Weak-Type (1,1) Inequality for Discrete Maximal Functions and Pointwise Ergodic Theorems Along Thin Arithmetic Sets

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Abstract

We establish weak-type $(1, 1)$ bounds for the maximal function associated with ergodic averaging operators modeled on a wide class of thin deterministic sets B. As a corollary we obtain the corresponding pointwise convergence result on L^1. This contributes yet another counterexample for the conjecture of Rosenblatt and Wierdl from 1991 asserting the failure of pointwise convergence on L^1 of ergodic averages along arithmetic sets with zero Banach density. The second main result is a multiparameter pointwise ergodic theorem in the spirit of Dunford and Zygmund along B on L^p, $p > 1$, which is derived by establishing uniform oscillation estimates and certain vector-valued maximal estimates.

1 Introduction

The work presented in this paper has two parts. The first part is a one-parameter pointwise ergodic theorem on L^1, which contributes another counterexample for a celebrated conjecture of Rosenblatt and Wierdl, and the second part is a multi-parameter pointwise result in the spirit of Dunford and Zygmund, see for example 1.11. Before precisely stating our results, some historical remarks are in order.

1.1 Brief Historical Remarks

In 1991 Rosenblatt and Wierdl [25, Conjecture 4.1] formulated a famous conjecture asserting that for any arithmetical set A with zero Banach density and for any $(X, \mathcal{B}, \mu, T)$ aperiodic probability dynamical system, there exists a function $f \in L^1_\mu(X)$, such that

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$$M_{A,N} f = \frac{1}{|A \cap [1,N]|} \sum_{n \in A \cap [1,N]} f \circ T^n \qquad \text{does not converge almost everywhere,}$$

i.e. the set of $x \in X$ such that $\lim_{N \to \infty} M_{A,N} f(x)$ does not exist has positive measure. This was disproven in 2006 by Buczolich [5], where he provided a counterexample by constructing inductively an appropriate set A of zero Banach density for which one gets the pointwise convergence of the ergodic averages $M_{A,N} f$ for all $f \in L^1$.

Calderón's transference principle suggests that such questions are closely related to the study of weak type $(1,1)$ estimates for the maximal function corresponding to those averages over the integer shift system, namely for the operator

$$\mathcal{M}_A f(x) = \sup_{N \in \mathbb{N}} \frac{1}{|A \cap [1,N]|} \sum_{n \in A \cap [1,N]} |f(x-n)|.$$

A year later it was shown [26] that for $A = \left\{ \lfloor n^c \rfloor : n \in \mathbb{N} \right\}$ with $c \in \left(1, \frac{1001}{1000} \right)$, the operator $\mathcal{M}_A$ is of weak type $(1,1)$, and as a corollary the authors proved pointwise convergence on L^1 for the corresponding ergodic averages along the set A, providing a counterexample of the aforementioned conjecture given by a concrete formula. This class of examples was extended in [20] where the author established the weak type $(1,1)$ bounds for $\mathcal{M}_A$ and the corresponding pointwise ergodic theorem on L^1 for sets of the form $\left\{ \lfloor n^c \ell(n) \rfloor : n \in \mathbb{N} \right\}$, where c is close to 1 and ℓ is a certain kind of slowly varying function, for example any iterate of log, see Definitions 1.1,1.2 below. Finally, another class of examples appeared in [7], where for any $r \in \mathbb{Q} \cap (1, \infty)$, certain sequences $\{a_n\}_{n \in \mathbb{N}}$ of algebraic nature were constructed such that $a_n \simeq n^r$ and such that the associated maximal operator is of weak type $(1,1)$. Comparing this with the failure of the estimate for $A = \{n^k : n \in \mathbb{N}\}$ for any $k \geq 2$, see [6] and [18], one sees that determining whether the weak type $(1,1)$ bound holds for $\mathcal{M}_A$ is a delicate problem which is rather sensitive to perturbations of the sequence. For example, even the case correspoding to $A = \{\lfloor n \log n \rfloor : n \in \mathbb{N}\}$ remains open.

One of the main results of the present work is a natural extension of the result from [20] and in order to formulate it, we must introduce two families of functions that one may think of as slowly varying and regularly varying functions respectively.

Definition 1.1 Fix $x_0 \geq 1$ and let $\mathcal{L}$ denote the set of all functions $\ell \colon [x_0, \infty) \to [1, \infty)$ such that

$$\ell(x) = \exp\left(\int_{x_0}^x \frac{\vartheta(t)}{t} dt \right)$$

where $\vartheta \in \mathcal{C}^2([x_0, \infty))$ is a real-valued function satisfying

$$\vartheta(x) \to 0 \, , \, x\vartheta'(x) \to 0 \, , \, x^2 \vartheta''(x) \to 0 \; \text{ as } x \to \infty.$$

Definition 1.2 Fix $x_0 \geq 1$ and let $\mathcal{L}_0$ denote the set of all functions $\ell \colon [x_0, \infty) \to [1, \infty)$ such that

$$\ell(x) = \exp\left(\int_{x_0}^x \frac{\vartheta(t)}{t} dt\right)$$

where $\vartheta \in \mathcal{C}^2([x_0, \infty))$ is a positive and decreasing function satisfying

$$\vartheta(x) \to 0 \, , \, \frac{x \vartheta'(x)}{\vartheta(x)} \to 0 \, , \, \frac{x^2 \vartheta''(x)}{\vartheta(x)} \to 0 \ \text{ as } x \to \infty,$$

and such that for all $\varepsilon > 0$ we have $\vartheta(x) \gtrsim_\varepsilon x^{-\varepsilon}$ and $\lim_{x \to \infty} \ell(x) = \infty$.

We note that $\mathcal{L}_0 \subseteq \mathcal{L}$ and we give some examples of these slowly varying functions. For $a > 0$, $b \in (0, 1)$, $k \in \mathbb{N}$ and $c \in \mathbb{R}$, we have

$$\log^a x \in \mathcal{L}_0, \quad e^{a \log^b x} \in \mathcal{L}_0, \quad \underbrace{\log \circ \cdots \circ \log x}_{k \text{ times}} \in \mathcal{L}_0, \quad \log^c x \in \mathcal{L}, \quad e^{c \log^b x} \in \mathcal{L}.$$

We proceed by defining a family of regularly varying functions.

Definition 1.3 Fix $x_0 \geq 1$, $c \in (1, \infty)$ and let $\mathcal{R}_c$ be the set of all functions $h \colon [x_0, \infty) \to [1, \infty)$ such that h is strictly increasing, convex and of the form $h(x) = x^c \ell(x)$ for some $\ell \in \mathcal{L}$. We define $\mathcal{R}_1$ analogously with the extra assumption that $\ell \in \mathcal{L}_0$.

We are now ready to give the definitions of the arithmetic sets we are interested in. Let $c_1, c_2 \in [1, 2)$ and let us fix h_1 and h_2 in $\mathcal{R}_{c_1}$ and $\mathcal{R}_{c_2}$ respectively. Let φ_1 and φ_2 be the compositional inverses of h_1 and h_2 and for convenience, let $\gamma_1 = 1/c_1$ and $\gamma_2 = 1/c_2$. Fix a function $\psi \colon [1, \infty) \to (0, 1/2]$, $\psi \in \mathcal{C}^2([1, \infty))$ such that

$$\psi(x) \sim \varphi_2'(x), \ \psi'(x) \sim \varphi_2''(x), \ \psi''(x) \sim \varphi_2'''(x) \text{ as } x \to \infty.$$

We can now define $B_+ = \{n \in \mathbb{N} : \{\varphi_1(n)\} < \psi(n)\}$ and $B_- = \{n \in \mathbb{N} : \{-\varphi_1(n)\} < \psi(n)\}$, where $\{x\} = x - \lfloor x \rfloor$.

Those sets have been introduced and studied in [17], where the authors proved that the Hardy-Littlewood majorant property holds for them, as a corollary of a restriction theorem. Recently, the author [9] proved an analogous result for $\mathbb{P} \cap B_\pm$, see Theorem 1.8 in [9], as well as Roth's theorem in these sets, namely, it was shown that any subset of the primes of the form $B_\pm$ with positive relative upper density contains infinitely many non-trivial three-term arithmetic progressions.

Following [9] we repeat and extend some comments on the sophisticated nature of the sets $B_\pm$. To motivate the definition of $B_\pm$, we note that

$$n \in B_- \iff \exists m \in \mathbb{N} \colon 0 \leq m - \varphi_1(n) < \psi(n)$$
$$\iff \exists m \in \mathbb{N} \colon \varphi_1(n) \leq m < \varphi_1(n) + \psi(n) \iff$$

$$\exists m \in \mathbb{N}\colon\ n \leq h_1(m) < h_1(\varphi_1(n) + \psi(n))$$
$$\iff \exists m \in \mathbb{N}\colon\ h_1(m) \in [n, h_1(\varphi_1(n) + \psi(n))).$$

For $\gamma \in (0, 1)$, $h_1(x) = h_2(x) = x^{1/\gamma}$ and $\psi(x) = \varphi_1(x+1) - \varphi_1(x)$, note that the last condition becomes $m^{1/\gamma} \in [n, n+1)$ or $n = \lfloor m^{1/\gamma} \rfloor$, thus $B_- = \{ \lfloor m^{1/\gamma} \rfloor : m \in \mathbb{N} \}$. It is not difficult to see that any set

$$\mathbb{N}_h := \{ \lfloor h(m) \rfloor : m \in \mathbb{N} \}, \quad h \in \mathcal{R}_c \tag{1.4}$$

can also be brought in the form B_- by similar appropriate choices. Thus the family of sets we consider includes the fractional powers with exponent close to 1 and even the more general sets considered in [20].

1.2 One-Parameter Ergodic Theorem on L^1

We state one of the main results of our paper. Due to some technical complications, we demand further that $\varphi_1 \simeq \varphi_2$, and note that this implies that $c_1 = c_2$.

Theorem 1.5 (Weak-type (1,1) inequality for $\mathcal{M}_{B_\pm}$) *Assume $c_1 \in (1, 30/29)$ and $\varphi_1 \simeq \varphi_2$. Then the maximal function*

$$\mathcal{M}_{B_\pm} f(x) = \sup_{N \in \mathbb{N}} \frac{1}{|B_\pm \cap [1, N]|} \sum_{n \in B_\pm \cap [1,N]} |f(x - n)|$$

is of weak-type (1,1), i.e.:

$$|\{x \in \mathbb{Z} : |\mathcal{M}_{B_\pm} f(x)| > \lambda\}| \leq C\lambda^{-1} \|f\|_{\ell^1(\mathbb{Z})}.$$

By interpolation, this implies that for all $p \in (1, \infty]$, there exists a constant $C_p > 0$ such that

$$\|\mathcal{M}_{B_\pm} f\|_{\ell^p(\mathbb{Z})} \leq C_p \|f\|_{\ell^p(\mathbb{Z})}.$$

We use this, together with 2-oscillation estimates, see Theorem 1.12, to obtain the following pointwise convergence result.

Theorem 1.6 (Pointwise ergodic theorem) *Assume $c \in (1, 30/29)$, $\varphi_1 \simeq \varphi_2$ and let $(X, \mathcal{B}, \mu, T)$ be an invertible σ-finite measure preserving system. For any $p \in [1, \infty)$ and any $f \in L^p_\mu(X)$, we have that*

$$M_{B_\pm, N} f(x) = \frac{1}{|B_\pm \cap [1, N]|} \sum_{n \in B_\pm \cap [1,N]} f(T^n x) \quad \text{converges } \mu\text{-a.e. on } X.$$

Before discussing the strategy of our proofs, we would like to further examine the sets $B_\pm$ and give some concrete examples. Note that

$$n \in B_+ \iff \exists m \in \mathbb{N} : 0 \le \varphi_1(n) - m < \psi(n)$$
$$\iff \exists m \in \mathbb{N} : m \in (\varphi_1(n) - \psi(n), \varphi_1(n)].$$

Now assume that $n \in B_+$, and $m \in \mathbb{N}$ is such that $m \in (\varphi_1(n) - \psi(n), \varphi_1(n)]$ and assume that n_0 is the smallest integer such that $m \in (\varphi_1(n_0) - \psi(n_0), \varphi_1(n_0)]$. Even in simple examples, we should expect that B_+ will contain a lot of consecutive integers after n_0. For example, for any φ_1 inverse of a function in $\mathcal{R}_c$, let $\varphi_2 = C100\varphi_1$, where C is the doubling constant of φ_1', namely, $\varphi_1'(x) \le C\varphi'(2x)$, and let $\psi = \varphi_2'$. Since φ_1 is increasing and $\psi = \varphi_2'$ is decreasing, we get that $\varphi_1 - \psi$ is increasing and thus since $m \notin (\varphi_1(n_0 - 1) - \psi(n_0 - 1), \varphi_1(n_0 - 1)]$, we get that $m > \varphi_1(n_0 - 1)$. We claim that for all $l \in \{0, \cdots, 99\}$, we get that $m \in (\varphi_1(n_0 + l) - \psi(n_0 + l), \varphi_1(n_0 + l)]$, which implies that B_+ contains 99 consecutive integers after n_0. Clearly, $m \le \varphi_1(n_0 + l)$ for all $l \in \mathbb{N}_0$. If we assume for the sake of a contradiction that for some $l \in \{0, \ldots, 99\}$ we have that $m < \varphi_1(n_0 + l) - \psi_1(n_0 + l)$, then

$$\varphi_1(n_0 - 1) < \varphi_1(n_0 + l) - \psi(n_0 + l)$$

and by the Mean Value Theorem, there exists a $\xi_{n_0,l} \in (n_0 - 1, n_0 + l)$ such that

$$\frac{C100\varphi_1'(n_0 + l)}{l + 1} < \frac{\varphi_1(n_0 + l) - \varphi_1(n_0 - 1)}{l + 1}$$
$$= \varphi_1'(\xi_{n_0,l}) \le \varphi_1'(n_0 - 1) \le C\varphi_1'(2n_0 - 2)$$
$$\le C\varphi_1'(n_0 + l).$$

Thus

$$100 < l + 1 \text{ which is a contradiction.}$$

This shows that the set B_+ considered here contains infinitely many full blocks of 100 consecutive integers. Such a set B_+ stands in sharp contrast to the sets of the form $\mathbb{N}_h = \{ \lfloor h(m) \rfloor : m \in \mathbb{N} \}$, $h \in \mathcal{R}_c$, as the gaps between members of such sets tend to infinity.

In general, the constant $\sup_{x \in [1,\infty)} \frac{\varphi_2'(x)}{\varphi_1'(x)}$ determines an important qualitative aspect of the form of the sets $B_\pm$, see Lemma 2.4. Loosely speaking, for big intervals of integers where the ratio is bigger than L, we expect that $B_\pm$ will contain blocks of length at least L/C, where C is the doubling constant of φ_1'. The technical issues that arose when trying to handle the case where $B_\pm$ contains arbitrarily long intervals of integers (specifically in the counting Lemma 5.1) forced the author to impose the restriction $\varphi_1' \simeq \varphi_2'$, which is equivalent to $\varphi_1 \simeq \varphi_2$. We note that $\frac{\varphi_2'(x)}{\varphi_1'(x)}$ could oscillate and thus $B_\pm$ could contain blocks of various oscillating lengths.

We give certain explicit examples to enhance and extend the heuristics described above as well as to illustrate the reach of the main theorems of the present work. We begin by considering the sets $\mathbb{N}_c = \{ \lfloor m^c \rfloor : m \in \mathbb{N} \}$, and by observing that

$$n \in \mathbb{N}_c \iff \exists m \in \mathbb{N} : n \le m^c < n + 1, \tag{1.7}$$

which means that $n \in \mathbb{N}_c$ if and only if the interval $[n, n+1)$ "contains a c-power". Introducing the sets $B_\pm$ allows us to consider intervals of various lengths.

Intervals of fixed length Let $c > 1$, $\gamma = 1/c$. With the choices

$$\varphi_1(x) = x^\gamma, \quad \varphi_2(x) = \frac{1}{100}x^\gamma, \quad \psi(x) = \left(x + \frac{1}{100}\right)^\gamma - x^\gamma,$$

we get

$$n \in B_- \iff \exists m \in \mathbb{N}: m^c \in \left[n, n + \frac{1}{100}\right),$$

or equivalently $B_- = \mathbb{N} \cap \left\{ \frac{\lfloor 100 m^c \rfloor}{100} : m \in \mathbb{N} \right\}$.

The previous example is readily generalizable. Let $c > 1$, $\gamma = 1/c$ and fix $A > 0$, $B \geq 0$ and $C \in \mathbb{N}$, and let

$$\varphi_1(x) = \left(\frac{Cx}{A} - \frac{B}{A}\right)^\gamma, \quad \varphi_2(x) = \frac{C^{\gamma-1}}{A^\gamma}x^\gamma,$$

$$\psi(x) = \left(\frac{Cx}{A} + \frac{1-B}{A}\right)^\gamma - \left(\frac{Cx}{A} - \frac{B}{A}\right)^\gamma.$$

Then

$$n \in B_- \iff \exists m \in \mathbb{N}: \frac{Cn}{A} - \frac{B}{A} \leq m^c < \frac{Cn}{A} + \frac{1-B}{A} \iff$$

$$\exists m \in \mathbb{N}: Cn \leq Am^c + B < Cn + 1 \iff \exists m \in \mathbb{N}: n = \frac{\lfloor Am^c + B \rfloor}{C},$$

and thus

$$B_- = \mathbb{N} \cap \left\{ \frac{\lfloor Am^c + B \rfloor}{C} : m \in \mathbb{N} \right\}.$$

Similarly, appropriate choices of parameters give B_- the following form

$$B_- = \{n \in \mathbb{N} : \exists m \in \mathbb{N} \text{ with } m^c \in [n + \alpha_1, n + \alpha_2)\}$$
$$\text{with } 0 < \alpha_1 < \alpha_2 \text{ and } c > 1. \tag{1.8}$$

For $\alpha_2 - \alpha_1 \geq 2$, we have that B_- is a union of intervals of length at least $\alpha_2 - \alpha_1 - 1$ "around c-powers", see also the proof of Lemma 2.4. This set highlights the aforementioned qualitative difference of $B_\pm$ with any $\mathbb{N}_h$ for $h \in \mathcal{R}_c$ considered in [20], namely, the sets $B_\pm$ can contain consecutive integers.

Intervals of oscillating lengths For certain choices of parameters one produces a set B_- which is comprised of intervals of oscillating lengths around c-powers. Let $c > 1$, $\gamma = 1/c$ and

$$\varphi_1(x) = x^\gamma, \quad \varphi_2(x) = x^\gamma (99 \sin(\log \log x) + 100),$$

$$\psi(x) = (x + 99\sin(\log\log x) + 100)^\gamma - x^\gamma.$$

Then $n \in B_- \iff n \le m^c < n + 99\sin(\log\log n) + 100$, and thus

$$B_- = \{n \in \mathbb{N} : \exists m \in \mathbb{N} \text{ with } m^c \in [n, n + 99\sin(\log\log n) + 100)\}.$$

We believe that the examples illustrate the flexibility in the choices of parameters defining $B_\pm$ and consequently the reach of Theorem 1.6.

Let us remark that the examples given are readily generalizable, for example, m^c can be replaced by $h_1(m)$ with $h_1 \in \mathcal{R}_{c_1}$. Routine but lengthy calculations show that all the choices we made for our parameters are compatible with the conditions in the definition of the set $B_\pm$, as well as the technical assumption $\varphi_1 \simeq \varphi_2$, except possibly for small values of x, which is insignificant since only the tail of the set $B_\pm$ plays a role in our results.

Let us restrict our attention to the sets B_+, which we call B from now on, as the results for B_- are of equal difficulty. We make the following remarks.

Remark 1.9 (Smooth dyadic maximal operator) To establish Theorem 1.5, it is convenient to work with a smooth dyadic variant of the maximal operator. More precisely, let us fix $\eta \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \le \eta(x) \le 1$ for all $x \in \mathbb{R}$, $\text{supp}(\eta) \subseteq (1/2, 4)$ and $\eta(x) = 1$ for all $x \in [1, 2]$. We define

$$\mathcal{M}_B^{(sd)} f(x) = \sup_{k \in \mathbb{N}_0} \left\{ \frac{1}{\varphi_2(2^k)} \sum_{n \in B} |f(x - n)| \eta\left(\frac{n}{2^k}\right) \right\}$$

and note that

$$\mathcal{M}_B f(x) \lesssim \sup_{k \in \mathbb{N}_0} \left\{ \frac{1}{|B \cap [2^k, 2^{k+1})|} \sum_{n \in B \cap [2^k, 2^{k+1})} |f(x - n)| \right\} \lesssim \mathcal{M}_B^{(sd)} f(x).$$

The first inequality is straightforward, and for the second one, note that Lemma 2.1 implies that

$$|B \cap [1, N]| \simeq |B \cap [N/2, N)| \simeq \varphi_2(N) \text{ for sufficiently large } N.$$

Thus it suffices to establish the weak type (1,1) bound for $\mathcal{M}_B^{(sd)}$ which we denote by $\mathcal{M}$, and let

$$K_N(x) = \frac{1}{\varphi_2(N)} \sum_{n \in B} \delta_n(x) \eta\left(\frac{n}{N}\right),$$

so that $\mathcal{M}f(x) = \sup_{k \in \mathbb{N}_0}\{K_{2^k} * |f|(x)\}$.

We give a brief overview of the main ideas of the proof of Theorem 1.5. We use a subtle variation of the Calderón-Zygmund decomposition that was introduced by Fefferman [11], see also [8], in a similar manner to [26] and [20]. More specifically,

after approximating $K_n * \widetilde{K}_n$ by suitable well-behaving functions, see Lemma 5.2, we employ a refined Calderón-Zygmund decomposition which allows us to use ℓ^2-estimates for the "very bad" part of the decomposition, see subsection 5.2. The aforementioned approximation is analogous to the one presented in section 5 of [20] and similar techniques are used here. The novelty lies in the sophisticated nature of the sets B which complicates the situation substancially. For example, bounding $|K_n * \widetilde{K}_n(x)|$ for small values of x, see Lemma 5.1, is precisely what forced the author to impose the extra assumption $\varphi_1 \simeq \varphi_2$. To carry out the approximation one needs to estimate certain exponential sums and the main tool is Van der Corput's inequality. Some of the necessary exponential sum estimates can be readily found in section 3 of [20] and suitable extensions are already established by the author in [9]. Finally, we formulate an abstract result, see Theorem 5.5, which is a generalization of Theorem 6.1 of [20], adapted to our approximation for $K_n * \widetilde{K}_n$. We give the full proof of Theorem 1.5 in Sect. 5 and the reader is encouraged to compare it with section 3 of [26], and section 5 and 6 of [20].

Combining Theorem 1.5 with the trivial estimate $\|\mathcal{M}_B\|_{\ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})} \lesssim 1$, we obtain by interpolation that $\|\mathcal{M}_B\|_{\ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})} \lesssim_p 1$, for all $p \in (1, \infty]$. Calderón's transference principle implies that for any invertible σ-finite measure preserving system $(X, \mathcal{B}, \mu, T)$ we have

$$\Big\| \sup_{N \in \mathbb{N}} M_{B,N} f \Big\|_{L_\mu^p(X)} \lesssim_p \|f\|_{L_\mu^p(X)} \text{ for } p \in (1, \infty), \text{ and}$$

$$\mu\Big(\big\{x \in X : |\sup_{N \in \mathbb{N}} M_{B,N} f(x)| > \lambda\big\}\Big) \lesssim \frac{\|f\|_{L_\mu^1(X)}}{\lambda},$$

and a standard argument shows that $\mathcal{L}_B^p = \big\{f \in L_\mu^p(X) : \lim_{N \to \infty} M_{B,N} f \text{ exists } \mu\text{-a.e.}\big\}$ is closed in $L_\mu^p(X)$, for all $p \in [1, \infty)$, and thus to establish Theorem 1.6 it suffices to exhibit an L_μ^p-dense class of functions $\mathcal{D}_p$ contained in $\mathcal{L}_B^p$. The exponential sum estimates of Lemma 2.1 together with a straightforward adaptation of the argument presented in section 3 of [20], which uses ideas from [4], shows that one may take $\mathcal{D}_p = L_\mu^p(X) \cap L_\mu^2(X)$ and conclude. Instead of this, we derive the pointwise convergence immediately by the much stronger uniform 2-oscillation L_μ^p-estimates of Theorem 1.12, which are also exploited in the sequel.

1.3 Multi-Parameter Ergodic Theorem

The second main result of our paper is a multi-parameter variant of Theorem 1.6. Here we discard the assumption $\varphi_1 \simeq \varphi_2$ and the acceptable range of c_1 and c_2 is considerably larger. In contrast to the one-parameter situation, weak-type $(1, 1)$ estimates do not hold here, which is comparable with the failure of Lebesgue's differentiation theorem for rectangles with sides parallel to the axes on L^1.

To make the exposition slightly cleaner, let us fix $k \in \mathbb{N}$ and $B_1, \ldots, B_k$ as in the introduction, where $(h_{1,i}, h_{2,i}) \in \mathcal{R}_{c_{1,i}} \times \mathcal{R}_{c_{2,i}}$ are as in the definition of B_i, and we

assume that $c_{1,i} \in [1, 2)$ and $c_{2,i} \in [1, 6/5)$ for all $i \in [k]$. We are ready to state the second main result.

Theorem 1.10 *Assume $(X, \mathcal{B}, \mu)$ is a σ-finite measure space and $\{S_i : i \in [k]\}$ is a family of invertible μ-invariant commuting transformations. Then for any $p \in (1, \infty)$ and any $f \in L_\mu^p(X)$ we have that*

$$\lim_{\min\{N_1,\dots,N_k\}\to\infty} \frac{1}{\prod_{i=1}^k |B_i \cap [1, N_i]|} \sum_{l \in \prod_{i=1}^k B_i \cap [1, N_i]} f \circ S^{(l)}(x) \text{ exists for } \mu\text{-a.e. } x \in X,$$

where $S^{(l_1,\dots,l_k)} = S_1^{l_1} \circ \cdots \circ S_k^{l_k}$.

Apart from the larger range of exponents, the absence of the condition $\varphi_1 \simeq \varphi_2$ allows us to consider a variety of additional sets here. We provide some examples.
Shrinking intervals The fact that we can choose $c_1 < c_2$ allows us to consider shrinking intervals. Let $1 < c_1 < c_2$, $\gamma_1 = 1/c_1$ and $\gamma_2 = 1/c_2$, and let

$$\varphi_1(x) = x^{\gamma_1}, \quad \varphi_2(x) = \frac{\gamma_1 x^{\gamma_2}}{\gamma_2},$$
$$\psi(x) = \left(x + x^{\gamma_2-\gamma_1}\right)^{\gamma_1} - x^{\gamma_1}.$$

Then

$$n \in B_- \iff \exists m \in \mathbb{N}: n \le m^{c_1} < n + n^{\gamma_2-\gamma_1}$$
$$\iff \exists m \in \mathbb{N}: m^{c_1} \in \left[n, n + n^{\gamma_2-\gamma_1}\right),$$

which gives B_- the following form,

$$B_- = \left\{n \in \mathbb{N} : \exists m \in \mathbb{N} \text{ with } m^c \in \left[n, n + n^{-\theta}\right)\right\},$$

for $c > 1$, $\theta > 0$ and close to 1 and 0 respectively, which in turn, means that we only consider integers that "approach c-powers" rapidly.
Expanding intervals Analogously, with appropriate choices with $c_1 > c_2$, we get

$$B_- = \left\{n \in \mathbb{N} : \exists m \in \mathbb{N} \text{ with } m^c \in \left[n, n + n^\theta\right)\right\},$$

for $c > 1$, $\theta > 0$ and close to 1 and 0 respectively, which is a union of intervals of expanding lengths, and again, such sets are not of the form $\mathbb{N}_h$. For an example of expanding intervals when $c_1 = c_2$, let $c > 1$, $\gamma = 1/c$ and let

$$\varphi_1(x) = x^\gamma, \quad \varphi_2(x) = x^\gamma(2\log x), \quad \psi(x) = (x + 2\log x)^\gamma - x^\gamma.$$

Note that then

$$B_- = \left\{n \in \mathbb{N} : \exists m \in \mathbb{N} \text{ with } m^c \in \left[n, n + 2\log n\right)\right\}.$$

We make some brief historical remarks. In 1951 Dunford [10] and Zygmund [28] independently showed that given a σ-finite measure space $(X, \mathcal{B}, \mu)$ and a family of μ-invariant transformations $\{T_i : i \in [k]\}$, for any $p \in (1, \infty)$ and any $f \in L^p_\mu(X)$, we have

$$\frac{1}{N_1 \cdots N_k} \sum_{l \in \prod_{i=1}^k [1, N_i]} f(T_1^{l_1} \cdots T_k^{l_k} x) \quad \text{converges } \mu\text{-a.e. on } X$$

and in L^p norm as $\min\{N_1, \ldots, N_k\} \to \infty$.

Notably, for $k \geq 2$, pointwise convergence fails on L^1. Motivated by that observation and after his seminal work on pointwise ergodic theory [2–4], Bourgain showed that for any $p \in (1, \infty)$ and any $f \in L^p_\mu(X)$, we have

$$\frac{1}{N_1 \cdots N_k} \sum_{l \in \prod_{i=1}^k [1, N_i]} f\left(T_1^{P_1(l_1)} \cdots T_k^{P_k(l_k)} x\right) \quad \text{converges } \mu\text{-a.e. on } X$$

as $\min\{N_1, \ldots, N_k\} \to \infty$,

where $P_1, \ldots, P_k \in \mathbb{Z}[x]$, $P_1(0) = \cdots = P_k(0) = 0$ and $\{T_i : i \in [k]\}$ is a family of commuting and invertible μ-invariant transformations. In contrast to Dunford and Zygmund's result, the commutativity assumption turns out to be indispensable for the polynomial case. For a more thorough exposition on the matter we refer the reader to Section 1.2 in [23] as well as the introduction from [1], see page 3. In the spirit of the above, Theorem 1.10 establishes the multi-parameter result for orbits along sets of the form B. For example, for appropriate choices of parameters, Theorem 1.10 implies that for any $p \in (1, \infty)$ and any $f \in L^p_\mu(X)$, we have

$$\frac{1}{N_1 \cdots N_k} \sum_{l \in \prod_{i=1}^k [1, N_i]} f\left(T_1^{\lfloor l_1^{c_1} \rfloor} \cdots T_k^{\lfloor l_k^{c_k} \rfloor} x\right) \quad \text{converges } \mu\text{-a.e. on } X$$

as $\min\{N_1, \ldots, N_k\} \to \infty$,   (1.11)

where $c_1, \ldots, c_k \in (1, 6/5)$ and $\{T_i : i \in [k]\}$ is a family of commuting and invertible μ-invariant transformations. To the best of the author's knowledge, this is the first time multi-parameter pointwise convergence is established for orbits along fractional powers in the spirit of Dunford and Zygmund.

Using an abstract multi-parameter oscillation result from [23], we reduce the task of proving the above theorem to establishing some useful quantitative uniform estimates, which may be of independent interest.

Before precisely formulating these estimates, we note that for any $Y \subseteq X \subseteq \mathbb{R}$, with $|X| > 2$, we have that

$$\mathfrak{S}_J(X) = \{\{I_0 < \cdots < I_J\} \subseteq X\} \text{ and } O^2_{I,J}(a_t(x) : t \in Y)$$
$$= \left(\sum_{j=0}^{J-1} \sup_{t \in [I_j, I_{j+1}) \cap Y} |a_t(x) - a_{I_j}(x)|^2 \right)^{1/2}.$$

For the basic properties of oscillations as well as the definition of multi-parameter oscillations we refer the reader to section 2 from [23].

Theorem 1.12 (Uniform 2-oscillation and vector-valued maximal estimates) *Assume $c_1 \in [1, 2)$, $c_2 \in [1, 6/5)$ and B as in the introduction. Assume $(X, \mathcal{B}, \mu)$ is a σ-finite measure space and T is an invertible μ-invariant transformation. Let*

$$M_{B,t} f(x) = \frac{1}{|B \cap [1, t]|} \sum_{n \in B \cap [1, t]} f(T^n x).$$

Then for any $p \in (1, \infty)$, there exists a constant C_p such that

$$\sup_{J \in \mathbb{N}} \sup_{I \in \mathfrak{S}_J(\mathbb{N})} \| O_{I,J}^2 (M_{B,t} f : t \in \mathbb{N}) \|_{L_\mu^p(X)}$$
$$\leq C_p \| f \|_{L_\mu^p(X)} \text{ for any } f \in L_\mu^p(X) \tag{1.13}$$

and such that

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left(\sup_{t \in \mathbb{N}} M_{B,t} |f_j| \right)^2 \right)^{1/2} \right\|_{L_\mu^p(X)}$$
$$\leq C_p \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{L_\mu^p(X)} \text{ for any } (f_j)_{j \in \mathbb{Z}} \in L_\mu^p(\mathbb{Z}; \ell^2(\mathbb{Z})). \tag{1.14}$$

We now comment on the proof of Theorem 1.12. Again, Calderón's transference principle suggests that it suffices to establish these estimates for the integer shift system. Ultimately, those estimates are derived from the analogous ones for the standard discrete Hardy–Littlewood averaging operator. For the vector-valued maximal inequality we use the exponential sum estimates of Lemma 2.1 together with the fact that ψ behaves "like a constant" in dyadic blocks in order to eventually be able to use the corresponding estimates for the Hardy–Littlewood averaging operator (for example see Theorem 1 in [12] or Theorem C in [21]). The situation is much more complicated for the oscillations. We follow the strategy from [22] and [27], and we break our analysis into short and long oscillations, and instead of opting to handle as our "long oscillations" the rather natural choice $\{2^n : n \in \mathbb{N}_0\}$, we choose a much denser set, namely $\{\lfloor 2^{n^\tau} \rfloor : n \in \mathbb{N}_0\}$, for τ small. This affords us to bound the short oscillations straightforwardly. Loosely speaking, the long oscillations are treated in similar manner to the vector-valued maximal inequality, but the fact that the 2-oscillations are not a positive operator makes the use of the fact that ψ behaves nicely in dyadic blocks difficult. Here, we adapt the argument from section 5 in [24] to our oscillation setting in order to compare averages with different weights. Again, we use the uniform oscillation estimates for the discrete Hardy–Littlewood averaging operator to conclude (which one may find for example in [15] or [19]). Finally, we mention that the exponential sum estimates help us understand some error terms on ℓ^2, and Riesz–Thorin interpolation together with trivial bounds coming from the fact that we deal with averaging operators help us establish the corresponding ℓ^p bounds.

1.4 Notation

We denote by C a positive constant that may change from occurrence to occurrence. If A, B are two non-negative quantities, we write $A \lesssim B$ or $B \gtrsim A$ to denote that there exists a positive constant C such that $A \leq CB$. Whenever $A \lesssim B$ and $A \gtrsim B$ we write $A \simeq B$. For two complex-valued functions f, g we write $f \sim g$ to denote that $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$. For any topological space X, we denote by $\mathcal{C}(X)$ the set of all complex-valued continuous functions. We denote the average value of a function $f \colon \mathbb{Z} \to \mathbb{C}$ over a finite set $Q \subseteq \mathbb{Z}$ by $[f]_Q = \frac{1}{|Q|} \sum_{x \in Q} f(x)$. For any natural number N, we let $[N] = \{1, 2, \ldots, N\}$.

2 Basic Properties of the Sets B

In this section we collect some useful properties of the sets B. We begin by stating an exponential sum estimate proven in [17]. Here, and for the following two sections, we fix two constants c_1, c_2 such that $c_1 \in [1, 2)$ and $c_2 \in [1, 6/5)$, as well as h_1, $h_2 \in \mathcal{R}_{c_1}$ and $\mathcal{R}_{c_2}$ respectively and $\psi, \gamma_1, \gamma_2, B$ are as in the introduction and all the implied constants may depend on them. We note that we use the basic properties of those functions as described in Lemma 2.6 and Lemma 2.14 from [20] without further mention.

Lemma 2.1 *Assume $\chi > 0$ is such that $(1 - \gamma_1) + 3(1 - \gamma_2) + 6\chi < 1$. Then there exists a real number $\chi' > 0$ such that*

$$\sum_{n \in B \cap [N]} \psi(n)^{-1} e^{2\pi i n \xi} = \sum_{n \in [N]} e^{2\pi i n \xi} + O(N^{1 - \chi - \chi'}) \tag{2.2}$$

as well as

$$\sum_{n \in B \cap [N]} e^{2\pi i n \xi} = \sum_{n \in [N]} \psi(n) e^{2\pi i n \xi} + O(\varphi_2(N) N^{-\chi - \chi'}) \tag{2.3}$$

where the implied constant does not depend on ξ or N.

Proof We note that one is derived from the other using summation by parts. The proof can be found in page 6 as well as Lemma 3.2 in [17]. □

Lemma 2.4 *If $\varphi_2 \lesssim \varphi_1$, then B does not contain arbitrarily long intervals in $\mathbb{Z}$.*

Proof Assume that $\varphi_2 \lesssim \varphi_1$ and let $\{n, n+1, \ldots, n+l-1\} \subseteq B$, we wish to bound l. First of all let us notice that B may be partitioned as follows

$$B = \bigcup_{m \in \mathbb{N}} B_m$$

where $B_m = \{n \in \mathbb{N} : 0 \leq \varphi_1(n) - m < \psi(n)\}$, note that $B_m \cap B_k = \emptyset$ for $m \neq k$. For sufficiently large m, k with $m < k$, we have that $dist(B_m, B_k) \geq 2$ since if we

assume for the sake of a contradiction that $dist(B_m, B_k) = 1$ then there exists $n \in B_m$ such that and $n + 1 \in B_k$. But then

$$0 \le \varphi_1(n) - m < \psi(n) \quad \text{and} \quad 0 \le \varphi_1(n+1) - k < \psi(n+1)$$

and thus

$$(k - m) - \psi(n) < \varphi_1(n+1) - \varphi_1(n),$$

which implies

$$\varphi_1(n+1) - \varphi_1(n) > 1/2.$$

By the Mean Value Theorem there exists $\xi_n \in (n, n+1)$ such that $\varphi_1'(\xi_n) > 1/2$, which will be a contradiction for sufficiently large n, since $\varphi_1'(x) \to 0$, as $x \to \infty$. Thus, ignoring some first few terms of B, the fact that $\{n, n+1, \ldots, n+l-1\} \subseteq B$, together with our previous observation, imply that there exists an $m \in \mathbb{N}$ such that $\{n, n+1, \ldots, n+l-1\} \subseteq B_m$. Then by combining $n, n+l-1 \in B_m$, we obtain

$$\varphi_1(n+l-1) - \varphi_1(n) < \psi(n+l)$$

which by the Mean Value Theorem becomes

$$(l-1)\varphi_1'(n+l) \le (l-1)\varphi_1'(\xi_{n,l}) < \psi(n+l) \lesssim \varphi_2'(n+l)$$

and thus $(l-1) \lesssim \frac{\varphi_2'(n+l)}{\varphi_1'(n+l)} \lesssim \frac{\varphi_2(n+l)}{\varphi_1(n+l)} \lesssim 1$, since $\varphi_2 \lesssim \varphi_1$. Thus B does not contain arbitrarily long intervals.　□

3 Uniform 2-oscillation Estimates

We prove the first half of Proposition 1.12, namely we establish the estimate (1.13). By the Calderón Transference Principle, in order to establish that estimate for any σ-finite measure preserving system, it suffices to establish it for the integer shift system, namely for $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), |\cdot|, S)$ where S is the shift map $S(x) = x - 1$ and the $|\cdot|$ is the counting measure. To simplify the notation, we let $B_t = B \cap [1, t]$ and write

$$M_t f(x) = \frac{1}{|B_t|} \sum_{n \in B_t} f(x - n).$$

We therefore wish to show that for any $p \in (1, \infty)$, there exists a contant C_p such that

$$\sup_{J \in \mathbb{N}_0} \sup_{I \in \mathfrak{S}_J(\mathbb{N})} \|O_{I,J}^2(M_t f : t \in \mathbb{N})\|_{\ell^p(\mathbb{Z})} \le C_p \|f\|_{\ell^p(\mathbb{Z})} \text{ for any } f \in \ell^p(\mathbb{Z}). \quad (3.1)$$

To obtain this result, we break the 2-oscillations into short and long ones, and we will need to carefully choose some parameters first. Let $p_0 \in (1, \infty)$ be such that $p \in (p_0, p_0')$, let $\tau \in (0, \min\{\frac{p_0-1}{2}, \frac{1}{2}\})$ and let $\mathbb{D}_\tau = \{2^{n^\tau} : n \in \mathbb{N}_0\}$. It is not difficult to see that

$$
\sup_{J \in \mathbb{N}} \sup_{I \in \mathfrak{S}_J(\mathbb{N})} \|O_{I,J}^2(M_t f : t \in \mathbb{N})\|_{\ell^p(\mathbb{Z})}
$$

$$
\lesssim \sup_{J \in \mathbb{N}} \sup_{I \in \mathfrak{S}_J(\mathbb{D}_\tau)} \|O_{I,J}^2(M_t f : t \in \mathbb{D}_\tau)\|_{\ell^p(\mathbb{Z})}
$$

$$
+ \left\| \left(\sum_{n=0}^{\infty} V^2\Big(M_t f : t \in \big[2^{n^\tau}, 2^{(n+1)^\tau}\big)\Big)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})} \tag{3.2}
$$

where

$$
V^2\Big(M_t f(x) : t \in \big[2^{n^\tau}, 2^{(n+1)^\tau}\big)\Big)
$$

$$
= \sup_{J \in \mathbb{N}} \sup_{\substack{t_0 < \cdots < t_J \\ t_j \in [2^{n^\tau}, 2^{(n+1)^\tau})}} \left(\sum_{j=0}^{J-1} |M_{t_{j+1}} f(x) - M_{t_j} f(x)|^2 \right)^{1/2},
$$

see [23] page 17. One may adapt the argument appearing in Lemma 1.3 in [16] to establish this. We deal with the second term of (3.2). Note that

$$
\left\| \left(\sum_{n=0}^{\infty} V^2\Big(M_t f : t \in \big[2^{n^\tau}, 2^{(n+1)^\tau}\big)\Big)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}
$$

$$
= \left\| \left(\sum_{n=0}^{\infty} \left(\sup_{J \in \mathbb{N}} \sup_{\substack{t_0 < \ldots t_J \\ t_j \in [2^{n^\tau}, 2^{(n+1)^\tau})}} \left(\sum_{j=0}^{J-1} |M_{t_{j+1}} f - M_{t_j} f|^2 \right)^{1/2} \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}
$$

$$
\lesssim \left\| \left(\sum_{n=1}^{\infty} \left(\sup_{J \in \mathbb{N}} \sup_{\substack{t_0 < \ldots t_J \\ t_j \in [2^{(n-1)^\tau}, 2^{(n+2)^\tau}) \cap B}} \left(\sum_{j=0}^{J-1} |M_{t_{j+1}} f - M_{t_j} f| \right) \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}
$$

where we have used the fact that $\|\cdot\|_{\ell^2} \le \|\cdot\|_{\ell^1}$, together with the fact that $M_t f(x) = M_s f(x)$ whenever $B_t = B_s$. For any $n \in \mathbb{N}$, let $\{\beta_m^{(n)} : m \in \{0, \ldots, l_n\}\}$ be an increasing enumeration of $[2^{(n-1)^\tau}, 2^{(n+2)^\tau}) \cap B$. Then we use the triangle inequality to bound the last expression by

$$
\left\| \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=1}^{l_n} |M_{\beta_m^{(n)}} f - M_{\beta_{m-1}^{(n)}} f| \right) \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}.
$$

Let $K_t(x) = \frac{1}{|B_t|} \sum_{n \in B_t} \delta_n(x)$, where $\delta_n(x) = 1_{\{n\}}(x)$, and note that $M_t f(x) = K_t * f(x)$. Thus we rewrite the expression above as

$$\left\| \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=1}^{l_n} |(K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}}) * f| \right) \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}.$$

We firstly consider the case $p \in (2, \infty)$. We get

$$\left\| \left(\sum_{n=0}^{\infty} V^2 \left(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$

$$\lesssim \left\| \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=1}^{l_n} |K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}}| * |f| \right) \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$

$$= \left(\sum_{x \in \mathbb{Z}} \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=1}^{l_n} |K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}}| * |f|(x) \right) \right)^2 \right)^{p/2} \right)^{1/p}$$

$$\leq \left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{l_n} \left(\sum_{x \in \mathbb{Z}} \left(|K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}}| * |f|(x) \right)^p \right)^{1/p} \right)^2 \right)^{1/2}$$

$$= \left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{l_n} \left\| |K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}}| * |f| \right\|_{\ell^p(\mathbb{Z})} \right)^2 \right)^{1/2}$$

$$\leq \left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{l_n} \left\| K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}} \right\|_{\ell^1(\mathbb{Z})} \|f\|_{\ell^p(\mathbb{Z})} \right)^2 \right)^{1/2}$$

$$= \left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{l_n} \left\| K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}} \right\|_{\ell^1(\mathbb{Z})} \right)^2 \right)^{1/2} \|f\|_{\ell^p(\mathbb{Z})}$$

where we have used Minkowski's inequality for $p/2 > 1$ and $p > 1$, and then Young's convolution inequality. In the case where $p \in (1, 2]$, we note

$$\left\| \left(\sum_{n=0}^{\infty} V^2 \left(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$

$$\lesssim \left\| \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=1}^{l_n} |K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}}| * |f| \right) \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$

$$\leq \left\| \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=1}^{l_n} |K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}}| * |f| \right) \right)^p \right)^{1/p} \right\|_{\ell^p(\mathbb{Z})}$$

$$\leq \left(\sum_{x \in \mathbb{Z}} \sum_{n=1}^{\infty} \left(\left(\sum_{m=1}^{l_n} |K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}}| * |f|(x) \right) \right)^p \right)^{1/p}$$

$$\leq \left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{l_n} \left(\sum_{x \in \mathbb{Z}} \left(|K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}}| * |f|(x) \right)^p \right)^{1/p} \right)^p \right)^{1/p}$$

$$= \left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{l_n} \left\| |K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}}| * |f| \right\|_{\ell^p(\mathbb{Z})} \right)^p \right)^{1/p}$$

$$\leq \left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{l_n} \left\| K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}} \right\|_{\ell^1(\mathbb{Z})} \|f\|_{\ell^p(\mathbb{Z})} \right)^p \right)^{1/p}$$

$$= \left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{l_n} \left\| K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}} \right\|_{\ell^1(\mathbb{Z})} \right)^p \right)^{1/p} \|f\|_{\ell^p(\mathbb{Z})}$$

where we have used Minkowski's inequality and Young's convolution inequality. Combining the two cases gives

$$\left\| \left(\sum_{n=0}^{\infty} V^2 \left(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$

$$\lesssim \left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{l_n} \left\| K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}} \right\|_{\ell^1(\mathbb{Z})} \right)^q \right)^{1/q} \|f\|_{\ell^p(\mathbb{Z})} \qquad (3.3)$$

where $q = \min\{2, p\}$. We focus on $\left\| K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}} \right\|_{\ell^1(\mathbb{Z})}$, note that

$$\left\| K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}} \right\|_{\ell^1(\mathbb{Z})} = \sum_{x \in B_{\beta_{m-1}^{(n)}}} \frac{1}{|B_{\beta_{m-1}^{(n)}}|} - \frac{1}{|B_{\beta_m^{(n)}}|}$$

$$+ \sum_{x \in B_{\beta_m^{(n)}} \setminus B_{\beta_{m-1}^{(n)}}} \frac{1}{|B_{\beta_m^{(n)}}|}$$

$$= \frac{|B_{\beta_m^{(n)}}| - |B_{\beta_{m-1}^{(n)}}|}{|B_{\beta_m^{(n)}}|} + \frac{|B_{\beta_m^{(n)}} \setminus B_{\beta_{m-1}^{(n)}}|}{|B_{\beta_m^{(n)}}|}$$

$$= 2 \frac{|B_{\beta_m^{(n)}}| - |B_{\beta_{m-1}^{(n)}}|}{|B_{\beta_m^{(n)}}|}$$

thus

$$\sum_{m=1}^{l_n} \left\| K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}} \right\|_{\ell^1(\mathbb{Z})} \lesssim \sum_{m=1}^{l_n} \frac{|B_{\beta_m^{(n)}}| - |B_{\beta_{m-1}^{(n)}}|}{|B_{\beta_m^{(n)}}|} \lesssim \frac{|B_{2^{(n+2)^\tau}}| - |B_{2^{(n-1)^\tau}}|}{|B_{2^{(n-1)^\tau}}|}.$$

We know that there exists $\varepsilon > 0$ such that $|B_t| = \varphi_2(t)(1 + O(t^{-\varepsilon}))$, (see [17], page 5), thus we get

$$
\sum_{m=1}^{l_n} \left\| K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}} \right\|_{\ell^1(\mathbb{Z})}
$$
$$
\lesssim \frac{\varphi_2(2^{(n+2)^\tau}) - \varphi_2(2^{(n-1)^\tau}) + \varphi_2(2^{(n+2)^\tau}) O(2^{-\varepsilon(n-1)^\tau})}{\varphi_2(2^{(n-1)^\tau})}.
$$

Note that $\frac{\varphi_2(2^{(n+2)^\tau}) 2^{-\varepsilon(n-1)^\tau}}{\varphi_2(2^{(n-1)^\tau})} \leq \frac{\varphi_2(2^{(n-1)^\tau + 3^\tau}) 2^{-\varepsilon(n-1)^\tau}}{\varphi_2(2^{(n-1)^\tau})} \lesssim 2^{-\varepsilon(n-1)^\tau}$, since $0 < \tau \leq 1/2$, φ_2 is increasing and $\varphi_2(4x) \lesssim \varphi_2(x)$. We also note that by the Mean Value Theorem for $f(x) = \varphi_2(2^{x^\tau})$ on the interval $[n-1, n+2]$ we get that there exists $x_n \in (n-1, n+2)$ such that

$$
\frac{\varphi_2(2^{(n+2)^\tau}) - \varphi_2(2^{(n-1)^\tau})}{3} = \frac{f(n+2) - f(n-1)}{3}
$$
$$
= f'(x_n) = \varphi_2'(2^{x_n^\tau}) 2^{x_n^\tau} \log(2) \tau x_n^{\tau-1}
$$
$$
\lesssim \varphi_2(2^{x_n^\tau})(n-1)^{\tau-1} \leq \varphi_2(2^{(n+2)^\tau})(n-1)^{\tau-1}
$$
$$
\leq \varphi_2(2^{(n-1)^\tau + 3^\tau})(n-1)^{\tau-1}
$$
$$
\lesssim \varphi_2(2^{(n-1)^\tau})(n-1)^{\tau-1}.
$$

Thus

$$
\sum_{m=1}^{l_n} \left\| K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}} \right\|_{\ell^1(\mathbb{Z})} \lesssim (n-1)^{\tau-1} + 2^{-\varepsilon(n-1)^\tau},
$$

which implies

$$
\left(\sum_{n=1}^{\infty} \left(\sum_{m=1}^{l_n} \left\| K_{\beta_m^{(n)}} - K_{\beta_{m-1}^{(n)}} \right\|_{\ell^1(\mathbb{Z})} \right)^q \right)^{1/q}
$$
$$
\lesssim 1 + \left(\sum_{n=2}^{\infty} (n-1)^{q(\tau-1)} \right)^{1/q} + \left(\sum_{n=1}^{\infty} 2^{-\varepsilon q(n-1)^\tau} \right)^{1/q}.
$$

Note that if $p > 2$ then $q = 2$ and then $q(1 - \tau) > 2(1 - 1/2) = 1$ thus the first sum converges. Similarly, if $p \leq 2$, then $q = p$. Note that $\tau < (p_0 - 1)/2 < (p_0 - 1)/p_0 = 1/p_0'$, but then $q(1 - \tau) > p(1 - 1/p_0') = p/p_0 > 1$, as desired. In either case, the first series is summable. The second series is also summable, since for example $2^{-q\varepsilon n^\tau} \lesssim_{p,\tau} n^{-2}$, and thus (3.3) becomes

$$
\left\| \left(\sum_{n=0}^{\infty} V^2 \left(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})} \lesssim_{p,\tau} \| f \|_{\ell^p(\mathbb{Z})}.
$$

To establish the estimate (3.2), it remains to bound the first term. Note that

$$\sup_{J \in \mathbb{N}} \sup_{I \in \mathfrak{S}_J(\mathbb{D}_\tau)} \| O_{I,J}^2 (M_t f : t \in \mathbb{D}_\tau) \|_{\ell^p(\mathbb{Z})}$$
$$= \sup_{J \in \mathbb{N}_0} \sup_{I \in \mathfrak{S}_J(\mathbb{N}_0)} \| O_{I,J}^2 (M_{2^{n\tau}} f : n \in \mathbb{N}_0) \|_{\ell^p(\mathbb{Z})},$$

and we now establish the following estimate.

Claim There exists a constant $C_{p,\tau} > 0$ such that

$$\sup_{J \in \mathbb{N}} \sup_{I \in \mathfrak{S}_J(\mathbb{N}_0)} \| O_{I,J}^2 (M_{2^{n\tau}} f : n \in \mathbb{N}_0) \|_{\ell^p(\mathbb{Z})} \le C_{p,\tau} \| f \|_{\ell^p(\mathbb{Z})}.$$

Proof We introduce some auxiliary averaging operators. Let

$$H_t f(x) = \frac{1}{t} \sum_{1 \le s \le t} f(x - s) \quad \text{and} \quad A_t f(x) = \frac{1}{|B_t|} \sum_{1 \le s \le t} \psi(s) f(x - s) = (L_t * f)(x)$$

where $L_t(x) = \frac{1}{|B_t|} \sum_{1 \le s \le t} \psi(s) \delta_s(x)$. We may compare M_t with A_t as follows

$$\sup_{J \in \mathbb{N}_0} \sup_{I \in \mathfrak{S}_J(\mathbb{N}_0)} \| O_{I,J}^2 (M_{2^{n\tau}} f : n \in \mathbb{N}_0) \|_{\ell^p(\mathbb{Z})}$$
$$\le \sup_{J \in \mathbb{N}_0} \sup_{I \in \mathfrak{S}_J(\mathbb{N}_0)} \| O_{I,J}^2 (M_{2^{n\tau}} f - A_{2^{n\tau}} f : n \in \mathbb{N}_0) \|_{\ell^p(\mathbb{Z})}$$
$$+ \sup_{J \in \mathbb{N}_0} \sup_{I \in \mathfrak{S}_J(\mathbb{N}_0)} \| O_{I,J}^2 (A_{2^{n\tau}} f : n \in \mathbb{N}_0) \|_{\ell^p(\mathbb{Z})}$$
$$\lesssim \left\| \left(\sum_{n \in \mathbb{N}_0} | M_{2^{n\tau}} f - A_{2^{n\tau}} f |^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$
$$+ \sup_{J \in \mathbb{N}_0} \sup_{I \in \mathfrak{S}_J(\mathbb{N}_0)} \| O_{I,J}^2 (A_{2^{n\tau}} f : n \in \mathbb{N}_0) \|_{\ell^p(\mathbb{Z})}. \tag{3.4}$$

The first term of the expression (3.4) will be bounded by using Lemma 2.1 and interpolation. More specifically, we start with $p = 2$, and we note

$$\left\| \left(\sum_{n \in \mathbb{N}_0} | M_{2^{n\tau}} f - A_{2^{n\tau}} f |^2 \right)^{1/2} \right\|_{\ell^2(\mathbb{Z})} = \left(\sum_{n \in \mathbb{N}_0} \| M_{2^{n\tau}} f - A_{2^{n\tau}} f \|_{\ell^2(\mathbb{Z})}^2 \right)^{1/2}$$

and for each $n \in \mathbb{N}_0$ we have

$$\| M_{2^{n\tau}} f - A_{2^{n\tau}} f \|_{\ell^2(\mathbb{Z})} = \| (K_{2^{n\tau}} - L_{2^{n\tau}}) * f \|_{\ell^2(\mathbb{Z})}$$
$$= \| (\widehat{K}_{2^{n\tau}} - \widehat{L}_{2^{n\tau}}) \widehat{f} \|_{L^2(\mathbb{T})} \le \| \widehat{K}_{2^{n\tau}} - \widehat{L}_{2^{n\tau}} \|_{L^\infty(\mathbb{T})} \| f \|_{\ell^2(\mathbb{Z})}$$

and note that there exists $\chi > 0$ such that for any $\xi \in \mathbb{T}$ we get

$$|\widehat{K}_{2^{n^\tau}}(\xi) - \widehat{L}_{2^{n^\tau}}(\xi)| = \left| \frac{1}{|B_{2^{n^\tau}}|} \sum_{s \in B_{2^{n^\tau}}} e^{2\pi i s \xi} - \frac{1}{|B_{2^{n^\tau}}|} \sum_{1 \le s \le 2^{n^\tau}} \psi(s) e^{2\pi i s \xi} \right|$$

$$\lesssim \frac{\varphi_2(2^{n^\tau}) 2^{-\chi n^\tau}}{|B_{2^{n^\tau}}|} \lesssim 2^{-\chi n^\tau}$$

and thus

$$\|M_{2^{n^\tau}} f - A_{2^{n^\tau}} f\|_{\ell^2(\mathbb{Z})} \lesssim 2^{-\chi n^\tau} \|f\|_{\ell^2(\mathbb{Z})}$$

which gives

$$\left\| \left(\sum_{n \in \mathbb{N}_0} |M_{2^{n^\tau}} f - A_{2^{n^\tau}} f|^2 \right)^{1/2} \right\|_{\ell^2(\mathbb{Z})}$$

$$\lesssim \left(\sum_{n \in \mathbb{N}_0} 2^{-2\chi n^\tau} \right)^{1/2} \|f\|_{\ell^2(\mathbb{Z})} \lesssim_\tau \|f\|_{\ell^2(\mathbb{Z})}.$$

For the case of $p \ne 2$, firstly, let us assume that $p \in (2, \infty)$. Note that there exists a positive constant C such that

$$\|M_{2^{n^\tau}} f - A_{2^{n^\tau}} f\|_{\ell^{p'_0}(\mathbb{Z})} \le \|M_{2^{n^\tau}} f\|_{\ell^{p'_0}(\mathbb{Z})} + \|A_{2^{n^\tau}} f\|_{\ell^{p'_0}(\mathbb{Z})} \le C \|f\|_{\ell^{p'_0}(\mathbb{Z})}$$

since

$$\left\| \frac{1}{|B_t|} \sum_{s \in B_t} f(\cdot - s) \right\|_{\ell^{p'_0}(\mathbb{Z})} \le \|f\|_{\ell^{p'_0}(\mathbb{Z})} \quad \text{and}$$

$$\left\| \frac{1}{|B_t|} \sum_{1 \le s \le t} \psi(s) f(\cdot - s) \right\|_{\ell^{p'_0}(\mathbb{Z})} \le \frac{1}{|B_t|} \sum_{1 \le s \le t} \psi(s) \|f\|_{\ell^{p'_0}(\mathbb{Z})} \lesssim \|f\|_{\ell^{p'_0}}$$

where we have used the trigonometric estimates [17], page 6, for $\xi = 0$. We may choose $\theta \in (0, 1)$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p'_0}$ and use Riesz–Thorin interpolation theorem. Since for any $n \in \mathbb{N}_0$ we have

$$\|M_{2^{n^\tau}} f - A_{2^{n^\tau}} f\|_{\ell^2(\mathbb{Z})} \le C 2^{-\chi n^\tau} \quad \text{and} \quad \|M_{2^{n^\tau}} f - A_{2^{n^\tau}} f\|_{\ell^{p'_0}(\mathbb{Z})} \le C \|f\|_{\ell^{p'_0}(\mathbb{Z})},$$

we interpolate to obtain

$$\|M_{2^{n^\tau}} f - A_{2^{n^\tau}} f\|_{\ell^p(\mathbb{Z})} \le (C 2^{-\chi n^\tau})^\theta C^{1-\theta} \|f\|_{\ell^p(\mathbb{Z})} = C (2^{-\chi \theta})^{n^\tau} \|f\|_{\ell^p(\mathbb{Z})}.$$

Note that since $p/2 > 1$, we have

$$
\left\| \left(\sum_{n \in \mathbb{N}_0} |M_{2^{n^\tau}} f - A_{2^{n^\tau}} f|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}
$$

$$
= \left(\sum_{x \in \mathbb{Z}} \left(\sum_{n \in \mathbb{N}_0} |M_{2^{n^\tau}} f(x) - A_{2^{n^\tau}} f(x)|^2 \right)^{p/2} \right)^{1/p}
$$

$$
\leq \left(\sum_{n \in \mathbb{N}_0} \left(\sum_{x \in \mathbb{Z}} |M_{2^{n^\tau}} f(x) - A_{2^{n^\tau}} f(x)|^p \right)^{2/p} \right)^{1/2}
$$

$$
= \left(\sum_{n \in \mathbb{N}_0} \| M_{2^{n^\tau}} f - A_{2^{n^\tau}} f \|_{\ell^p(\mathbb{Z})}^2 \right)^{1/2}
$$

$$
\leq C \left(\sum_{n \in \mathbb{N}_0} (2^{-2\chi\theta})^{n^\tau} \right)^{1/2} \| f \|_{\ell^p(\mathbb{Z})} \leq C_{p,\tau} \| f \|_{\ell^p(\mathbb{Z})}.
$$

The case of $p \in (1, 2)$ is similar; we choose $\theta \in (0, 1)$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_0}$. Riesz-Thorin interpolation theorem yields the same estimate as before

$$
\| M_{2^{n^\tau}} f - A_{2^{n^\tau}} f \|_{\ell^p(\mathbb{Z})} \leq C (2^{-\chi\theta})^{n^\tau} \| f \|_{\ell^p(\mathbb{Z})},
$$

and we note that

$$
\left\| \left(\sum_{n \in \mathbb{N}_0} |M_{2^{n^\tau}} f - A_{2^{n^\tau}} f|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})} \leq \left\| \left(\sum_{n \in \mathbb{N}_0} |M_{2^{n^\tau}} f - A_{2^{n^\tau}} f|^p \right)^{1/p} \right\|_{\ell^p(\mathbb{Z})}
$$

$$
\leq \left(\sum_{n \in \mathbb{N}_0} \| M_{2^{n^\tau}} f - A_{2^{n^\tau}} f \|_{\ell^p(\mathbb{Z})}^p \right)^{1/p} \leq C \left(\sum_{n \in \mathbb{N}_0} (2^{-p\chi\theta})^{n^\tau} \right)^{1/p} \| f \|_{\ell^p(\mathbb{Z})}
$$

$$
\leq C_{p,\tau} \| f \|_{\ell^p(\mathbb{Z})}.
$$

We have appropriately bounded the first term of equation (3.4). We now focus on the second term. We will reduce the 2-oscillation ℓ^p estimates for $A_{2^{n^\tau}}$ to the corresponding ones for the standard $H_{2^{n^\tau}}$. Firstly, the analysis of the 2-oscillations will be made easier if we adjust A_t to the following very similar operator

$$
D_t f(x) = \frac{1}{\sum_{1 \leq s \leq t} \psi(s)} \sum_{1 \leq s \leq t} \psi(s) f(x - s).
$$

Note that

$$
|A_{2^{n^\tau}} f(x) - D_{2^{n^\tau}} f(x)| = \left| \left(\frac{1}{|B_{2^{n^\tau}}|} - \frac{1}{\sum_{1 \leq s \leq 2^{n^\tau}} \psi(s)} \right) \sum_{1 \leq s \leq 2^{n^\tau}} \psi(s) f(x - s) \right|.
$$

Thus

$$\|A_{2^{n^\tau}} f - D_{2^{n^\tau}} f\|_{\ell^p(\mathbb{Z})} \leq \frac{\left|\sum_{1 \leq s \leq 2^{n^\tau}} \psi(s) - |B_{2^{n^\tau}}|\right|}{|B_{2^{n^\tau}}| \sum_{1 \leq s \leq 2^{n^\tau}} \psi(s)} \sum_{1 \leq s \leq 2^{n^\tau}} \psi(s) \|f\|_{\ell^p(\mathbb{Z})}$$

$$\lesssim \frac{\varphi_2(2^{n^\tau}) 2^{-\chi n^\tau}}{|B_{2^{n^\tau}}|} \|f\|_{\ell^p(\mathbb{Z})} \lesssim 2^{-\chi n^\tau} \|f\|_{\ell^p(\mathbb{Z})}.$$

One may use a similar argument to the one presented earlier to compare $A_{2^{n^\tau}}$ with $D_{2^{n^\tau}}$ and obtain

$$\sup_{J \in \mathbb{N}_0} \sup_{I \in \mathfrak{S}_J(\mathbb{N}_0)} \|O_{I,J}^2(A_{2^{n^\tau}} f : n \in \mathbb{N}_0)\|_{\ell^p(\mathbb{Z})}$$

$$\leq \sup_{J \in \mathbb{N}_0} \sup_{I \in \mathfrak{S}_J(\mathbb{N}_0)} \|O_{I,J}^2(A_{2^{n^\tau}} f - D_{2^{n^\tau}} f : n \in \mathbb{N}_0)\|_{\ell^p(\mathbb{Z})}$$

$$+ \sup_{J \in \mathbb{N}_0} \sup_{I \in \mathfrak{S}_J(\mathbb{N}_0)} \|O_{I,J}^2(D_{2^{n^\tau}} f : n \in \mathbb{N}_0)\|_{\ell^p(\mathbb{Z})}$$

$$\lesssim \left\|\left(\sum_{n \in \mathbb{N}_0} |A_{2^{n^\tau}} f - D_{2^{n^\tau}} f|^2\right)^{1/2}\right\|_{\ell^p(\mathbb{Z})}$$

$$+ \sup_{J \in \mathbb{N}_0} \sup_{I \in \mathfrak{S}_J(\mathbb{N}_0)} \|O_{I,J}^2(D_{2^{n^\tau}} f : n \in \mathbb{N}_0)\|_{\ell^p(\mathbb{Z})}. \tag{3.5}$$

Using the estimate $\|A_{2^{n^\tau}} f - D_{2^{n^\tau}} f\|_{\ell^p(\mathbb{Z})} \lesssim 2^{-\chi n^\tau} \|f\|_{\ell^p(\mathbb{Z})}$, we can bound the first term of equation (3.5) by $C_{p,\tau} \|f\|_{\ell^p(\mathbb{Z})}$ and our task has been reduced to estimating the 2-oscillations of $D_{2^{n^\tau}}$. In fact, we will be able to estimate the 2-oscillations of D_n by comparing it with H_n. For convenience, let $\Psi(k) = \sum_{1 \leq s \leq k} \psi(s)$. We perform summation by parts

$$D_k f(x) = \frac{1}{\Psi(k)} \sum_{1 \leq s \leq k} \psi(s) f(x-s)$$

$$= \frac{1}{\Psi(k)} \left(\psi(k) \sum_{1 \leq s \leq k} f(x-s) - \sum_{1 \leq s \leq k-1} \left(\sum_{1 \leq l \leq s} f(x-l)\right)(\psi(s+1) - \psi(s))\right)$$

$$= \frac{k\psi(k)}{\Psi(k)} \sum_{1 \leq s \leq k} \frac{f(x-s)}{k} - \sum_{1 \leq s \leq k-1} \frac{s(\psi(s+1) - \psi(s))}{\Psi(k)} \sum_{1 \leq l \leq s} \frac{f(x-l)}{s}$$

$$= \frac{k\psi(k)}{\Psi(k)} H_k f(x) - \sum_{1 \leq s \leq k-1} \frac{s(\psi(s+1) - \psi(s))}{\Psi(k)} H_s f(x)$$

$$= \sum_{s=1}^{\infty} \lambda_s^k H_s f(x)$$

where

$$\lambda_s^k = \begin{cases} \frac{s(\psi(s)-\psi(s+1))}{\Psi(k)} & \text{if } 1 \le s \le k-1 \\ \frac{k\psi(k)}{\Psi(k)} & \text{if } s = k \\ 0 & \text{if } s > k. \end{cases}$$

We have shown $D_k f(x) = \sum_{s=1}^{\infty} \lambda_s^k H_s f(x)$. Without loss of generality, $\psi(s)$ is decreasing, and thus $(\lambda_s^k)_{s,k \in \mathbb{N}}$ is a family of non-negative real numbers (in the spirit of Lemma 2 from [24]) and we note that for any $k \in \mathbb{N}$ we have that

$$\sum_{s=1}^{\infty} \lambda_s^k = \sum_{s=1}^{k-1} \frac{s(\psi(s)-\psi(s+1))}{\Psi(k)} + \frac{k\psi(k)}{\Psi(k)} = 1$$

and we also have that for any fixed $N \in \mathbb{N}$ the sequence $\sum_{s=1}^{N} \lambda_s^k$ is decreasing in k, since

$$\sum_{s=1}^{N} \lambda_s^k = \begin{cases} \frac{1}{\Psi(k)} \sum_{s=1}^{N} s(\psi(s)-\psi(s+1)) & \text{if } 1 \le N \le k-1 \\ 1 & \text{if } N \ge k \end{cases}$$

and for any $1 \le N \le k-1$ we have

$$\frac{1}{\Psi(k)} \sum_{s=1}^{N} s(\psi(s)-\psi(s+1)) = \frac{1}{\Psi(k)} \left(\sum_{s=1}^{N} \psi(s) - N\psi(N+1) \right) \le 1.$$

For any $k \in \mathbb{N}$ we introduce the function $N_k : [0,1) \to \mathbb{N}$ such that $N_k(t) = \min\{N \in \mathbb{N} : \sum_{i=1}^{N} \lambda_i^k > t\}$ and we also introduce $I_s^k = N_k^{-1}(\{s\}) = \{t \in [0,1) : N_k(t) = s\}$. We note that for all $k \in \mathbb{N}$, N_k is increasing in t. Also, since for any fixed $N \in \mathbb{N}$ the sequence $\sum_{s=1}^{N} \lambda_s^k$ is decreasing in k, we have that for any fixed $t \in [0,1)$, the sequence $N_k(t)$ is increasing in k. Note that $|I_s^k| = \lambda_s^k$. Then

$$D_k f(x) = \sum_{s=1}^{\infty} \lambda_s^k H_s f(x) = \sum_{s=1}^{\infty} |I_s^k| H_s f(x)$$

$$= \sum_{s=1}^{\infty} \int_{I_s^k} H_{N_k(t)} f(x) dt = \int_0^1 H_{N_k(t)} f(x) dt.$$

Finally, for any $J \in \mathbb{N}_0$ and any $I = \{I_0, \dots, I_J\} \in \mathfrak{S}_J(\mathbb{N}_0)$, we have that

$$O_{I,J}^2 \big(D_k f(x) : k \in \mathbb{N} \big) = O_{I,J}^2 \Big(\sum_{s=1}^{\infty} \lambda_s^k H_s f(x) : k \in \mathbb{N} \Big)$$

$$= O_{I,J}^2 \Big(\int_0^1 H_{N_k(t)} f(x) dt : k \in \mathbb{N} \Big)$$

$$= \left(\sum_{j=0}^{J-1} \sup_{I_j \leq k < I_{j+1}} \left| \int_0^1 \left(H_{N_k(t)} f(x) - H_{N_{I_j}(t)} f(x) \right) dt \right|^2 \right)^{1/2}$$

$$\leq \left(\sum_{j=0}^{J-1} \left(\int_0^1 \sup_{I_j \leq k < I_{j+1}} |H_{N_k(t)} f(x) - H_{N_{I_j}(t)} f(x)| dt \right)^2 \right)^{1/2}$$

$$\leq \int_0^1 \left(\sum_{j=0}^{J-1} \sup_{I_j \leq k < I_{j+1}} |H_{N_k(t)} f(x) - H_{N_{I_j}(t)} f(x)|^2 \right)^{1/2} dt,$$

and we finish the argument by noting that

$$\| O_{I,J}^2 \left(D_k f(x) : k \in \mathbb{N} \right) \|_{\ell^p(\mathbb{Z})}$$

$$\leq \left(\sum_{x \in \mathbb{Z}} \left| \int_0^1 \left(\sum_{j=0}^{J-1} \sup_{I_j \leq k < I_{j+1}} |H_{N_k(t)} f(x) - H_{N_{I_j}(t)} f(x)|^2 \right)^{1/2} dt \right|^p \right)^{1/p}$$

$$\leq \int_0^1 \left(\sum_{x \in \mathbb{Z}} \left(\sum_{j=0}^{J-1} \sup_{I_j \leq k < I_{j+1}} |H_{N_k(t)} f(x) - H_{N_{I_j}(t)} f(x)|^2 \right)^{p/2} \right)^{1/p} dt$$

$$\leq \int_0^1 \left(\sum_{x \in \mathbb{Z}} \left(\sum_{j=0}^{J-1} \sup_{N_{I_j}(t) \leq m < N_{I_{j+1}}(t)} |H_m f(x) - H_{N_{I_j}(t)} f(x)|^2 \right)^{p/2} \right)^{1/p} dt$$

$$= \int_0^1 \| O_{\{N_{I_0(t)}, \ldots, N_{I_J(t)}\}, |\{N_{I_0(t)}, \ldots, N_{I_J(t)}\}| - 1}^2 \left(H_m f(x) : m \in \mathbb{N} \right) \|_{\ell^p(\mathbb{Z})} dt$$

$$\leq \int_0^1 \sup_{\tilde{J} \in \mathbb{N}} \sup_{\tilde{I} \in \mathfrak{S}(\mathbb{N})} \| O_{\tilde{I}, \tilde{J}}^2 \left(H_m f(x) : m \in \mathbb{N} \right) \|_{\ell^p(\mathbb{Z})} dt \leq C_p \| f \|_{\ell^p(\mathbb{Z})}$$

where we have used the uniform 2-oscillation ℓ^p-estimates for the standard averaging operator for $p \in (1, \infty)$, see [15] or [19], or more precisely

$$\sup_{\tilde{J} \in \mathbb{N}} \sup_{\tilde{I} \in \mathfrak{S}(\mathbb{N})} \| O_{\tilde{I}, \tilde{J}}^2 \left(H_m f(x) : m \in \mathbb{N} \right) \|_{\ell^p(\mathbb{Z})}$$

$$\lesssim_p \| f \|_{\ell^p(\mathbb{Z})}.$$

We note that

$$\sup_{J \in \mathbb{N}_0} \sup_{I \in \mathfrak{S}_J(\mathbb{N}_0)} \| O_{I,J}^2 \left(D_{2^{n\tau}} f : n \in \mathbb{N}_0 \right) \|_{\ell^p(\mathbb{Z})}$$

$$\leq \sup_{J \in \mathbb{N}} \sup_{I \in \mathfrak{S}(\mathbb{N})} \| O_{I,J}^2 \left(D_k f(x) : k \in \mathbb{N} \right) \|_{\ell^p(\mathbb{Z})} \lesssim_p \| f \|_{\ell^p(\mathbb{Z})}.$$

This establishes the estimate (1.13). □

4 Vector-Valued Maximal Estimates and Concluding the Proof of Theorem 1.10

In this section we establish the vector-valued estimates for the maximal function corresponding to M_t, namely the estimate (1.14). By the Calderón Transference Principle, it suffices to prove the following.

Proposition 4.1 *For any $p \in (1, \infty)$, there exists a constant C_p such that for any $(f_j)_{j \in \mathbb{Z}} \in \ell^p(\mathbb{Z}; \ell^2(\mathbb{Z}))$ we have*

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left(\sup_{t \in [1,\infty)} M_t |f_j| \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})} \leq C_p \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}.$$

Proof Firstly, we note that $\sup_{t \in [1,\infty)} M_t |f|(x) \lesssim \sup_{n \in \mathbb{N}_0} M_{2^n} |f|(x)$ and thus

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left(\sup_{t \in [1,\infty)} M_t |f_j| \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \left(\sup_{n \in \mathbb{N}_0} M_{2^n} |f_j| \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$

$$\leq \left\| \left(\sum_{j \in \mathbb{Z}} \left(\sup_{n \in \mathbb{N}_0} \left| M_{2^n} |f_j| - A_{2^n} |f_j| \right| + \sup_{n \in \mathbb{N}_0} A_{2^n} |f_j| \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$

$$\leq \left\| \left(\sum_{j \in \mathbb{Z}} \left(\sup_{n \in \mathbb{N}_0} \left| M_{2^n} |f_j| - A_{2^n} |f_j| \right|^2 \right) \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$

$$+ \left\| \left(\sum_{j \in \mathbb{Z}} \left(\sup_{n \in \mathbb{N}_0} A_{2^n} |f_j| \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$

$$\leq \left\| \left(\sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \left| M_{2^n} |f_j| - A_{2^n} |f_j| \right|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$

$$+ \left\| \left(\sum_{j \in \mathbb{Z}} \left(\sup_{n \in \mathbb{N}_0} A_{2^n} |f_j| \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}. \tag{4.2}$$

We focus on the first term, for $p = 2$ we note that

$$\left\| \left(\sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \left| M_{2^n} |f_j| - A_{2^n} |f_j| \right|^2 \right)^{1/2} \right\|_{\ell^2(\mathbb{Z})}$$

$$= \left(\sum_{x \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \left| M_{2^n} |f_j|(x) - A_{2^n} |f_j|(x) \right|^2 \right)^{1/2}$$

$$= \left(\sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{N}_0} \left\| M_{2^n} |f_j| - A_{2^n} |f_j| \right\|_{\ell^2(\mathbb{Z})}^2 \right)^{1/2}.$$

By Plancherel theorem combined with Lemma 2.1 we obtain the following

$$\left\| M_{2^n} |f_j| - A_{2^n} |f_j| \right\|_{\ell^2(\mathbb{Z})} \lesssim 2^{-\chi n} \|f_j\|_{\ell^2(\mathbb{Z})},$$

and thus

$$\left\| \left(\sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{N}_0} \left| M_{2^n}|f_j| - A_{2^n}|f_j| \right|^2 \right)^{1/2} \right\|_{\ell^2(\mathbb{Z})}$$

$$\lesssim \left(\sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{N}_0} 2^{-2\chi n} \|f_j\|^2_{\ell^2(\mathbb{Z})} \right)^{1/2} \lesssim \left(\sum_{j\in\mathbb{Z}} \|f_j\|^2_{\ell^2(\mathbb{Z})} \right)^{1/2}$$

$$= \left(\sum_{j\in\mathbb{Z}} \sum_{x\in\mathbb{Z}} |f_j(x)|^2 \right)^{1/2} = \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{\ell^2(\mathbb{Z})}.$$

For the case of $p \neq 2$, we proceed in a manner identical to the one of the previous section. Firstly, let us assume that $p \in (2, \infty)$. We fix a $p_0 > p$ and we note that there exists a positive constant C such that

$$\|M_{2^n} f - A_{2^n} f\|_{\ell^{p_0}(\mathbb{Z})} \leq \|M_{2^n} f\|_{\ell^{p_0}(\mathbb{Z})} + \|A_{2^n} f\|_{\ell^{p_0}(\mathbb{Z})} \leq C\|f\|_{\ell^{p_0}(\mathbb{Z})},$$

since

$$\left\| \frac{1}{|B_t|} \sum_{s\in B_t} f(\cdot - s) \right\|_{\ell^{p_0}(\mathbb{Z})} \leq \|f\|_{\ell^{p_0}(\mathbb{Z})} \& \left\| \frac{1}{|B_t|} \sum_{1\leq s\leq t} \psi(s) f(\cdot - s) \right\|_{\ell^{p_0}(\mathbb{Z})}$$

$$\leq \frac{1}{|B_t|} \sum_{1\leq s\leq t} \psi(s)\|f\|_{\ell^{p_0}(\mathbb{Z})} \lesssim \|f\|_{\ell^{p_0}}.$$

We may choose $\theta \in (0, 1)$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_0}$ and use Riesz-Thorin interpolation theorem. Since for any $n \in \mathbb{N}_0$ we have

$$\|M_{2^n} f - A_{2^n} f\|_{\ell^2(\mathbb{Z})} \leq C 2^{-\chi n}\|f\|_{\ell^2(\mathbb{Z})} \ \& \ \|M_{2^n} f - A_{2^n} f\|_{\ell^{p_0}(\mathbb{Z})} \leq C\|f\|_{\ell^{p_0}(\mathbb{Z})},$$

we interpolate to obtain

$$\|M_{2^n} f - A_{2^n} f\|_{\ell^p(\mathbb{Z})} \leq (C2^{-\chi n})^\theta C^{1-\theta}\|f\|_{\ell^p(\mathbb{Z})} = C(2^{-\chi\theta})^n\|f\|_{\ell^p(\mathbb{Z})}. \quad (4.3)$$

Thus if we let $T_n \colon \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ such that $T_n f = M_{2^n} f - A_{2^n} f$, then we know that T_n is a bounded linear operator with $\|T_n\|_{\ell^p(\mathbb{Z})\to\ell^p(\mathbb{Z})} \leq C(2^{\chi\theta})^{-n}$. Thus, we know that T_n has an ℓ^2-valued extension (see for example [13], page 386) with the same norm, that is

$$\left\| \left(\sum_{j\in\mathbb{Z}} |T_n(f_j)|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})} \leq C(2^{\chi\theta})^{-n} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}. \quad (4.4)$$

Finally, since $p/2 > 1$, we get

$$
\left\| \left(\sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{N}_0} \left| M_{2^n}|f_j| - A_{2^n}|f_j| \right|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}
$$

$$
= \left(\sum_{x\in\mathbb{Z}} \left(\sum_{n\in\mathbb{N}_0} \sum_{j\in\mathbb{Z}} \left| M_{2^n}|f_j|(x) - A_{2^n}|f_j|(x) \right|^2 \right)^{p/2} \right)^{1/p}
$$

$$
\leq \left(\sum_{n\in\mathbb{N}_0} \left(\sum_{x\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} |T_n|f_j|(x)|^2 \right)^{p/2} \right)^{2/p} \right)^{1/2}
$$

$$
\leq \left(\sum_{n\in\mathbb{N}_0} \left\| \left(\sum_{j\in\mathbb{Z}} |T_n|f_j||^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}^2 \right)^{1/2}
$$

$$
\leq \left(\sum_{n\in\mathbb{N}_0} C^2 (2^{2\chi\theta})^{-n} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}^2 \right)^{1/2}
$$

$$
\leq C \sum_{n\in\mathbb{N}_0} (2^{2\chi\theta})^{-n} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}
$$

$$
\lesssim_p \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}.
$$

For $p \in (1,2)$ the situation is similar, we choose $p_0 \in (1,p)$, and $\theta \in (0,1)$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_0}$ and Riesz-Thorin interpolation theorem yields the estimate of (4.3), which in turn implies the estimate (4.4). Since $p < 2$, we have

$$
\left\| \left(\sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{N}_0} \left| M_{2^n}|f_j| - A_{2^n}|f_j| \right|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}
$$

$$
= \left(\sum_{x\in\mathbb{Z}} \left(\sum_{n\in\mathbb{N}_0} \sum_{j\in\mathbb{Z}} \left| M_{2^n}|f_j|(x) - A_{2^n}|f_j|(x) \right|^2 \right)^{p/2} \right)^{1/p}
$$

$$
= \left(\sum_{x\in\mathbb{Z}} \left(\sum_{n\in\mathbb{N}_0} \left[\left(\sum_{j\in\mathbb{Z}} |T_n|f_j|(x)|^2 \right)^{1/2} \right]^2 \right)^{p/2} \right)^{1/p}
$$

$$
\leq \left(\sum_{x\in\mathbb{Z}} \sum_{n\in\mathbb{N}_0} \left[\left(\sum_{j\in\mathbb{Z}} |T_n|f_j|(x)|^2 \right)^{1/2} \right]^p \right)^{1/p}
$$

$$
= \left(\sum_{n\in\mathbb{N}_0} \left\| \left(\sum_{j\in\mathbb{Z}} |T_n|f_j||^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}^p \right)^{1/p}
$$

$$
\lesssim \left(\sum_{n\in\mathbb{N}_0} 2^{-p\chi\theta n} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}^p \right)^{1/p}
$$

$$
\lesssim_p \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}.
$$

We have bounded appropriately the first term of (4.2) and all is left is to bound the second term. We firstly observe that for any $n \in \mathbb{N}_0$ we get

$$A_{2^n}|f|(x) = \frac{1}{|B_{2^n}|} \sum_{1 \le s \le 2^n} \psi(s)|f(x-s)| \le \frac{1}{|B_{2^n}|} \sum_{k=0}^{n} \sum_{2^k \le s < 2^{k+1}} \psi(s)|f(x-s)|$$

$$\le \frac{1}{|B_{2^n}|} \sum_{k=0}^{n} |B \cap [2^k, 2^{k+1})| \frac{1}{|B \cap [2^k, 2^{k+1})|} \sum_{2^k \le s < 2^{k+1}} \psi(s)|f(x-s)|$$

$$\le \frac{|B_{2^{n+1}}|}{|B_{2^n}|} \sup_{k \in \mathbb{N}_0} \frac{1}{|B \cap [2^k, 2^{k+1})|} \sum_{2^k \le s < 2^{k+1}} \psi(s)|f(x-s)|$$

$$\lesssim \frac{\varphi_2(2^{n+1})}{\varphi_2(2^n)} \sup_{k \in \mathbb{N}_0} \frac{\varphi_2'(2^k)}{|B \cap [2^k, 2^{k+1})|} \sum_{2^k \le s < 2^{k+1}} |f(x-s)|$$

$$\lesssim \sup_{k \in \mathbb{N}_0} \frac{\varphi_2'(2^k) 2^k}{|B \cap [2^k, 2^{k+1})|} \frac{1}{2^k} \sum_{2^k \le s < 2^{k+1}} |f(x-s)|$$

$$\lesssim \sup_{k \in \mathbb{N}_0} \frac{1}{2^k} \sum_{2^k \le s < 2^{k+1}} |f(x-s)| \lesssim \sup_{k \in \mathbb{N}_0} \frac{1}{2^k} \sum_{1 \le s \le 2^k} |f(x-s)| = \sup_{k \in \mathbb{N}_0} H_{2^k}|f|(x)$$

since

$$\frac{\varphi_2'(2^k) 2^k}{|B \cap [2^k, 2^{k+1})|} \simeq \frac{\varphi_2(2^k)}{|B \cap [2^k, 2^{k+1})|} \simeq 1.$$

Since $n \in \mathbb{N}_0$ was arbitrary, we have shown that $\sup_{n \in \mathbb{N}_0} A_{2^n}|f|(x) \lesssim \sup_{n \in \mathbb{N}_0} H_{2^n}|f|(x)$, and thus the second term of 4.2 may dominated by

$$\left\| \left(\sum_{j \in \mathbb{Z}} \left(\sup_{n \in \mathbb{N}_0} A_{2^n}|f_j| \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \left(\sup_{n \in \mathbb{N}_0} H_{2^n}|f_j| \right)^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}$$

$$\lesssim_p \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{\ell^p(\mathbb{Z})}.$$

This completes the proof. ∎

The work of Sects. 3 and 4 proves Proposition 1.12. We describe how Proposition 1.12 implies Theorem 1.10.

Proof of Theorem 1.10 assuming Proposition 1.12 We simply apply Proposition 4.1 from [23] to establish a multi-parameter uniform 2-oscillation estimate, which according to Remark 2.4 together with Proposition 2.8, page 15, from the same paper yield the desired result. ∎

5 Proof of the Weak-Type (1,1) Inequality

Throughout this section we have fixed a set B with $\varphi_1 \simeq \varphi_2$ and $c_1 = c_2 = c \in (1, 30/29)$. All constants may depend on φ_1, φ_2 and ψ but on nothing else, unless stated otherwise. We remind the reader that it suffices to establish the weak type (1,1) bound for the smooth dyadic maximal function

$$\mathcal{M}(f)(x) = \sup_{k \in \mathbb{N}_0} \{K_{2^k} * |f|(x), \text{ where } K_N(x)$$

$$= \frac{1}{\varphi_2(N)} \sum_{n \in B} \delta_n(x) \eta\left(\frac{n}{N}\right), \text{ see Remark } 1.9.$$

The next two lemmas are devoted to studying the properties of $K_N * \widetilde{K}_N$ and they will be key ingredients for establishing the weak-type (1,1) bound.

Lemma 5.1 *There exists a positive constant C such that for all $N \in \mathbb{N}$ and all $x \in \mathbb{Z}$ with $C \leq |x| \leq \varphi_1(N)$, we have that $|K_N * \widetilde{K}_N(x)| \leq CN^{-1}$.*

Proof We have assumed that $\varphi_2 \simeq \varphi_1$ and thus, by Lemma 2.4, there exists a uniform bound C_B for the length of intervals contained in B. For any $x \in \mathbb{Z}$ such that $|x| \geq C_B + 1$, we get

$$K_N * \widetilde{K}_N(x) = \frac{1}{\varphi_2(N)^2} \sum_{n \in \mathbb{Z}} 1_B(n) 1_B(n+x) \eta\left(\frac{n}{N}\right) \eta\left(\frac{n+x}{N}\right)$$

and with a change of variables we see that $K_N * \widetilde{K}_N(x) = K_N * \widetilde{K}_N(-x)$, and thus, without loss of generality, let us assume that $x \geq C_B + 1$. Since $\text{supp}(\eta) \subseteq (1/2, 4)$ and $0 \leq \eta(x) \leq 1$, we have

$$K_N * \widetilde{K}_N(x) \leq \frac{1}{\varphi_2(N)^2} |\{n \in \mathbb{Z} : n, n+x \in B \cap (N/2, 4N)\}|$$

and all is left to do is to estimate the cardinality of that set. Let $\mathcal{A}_N^x = \{n \in \mathbb{Z} : n, n+x \in B \cap (N/2, 4N)\}$, and notice that for any $n \in \mathcal{A}_N^x$, we have that there exists a unique $s, m \in \mathbb{N}_0$ such that

$$0 \leq \varphi_1(n+x) - (m+s) < \psi(n+x) \text{ and } 0 \leq \varphi_1(n) - m < \psi(n).$$

Notice that since $x > C_B$, we have that $s \geq 1$, since n and $n+x$ cannot correspond to the same m. By combining the previous set of inequalities we obtain

$$\varphi_1(n+x) - \varphi_1(n) - \psi(n+x) < s < \varphi_1(n+x) - \varphi_1(n) + \psi(n).$$

For constants depending only on φ_1, φ_2 and ψ, we have

$$\varphi_1(n+x) - \varphi_1(n) + \psi(n) \le Cx\varphi_1'(N) + C\varphi_2'(N) \le Cx\varphi_1'(N) + C\varphi_1'(N)$$
$$= C(x+1)\frac{\varphi_1(N)}{N} \le Cx\frac{\varphi_1(N)}{N}$$

and similarly

$$\varphi_1(n+x) - \varphi_1(n) - \psi(n+x) \ge cx\varphi_1'(N) - c\varphi_2'(N) \ge cx\varphi_1'(N) - c\varphi_1'(N)$$
$$\ge c(x-1)\frac{\varphi_1(N)}{N} \ge cx\frac{\varphi_1(N)}{N}.$$

Thus $s \simeq \frac{x\varphi_1(N)}{N}$. We have that $m \le \varphi_1(4N) \lesssim \varphi_1(N)$ and $m > \varphi_1(n) - \psi(n) \ge \varphi_1(n) - C\varphi_1'(n) \ge \varphi_1(n)(1 - C/n) \ge C\varphi_1(N)$, when $n \in (N/2, 4N)$ thus $m \simeq \varphi_1(N)$. We also note that

$$h_1(\varphi_1(n) - \psi(n)) < h_1(m) \le n \text{ and}$$
$$h_1(\varphi_1(n+x) - \psi(n+x)) < h_1(m+s) \le n + x$$

and thus

$$x - \big(n + x - h_1(\varphi_1(n+x) - \psi(n+x))\big)$$
$$< h_1(m+s) - h_1(m) < x + \big(n - h_1(\varphi_1(n - \psi(n))))\big).$$

Note that for all l we get that $l - h_1(\varphi_1(l) - \psi(l)) = h_1(\varphi_1(l)) - h_1(\varphi_1(l) - \psi(l)) = \psi(l)h_1'(\xi_l)$, for some $\xi_l \in (\varphi_1(l) - \psi(l), \varphi_1(l))$, and thus $l - h_1(\varphi_1(l) - \psi(l)) \lesssim \varphi_2'(l)h_1'(\varphi_1(l)) = \frac{\varphi_2'(l)}{\varphi_1'(l)}$. Since $\varphi_2 \simeq \varphi_1$, we get that there exists an absolute constant T, such that $l - h_1(\varphi_1(l) - \psi(l)) \le T$, and thus

$$h_1(m+s) - h_1(m) \in B(x, T).$$

Consider the set $\mathcal{B}_N^x = \{(s, m) \in \mathbb{N} \times \mathbb{N} : s \simeq \frac{x\varphi_1(N)}{N}, m \simeq \varphi_1(N), h_1(m+s) - h_1(m) \in B(x, T)\}$ and note that for any $(s, m) \in \mathcal{B}_N^x$ there are at most C_B number of n's in $\mathcal{A}_N^x$ corresponding to m. Therefore $|\mathcal{A}_N^x| \lesssim |\mathcal{B}_N^x|$, and everything reduces to estimating $|\mathcal{B}_N^x|$. For every $s \ge 1$ such that $s \simeq \frac{x\varphi_1(N)}{N}$, we wish to estimate the number of m's such that $(s, m) \in \mathcal{B}_N^x$. Notice that if we define $g(m) = h_1(m+s) - h_1(m)$ then by the Mean Value Theorem we get

$$g(m+1) - g(m) = h_1(m+1+s) - h_1(m+1) - h_1(m+s) + h_1(m)$$
$$= h_1'(m+s+\xi_1) - h_1'(m+\xi_2)$$
$$= (s + \xi_1 - \xi_2)h_1''(m+\xi_3)$$

for some $\xi_1, \xi_2 \in (0, 1)$ and $\xi_3 \in (m, m+s+1)$. Thus

$$g(m+1) - g(m) \simeq sh_1''(\varphi_1(N)) \simeq \frac{sN}{\varphi_1(N)^2} \simeq \frac{x}{\varphi_1(N)}.$$

Since $x - T < h_1(m+s) - h_1(m) < x + T$, according to the previous calculation, for any fixed $s \simeq \frac{x\varphi_1(N)}{N}$, we have that there are at most $1 + C\frac{T\varphi_1(N)}{x} \lesssim \frac{\varphi_1(N)}{x}$, where in the last estimate we used that $x \leq \varphi_1(N)$. The number of s's in $[1, C\frac{x\varphi_1(N)}{N}]$, where C is the implied fixed constant appearing in $s \simeq \frac{x\varphi_1(N)}{N}$, are bounded by $C\frac{x\varphi_1(N)}{N}$, and therefore

$$|\mathcal{B}_N^x| \lesssim \frac{x\varphi_1(N)}{N} \frac{\varphi_1(N)}{x} = \frac{\varphi_1(N)^2}{N}.$$

This implies that

$$K_N * \widetilde{K}_N(x) \lesssim \frac{\varphi_1(N)^2}{N\varphi_2(N)^2} \lesssim \frac{1}{N}$$

and the proof is complete. $\qquad\square$

Lemma 5.2 *There exists a real number $\chi > 0$ such that $K_N * \widetilde{K}_N(x) = G_N(x) + E_N(x)$ for all $|x| > \varphi_1(N)$ where*

$$G_N(x) = \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{Z}} \psi(n)\psi(n+|x|)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right) \text{ and}$$
$$E_N(x) = K_N * \widetilde{K}_N(x) - G_N(x).$$

We also have that $G_N(x) \lesssim N^{-1}$, $|G_N(x+h) - G_N(x)| \lesssim N^{-2}|h|$ and $|E_N(x)| \lesssim N^{-1-\chi}$.

Proof We note that for all $n \in \mathbb{N}$ we have that $1_B(n) = \lfloor\varphi_1(n)\rfloor - \lfloor\varphi_1(n) - \psi(n)\rfloor$, see [17, Lemma 2.2]. We can therefore split our kernel to several manageable pieces.

$$K_N * \widetilde{K}_N(x) = \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{Z}} 1_B(n)1_B(n+x)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$$
$$= \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \left(\lfloor\varphi_1(n)\rfloor - \lfloor\varphi_1(n) - \psi(n)\rfloor\right)\left(\lfloor\varphi_1(n+x)\rfloor\right)$$
$$- \lfloor\varphi_1(n+x) - \psi(n+x)\rfloor\right)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right).$$

We will exploit a famous truncated Fourier Series. More precisely, we know that if $\Phi(x) = \{x\} - 1/2$ then for all $M \in \mathbb{N}$ we get

$$\Phi(x) = \sum_{0<|m|\leq M} \frac{1}{2\pi i m}e^{-2\pi i m x} + O\left(\min\left\{1, \frac{1}{M||x||}\right\}\right) \text{ (see section 2 from [14]).}$$

Importantly, we also have

$$\min\left\{1, \frac{1}{M||x||}\right\} = \sum_{m\in\mathbb{Z}} b_m e^{2\pi i m x}$$

and

$$b_m \lesssim \min\left\{ \frac{\log(M)}{M}, \frac{1}{|m|}, \frac{M}{|m|^2} \right\}.$$

Finally, we can rewrite

$$
\begin{aligned}
1_B(n) &= \lfloor \varphi_1(n) \rfloor - \lfloor \varphi_1(n) - \psi(n) \rfloor = \varphi_1(n) - \{\varphi_1(n)\} \\
&\quad - \big(\varphi_1(n) - \psi(n) - \{\varphi_1(n) - \psi(n)\} \big) \\
&= \psi(n) + \{\varphi_1(n) - \psi(n)\} - \{\varphi_1(n)\} \\
&= \psi(n) + (\{\varphi_1(n) - \psi(n)\} - 1/2) - (\{\varphi_1(n)\} - 1/2) \\
&= \psi(n) + \Phi(\varphi_1(n) - \psi(n)) - \Phi(\varphi_1(n)).
\end{aligned}
$$

Let us use the truncated Fourier Series and define

$$
\begin{aligned}
\Delta_M(n) &= \sum_{0<|m|\le M} \frac{1}{2\pi i m} e^{-2\pi i m(\varphi_1(n)-\psi(n))} - \sum_{0<|m|\le M} \frac{1}{2\pi i m} e^{-2\pi i m \varphi_1(n)} \\
&= \sum_{0<|m|\le M} \frac{e^{-2\pi i m \varphi_1(n)}}{2\pi i m} \left(e^{2\pi i m \psi(n)} - 1 \right)
\end{aligned}
$$

and

$$\Pi_M(n) = \Big(\Phi(\varphi_1(n) - \psi(n)) - \Phi(\varphi_1(n)) \Big) - \Delta_M(n)$$

and thus

$$\Pi_M(n) = O\left(\min\left\{ 1, \frac{1}{M\|\varphi_1(n) - \psi(n)\|} \right\} \right) + O\left(\min\left\{ 1, \frac{1}{M\|\varphi_1(n)\|} \right\} \right).$$

Thus

$$1_B(n) = \psi(n) + \Delta_M(n) + \Pi_M(n).$$

Returning back to the splitting

$$
\begin{aligned}
K_N * \widetilde{K}_N(x) &= \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \big(\psi(n) + \Delta_M(n) + \Pi_M(n) \big)\big(\psi(n+x) + \Delta_M(n+x) \\
&\quad + \Pi_M(n+x) \big) \eta\left(\frac{n}{N}\right) \eta\left(\frac{n+x}{N}\right) \\
&= \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \psi(n)\psi(n+x) \eta\left(\frac{n}{N}\right) \eta\left(\frac{n+x}{N}\right) \\
&\quad + \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \psi(n)\Delta_M(n+x) \eta\left(\frac{n}{N}\right) \eta\left(\frac{n+x}{N}\right)
\end{aligned}
$$

$$+ \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \psi(n)\Pi_M(n+x)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$$

$$+ \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \Delta_M(n)\psi(n+x)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$$

$$+ \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \Pi_M(n)\psi(n+x))\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$$

$$+ \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \Delta_M(n)\Delta_M(n+x)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$$

$$+ \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \Delta_M(n)\Pi_M(n+x)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$$

$$+ \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \Pi_M(n)\Delta_M(n+x)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$$

$$+ \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \Pi_M(n)\Pi_M(n+x)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right) =$$

$= I_1(x) + I_2(x) + \cdots + I_9(x)$. Let $G_N(x) = I_1(x)$ and $E_N(x) = \sum_{i=2}^{9} I_i(x)$. Let us firstly estimate I_1, we have

$$I_1(x) = \frac{1}{\varphi_2(N)^2} \sum_{\substack{N/2\leq n\leq 4N \\ N/2\leq n+x\leq 4N}} \psi(n)\psi(n+x)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right) \lesssim \frac{N\varphi_2'(N)^2}{\varphi_2(N)^2} \lesssim N^{-1},$$

and for any $h \in \mathbb{Z}$ we have

$$|I_1(x+h) - I_1(x)| \leq \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \psi(n)\eta\left(\frac{n}{N}\right)\left|\psi(n+x+h)\eta\left(\frac{n+x+h}{N}\right)\right.$$

$$\left. -\psi(n+x)\eta\left(\frac{n+x}{N}\right)\right|$$

$$= \frac{1}{\varphi_2(N)^2} \sum_{n\in\mathbb{N}} \psi(n)\eta\left(\frac{n}{N}\right)$$

$$\left|\int_{n+x}^{n+x+h} \left(\psi'(t)\eta(t/N) - \psi(t)\eta'(t/N)\frac{1}{N}\right)dt\right|.$$

It suffices to consider x, h such that $n+x, n+x+h \in [N/2, 4N]$ since the integrand is zero outside that interval. We get

$$|I_1(x+h) - I_1(x)| \lesssim \frac{\varphi_2'(N)}{\varphi_2(N)^2} N|h|(\varphi_2''(N) + \varphi_2'(N)/N) \lesssim N^{-2}|h|.$$

This shows that G_N satisfies the desired properties. Now we bound E_N. Let us start with I_2. We can rewrite I_2 as

$$
I_2(x) = \frac{1}{\varphi_2(N)^2} \sum_{n \in \mathbb{N}} \psi(n) \Delta_M(n+x) \eta\left(\frac{n}{N}\right) \eta\left(\frac{n+x}{N}\right)
$$

$$
= \frac{1}{\varphi_2(N)^2} \sum_{n \in \mathbb{N}} \psi(n) \sum_{0 < |m| \le M} \frac{e^{-2\pi i m \varphi_1(n+x)}}{2\pi i m} \left(e^{2\pi i m \psi(n+x)} - 1\right) \eta\left(\frac{n}{N}\right) \eta
$$

$$
\left(\frac{n+x}{N}\right)
$$

$$
= \frac{1}{\varphi_2(N)^2} \sum_{0 < |m| \le M} \frac{1}{2\pi i m} \sum_{\substack{N/2 \le n \le 4N \\ N/2 \le n+x \le 4N}} e^{2\pi i (-m) \varphi_1(n+x)} \left(\psi(n) \eta\left(\frac{n}{N}\right) \eta\left(\frac{n+x}{N}\right)\right.
$$

$$
\left(e^{2\pi i m \psi(n+x)} - 1\right)\biggr).
$$

According to Corollary 3.12 in [20], if we let $F_m^x(t) = \psi(t) \eta\left(\frac{t}{N}\right) \eta\left(\frac{t+x}{N}\right) \left(e^{2\pi i m \psi(t+x)} - 1\right)$, we have that for all $m \in \mathbb{Z} \setminus \{0\}$

$$
\left| \sum_{\substack{N/2 < n \le 4N \\ N/2 < n+x \le 4N}} e^{2\pi i (-m) \varphi_1(n+x)} F_m^x(n) \right|
$$

$$
\lesssim |m|^{1/2} N (\varphi_1(N) \sigma_1(N))^{-1/2}
$$

$$
\left(\sup_{\substack{N/2 < n \le 4N \\ N/2 < n+x \le 4N}} |F_m^x(n)| + N \sup_{\substack{N/2 < n \le 4N \\ N/2 < n+x \le 4N}} |F_m^x(n+1) - F_m^x(n)| \right).
$$

Let us follow the notation of [20] and write $N_{1,x} = \max\{N/2, N/2 - x\}$ and $N_{2,x} = \min\{4N, 4N - x\}$. For all $n \in (N_{1,x}, N_{2,x}]$ we get

$$
|F_m^x(n)| = \psi(n) \eta\left(\frac{n}{N}\right) \eta\left(\frac{n+x}{N}\right) \left| e^{2\pi i m \psi(n+x)} - 1 \right|
$$

$$
\lesssim \frac{\varphi_2(N)}{N} |2\pi i m \psi(n+x)| \lesssim \frac{\varphi_2(N)^2}{N^2} |m|
$$

where we used that for all real numbers x we have $|e^{ix} - 1| \le |x|$. Similarly, after applying the mean value theorem to obtain $|F_m^x(n+1) - F_m^x(n)| \le \sup_{t \in [n, n+1]} \left| \frac{d(F_m^x(t))}{dt} \right|$, we can use the following bound

$$
\left| \frac{d(F_m^x(t))}{dt} \right| \le \left| \psi'(t) \eta\left(\frac{t}{N}\right) \eta\left(\frac{t+x}{N}\right) \left(e^{2\pi i m \psi(t+x)} - 1\right) \right|
$$

$$
+ \left| \psi(t) \eta'\left(\frac{t}{N}\right) \frac{1}{N} \eta\left(\frac{t+x}{N}\right) \left(e^{2\pi i m \psi(t+x)} - 1\right) \right|
$$

$$+\left|\psi(t)\eta\left(\frac{t}{N}\right)\eta'\left(\frac{t+x}{N}\right)\frac{1}{N}\left(e^{2\pi im\psi(t+x)}-1\right)\right|$$

$$+\left|\psi(t)\eta\left(\frac{t}{N}\right)\eta\left(\frac{t+x}{N}\right)\left((2\pi im\psi'(t+x))e^{2\pi im\psi(t+x)}\right)\right|$$

$$\lesssim \frac{\varphi_2(N)^2}{N^3}|m|.$$

Therefore we can bound I_2

$$|I_2(x)| \lesssim \frac{1}{\varphi_2(N)^2}\sum_{0<|m|\leq M}\frac{1}{|m|}|m|^{1/2}N(\varphi_1(N)\sigma_1(N))^{-1/2}\left(\frac{\varphi_2(N)^2}{N^2}|m|\right).$$

Since $\gamma \in (29/30, 1)$ and $\varphi_1 \simeq \varphi_2$, we get that for $M = N^{1+2\chi+\varepsilon}\varphi_2(N)^{-1}$, $\chi = 1-\gamma$ and $\varepsilon < \chi/10$

$$|I_2(x)| \lesssim \frac{M^{3/2}}{N\varphi_2(N)^{1/2}\sigma_1(N)^{1/2}} = \frac{N^{3/2+5/2\chi+3\varepsilon}}{N^{1+\chi}\varphi_2(N)^2\sigma_1(N)^{1/2}}.$$

We use that for all $\varepsilon_1 > 0$ we have $\sigma_1(x) \gtrsim_{\varepsilon_1} x^{-\varepsilon_1}$ and $\varphi_2(x) \gtrsim_{\varepsilon_1} x^{\gamma-\varepsilon_1}$ to get

$$|I_2(x)| \lesssim \frac{1}{N^{1+\chi}}N^{3/2+5/2\chi+3\varepsilon-2\gamma+2\varepsilon_1+\varepsilon_1/2}.$$

For a fixed $\varepsilon_1 = \varepsilon \in (0, \chi/10)$ we get

$$|I_2(x)| \lesssim N^{-1-\chi}N^{3/2-2\gamma+6\varepsilon} \lesssim N^{-1-\chi}$$

since

$$3/2 + 5/2\chi - 2\gamma + 6\varepsilon < 0 \iff 4(1-\gamma) + 5(1-\gamma) + 6/10(1-\gamma) < 1$$
$$\iff 96/10(1-\gamma) < 1$$

which is true since $\gamma > 29/30$. Therefore we have shown that $|I_2(x)| \lesssim N^{-1-\chi}$, as desired. The term I_4 is treated similarly

$$I_4(x) = \frac{1}{\varphi_2(N)^2}\sum_{n\in\mathbb{N}}\psi(n+x)\Delta_M(n)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$$

$$= \frac{1}{\varphi_2(N)^2}\sum_{n\in\mathbb{N}}\psi(n+x)\sum_{0<|m|\leq M}\frac{e^{-2\pi im\varphi_1(n)}}{2\pi im}\left(e^{2\pi im\psi(n)}-1\right)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$$

$$= \frac{1}{\varphi_2(N)^2}\sum_{0<|m|\leq M}\frac{1}{2\pi im}\sum_{\substack{N/2\leq n\leq 4N \\ N/2\leq n+x\leq 4N}}e^{2\pi i(-m)\varphi_1(n)}\left(\psi(n+x)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)\right.$$

$$\left.\left(e^{2\pi im\psi(n)}-1\right)\right).$$

Using Corollary 3.12 in [20], for $G_m^x(n) = \psi(n+x)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)\left(e^{2\pi im\psi(n)} - 1\right)$, we obtain in an almost identical fashion the bound $|I_4(x)| \lesssim N^{-1-\chi}$. To bound the remaining terms, we use Lemma 3.18 [20], which we state here.

Lemma 5.3 *Let $N \geq 1$, $p, q \in \{0, 1\}$, $x \in \mathbb{Z}$ and $M \geq 1$. Then*

$$\sum_{n \in \mathbb{N}} \min\left\{1, \frac{1}{M||\varphi_1(n + px + q)||}\right\}\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right) \lesssim \frac{N\log(M)}{M} + \frac{NM^{1/2}\log(M)}{(\sigma_1(N)\varphi_1(N))^{1/2}}.$$

We will also use an appropriate extension of the lemma.

Lemma 5.4 *Let $N \geq 1$, $p \in \{0, 1\}$, $x \in \mathbb{Z}$ and $M \geq 1$. Then*

$$\sum_{n \in \mathbb{N}} \min\left\{1, \frac{1}{M||\varphi_1(n + px) - \psi(n + px)||}\right\}\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$$
$$\lesssim \frac{N\log(M)}{M} + \frac{NM^{1/2}\log(M)}{(\sigma_1(N)\varphi_1(N))^{1/2}}.$$

Proof We have that

$$\min\left\{1, \frac{1}{M||x||}\right\} = \sum_{m \in \mathbb{Z}} b_m e^{2\pi imx}$$

and

$$b_m \lesssim \min\left\{\frac{\log(M)}{M}, \frac{1}{|m|}, \frac{M}{|m|^2}\right\}.$$

Thus

$$\sum_{n \in \mathbb{N}} \min\left\{1, \frac{1}{M||\varphi_1(n + px) - \psi(n + px)||}\right\}\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$$
$$\lesssim \sum_{n+px \in (N/2, 4N]} \min\left\{1, \frac{1}{M||\varphi_1(n + px) - \psi(n + px)||}\right\}$$
$$= \sum_{n+px \in (N/2, 4N]} \sum_{m \in \mathbb{Z}} b_m e^{2\pi im(\varphi_1(n+px) - \psi(n+px))} \lesssim \sum_{m \in \mathbb{Z}} |b_m| \left| \sum_{n+px \in (N/2, 4N]} \right.$$
$$\left. e^{2\pi im(\varphi_1(n+px) - \psi(n+px))} \right|$$
$$\lesssim \frac{N\log(M)}{M} + \sum_{m \in \mathbb{Z}} |b_m||m|^{1/2} N\left(\varphi_1(N)\sigma_1(N)\right)^{-1/2} \lesssim \frac{N\log(M)}{M}$$
$$+ \sum_{0 < |m| \leq M} \frac{\log(M)}{M}|m|^{1/2} N\left(\varphi_1(N)\sigma_1(N)\right)^{-1/2}$$

$$+ \sum_{|m|>M} \frac{M}{|m|^2} |m|^{1/2} N \left(\varphi_1(N)\sigma_1(N)\right)^{-1/2} \lesssim \frac{N \log(M)}{M}$$

$$+ N \log(M) M^{1/2} \left(\varphi_1(N)\sigma_1(N)\right)^{-1/2}$$

where we used Lemma 4.1 from [9] to obtain the estimate for $\left| \sum_{n+px \in (N/2, 4N]} e^{2\pi i m(\varphi_1(n+px) - \psi(n+px))} \right|$. $\qquad \square$

Using the two lemmas above, together with the trivial estimates $\psi(x), |\Delta_M(x)|, |\Pi_M(x)| \lesssim 1$ (since $1_B(n) = \psi(n) + \Delta_M(n) + \Pi_M(n)$), we may estimate

$$|I_3(x)| + |I_5(x)| + |I_7(x)| + |I_8(x)| + |I_9(x)|$$

$$\lesssim \frac{1}{\varphi_2(N)^2} \sum_{n \in \mathbb{N}} \sum_{p \in \{0,1\}} \left(\min\left\{1, \frac{1}{M \|\varphi_1(n+px) - \psi(n+px)\|}\right\} \right.$$

$$\left. + \min\left\{1, \frac{1}{M \|\varphi_1(n+px)\|}\right\} \right) \eta\left(\frac{n}{N}\right) \eta\left(\frac{n+x}{N}\right)$$

$$\lesssim \frac{1}{\varphi_2(N)^2} \left(\frac{N \log(M)}{M} + \frac{N M^{1/2} \log(M)}{(\sigma_1(N)\varphi_1(N))^{1/2}} \right)$$

$$\lesssim \frac{N \log(N)}{\varphi_2(N) N^{1+2\chi+\varepsilon}} + \frac{N^{3/2 + \chi + \varepsilon/2} \log(N)}{\varphi_2(N)^3 \sigma_1(N)^{1/2}}$$

$$\lesssim N^{-1-\chi} \log(N) N^{1-\varepsilon-\gamma-\chi+\varepsilon/2} + N^{-1-\chi}$$

$$\left(N^{5/2 + 2\chi + \varepsilon/2 - 3\gamma + 3\varepsilon/2 + \varepsilon} \log(N) \right) \lesssim N^{-1-\chi}$$

since $\chi = 1 - \gamma$ and $5/2 + 2\chi - 3\gamma + 3\varepsilon < 0 \iff 10(1-\gamma) + 6\varepsilon < 1$ and since $\gamma \in (29/30, 1)$, one may choose such an appropriately small $\varepsilon > 0$. Finally, for I_6 we get

$$I_6(x) = \frac{1}{\varphi_2(N)^2} \sum_{n \in \mathbb{N}} \Delta_M(n) \Delta_M(n+x) \eta\left(\frac{n}{N}\right) \eta\left(\frac{n+x}{N}\right)$$

$$= \frac{1}{\varphi_2(N)^2} \sum_{n \in \mathbb{N}} \sum_{0 < |m_1|, |m_2| \le M} \frac{e^{-2\pi i m_1 \varphi_1(n)}}{2\pi i m_1}$$

$$\left(e^{2\pi i m_1 \psi(n)} - 1\right) \frac{e^{-2\pi i m_2 \varphi_1(n+x)}}{2\pi i m_2} \left(e^{2\pi i m_2 \psi(n)} - 1\right) \eta\left(\frac{n}{N}\right) \eta\left(\frac{n+x}{N}\right)$$

$$= \frac{1}{\varphi_2(N)^2} \sum_{0 < |m_1|, |m_2| \le M} \frac{1}{(2\pi i)^2 m_1 m_2} \sum_{N_{1,x} < n \le N_{2,x}}$$

$$e^{-2\pi i m_1 \varphi_1(n) - 2\pi i m_2 \varphi_1(n+x)} F^x_{m_1, m_2}(n)$$

where $F^x_{m_1,m_2}(n) = \left(e^{2\pi i m_1 \psi(n)} - 1\right)\left(e^{2\pi i m_2 \psi(n)} - 1\right)\eta\left(\frac{n}{N}\right)\eta\left(\frac{n+x}{N}\right)$. Therefore we have that

$$|I_6(x)| \lesssim \frac{1}{\varphi_2(N)^2} \sum_{0 < |m_1|, |m_2| \le M} \frac{1}{|m_1 m_2|}$$

$$\left| \sum_{N_{1,x} < n \le N_{2,x}} e^{-2\pi i m_1 \varphi_1(n) - 2\pi i m_2 \varphi_1(n+x)} F^x_{m_1,m_2}(n) \right|.$$

For all $n \in (N_{1,x}, N_{2,x}]$ we have $|F^x_{m_1,m_2}(n)| \lesssim |m_1 m_2|\psi(N)^2 \lesssim |m_1 m_2|\frac{\varphi_2(N)^2}{N^2}$ and also by the mean value theorem together with the following calculation

$$\left| \frac{d\left(F^x_{m_1,m_2}(t)\right)}{dt} \right| \lesssim |m_1||\psi'(N)||m_2|\psi(N)$$

$$+ |m_1|\psi(N)|m_2|\psi'(N) + 1/N|m_1 m_2|\psi(N)^2$$

$$\lesssim |m_1 m_2|\frac{\varphi_2(N)^2}{N^3}$$

we get $\sup_{N_{1,x} < n \le N_{2,x}}\{|F^x_{m_1,m_2}(n+1) - F^x_{m_1,m_2}(n)|\} \lesssim |m_1 m_2|\frac{\varphi_2(N)^2}{N^3}$. We let $m = \max\{m_1, m_2\}$ and we use Corollary 3.12 from [20] for $\alpha = 0$ and $\kappa = 1$ to obtain

$$\left| \sum_{N_{1,x} < n \le N_{2,x}} e^{-2\pi i m_1 \varphi_1(n) - 2\pi i m_2 \varphi_1(n+x)} F^x_{m_1,m_2}(n) \right|$$

$$\lesssim \max\{m_1, m_2\}^{2/3} N^{4/3} \sigma_1(N)^{-1/3} \varphi_1(N)^{-2/3} |m_1 m_2| \varphi_2(N)^2 N^{-2}.$$

We can now finish our estimates for I_6

$$|I_6(x)| \lesssim \sum_{0 < |m_1|, |m_2| \le M} \max\{m_1, m_2\}^{2/3} N^{4/3} \sigma_1(N)^{-1/3} \varphi_1(N)^{-2/3} N^{-2}$$

$$= \frac{\sum_{0 < |m_1|, |m_2| \le M} \max\{m_1, m_2\}^{2/3}}{\varphi_1(N)^{2/3} N^{2/3} \sigma_1(N)^{1/3}}$$

$$\lesssim \frac{M^{8/3}}{\varphi_1(N)^{2/3} N^{2/3} \sigma_1(N)^{1/3}} \lesssim_{\varepsilon_1} \frac{N^{8/3 + 16/3\chi + 8/3\varepsilon}}{\varphi_1(N)^{10/3} N^{2/3} N^{-\varepsilon_1}}$$

$$\lesssim_{\varepsilon_1} \frac{N^{8/3 + 16/3\chi + 8/3\varepsilon}}{N^{10/3\gamma - \varepsilon_1} N^{2/3} N^{-\varepsilon_1}}.$$

We wish to have that $8/3 + 16/3\chi + 8/3\varepsilon - 10/3\gamma + 2\varepsilon_1 - 2/3 \le -1 - \chi$, but we have that

$$8/3 + 16/3\chi + 8/3\varepsilon - 10/3\gamma + 2\varepsilon_1 - 2/3 \le -1 - \chi \iff$$
$$10(1 - \gamma) + 19\chi + 8\varepsilon + 6\varepsilon_1 \le 1,$$

and we can choose $\varepsilon_1 > 0$ to make this true. $\qquad\square$

We use Lemma 5.1 and Lemma 5.2 to prove the weak type (1,1) estimates of Theorem 1.5. We state and prove a general Theorem that allows us to conclude. It is a natural extension of Theorem 6.1 in [20], and the novelty lies in our handling of the problematic initial part of $K_N * \widetilde{K}_N$.

Theorem 5.5 *Let $\mathcal{M}(f)(x) = \sup_{n \in \mathbb{N}} |K_n * f(x)|$ be the maximal function corresponding to a family of nonnegative kernels $(K_n)_{n \in \mathbb{N}} \subseteq \ell^1(\mathbb{Z})$ such that $\|\mathcal{M}(f)\|_{\ell^\infty(\mathbb{Z})} \lesssim \|f\|_{\ell^\infty(\mathbb{Z})}$ for all $f \in \ell^\infty(\mathbb{Z})$ and let $(F_n)_{n \in \mathbb{Z}}$ be a family of nonnegative functions. Assume that there are sequences $(d_n)_{n \in \mathbb{N}}, (D_n)_{n \in \mathbb{N}} \subseteq [1, \infty)$ such that $|supp(K_n)| = d_n$, $supp(K_n) \subseteq [0, D_n]$, $supp(F_n) \subseteq [-D_n, D_n]$, $d_n \leq D_n^{\varepsilon_0}$ for some $\varepsilon_0 \in (0, 1)$ and assume there exists a finite constant $M > 1$ such that $M d_n \leq d_{n+1}$ and $M D_n \leq D_{n+1} \leq 2^{n+1}$ for all $n \in \mathbb{N}$. Also, assume that exists a real number $\varepsilon_1 > 0$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}$ we have*

$$|K_n * \widetilde{K}_n(x) - F_n(x)| \lesssim D_n^{-1-\varepsilon_1}$$

and assume that there exists a constant $A > 0$ such that

$$
\begin{aligned}
F_n(x) &\gtrsim d_n^{-1} \text{ for all } x \text{ with } |x| \leq A \text{ and } |F_n(x)| \\
&\lesssim D_n^{-1} \text{ for all } x \text{ with } |x| > A.
\end{aligned}
\tag{5.6}
$$

Finally, assume that there exists an $\varepsilon_2 \in (0, 1]$ such that

$$|F_n(x + y) - F_n(x)| \lesssim D_n^{-2}|y| \text{ whenever } |x|, |x + y| \gtrsim d_n^{\varepsilon_2}. \tag{5.7}$$

Then we have that there exists a constant $C > 0$ such that

$$\|\mathcal{M}(f)\|_{\ell^{1,\infty}(\mathbb{Z})} \leq C \|f\|_{\ell^1(\mathbb{Z})} \text{ for all } f \in \ell^1(\mathbb{Z}).$$

Before proving the theorem let us briefly show how it implies the weak-type (1,1) bound.

Proof of Theorem 1.5 By letting $K_n(x) = \frac{1}{\varphi_2(2^n)} \sum_{k \in B} \delta_k(x) \eta\left(\frac{k}{2^n}\right)$, $d_n \simeq \varphi_1(2^n)$, $D_n \simeq 2^n$ we can apply the theorem for

$$
F_n(x) = \begin{cases} (K_n * \widetilde{K}_n)(x), & 0 \leq |x| \leq \varphi_1(2^n) \\ G_{2^n}(x), & |x| > \varphi_1(2^n) \end{cases}.
$$

Lemma 5.2 guarantees the existence of a real number $\varepsilon_1 > 0$ such that $|K_n * \widetilde{K}_n(x) - F_n(x)| \lesssim D_n^{-1-\varepsilon_1}$ for $|x| > \varphi_1(2^n)$ and for smaller values of x the estimate is trivially established from the definition of F_n. We also have

$$K_n * \widetilde{K}_n(x) = \frac{1}{\varphi_2(2^n)^2} \sum_{k \in \mathbb{N}} 1_B(k) 1_B(k + x) \eta\left(\frac{k}{2^n}\right) \eta\left(\frac{k + x}{2^n}\right)$$

and thus if C is the constant appearing in Lemma 5.1, for all integers x such that $|x| \leq C$, we get that

$$|F_n(x)| = |K_n * \widetilde{K}_n(x)| \lesssim \frac{1}{\varphi_1(2^n)} \simeq d_n^{-1}$$

and for all x such that $C \leq |x| \leq \varphi_1(2^n)$ we get

$$|F_n(x)| = |G_{2^n}(x)| \lesssim 2^{-n} \simeq D_n^{-1}.$$

We can conclude by letting $\varepsilon_2 = 1$, and using the estimates from Lemmas 5.1 and 5.2. This completes the proof. $\qquad\qquad\square$

Proof of Theorem 5.5 Let $f \in \ell^1(\mathbb{Z})$ such that $f \geq 0$ and let $\alpha > 0$. We will perform a subtle variation of the Calderón-Zygmund decomposition. There exists a family of disjoint dyadic cubes $(Q_{s,j})_{(s,j) \in \mathcal{B}}$, where $\mathcal{B} \subseteq \mathbb{N}_0 \times \mathbb{Z}$ and $Q_{s,j} = [j2^s, (j+1)2^s) \cap \mathbb{Z}$ and functions g, b such that

- $f = g + b$,
- $\|g\|_{\ell^1(\mathbb{Z})} \leq \|f\|_{\ell^1(\mathbb{Z})}$ and $\|g\|_{\ell^\infty(\mathbb{Z})} \leq 2\alpha$,
- $b = \sum_{(s,j) \in \mathcal{B}} b_{s,j}$ where $b_{s,j}$ is supported on $Q_{s,j}$,
- $\sum_{x \in Q_{s,j}} b_{s,j}(x) = 0$,
- $\|b_{s,j}\|_{\ell^1(\mathbb{Z})} \leq 4\alpha|Q_{s,j}|$,
- $\sum_{(s,j) \in \mathcal{B}} |Q_{s,j}| \leq \alpha^{-1} \|f\|_{\ell^1(\mathbb{Z})}$.

For every $s \geq 0$ we let

$$b_s = \sum_{\substack{j \in \mathbb{Z}: \\ (s,j) \in \mathcal{B}}} b_{s,j}$$

and for every $n \in \mathbb{N}_0$ we decompose further

- $b_s^n(x) = b_s(x) 1_{\{y \in \mathbb{Z}: |b_s(y)| > \alpha d_n\}}(x)$,
- $h_s^n(x) = b_s(x) - b_s^n(x) = b_s(x) 1_{\{y \in \mathbb{Z}: |b_s(y)| \leq \alpha d_n\}}(x)$,
- $g_s^n(x) = \sum_{\substack{j \in \mathbb{Z}: \\ (s,j) \in \mathcal{B}}} [h_s^n]_{Q_{s,j}} 1_{Q_{s,j}}$,
- $B_s^n(x) = h_s^n(x) - g_s^n(x) = \sum_{\substack{j \in \mathbb{Z}: \\ (s,j) \in \mathcal{B}}} (h_s^n - [h_s^n]_{Q_{s,j}}) 1_{Q_{s,j}}$.

Let $s(n) = \min\{s \in \mathbb{N}_0 : 2^s \geq D_n\}$ and decompose f as $g + \sum_{s \geq 0} b_s = g + \sum_{s \geq 0} (b_s^n + g_s^n + B_s^n) = \left(g + \sum_{s \geq 0} g_s^n\right) + \sum_{s \geq 0} b_s^n + \sum_{s=0}^{s(n)-1} B_s^n + \sum_{s=s(n)}^{\infty} B_s^n$. We have that

$$\left|\left\{x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} |K_n * f(x)| > C\alpha\right\}\right|$$

$$\leq \left|\left\{x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} \left|K_n * \left(g + \sum_{s \geq 0} g_s^n\right)(x)\right| > C\alpha/4\right\}\right|$$

$$+ \left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} |K_n * \left(\sum_{s \geq 0} b_s^n \right)(x)| > C\alpha/4 \right\} \right|$$

$$+ \left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} |K_n * \left(\sum_{s=s(n)}^{\infty} B_s^n \right)(x)| > C\alpha/4 \right\} \right|$$

$$+ \left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} |K_n * \left(\sum_{s=0}^{s(n)-1} B_s^n \right)(x)| > C\alpha/4 \right\} \right|$$

and our treatment will be different for each summand. The following subsections are devoted to this task, and the most difficult part will be to bound the final one, where we will exploit the cancellation properties of B_s^n together with the properties of F_n.

5.1 Estimates for the First Three Summands

For the good part we will use ℓ^∞-bounds together with the fact that $\|\mathcal{M}\|_{\ell^\infty \to \ell^\infty} = T < \infty$. Note that

$$\left| \sum_{s \geq 0} g_s^n(x) \right| \leq \sum_{(s,j) \in \mathcal{B}} |[h_s^n]_{Q_{s,j}} 1_{Q_{s,j}}(x)| \leq \sum_{(s,j) \in \mathcal{B}} [|h_s^n|]_{Q_{s,j}} 1_{Q_{s,j}}(x)$$

$$\leq \sum_{(s,j) \in \mathcal{B}} [|b_{s,j}|]_{Q_{s,j}} 1_{Q_{s,j}}(x)$$

$$\leq \sum_{(s,j) \in \mathcal{B}} \|b_{s,j}\|_{\ell^1(\mathbb{Z})} |Q_{s,j}|^{-1} 1_{Q_{s,j}}(x) \leq \sum_{(s,j) \in \mathcal{B}} 4\alpha 1_{Q_{s,j}}(x) \leq 4\alpha$$

and $g(x) \leq 2\alpha$ for all $x \in \mathbb{Z}$, and thus $\|g + \sum_{s \geq 0} g_s^n\|_{\ell^\infty(\mathbb{Z})} \leq 6\alpha$ and thus $\left| K_n * \left(g + \sum_{s \geq 0} g_s^n \right)(x) \right| \leq T 6\alpha$ for all $x \in \mathbb{Z}$ and $n \in \mathbb{N}$. Thus for any $C > 24T$ we get

$$\left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} |K_n * \left(g + \sum_{s \geq 0} g_s^n \right)(x)| > C\alpha/4 \right\} \right| = 0$$

since it is the empty set.

For the second summand we use the lacunary nature of $(d_n)_{n \in \mathbb{N}}$ as well as the bounds for the cardinality of the support of K_n. Specifically, we have

$$\left| \left\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} |K_n * \left(\sum_{s \geq 0} b_s^n \right)(x)| > C\alpha/4 \right\} \right|$$

$$\leq \left| \bigcup_{n \in \mathbb{N}} \bigcup_{s \in \mathbb{N}_0} \operatorname{supp} K_n * |b_s^n| \right| \leq \sum_{n \in \mathbb{N}} \sum_{s \in \mathbb{N}_0} |\operatorname{supp} K_n * |b_s^n||$$

$$\leq \sum_{n \in \mathbb{N}} \sum_{s \in \mathbb{N}_0} |\operatorname{supp} K_n| \cdot |\operatorname{supp} b_s^n| \leq \sum_{n \in \mathbb{N}} \sum_{s \in \mathbb{N}_0} d_n |\{ x \in \mathbb{Z} : |b_s(x)| > \alpha d_n \}|$$

$$= \sum_{n \in \mathbb{N}} \sum_{s \in \mathbb{N}_0} d_n \sum_{k \geq n} |\{ x \in \mathbb{Z} : \alpha d_{k+1} \geq |b_s(x)| > \alpha d_k \}|$$

$$
= \sum_{s \in \mathbb{N}_0} \sum_{k \in \mathbb{N}} \Big(\sum_{n=1}^{k} d_n \Big) |\{x \in \mathbb{Z} : \alpha d_{k+1} \geq |b_s(x)| > \alpha d_k \}| \lesssim_M
$$

$$
\alpha^{-1} \sum_{s \in \mathbb{N}_0} \sum_{k \in \mathbb{N}} \alpha d_k |\{x \in \mathbb{Z} : \alpha d_{k+1} \geq |b_s(x)| > \alpha d_k \}|
$$

$$
\leq \alpha^{-1} \sum_{s \in \mathbb{N}_0} \|b_s\|_{\ell^1(\mathbb{Z})} \lesssim \alpha^{-1} \|f\|_{\ell^1(\mathbb{Z})}.
$$

For the third summand we simply use the fact for any $n \in \mathbb{N}$ and any $s \geq s(n)$ we get that $2^s \geq D_n$ and thus

$$
\mathrm{supp}(K_n * B_s^n) \subseteq \mathrm{supp}(K_n) + \mathrm{supp}(B_s^n) \subseteq [0, D_n] + \bigcup_{\substack{j \in \mathbb{Z} \\ (s,j) \in \mathcal{B}}} Q_{s,j} \subseteq [0, 2^s]
$$

$$
+ \bigcup_{\substack{j \in \mathbb{Z} \\ (s,j) \in \mathcal{B}}} Q_{s,j} \subseteq \bigcup_{\substack{j \in \mathbb{Z} \\ (s,j) \in \mathcal{B}}} 3 Q_{s,j}
$$

where $3Q$ denotes the interval with the same center as Q and three times its radius. Therefore

$$
\Big| \Big\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} \Big| K_n * \Big(\sum_{s=s(n)}^{\infty} B_s^n \Big)(x) \Big| > C\alpha/4 \Big\} \Big|
$$

$$
= \Big| \bigcup_{n \in \mathbb{N}} \bigcup_{s \geq s(n)} \mathrm{supp}(K_n * B_s^n) \Big| \leq \Big| \bigcup_{n \in \mathbb{N}} \bigcup_{s \geq s(n)} \bigcup_{\substack{j \in \mathbb{Z} \\ (s,j) \in \mathcal{B}}} 3 Q_{s,j} \Big|
$$

$$
\leq \Big| \bigcup_{(s,j) \in \mathcal{B}} 3 Q_{s,j} \Big| \lesssim \alpha^{-1} \|f\|_{\ell^1(\mathbb{Z})}.
$$

5.2 Estimates for the Fourth Summand

The fourth summand is the most difficult to estimate and here we use the regularity of $K_n * \widetilde{K}_n$. We have

$$
\Big| \Big\{ x \in \mathbb{Z} : \sup_{n \in \mathbb{N}} \Big| K_n * \Big(\sum_{s=0}^{s(n)-1} B_s^n \Big)(x) \Big| > C\alpha/4 \Big\} \Big| \lesssim \alpha^{-2} \sum_{x \in \mathbb{Z}} \sup_{n \in \mathbb{N}} \Big| K_n * \Big(\sum_{s=0}^{s(n)-1} B_s^n \Big)(x) \Big|^2
$$

$$
\leq \alpha^{-2} \sum_{x \in \mathbb{Z}} \sum_{n \in \mathbb{N}} \Big| K_n * \Big(\sum_{s=0}^{s(n)-1} B_s^n \Big)(x) \Big|^2 = \alpha^{-2} \sum_{n \in \mathbb{N}} \Big\| \sum_{s=0}^{s(n)-1} K_n * B_s^n \Big\|_{\ell^2(\mathbb{Z})}^2
$$

$$
= \alpha^{-2} \sum_{n \in \mathbb{N}} \Big(\sum_{s=0}^{s(n)-1} \| K_n * B_s^n \|_{\ell^2(\mathbb{Z})}^2 + 2 \sum_{0 \leq s_1 < s_2 \leq s(n)-1} \langle K_n * B_{s_1}^n, K_n * B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \Big).
$$

We need the following result to conclude.

Claim There exists $0 < \lambda < 1$ such that for all $n \in \mathbb{N}$ and $0 \le s_1 \le s_2 \le s(n) - 1$ we get

$$
\left| \langle K_n * B_{s_1}^n, K_n * B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| \lesssim \lambda^{s(n) - s_1} \alpha \| B_{s_2}^n \|_{\ell^1(\mathbb{Z})}
$$
$$
+ d_n^{-1} \sum_{|j| \le A} |\langle \delta_j * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})}|. \tag{5.8}
$$

Assuming that (5.8) holds, let us see how we can deduce the desired estimate. We have

$$
\alpha^{-2} \sum_{n \in \mathbb{N}} \left(\sum_{s=0}^{s(n)-1} \| K_n * B_s^n \|_{\ell^2(\mathbb{Z})}^2 + 2 \sum_{0 \le s_1 < s_2 \le s(n)-1} \langle K_n * B_{s_1}^n, K_n * B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right)
$$

$$
\lesssim \alpha^{-2} \sum_{n \in \mathbb{N}} \sum_{s=0}^{s(n)-1} \left(\lambda^{s(n)-s} \alpha \| B_s^n \|_{\ell^1(\mathbb{Z})} + d_n^{-1} \sum_{|j| \le A} |\langle \delta_j * B_s^n, B_s^n \rangle_{\ell^2(\mathbb{Z})}| \right)
$$

$$
+ \alpha^{-2} \sum_{n \in \mathbb{N}} \sum_{0 \le s_1 < s_2 \le s(n)-1} \left(\lambda^{s(n)-s_1} \alpha \| B_{s_2}^n \|_{\ell^1(\mathbb{Z})} \right.
$$

$$
\left. + d_n^{-1} \sum_{|j| \le A} |\langle \delta_j * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})}| \right)
$$

$$
\lesssim \alpha^{-1} \sum_{n \in \mathbb{N}} \sum_{s=0}^{s(n)-1} \lambda^{s(n)-s} \| B_s^n \|_{\ell^1(\mathbb{Z})} + \alpha^{-2} \sum_{n \in \mathbb{N}} \sum_{s=0}^{s(n)-1} d_n^{-1}
$$

$$
\sum_{|j| \le A} |\langle \delta_j * B_s^n, B_s^n \rangle_{\ell^2(\mathbb{Z})}|
$$

$$
+ \alpha^{-1} \sum_{n \in \mathbb{N}} \sum_{0 \le s_1 < s_2 \le s(n)-1} \lambda^{s(n)-s_1} \| B_{s_2}^n \|_{\ell^1(\mathbb{Z})} + \alpha^{-2} \sum_{n \in \mathbb{N}}
$$

$$
\sum_{0 \le s_1 < s_2 \le s(n)-1} d_n^{-1} \sum_{|j| \le A} |\langle \delta_j * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})}|.
$$

For the first and the third term note that

$$
\alpha^{-1} \sum_{n \in \mathbb{N}} \sum_{0 \le s_1 < s_2 \le s(n)-1} \lambda^{s(n)-s_1} \| B_{s_2}^n \|_{\ell^1(\mathbb{Z})}
$$

$$
= \alpha^{-1} \sum_{n \in \mathbb{N}} \sum_{1 \le s_2 \le s(n)-1} \| B_{s_2}^n \|_{\ell^1(\mathbb{Z})} \left(\sum_{0 \le s_1 \le s_2 - 1} \lambda^{s(n)-s_1} \right)
$$

$$
\lesssim \alpha^{-1} \sum_{n \in \mathbb{N}} \sum_{1 \le s_2 \le s(n)-1} \lambda^{s(n)-s_2} \| B_{s_2}^n \|_{\ell^1(\mathbb{Z})} \le \alpha^{-1} \sum_{n \in \mathbb{N}} \sum_{s=0}^{s(n)-1} \lambda^{s(n)-s} \| B_s^n \|_{\ell^1(\mathbb{Z})} \lesssim_\lambda
$$

$$\alpha^{-1} \sum_{n\in\mathbb{N}} \sum_{s=0}^{s(n)} \lambda^s \|B^n_{s(n)-s-1}\|_{\ell^1(\mathbb{Z})} \lesssim \alpha^{-1} \sum_{s\in\mathbb{N}_0} \lambda^s \sum_{(t,j)\in\mathcal{B}} \|b_{t,j}\|_{\ell^1(\mathbb{Z})}$$

$$\lesssim \alpha^{-1} \sum_{(t,j)\in\mathcal{B}} 4\alpha |Q_{t,j}| \lesssim \alpha^{-1}\|f\|_{\ell^1(\mathbb{Z})}$$

where we have used the estimates from the Calderón-Zygmund decomposition in the beginning of the proof.

The fourth term is bounded as follows.

$$\alpha^{-2} \sum_{n\in\mathbb{N}} \sum_{0\le s_1 < s_2 \le s(n)-1} d_n^{-1} \sum_{1\le |j| \le A} |\langle \delta_j * B^n_{s_1}, B^n_{s_2}\rangle_{\ell^2(\mathbb{Z})}|$$

$$\le \alpha^{-2} \sum_{1\le |j| \le A} \sum_{n\in\mathbb{N}} \sum_{0\le s_1 < s_2 \le s(n)-1} d_n^{-1} \sum_{x\in\mathbb{Z}} |B^n_{s_1}(x-j)||B^n_{s_2}(x)|$$

$$= \alpha^{-2} \sum_{x\in\mathbb{Z}} \sum_{1\le |j| \le A} \sum_{n\in\mathbb{N}} \sum_{0\le s_1 < s_2 \le s(n)-1} d_n^{-1} |B^n_{s_1}(x-j)||B^n_{s_2}(x)|.$$

Fix $|j| \le A$ and $x \in \mathbb{Z}$ such that $x - j \in \mathrm{supp}(b_{s_0})$ for some integer s_0. Since the supports of b_s's are disjoint we have that there can be at most one integer s_0' such that $x \in \mathrm{supp}(b_{s_0'})$. Note also that $[|h^n_s|]_{Q_{s,j}} \lesssim \alpha$, as we observed earlier. We have

$$\sum_{n\in\mathbb{N}} \sum_{0\le s_1 < s_2 \le s(n)-1} d_n^{-1} |B^n_{s_1}(x-j)||B^n_{s_2}(x)| \le \sum_{n\in\mathbb{N}} d_n^{-1} |B^n_{s_0}(x)||B^n_{s_0'}(x-j)|$$

$$\lesssim \sum_{n\in\mathbb{N}} d_n^{-1}\Big(|b_{s_0}(x)|1_{\{y\in\mathbb{Z}:\, |b_{s_0}(y)|\le \alpha d_n\}}(x) + \alpha 1_{\mathrm{supp}\, b_{s_0}}(x)\Big)$$

$$\Big(|b_{s_0'}(x-j)|1_{\{y\in\mathbb{Z}:\, |b_{s_0'}(y)|\le \alpha d_n\}}(x-j) + \alpha 1_{\mathrm{supp}\, b_{s_0'}}(x-j)\Big)$$

$$\le \sum_{\substack{n\in\mathbb{N}: \\ d_n \ge |b_{s_0}(x)|/\alpha \\ d_n \ge |b_{s_0'}(x-j)|/\alpha}} d_n^{-1} |b_{s_0}(x)||b_{s_0'}(x-j)| + \alpha^2 \sum_{n\in\mathbb{N}} d_n^{-1} 1_{\mathrm{supp}\, b_{s_0}}(x)$$

$$+ \sum_{\substack{n\in\mathbb{N}: \\ d_n \ge |b_{s_0}(x)|/\alpha}} d_n^{-1} |b_{s_0}(x)| \alpha 1_{\mathrm{supp}\, b_{s_0'}}(x-j)$$

$$+ \sum_{\substack{n\in\mathbb{N}: \\ d_n \ge |b_{s_0'}(x-j)|/\alpha}} d_n^{-1} |b_{s_0'}(x-j)| \alpha 1_{\mathrm{supp}\, b_{s_0}}(x)$$

$$\lesssim \max\{|b_{s_0}(x)|^2, |b_{s_0'}(x-j)|^2\} \min\{\alpha/|b_{s_0}(x)|, \alpha/|b_{s_0'}(x-j)|\} + \alpha^2 1_{\mathrm{supp}\, b_{s_0}}(x)$$

$$\le \sum_{s\in\mathbb{N}_0} \alpha |b_s(x)| + \alpha |b_s(x-j)| + \sum_{s\in\mathbb{N}_0} \alpha^2 1_{\mathrm{supp}\, b_s}(x) \tag{5.9}$$

where we have used the existence of a finite constant $M > 1$ such that $M d_n \leq d_{n+1}$. We get that

$$
\begin{aligned}
&\alpha^{-2} \sum_{n \in \mathbb{N}} \sum_{0 \leq s_1 < s_2 \leq s(n)-1} d_n^{-1} \sum_{1 \leq |j| \leq A} |\langle \delta_j * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})}| \\
&\lesssim \alpha^{-2} \sum_{x \in \mathbb{Z}} \sum_{1 \leq |j| \leq A} \left(\sum_{s \in \mathbb{N}_0} \alpha |b_s(x)| + \alpha |b_s(x-j)| + \sum_{s \in \mathbb{N}_0} \alpha^2 1_{\operatorname{supp} b_s}(x) \right) \\
&\lesssim \alpha^{-1} \sum_{1 \leq |j| \leq A} \sum_{s \in \mathbb{N}_0} 2 \|b_s\|_{\ell^1(\mathbb{Z})} + A \sum_{s \in \mathbb{N}_0} |\operatorname{supp} b_s| \\
&\lesssim \alpha^{-1} A \|b\|_{\ell^1(\mathbb{Z})} + A \left| \bigcup_{(s,j) \in \mathcal{B}} Q_{s,j} \right| \lesssim \alpha^{-1} \|f\|_{\ell^1(\mathbb{Z})}.
\end{aligned}
$$

A similar argument may be used to bound the second term. For the sake of completeness we note that

$$
\begin{aligned}
&\alpha^{-2} \sum_{n \in \mathbb{N}} \sum_{s=0}^{s(n)-1} d_n^{-1} \sum_{1 \leq |j| \leq A} |\langle \delta_j * B_s^n, B_s^n \rangle_{\ell^2(\mathbb{Z})}| \\
&\leq \alpha^{-2} \sum_{1 \leq |j| \leq A} \sum_{n \in \mathbb{N}} \sum_{s=0}^{s(n)-1} d_n^{-1} \sum_{x \in \mathbb{Z}} |B_s^n(x-j)||B_s^n(x)| \\
&= \alpha^{-2} \sum_{x \in \mathbb{Z}} \sum_{1 \leq |j| \leq A} \sum_{n \in \mathbb{N}} \sum_{s=0}^{s(n)-1} d_n^{-1} |B_s^n(x-j)||B_s^n(x)|.
\end{aligned}
$$

Fix $|j| \leq A$ and $x \in \mathbb{Z}$ such that $x - j \in \operatorname{supp}(b_{s_0})$ for some integer s_0. Since the supports of b_s's are disjoint we have that there can be at most one integer s_0' such that $x \in \operatorname{supp}(b_{s_0'})$. We have

$$
\sum_{n \in \mathbb{N}} \sum_{s=0}^{s(n)-1} d_n^{-1} |B_s^n(x-j)||B_s^n(x)| \leq \sum_{n \in \mathbb{N}} d_n^{-1} |B_{s_0}^n(x)||B_{s_0'}^n(x-j)|.
$$

By comparing the inequality above with (5.9), we see that an argument identical to the one used previously may be used here. The proof will be completed once we have established the estimate (5.8) of the claim. We do this in the following subsection.

5.3 Proof of the Estimate (5.8)

Let $n \in \mathbb{N}$ and $0 \leq s_1 \leq s_2 \leq s(n) - 1$ and let us note that

$$
\begin{aligned}
\left| \langle K_n * B_{s_1}^n, K_n * B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| &= \left| \langle K_n * \widetilde{K}_n * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| \\
&\leq \left| \langle F_n * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| \\
&\quad + \left| \langle (K_n * \widetilde{K}_n - F_n) * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right|.
\end{aligned}
$$

Now decompose $F_n(x) = F_n(x)1_{|x| \leq A} + F_n(x)1_{|x| > A} = \sum_{-A \leq j \leq A} F_n(j)\delta_j(x) + F_n(x)1_{|x| > A}$ and let $G_n(x) = F_n(x)1_{|x| > A}$. We obtain

$$
\begin{aligned}
\left| \langle K_n * B_{s_1}^n, K_n * B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| &\leq \sum_{-A \leq j \leq A} |F_n(j)| \left| \langle \delta_j * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| \\
&\quad + \left| \langle G_n * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| \\
&\quad + \left| \langle (K_n * \widetilde{K}_n - F_n) * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| \\
&\lesssim d_n^{-1} \sum_{|j| \leq A} \left| \langle \delta_j * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| \\
&\quad + \left| \langle G_n * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| + \\
&\quad + \left| \langle (K_n * \widetilde{K}_n - F_n) * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right|.
\end{aligned}
$$

In the right hand side of our inequality one of the terms of the desired estimate already appeared and thus we can now focus on the other two summands. Let $Z_{m,n} = \left[m2^{s(n)}, (m+1)2^{s(n)} \right) \cap \mathbb{Z}$, $\widetilde{Z}_{m,n} = \left[(m-1)2^{s(n)}, (m+2)2^{s(n)} \right) \cap \mathbb{Z}$ and $E_n = K_n * \widetilde{K}_n - F_n$. Note that $\mathrm{supp}(E_n) \subseteq [-D_n, D_n] \subseteq [-2^{s(n)}, 2^{s(n)}]$. We have that

$$
\begin{aligned}
\left| \langle (K_n * \widetilde{K}_n - F_n) * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| &= \left| \sum_{x \in \mathbb{Z}} (E_n * B_{s_1}^n)(x) B_{s_2}^n(x) \right| \\
&= \left| \sum_{x \in \mathbb{Z}} \left(\sum_{y \in \mathbb{Z}} E_n(y) B_{s_1}^n(x-y) \right) B_{s_2}^n(x) \right| \\
&\leq \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} |E_n(y) B_{s_1}^n(x-y) B_{s_2}^n(x)| \leq \sum_{y \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \sum_{x \in Z_{m,n}} 1_{\widetilde{Z}_{m,n}}(x-y) \\
&\quad |E_n(y) B_{s_1}^n(x-y) B_{s_2}^n(x)| \\
&\leq D_n^{-1-\varepsilon_1} \sum_{m \in \mathbb{Z}} \sum_{x \in Z_{m,n}} |B_{s_2}^n(x)| \left(\sum_{y \in \mathbb{Z}} 1_{\widetilde{Z}_{m,n}}(x-y) |B_{s_1}^n(x-y)| \right) \\
&\leq D_n^{-1-\varepsilon_1} \| B_{s_2}^n \|_{\ell^1(\mathbb{Z})} \sup_{m \in \mathbb{Z}} \| B_{s_1}^n 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})}.
\end{aligned}
$$

Now we note that for any $m \in \mathbb{Z}$ we have

$$
\begin{aligned}
\| B_{s_1}^n 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})} &= \sum_{\substack{k \in \mathbb{Z} \\ (s_1,k) \in \mathcal{B}}} \sum_{x \in \mathbb{Z}} |B_{s_1}^n 1_{\widetilde{Z}_{m,n}}(x) 1_{Q_{s_1,k}}(x)| \\
&\leq \sum_{\substack{k \in \mathbb{Z}: (s_1,k) \in \mathcal{B} \\ Q_{s_1,k} \cap \widetilde{Z}_{m,n} \neq \emptyset}} \| B_{s_1}^n 1_{Q_{s_1,k}} \|_{\ell^1(\mathbb{Z})}. \tag{5.10}
\end{aligned}
$$

On the one hand

$$\|B_{s_1}^n 1_{Q_{s_1,k}}\|_{\ell^1(\mathbb{Z})} \le \|h_{s_1}^n 1_{Q_{s_1,k}}\|_{\ell^1(\mathbb{Z})} + \|[h_{s_1}^n]_{Q_{s_1,k}} 1_{Q_{s_1,k}}\|_{\ell^1(\mathbb{Z})}$$
$$\le 2\|b_{s_1} 1_{Q_{s_1,k}}\|_{\ell^1(\mathbb{Z})} \le 4\alpha|Q_{s_1,k}| \le 4\alpha 2^{s_1} \qquad (5.11)$$

and on the other hand $|\{k \in \mathbb{Z} : (s_1,k) \in \mathcal{B} \ \& \ Q_{s_1,k} \cap \widetilde{Z}_{m,n} \ne \emptyset\}| \lesssim 2^{s(n)-s_1}$ since $\widetilde{Z}_{m,n}$ can be partitioned into 3 dyadic intervals, $Q_{s_1,k}$ is a dyadic interval and $s_1 \le s(n)$. Therefore

$$\|B_{s_1}^n 1_{\widetilde{Z}_{m,n}}\|_{\ell^1(\mathbb{Z})} \lesssim 2^{s(n)-s_1} 4\alpha 2^{s_1} = 8\alpha 2^{s(n)-1} \le 8\alpha D_n, \qquad (5.12)$$

and finally

$$\left|\langle (K_n * \widetilde{K}_n - F_n) * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})}\right| \lesssim D_n^{-\varepsilon_1} 8\alpha \|B_{s_2}^n\|_{\ell^1(\mathbb{Z})}$$
$$\lesssim 2^{-\varepsilon_1(s(n)-s_1)}\alpha \|B_{s_2}^n\|_{\ell^1(\mathbb{Z})},$$

since $D_n \ge 2^{s(n)-1} \ge 2^{s(n)-s_1-1}$, we get

$$\left|\langle (K_n * \widetilde{K}_n - F_n) * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})}\right| \lesssim (2^{-\varepsilon_1})^{s(n)-s_1}\alpha \|B_{s_2}^n\|_{\ell^1(\mathbb{Z})}$$

as desired.

Now we focus on the last term $\left|\langle G_n * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})}\right|$. Note that $\text{supp}(G_n) \subseteq [-2^{s(n)}, 2^{s(n)}]$. We have

$$\left|\langle G_n * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})}\right| = \left|\sum_{x \in \mathbb{Z}} (G_n * B_{s_1}^n)(x) B_{s_2}^n(x)\right|$$
$$= \left|\sum_{x \in \mathbb{Z}} \left(\sum_{y \in \mathbb{Z}} G_n(y) B_{s_1}^n(x-y)\right) B_{s_2}^n(x)\right|$$
$$\le \left|\sum_{m \in \mathbb{Z}} \sum_{x \in Z_{m,n}} \left(\sum_{y \in \mathbb{Z}} G_n(y) B_{s_1}^n(x-y) 1_{\widetilde{Z}_{m,n}}(x-y)\right) B_{s_2}^n(x)\right|$$
$$\le \sum_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} \left|G_n * (B_{s_1}^n 1_{\widetilde{Z}_{m,n}})(x)\right| \sum_{x \in Z_{m,n}} |B_{s_2}^n(x)|$$
$$\le \sup_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} \left|G_n * (B_{s_1}^n 1_{\widetilde{Z}_{m,n}})(x)\right| \|B_{s_2}^n(x)\|_{\ell^1(\mathbb{Z})}.$$

Let us define $B_{s,j}^n = B_s^n 1_{Q_{s,j}}$, and note that for any $m \in \mathbb{Z}$, $x \in Z_{m,n}$, we have

$$\left|G_n * (B_{s_1}^n 1_{\widetilde{Z}_{m,n}})(x)\right| \le \sum_{\substack{j \in \mathbb{Z}: \\ (s_1,j) \in \mathcal{B}}} \left|G_n * (B_{s_1,j}^n 1_{\widetilde{Z}_{m,n}})(x)\right|, \text{ since } \sum_{\substack{j \in \mathbb{Z}: \\ (s_1,j) \in \mathcal{B}}} B_{s,j}^n = B_s^n.$$

We also note that $\sum_{x \in \mathbb{Z}} B_{s_1,j}^n(x) 1_{\widetilde{Z}_{m,n}}(x) = 0$. To see this note that if $\text{supp}(B_{s_1,j}^n) \cap \widetilde{Z}_{m,n} = \emptyset$, then it is trivial, and if they intersect, we must have that $\text{supp}(B_{s_1,j}^n) \subseteq \widetilde{Z}_{m,n}$,

since $\operatorname{supp}(B^n_{s_1,j}) \subseteq Q_{s_1,j}$ which is a dyadic interval of length 2^{s_1} and $\widetilde{Z}_{m,n}$ is the union of three dyadic intervals of larger length. In the second case we get

$$\sum_{x \in \mathbb{Z}} B^n_{s_1,j}(x) 1_{\widetilde{Z}_{m,n}}(x) = \sum_{x \in \mathbb{Z}} B^n_{s_1,j}(x) = 0$$

from the definition of $B^n_{s_1,j}(x)$. Fix $m \in \mathbb{Z}$ and $j \in \mathbb{Z}$ such that $(s_1, j) \in \mathcal{B}$ and let $x_{s_1,j}$ be the center of the cube $Q_{s_1,j}$. Assume $x \in Z_{m,n}$ is such that $|x - x_{s_1,j}| \geq Cd_n^{\varepsilon_2} + C2^{s_1}$. Using the cancellation property we have established together with the regularity assumptions for F_n, we get

$$\left| G_n * \left(B^n_{s_1,j} 1_{\widetilde{Z}_{m,n}} \right)(x) \right| = \left| \sum_{y \in \mathbb{Z}} \left(G_n(x - y) - G_n(x - x_{s_1,j}) \right) B^n_{s_1,j}(y) 1_{\widetilde{Z}_{m,n}}(y) \right|$$

$$\lesssim \sum_{y \in \mathbb{Z}} D_n^{-2} |y - x_{s_1,j}| |B^n_{s_1,j}(y) 1_{\widetilde{Z}_{m,n}}(y)|$$

$$\lesssim D_n^{-2} 2^{s_1} \| B^n_{s_1,j} 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})}.$$

Here we have used the fact that $|x - y| \geq |x - x_{s_1,j}| - |x_{s_1,j} - y| \geq Cd_n^{\varepsilon_2} + C2^{s_1} - 2^{s_1} \gtrsim d_n^{\varepsilon_2}$ and thus we may use (5.7). Taking into account (5.6) and the definition of G_n, we get that for any $x \in \mathbb{Z}$

$$\left| G_n * \left(B^n_{s_1,j} 1_{\widetilde{Z}_{m,n}} \right)(x) \right| \lesssim D_n^{-1} \| B^n_{s_1,j} 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})}.$$

Now we may estimate as follows

$$\sup_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} \left| G_n * \left(B^n_{s_1} 1_{\widetilde{Z}_{m,n}} \right)(x) \right|$$

$$\leq \sup_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} \sum_{\substack{j \in \mathbb{Z}: \\ (s_1,j) \in \mathcal{B}}} \left| G_n * \left(B^n_{s_1,j} 1_{\widetilde{Z}_{m,n}} \right)(x) \right|$$

$$\leq \sup_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} \sum_{\substack{j \in \mathbb{Z}: (s_1,j) \in \mathcal{B} \\ |x - x_{s_1,j}| \geq Cd_n^{\varepsilon_2} + C2^{s_1}}} D_n^{-2} 2^{s_1} \| B^n_{s_1,j} 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})}$$

$$+ \sup_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} \sum_{\substack{j \in \mathbb{Z}: (s_1,j) \in \mathcal{B} \\ |x - x_{s_1,j}| < Cd_n^{\varepsilon_2} + C2^{s_1}}} D_n^{-1} \| B^n_{s_1,j} 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})}.$$

For the first summand note that

$$\sup_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} \sum_{\substack{j \in \mathbb{Z}: (s_1,j) \in \mathcal{B} \\ |x - x_{s_1,j}| \geq Cd_n^{\varepsilon_2} + C2^{s_1}}} D_n^{-2} 2^{s_1} \| B^n_{s_1,j} 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})}$$

$$\leq D_n^{-2} 2^{s_1} \sup_{m \in \mathbb{Z}} \sum_{j \in \mathbb{Z}: (s_1,j) \in \mathcal{B}} \| B^n_{s_1,j} 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})}$$

$$\leq D_n^{-2} 2^{s_1} \sup_{m \in \mathbb{Z}} \| B_{s_1}^n 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})}.$$

The calculations from (5.10), (5.12) show that $\sup_{m \in \mathbb{Z}} \| B_{s_1}^n 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})} \lesssim \alpha D_n$ and thus the first summand is bounded by a constant multiple of

$$\alpha D_n^{-1} 2^{s_1} \leq \alpha 2^{1-s(n)} 2^{s_1} \lesssim \alpha (1/2)^{s(n)-s_1}.$$

For the second summand we consider two cases. In the first case we assume that $2^{s_1} \leq d_n^{\varepsilon_2}$. Then, any interval of radius $\lesssim d_n^{\varepsilon_2}$ contains at most $\lesssim 2^{-s_1} d_n^{\varepsilon_2}$ sets of the form $Q_{s_1,j}$. Thus we have

$$
\begin{aligned}
&\sup_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} \sum_{\substack{j \in \mathbb{Z}: (s_1,j) \in \mathcal{B} \\ |x - x_{s_1,j}| < C d_n^{\varepsilon_2} + C 2^{s_1}}} D_n^{-1} \| B_{s_1,j}^n 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})} \\
&\lesssim \sup_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} |\{ j \in \mathbb{Z}: (s_1,j) \in \mathcal{B} \text{ and } |x - x_{s_1,j}| < 2 C d_n^{\varepsilon_2} \}| D_n^{-1} \alpha 2^{s_1} \\
&\lesssim 2^{-s_1} d_n^{\varepsilon_2} D_n^{-1} \alpha 2^{s_1} \\
&\lesssim \frac{\alpha d_n}{D_n^{\varepsilon_0} D_n^{1-\varepsilon_0}} \lesssim \frac{\alpha}{(2^{s(n)})^{1-\varepsilon_0}} \lesssim \alpha (1/2^{1-\varepsilon_0})^{s(n)-s_1}
\end{aligned}
$$

where we used the estimate (5.11), the fact that $d_n \leq D_n^{\varepsilon_0}$ and the fact that $2^{s(n)-1} < D_n \leq 2^{s(n)}$. We have established the appropriate bound for the first case.

In the second case, we assume that $2^{s_1} > d_n^{\varepsilon_2}$. In that case, any interval of radius $\lesssim 2^{s_1}$ contains at most $\lesssim 2^{s_1} 2^{-s_1} = 1$ sets of the form $Q_{s_1,j}$. Thus we have

$$
\begin{aligned}
&\sup_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} \sum_{\substack{j \in \mathbb{Z}: (s_1,j) \in \mathcal{B} \\ |x - x_{s_1,j}| < C d_n^{\varepsilon_2} + C 2^{s_1}}} D_n^{-1} \| B_{s_1,j}^n 1_{\widetilde{Z}_{m,n}} \|_{\ell^1(\mathbb{Z})} \\
&\lesssim \sup_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} |\{ j \in \mathbb{Z}: (s_1,j) \in \mathcal{B} \text{ and } |x - x_{s_1,j}| < 2 C 2^{s_1} \}| D_n^{-1} \alpha 2^{s_1} \\
&\lesssim D_n^{-1} \alpha 2^{s_1} \lesssim \alpha (1/2)^{s(n)-s_1}.
\end{aligned}
$$

This concludes the second case.

Combining everything we get

$$
\begin{aligned}
\left| \langle G_n * B_{s_1}^n, B_{s_2}^n \rangle_{\ell^2(\mathbb{Z})} \right| &\lesssim \sup_{m \in \mathbb{Z}} \sup_{x \in Z_{m,n}} \left| G_n * \left(B_{s_1}^n 1_{\widetilde{Z}_{m,n}} \right)(x) \right| \| B_{s_2}^n(x) \|_{\ell^1(\mathbb{Z})} \\
&\lesssim \left(2^{\varepsilon_0 - 1} \right)^{s(n)-s_1} \| B_{s_2}^n(x) \|_{\ell^1(\mathbb{Z})}
\end{aligned}
$$

since $2^{-1} \lesssim 2^{\varepsilon_0 - 1}$. For $\lambda = \min\{2^{-\varepsilon_1}, 2^{\varepsilon_0 - 1}\} \in (0,1)$, we obtain the estimate (5.8) and the proof of Theorem 5.5 is complete. $\square$

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