_0^{+\infty} f_n(t)e^{-zt}dt \overset{(3.18)}{\to} \int_0^{+\infty} f(t)e^{-zt}dt = F(z), \\ G_n(z) &= \int_0^{+\infty} g_n(t)e^{-zt}dt \overset{(3.18)}{\to} \int_0^{+\infty} g(t)e^{-zt}dt = G(z), \end{aligned} \tag{3.19}$$

when $n \to +\infty$, and at the same time

$$\begin{aligned} \iint_\Pi F_n\left(z + \overline{w}\right) G_n(w)dwd\overline{w} &\overset{(3.4)}{=} \int_0^{+\infty} f_n(t)g_n(t)\frac{e^{-zt}}{t}dt \to \\ &\overset{(3.18)}{\to} \int_0^{+\infty} f(t)g(t)\frac{e^{-zt}}{t}dt, \end{aligned} \tag{3.20}$$

also when $n \to +\infty$. By Fatou's theorem, from conditions (3.19) and (3.20) it follows that for every $z \in \Pi$ the function $w \to F\left(z + \overline{w}\right) G(w)$ is Lebesgue integrable on the half-plane $w \in \Pi$, and

$$\iint_\Pi F\left(z + \overline{w}\right) G(w)dwd\overline{w} = \int_0^{+\infty} f(t)g(t)\frac{e^{-zt}}{t}dt,$$

or in other words

$$I(F, G) = P(f, g).$$

This finishes the proof. □

Lemma 3.4 could be compared to the following known result ([12, Theorem 10, p. 511]):

Theorem 3.8 ([12, Theorem 10, p. 511]). *Let $f, g \in \mathcal{E}$. Then*

$$\mathrm{L}(fg)(z) = \int_{a+i\mathbb{R}} (\mathrm{L}f)(w) \cdot (\mathrm{L}g)(z-w)dw, \quad \textit{for } Re(z) > a, \textit{ and } a > 0.$$

4 Proofs of Main Results

4.1 Proof of Theorem 1.5

Let the functions F, G belong to the Bergman space $\mathcal{A}$. Then by Theorem 2.2 the functions F and G are Laplace transforms of some functions f, g,

$$F = \mathrm{L}f, \quad G = \mathrm{L}g, \tag{4.1}$$

satisfying conditions

$$\int_0^{+\infty} \frac{|f(t)|^2}{t} dt < +\infty, \quad \int_0^{+\infty} \frac{|g(t)|^2}{t} dt < +\infty. \tag{4.2}$$

We now check conditions of Lemma 3.4. By Holder's inequality, for all $\varepsilon > 0$ we have

$$\begin{aligned} \int_0^{+\infty} |f(t)|e^{-\varepsilon t} dt &= \int_0^{+\infty} \frac{|f(t)|}{\sqrt{t}} \sqrt{t} e^{-\varepsilon t} dt \leq \\ &\leq \left(\int_0^{+\infty} \frac{|f(t)|^2}{t} dt \right)^{1/2} \left(\int_0^{+\infty} t e^{-2\varepsilon t} dt \right)^{1/2} \overset{(4.2)}{<} +\infty. \end{aligned} \tag{4.3}$$

Also by Holder's inequality

$$\frac{f(t)g(t)}{t} \overset{(4.2)}{\in} L^1([0, +\infty)) \subset \mathcal{E}. \tag{4.4}$$

Consequently, by (4.3) and (4.4) the assumption of Lemma 3.4 holds. Thus, the conditions of Lemma 3.4 hold. Consequently by Lemma 3.4, Theorem 1.5 follows.

4.2 Proof of Theorem 1.9

We divide the proof of Theorem 1.9 in two steps.

Step 1. Assume that $F \in \mathcal{A}_\alpha$. By Theorem 2.2, this is equivalent to claiming that $F = \mathrm{L} f$ for some

$$f \in L^2\left([0,+\infty), t^{-(\alpha+1)}dt\right). \tag{4.5}$$

Given $\sigma \geq 0$, introduce the auxiliary function

$$f_\sigma : [0,+\infty) \to \mathbb{C}$$

by the following two relations:

$$\begin{aligned} f_\sigma(t) &= f(t-\sigma), \quad \text{for } t \geq \sigma, \\ f_\sigma(t) &= 0, \quad \text{for } 0 \leq t < \sigma. \end{aligned}$$

We have that condition (4.5) is equivalent to

$$\int_0^{+\infty} f_\sigma^2(t) \cdot t^{-(\alpha+1)} dt < +\infty,$$

or equivalently

$$\int_0^{+\infty} \left[f_\sigma(t) t^{\beta-2}\right]^2 t^{-\alpha-2\beta+3} dt. \tag{4.6}$$

Recalling the assumption $\alpha+2\beta-4 > -1$ of Theorem 1.9, the conditions of Theorem 2.2 are satisfied for the function $f_\sigma(t)t^{\beta-2}$. Hence, condition (4.6) is equivalent to

$$L\left(f_\sigma(t) \cdot t^{\beta-2}\right) \in \mathcal{A}_{\alpha+2\beta-4}. \tag{4.7}$$

Step 2. We now express the function $w \to F(w)e^{-\sigma w}$ in terms of the auxiliary function f_σ as follows. For $w \in \Pi$ we have

$$\begin{aligned} F(w)e^{-\sigma w} &= e^{-\sigma w} \int_0^{+\infty} f(t) e^{-wt} dt \overset{t \to t-\sigma}{=} \int_\sigma^{+\infty} f(t-\sigma) e^{-wt} dt = \\ &= \int_0^{+\infty} f_\sigma(t) e^{-wt} dt = \mathrm{L}\left(f_\sigma\right)(w). \end{aligned} \tag{4.8}$$

We write the expression $\mathrm{S}_{\beta,\sigma} F$ as follows:

$$\left(\mathrm{S}_{\beta,\sigma} F\right)(z) \overset{(1.7)}{=} \iint_\Pi \frac{F(w)e^{-\sigma w}}{(z+\overline{w})^\beta} dw d\overline{w} \overset{(1.4)}{=} I\left(F(w)e^{-\sigma w}, \frac{1}{w^\beta}\right)(z) =$$

$$\overset{(4.8),(2.4)}{=} I\left(\mathrm{L} f_\sigma, \mathrm{L} g_\beta\right)(z) \overset{(3.4)}{=} P\left(f_\sigma, g_\beta\right) \overset{(3.3)}{=} \int_0^{+\infty} f_\sigma(t) g_\beta(t) e^{-zt} \frac{dt}{2t} =$$

$$\overset{(2.4)}{=} \int_0^{+\infty} f_\sigma(t) t^{\beta-2} e^{-zt} dt = \mathrm{L}\left(f_\sigma(t) t^{\beta-2}\right)(z).$$

Here g_β is as given in Example 2.4. Consequently, condition (4.7) is equivalent to

$$\mathrm{S}_{\beta,\sigma} F \in \mathcal{A}_{\alpha+2\beta-4}.$$

as claimed by Theorem 1.9.

4.3 Proof of Theorem 1.11

Assume that

$$F \in \mathcal{A}_\alpha. \tag{4.9}$$

By Theorem 2.2, this is equivalent to claiming that $F = \mathrm{L} f$ for some

$$f \in L^2\left([0, +\infty), t^{-(\alpha'+1)} dt\right). \tag{4.10}$$

Additionally assume that $f \in \mathcal{T}$. Integrate formula (4.10) by parts to get

$$zF(z) = -\int_0^{+\infty} f'(t) e^{-zt} dt. \tag{4.11}$$

Assume that

$$zF(z) \in \mathcal{A}_{\alpha'}. \tag{4.12}$$

By Theorem 2.2, statements (4.11) and (4.12) are equivalent to claiming that

$$f' \in L^2\left([0, +\infty), t^{-(\alpha'+1)} dt\right). \tag{4.13}$$

From Corollary 2.17 we obtain that if α'' satisfies the conditions of Theorem 1.11, then Assumptions (4.9) and (4.12) imply

$$F \in \mathcal{A}_{\alpha''}.$$

5 Summary

Summarizing, we state that in this paper, by analogy with the known results for the disc, Hadamard–Bergman operators on the half-plane are introduced and studied for the first time. However, the analogy is only formal, while the case of a half-plane is essentially different from the case of a disc. The main results are as follows. We give sufficient conditions under which these operators are well defined (Theorem 1.5). More general conditions are obtained in Theorem 3.7, however, it is not given in constructive terms. We state, as an open question, the finding of constructive general conditions. Further, we present results on mapping properties of operators R_β and $S_{\beta,\sigma}$ in weighted Bergman spaces (see Theorem 1.9 and Corollary 1.10). As an application of the above-mentioned results, in Theorem 1.11 we relate sharp inclusions for weighted Bergman spaces with some Sobolev-type inequalities.

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Declarations

Conflict of interest This work does not have any conflict of interest.

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